TT2 Review: Continuous Distributions and Order Statistics

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Plan for Today

1. Continuous Random Variables
2. Order Statistics

What is a Continuous Random Variable?

Definition

A random variable X is **continuous** if there exists a function f_X such that

$$P(a < X \leq b) = \int_a^b f_X(x)\,dx$$

for all real $a < b$, and

$$P(X = x) = 0 \quad \text{for every } x \in \mathbb{R}.$$

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- f_X is the **probability density function (PDF)**.
- The **cumulative distribution function (CDF)** is

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt.$$

Properties of CDF and PDF

CDF $F_X(x)$

- Non–decreasing, right–continuous.
- $\lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to \infty} F_X(x) = 1.$
- For $a < b$,

$$P(a < X \leq b) = F_X(b) - F_X(a).$$

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PDF $f_X(x)$

- $f_X(x) \geq 0$ for all x.
- $\int_{-\infty}^{\infty} f_X(x)\, dx = 1.$
- When F_X is differentiable:

$$f_X(x) = F_X'(x).$$

Common Continuous Distributions

Uniform(ℓ, u)

$$f(x) = \frac{1}{u - \ell}, \quad \ell \leq x \leq u.$$

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Memoryless:

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Normal(μ, σ^2)

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(- \frac{(x-\mu)^2}{2\sigma^2} \Big).$$

No closed–form CDF in terms of elementary functions.

Example 1: Custom PDF

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} c(x + x^2), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Tasks

1. Find c.
2. Find the CDF $F_X(x)$.
3. Compute $P\left(\frac{1}{4} < X < \frac{3}{4}\right)$.

Example 1: Solution

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So $c = \frac{6}{5}$.

2. CDF. For $x \le 0$: $F_X(x) = 0$.
For $0 < x < 1$:

$$F_X(x) = \int_0^x \frac{6}{5}(t + t^2)\,dt = \frac{6}{5}\left(\frac{t^2}{2} + \frac{t^3}{3}\right)_0^x = \frac{3x^2 + 2x^3}{5}.$$

For $x \ge 1$: $F_X(x) = 1$.

Example 1: Solution (cont.)

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$$P\left(\frac{1}{4} < X < \frac{3}{4}\right) = F_X\left(\frac{3}{4}\right) - F_X\left(\frac{1}{4}\right).$$

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$$P\left(\frac{1}{4} < X < \frac{3}{4}\right) = \frac{81}{160} - \frac{7}{160} = \boxed{\frac{74}{160} = \frac{37}{80} \approx 0.4625.}$$

Order Statistics: Definitions

Let $X_1, \ldots, X_n$ be i.i.d. continuous random variables with CDF F and PDF f.

Order statistics

$$X_{(1)} = \min(X_1, \ldots, X_n), \quad X_{(n)} = \max(X_1, \ldots, X_n).$$

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$$X_{(1)} = \min(X_1, \ldots, X_n), \quad X_{(n)} = \max(X_1, \ldots, X_n).$$

Because the X_i are continuous and i.i.d., ties occur with probability 0. The distributions of $X_{(1)}$ and $X_{(n)}$ can be written in terms of F and f.

Distribution of Maximum and Minimum

For $X_1, \ldots, X_n$ i.i.d. with CDF F and PDF f:

Maximum $X_{(n)}$

$$F_{X_{(n)}}(x) = P(X_{(n)} \leq x) = P(X_1 \leq x, \ldots, X_n \leq x) = [F(x)]^n.$$

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$$f_{X_{(n)}}(x) = \frac{d}{dx} F_{X_{(n)}}(x) = n[F(x)]^{n-1} f(x).$$

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Example 2: Uniform Order Statistics

Let $X_1, \ldots, X_4$ be i.i.d. Uniform$(0, 1)$.

Tasks

1. Find the CDF and PDF of $M = X_{(4)} = \max(X_1, \ldots, X_4)$.
2. Find $P(M < 0.7)$.
3. Find $P(X_{(1)} > 0.2)$.

Example 2: Solution

For Uniform$(0, 1)$, $F(x) = x$ on $0 < x < 1$, and $f(x) = 1$.

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$$f_M(x) = \frac{d}{dx} x^4 = 4x^3, \quad 0 < x < 1.$$

2. **Probability that $M < 0.7$.**

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2. Probability that $M < 0.7$.

$$P(M < 0.7) = F_M(0.7) = (0.7)^4 \approx 0.2401.$$

Example 2: Solution (cont.)

3. Probability that the minimum exceeds 0.2**.**

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3. Probability that the minimum exceeds 0.2**.**
First find the CDF of $X_{(1)}$:

$$F_{X_{(1)}}(x) = 1 - [1 - F(x)]^4 = 1 - (1 - x)^4, \quad 0 < x < 1.$$

Example 2: Solution (cont.)

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Then

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So

$$\boxed{P(M < 0.7) \approx 0.24, \quad P(X_{(1)} > 0.2) \approx 0.41.}$$

Summary

- Continuous RVs are described by CDFs and PDFs; probabilities are integrals of the PDF.
- Uniform, Exponential, Gamma, and Normal distributions are core models you should recognize.
- Order statistics $X_{(1)}$ and $X_{(n)}$ give the minimum and maximum of a sample.
- For i.i.d. samples:

$$F_{X_{(n)}}(x) = F(x)^n, \qquad F_{X_{(1)}}(x) = 1 - [1 - F(x)]^n.$$

- Practice: derive similar formulas for general kth order statistic $X_{(k)}$.

Questions?

Thank you!

Any questions, comments or concerns?