STAB52 TT1 Seminar

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Welcome to the STAB52 TT1 Review Seminar!

If you have any questions throughout the seminar, please raise your hand (on Zoom) or send them in the chat!

Probability Spaces

- **Sample Space (S)**: set of all possible outcomes (individual outcomes often denoted by lowercase letters).
- **Event ($A, B, C, \ldots$)**: a subset of the sample space. Probabilities are assigned to events.
- **Mutually Exclusive (Disjoint)**: two events with no outcomes in common (i.e. $A \cap B = \emptyset$).
- **Partition**: two or more events that are disjoint and whose union is the entire sample space. Example: A and A^c.

Axioms of Probability

A probability function P must satisfy three axioms:

1. For all events $A \subseteq S$, $0 \leq P(A) \leq 1$.
2. $P(S) = 1$.
3. If $A_1, A_2, \ldots$ are disjoint, then

$$P\left(\bigcup_{\forall i} A_i\right) = \sum_{\forall i} P(A_i).$$

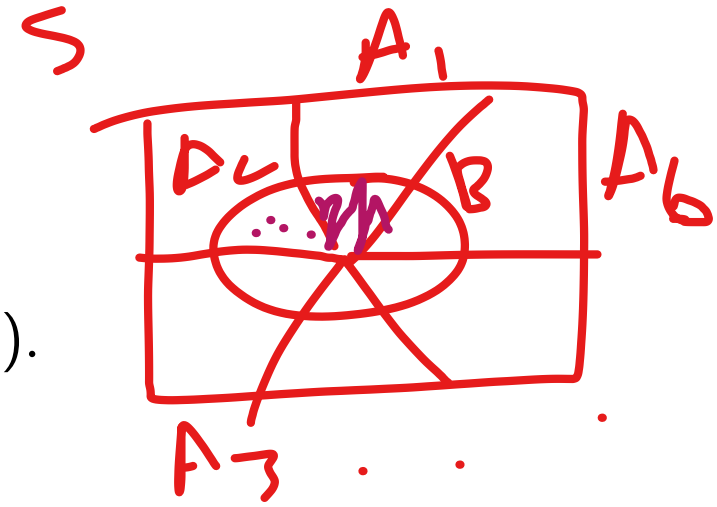
Note: We use axiom 3 all the time without citing — this is why it's an axiom.

Probability Function and PMF are different

Warning

- The **probability function** P takes in **events (sets)**. Example: $P(A)$ where $A \subseteq S$.
- The **probability mass function (PMF)** p takes in **outcomes (elements)**. Example: $p_X(x)$ where $x \in \mathbb{R}$ is a possible value of RV X.

Probability Rules



- **Complement Rule:** For any event A,

$$P(A^c) = 1 - P(A).$$

- **Law of Total Probability (LoTP):**
Let $\{A_1, \ldots, A_n\}$ be a partition of the sample space S. Then for any event $B \subseteq S$:

$$P(B) = P(B \cap A_1) + \cdots + P(B \cap A_n)$$

- **Inclusion–Exclusion Principle:**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This is like a generalization of axiom 3 for non-disjoint events.

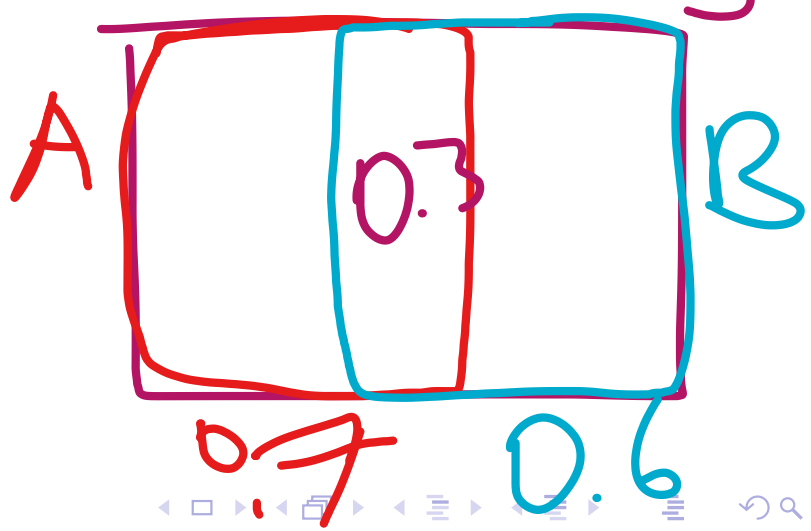
Example: Maximum Probability

Problem. We are given events A, B, C such that $P(A \cup B \cup C) = 1$, with

$$P(A) = 0.7, \quad P(B) = 0.6, \quad P(C) = 0.2.$$

Find the maximum possible value of $P(A \cap B \cap C)$.

