STAB52 AMACSS Review Seminar (Part 1)

December 2025

1. Consider a group of 8 individuals, each of whom has their birthday on a randomly chosen day of the year, out of 365 days.

   (a) Find the probability that all these individuals have birthdays in a sequence; for example, (Dec 30, Dec 31, Jan 1, Jan 2, ..., Jan 5). Note that sequences can "wrap around" the year, i.e. Jan 1 is considered as coming after Dec 31, in a loop.

Let S = 8-tuples of birthdays, $|S| = 365^8$

Note this is a discrete unif dist.

Let E = all 8 birthdays are in a row

$|E| = 365 \cdot 8!$

$$P(E) = \frac{|E|}{|S|} = \frac{365 \cdot 8!}{365^8} = \frac{8!}{365^7}$$

   (b) Find the probability that exactly 4 individuals have their birthdays in September.

Let A = that event

$$|A| = \binom{8}{4} \cdot 30^4 (365-30)^4$$

$\binom{8}{4}$ ways to choose who's in Sept

30^4 assign Sept

335^4 assign non-sept

$$P(A) = \frac{|A|}{|S|} = \frac{\binom{8}{4} \cdot 30^4 \cdot 335^4}{365^8}$$

disc unif

2. Suppose we roll two fair six-sided dice, one red and one blue.

- Let A be the event that the two dice show the same value.
- Let B be the event that the sum of the two dice is equal to 12.
- Let C be the event that the red die shows 4.
- Let D be the event that the blue die shows 4.

(a) Are A and B independent? $P(A) \cdot P(B) = ? P(A \cap B)$

Note that outcomes follow a disc unif dist.
S = ordered pairs of dice rolls
$|S| = 6^2 = 36$

$P(A) = \frac{|A|}{|S|} = \frac{6}{36} = \boxed{\frac{1}{6}}$

$A = \{(1,1), (2,2), \ldots, (6,6)\}$

$P(B) = \frac{1}{36}$

$B = \{(6,6)\}$

$P(A \cap B) = \frac{1}{36}$

$A \cap B = \{(6,6)\}$

$P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{36} \neq \frac{1}{36} = P(A \cap B)$

$\therefore$ A, B not ind by def

(b) Are A and C independent?

$P(C) = \frac{1}{6}$

$P(A \cap C) = \frac{1}{36}$

$\{(4,4)\}$

$\therefore$ A, C indep

$P(A)P(C) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = P(A \cap C)$

(c) Are C and D independent?

note $P(D) = \frac{1}{6}$

$P(C \cap D) = \frac{1}{36}$

$\{(4,4)\}$

$P(C) \cdot P(D) = P(C \cap D)$ $\therefore$ C, D indep.

(d) Are A, C, and D all independent?

$P(A \mid C \cap D) = 1 \neq \frac{1}{6} = P(A)$

$\therefore$ A, C, D not mutually indep.

3. A test consists of 10 multiple choice questions, each of which has 5 choices. A student takes the test and answers each question at random (picks one of the choices with equal probability). The test outcome can be represented as a sequence of 10 Right or Wrong answers, for example (W, R, W, W, W, W, W, W, R, R).

(a) Calculate the probability that the student gets 7 or more right answers (leave answer unsimplified).

Let X = # correct answers, $X \sim \text{Binom}(10, \frac{1}{5})$

$$P(X \geq 7) = P(X=7) + \cdots + P(X=10)$$

$$= \boxed{\sum_{k=7}^{10} \binom{10}{k} \cdot \left(\frac{1}{5}\right)^k \left(1-\frac{1}{5}\right)^{10-k}}$$

(b) Find the probability that the first question the student gets right is question 3.

↳ Q1 wrong, Q2 wrong, Q3 right

$$= \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} = \frac{16}{125}$$

(c) Suppose you know that the student got 3 out of 10 questions right, but you don't know which ones. Find the probability that exactly 2 out the first 5 questions were right, given that the student scored 3/10.

A = 3/10 Qs right, E = 2 Qs / first 5 correct

↳ sample space, disc uniform space

$E \subseteq A$ = 2 Qs / first 5 correct & 3/10 Qs correct

$$|A| = \binom{10}{3}$$

$$|E| = \binom{5}{2}\binom{5}{1}$$

$$P(E|A) = \frac{|E|}{|A|} = \frac{\binom{5}{2}\binom{5}{1}}{\binom{10}{3}}$$

4. Consider an election with two candidates, candidate A and candidate B. Every voter is invited to participate in an exit poll, which they can accept or refuse to participate in. For a randomly selected voter, let A be the event that they voted for A, and W be the event that they are willing to participate in the exit poll. Suppose $P(W|A) = 0.7$ but $P(W|A^C) = 0.3$. In the exit poll, 60 percent of respondents say they voted for A, and assume that they are all honest. Find $P(A)$, a.k.a. the true proportion of people who voted for A.

$P(A|W) = 0.6$ $\quad P(W|A) = 0.7$ $\quad P(W|A^c) = 0.3$

$\Downarrow$ $\quad \Downarrow$ $\quad \Downarrow$

$\frac{P(A \cap W)}{P(W)} = 0.6$ $\quad \frac{P(A \cap W)}{P(A)} = 0.7$ $\quad \frac{P(A^c \cap W)}{P(A^c)} = 0.3$ $\quad = P(W) - P(A \cap W)$

$\quad 1 - P(A)$ $\quad$ LoTP

$\Downarrow$ $\quad \Downarrow$ $\quad \Downarrow$

$P(A \cap W) = 0.6\,P(W)$ $\quad P(A \cap W) = 0.7\,P(A)$ $\quad P(W) - P(A \cap W) = 0.3(1 - P(A))$

$\Downarrow$

$P(A \cap W) = P(W) - 0.3 + 0.3P(A)$

Let $a = P(A)$, $w = P(W)$

$$\begin{cases} 0.6w = 0.7a \\ 0.7a = w - 0.3 + 0.3a \end{cases}$$

$\hookrightarrow \quad 0.4a = w - 0.3$

$\Rightarrow \quad w = 0.4a + 0.3$

$0.6(0.4a + 0.3) = 0.7a$

$0.24a + 0.18 = 0.7a$

$0.18 = 0.46a$

$P(A) = a \approx 0.391$

Summary: set all $P(A \cap W)$ equal to eachother to create a linear system in $P(A)$ and $P(W)$, can solve for $P(A)$

[not shown during review seminar]

5. Let $n \in \mathbb{Z}^+$. Let XBinomial$(n, 1/2)$ and YBinomial$(n+1, 1/2)$ independently. Let $V = \min(X, Y)$ and $W = \max(X, Y)$ $X \sim Bin(n, 1/2)$ $Y \sim Bin(n+1, 1/2)$

(a) Show that $E|X - Y| = E(W) - E(V)$.

Use property of min/max: $\forall x, y \in \mathbb{R}$, $\max(x,y) - \min(x,y) = |x-y|$

Then notice $W - V = \max(X,Y) - \min(X,Y) = |X-Y|$

$\therefore E|X-Y| = E(W-V)$, above

$= E(W) - E(V)$, linearity of expectations

as needed.

(b) Compute $V(n - X)$. variance

Notice that n is fixed, so

$V(n-X) = V(X)$ (*)

$= np(1-p)$, binom variance formula

$= n \cdot \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{1}{4}n$

(*) holds since we're just reflecting & translating X, which doesn't change the variance.

Formally, $Var(n - X) = Var(n) + Var(-X)$, } independent since n is a constant

$= 0 + (-1)^2 Var(X)$ } Variance of linear func's

$= Var(X)$