Inequalities

Markov's Inequality (only for $X \geq 0$)

$$P(X \geq a) = \frac{E(X)}{a}$$

Cauchy-Schwarz Inequality

$$(E(XY))^2 \leq E(X^2)E(Y^2)$$

Chebyshev's Inequality

$$P(|X-\mu| \geq a) \leq \frac{V(X)}{a^2}$$

Jensen's Inequality (for convex $g(\cdot)$, i.e. concave up)

$$E(g(X)) \geq g(E(X))$$

Q) A red-lipped batfish has an expected lifespan of 1000hrs. Its lifespan has a standard deviation of 50hrs. Find the tightest upper bound you can on the probability that a red-lipped batfish will live 1250hrs or more.

let X be the lifespan of a red-lipped batfish

start with Markov's inequality

$$E(X) = 1000$$

so $P(X \geq 1250) \leq \frac{1000}{1250} = 0.8$

next we try Chebyshev's inequality

$$\mu = 1000 \qquad V(X) = \sigma^2 = 50^2 = 2500$$

$$P(X \geq 1250) \leq P(|X-1000| \leq 250) \leq \frac{2500}{250^2} = 0.04$$

the tighest bound we found was $P(X \geq 1250) \leq 0.04$ (from Chebyshev's)

Moment Generating Function

Formulas:

$\underbrace{m(t)=E(e^{tX})}_{\text{find the MGF}}$ $\qquad$ $\underbrace{E(X^k)=m^{(k)}(0)}_{\text{use MGF to find } E(X^k)}$

MGF Method Steps:

Show X, Y have the same MGF, ie. $m_X(t)=m_Y(t)$

Then $X \sim Y$

Q1) let $X \sim \text{Gamma}(a,\lambda)$

find $E(X)$ using its MGF

$$X \sim \text{Gamma}(a,\lambda) \Rightarrow m_X(t)=\left(\frac{\lambda}{\lambda-t}\right)^a=\frac{\lambda^a}{(\lambda-t)^a}$$

$$\Rightarrow m^{(1)}(t)=m'(t)=\frac{0(\lambda-t)^a-\lambda^a\cdot a(\lambda-t)^{a-1}(-1)}{(\lambda-t)^{2a}}=\frac{a\lambda^a(\lambda-t)^a(\lambda-t)^{-1}}{(\lambda-t)^a(\lambda-t)^a}=\frac{a\lambda^a}{(\lambda-t)^{a+1}}$$

$$\Rightarrow E(X^1)=m^{(1)}(0)=\frac{a\lambda^a}{\lambda^{a+1}}=\boxed{\frac{a}{\lambda}}$$

Q2) let $X_1,\dots,X_n \overset{iid}{\sim} \text{Bernoulli}(p)$

let $Y \sim \text{Binomial}(n,p)$

prove that $\sum_{i=1}^n X_i \sim Y$ using MGF method

$$X_i \sim \text{Bernoulli}(p) \Rightarrow m_{X_i}(t)=E(e^{tX_i})=\sum_{x=0}^{1} e^{tx}p_X(x)=e^{t\cdot 0}(1-p)+e^{t\cdot 1}p=1-p+pe^t$$

$$m_{\sum_{i=1}^n X_i}(t)=\prod_{i=1}^n m_{X_i}(t)=\prod_{i=1}^n (1-p+pe^t)=(1-p+pe^t)^n$$

(by indep.)

$$Y \sim \text{Binomial}(n,p) \Rightarrow m_Y(t)=(1-p+pe^t)^n$$

since $m_{\sum_{i=1}^n X_i}(t)=m_Y(t)$ (same MGF) $\therefore \sum_{i=1}^n X_i \sim Y$

Central Limit Theorem

Simplified CLT:

IF RVs $X_1, \dots, X_n$ are i.i.d

THEN $S = X_1 + \dots + X_n \sim \text{Normal}(\mu, \sigma^2)$

SUPER IMPORTANT CLT RESULT

① $\bar{X}_n = \frac{X_1 + \dots + X_n}{n} \sim \text{Normal}(\mu, \sigma^2/n)$ ② $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \overset{\text{appr.}}{=} Z \sim \text{Normal}(0,1)$

Cauchy is a French mathematician taking a 36-week-long probability course under prof. Sotirios. Sotirios gives him weekly quizzes scored out of 20pts.

Since each quiz is the same difficulty and Cauchy is locked in for the whole semester, Cauchy's quiz scores X_i are independent with mean 15 and standard deviation 3.

Using CLT, approximate the probability that his avg quiz score $\bar{X}_{36}$ is between 14 and 16
Express your answer in terms of Φ, the standard normal CDF fcn. (ie. $\Phi(z) = P(Z \le z)$ where $Z \sim \text{Normal}(0,1)$)

want: $P(14 \le \bar{X}_{36} \le 16)$

given: mean $= \mu = 15$
stddev $= \sigma = 3 \Rightarrow \sigma^2 = 9$
$n = 36$

by CLT, $\bar{X}_{36} \overset{\text{appr.}}{\sim} \text{Normal}(\mu, \sigma^2/n) = \text{Normal}(15, 9/36)$ ①

since $\sigma^2/n = 9/36 = 1/4$, $\sigma/\sqrt{n} = 1/2$

$\therefore P(14 \le \bar{X}_{36} \le 16) = P\left(\frac{14-\mu}{\sigma/\sqrt{n}} \le \underbrace{\frac{\bar{X}_{36}-\mu}{\sigma/\sqrt{n}}}_{Z \sim \text{Normal}(0,1)\ ②} \le \frac{16-\mu}{\sigma/\sqrt{n}}\right) = P\left(\frac{14-15}{1/2} \le Z \le \frac{16-15}{1/2}\right)$ where $Z \sim \text{Normal}(0,1)$

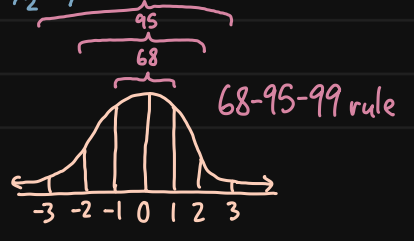
$= P(-2 \le Z \le 2)$

$= \Phi(2) - \Phi(-2)$

$\doteq 0.95$

steps for applying CLT:

1. what is given?
2. appr. distribution of $\bar{X}_n$ (use formulas above)
3. normalize the inequality
   ↳ subtract μ
   divide by $\sigma/\sqrt{n}$
   ↳ $\bar{X}_n$ become Z
4. use "formula" ②



Red-lipped batfish



Good Luck!