$A \cup B \cup C = S$

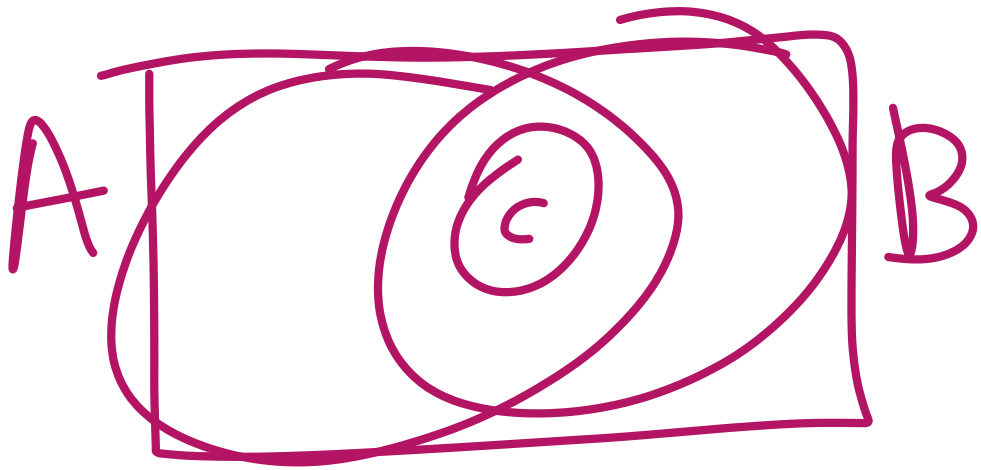


Solution: Maximum Probability

Solution.

Solution: Maximum Probability

- Claim: the maximum possible value of $P(A \cap B \cap C)$ is 0.2.
- Consider the construction with $S = A \cup B$ and $C \subseteq A \cap B$. Note that $P(A \cap B) = 0.3 > P(C)$ by Inclusion-Exclusion.
- Then we have $P(A \cap B \cap C) = P(S \cap C) = P(C) = 0.2$, which must be the maximum possible value since $P(A \cap B \cap C)$ is bounded above by $P(C)$.



Counting

- **Permutations:** The number of ordered arrangements of k objects chosen from n:

$$P(n,k) = \frac{n!}{(n-k)!}. = \binom{n}{k} \cdot k!$$

- **Combinations:** The number of unordered selections of k objects chosen from n:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Note: For the same n, k, the number of permutations is $k!$ times the number of combinations.

Example: Poker hands

Problem. Draw a random 5-card hand from a standard 52-card deck. How many hands contain exactly two non-face cards whose ranks multiply to 9 (Ace counts as 1)? The order of the hand does not matter.

Solution: Poker hands

Solution.

- Note that face cards are J,Q,K. There are four suits, so there are $3 \cdot 4 = 12$ face cards total.
- We require exactly two non-face cards, so the other three cards must be face cards. There are $\binom{12}{3}$ ways to choose these three face cards.
- Looking at the two non-face cards, they must be either an ace 9 or two 3's in order to have a product of 9. Consider these as two cases:
- Case 1 (Ace & 9): $4 \cdot 4 = 16$
  Case 2 (two 3's): $\binom{4}{2} = 6$
- Therefore, there are 16+6=22 total ways to choose the two non-face cards.
- By Multiplication Rule, the total number of hands is

$$\binom{12}{3} \cdot 22 = 4840$$

Discrete Uniform Probability

- If we have a finite sample space S where each outcome has equal probability, then for any event $A \subseteq S$:

$$P(A) = \frac{|A|}{|S|}.$$

$\frac{1}{|S|} \cdot |A|$

- This turns a probability problem into a **counting problem** :D

Example: Random 4-digit PIN

Problem. A phone generates a random PIN from 0000 to 9999. What is the probability that the four digits are in *strictly increasing* order?

Solution: Random 4-digit PIN

Problem. A phone generates a random PIN from 0000 to 9999. What is the probability that the four digits are in *strictly increasing* order?

Solution.

- Let $S = \{0000, 0001, \ldots, 9999\}$ be the set of all 4-digit PINs. Every PIN is equally likely to be generated, so we have a discrete uniform space.
- There are 10 options for each digit and 4 digits, so $|S| = 10^4$.
- Let $A \subseteq S$ be the event that the four digits are strictly increasing. To create an increasing PIN, all four digits must be distinct and chosen from $\{0, 1, \ldots, 9\}$, then arranged in increasing order.
- $|A|$ is the number of ways to choose 4 distinct digits: $\binom{10}{4}$.
- Since we have a discrete uniform sample space,

$$P(A) = \frac{|A|}{|S|} = \frac{\binom{10}{4}}{10^4} = \frac{210}{10000} = 0.021.$$

Final Note on Binomial Coefficients

- **Special Case of Binomial Theorem:**

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

- The LHS and RHS are different approaches to answering the question: how many subsets does an n-element set have?
- LHS: Let k be the number of elements in a subset. Then the number of k-element subsets is $\binom{n}{k}$. We then sum over all possible values of k, from 0 to n.
- RHS: To construct a subset, consider each element of the original set and decide whether you put it in the subset or not. (i.e, each element has two choices, include or exclude, and there are n elements).