problem one

Let continuous RVs X,Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c(2x^2y + y^2), & 0 \leq x, y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

a. Find the value of constant c.
b. Find the joint CDF of X,Y for $0 \leq x, y \leq 1$.
c. Find the marginal CDF of Y.
d. Are X and Y independent?
e. Find $P(4X^2 > Y)$
f. Find $P(\frac{1}{2} \leq \max\{X, Y\} \leq 1)$

Explicitly state the support for all CDFs when applicable.

problem one - part a

a. Find the value of constant c.

By PDF rules, we must have

$$
\begin{aligned}
1 &= \int_0^1 \int_0^1 f_{X,Y}(x,y)\,dx\,dy \\
&= \int_0^1 \int_0^1 c(2x^2y + y^2)\,dx\,dy && \text{def'n f} \\
&= c\int_0^1 \int_0^1 (2x^2y + y^2)\,dx\,dy && \text{integral prop.} \\
&= c\int_0^1 \left(\tfrac{2}{3}yx^3 + y^2x\right)|_{x=0}^{x=1}\,dy \\
&= c\int_0^1 \left(\tfrac{2}{3}y + y^2\right)\,dy \\
&= c\left(\tfrac{1}{3}y^2 + \tfrac{1}{3}y^3\right)|_{y=0}^{y=1} \\
&= \tfrac{2}{3}c.
\end{aligned}
$$

Therefore $c = \frac{3}{2}$.

problem one - part b

b. Find the joint CDF of X,Y for $0 \leq x, y \leq 1$.

Let $0 \leq x, y \leq 1$. Then

$$
\begin{aligned}
F_{X,Y}(x,y) &= \int_0^y \int_0^x f_{X,Y}(s,t)\,ds\,dt && \text{def'n F} \\
&= \int_0^y \int_0^x \tfrac{3}{2}(2s^2t + t^2)\,ds\,dt \\
&= \tfrac{3}{2} \int_0^y \left(\tfrac{2}{3}x^3t + xt^2\right) dt \\
&= \tfrac{3}{2}\left(\tfrac{1}{3}x^3y^2 + \tfrac{1}{3}xy^3\right) = \tfrac{1}{2}(x^3y^2 + xy^3).
\end{aligned}
$$

This holds for $0 \leq x, y \leq 1$.

problem one - part c

c. Find the marginal CDF of Y.

We can shortcut this problem by using $F(x, y)$, which we computed in part (b).
Let $0 \leq y \leq 1$. Then

$$\begin{aligned} F_Y(y) &= \lim_{x \to \infty} F_{X,Y}(x, y) && \text{def'n CDF} \\ &= F_{X,Y}(1, y) \\ &= \tfrac{1}{2}(1^3 y^2 + 1 y^3) && \text{def'n F} \\ &= \tfrac{1}{2}(y^2 + y^3) \end{aligned}$$

We must also consider cases $y < 0$ and $y > 1$. Therefore, our final answer is

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{1}{2}(y^2 + y^3), & 0 \leq y \leq 1 \\ 1, & y > 1. \end{cases}$$

problem one - part d

d. Are X and Y independent?

We can verify independence by checking whether or not

$$F_{X,Y}(x,y) = F_X(x)F_Y(y).$$

Similarly to part (c), we can compute $F_X(x)$ for $0 \leq x \leq 1$:

$$F_X(x) = F_{X,Y}(x,1) = \tfrac{1}{2}(x^3 \cdot 1^2 + x \cdot 1^3) = \tfrac{1}{2}(x^3 + x)$$

Recall $F_Y(y) = \frac{1}{2}(y^2 + y^3)$ and $F_{X,Y}(x,y) = \frac{1}{2}(x^3y^2 + xy^3)$.
Consider $(x,y) = (1/2, 1/2)$ as a counterexample. Then

$$F_{X,Y}(1/2,1/2) = \tfrac{1}{2}\big((1/2)^3(1/2)^2 + (1/2)(1/2)^3\big) = \tfrac{3}{64},$$

$$F_X(1/2)F_Y(1/2) = \tfrac{1}{2}\big((1/2)^3 + 1/2\big) \cdot \tfrac{1}{2}\big((1/2)^2 + (1/2)^3\big) = \tfrac{15}{256}.$$

Since $F_{X,Y}(1/2,1/2) \neq F_X(1/2)F_Y(1/2)$, X and Y are not independent.

problem one - part e

e. Find $P(4X^2 > Y)$.

Note that the function $y = 4x^2$ intersects the boundary of square $[0,1]^2$ at $(0,0)$ and $(1/2,1)$.

$$
\begin{aligned}
P(4X^2 > Y) &= 1 - P(Y \geq 4X^2) \\
&= 1 - \int_0^{1/2} \int_{4x^2}^{1} \frac{3}{2}(2x^2y + y^2)\, dy\, dx \\
&= 1 - \frac{3}{2} \int_0^{1/2} \left(x^2y^2 + \frac{1}{3}y^3 \right) \Big|_{y=4x^2}^{y=1} dx \\
&= 1 - \frac{3}{2} \int_0^{1/2} \left(\tfrac{1}{2} + \tfrac{3}{2}x^2 - 56x^6 \right) dx \\
&= 1 - \left[\tfrac{1}{2}x + \tfrac{1}{2}x^3 - 8x^7 \right]_0^{1/2} \\
&= 1 - \left(\tfrac{1}{4} + \tfrac{1}{16} - \tfrac{1}{16} \right) = 1 - \tfrac{1}{4} = \tfrac{3}{4}.
\end{aligned}
$$

problem one - part f

f. Find $P(\frac{1}{2} \leq \max\{X, Y\} \leq 1)$.

Note that

$$\max\{X, Y\} > \tfrac{1}{2} \iff \neg(X < \tfrac{1}{2} \text{ and } Y < \tfrac{1}{2})$$

Then

$$\begin{aligned} P\big(\tfrac{1}{2} < \max\{X, Y\} < 1\big) &= 1 - P(X < \tfrac{1}{2}, Y < \tfrac{1}{2}) \\ &= 1 - F_{X,Y}(\tfrac{1}{2}, \tfrac{1}{2}) \\ &= 1 - \tfrac{1}{2}(\tfrac{1}{2}^3 \cdot \tfrac{1}{2}^2 + \tfrac{1}{2} \cdot \tfrac{1}{2}^3) \\ &= 1 - \tfrac{3}{64} \\ &= \frac{61}{64} \end{aligned}$$

Exercise: Same question but with min instead of max. Solve with a double integral of $f_{X,Y}$ over $1/2 \leq x \leq y$, or use Inclusion-Exclusion with $F_{X,Y}(x, y)$.

problem two

Let X,Y be continuous RVs such that

$$F_{X|Y}(x|y) = 1 - e^{-x(1+y)}, \quad x, y \geq 0,$$

$$f_Y(y) = e^{-y}, \quad y \geq 0.$$

a. Find $E[X^2|Y = 2]$.
b. Find the joint PDF of X,Y.
Hint: X follows a particular continuous distribution.

problem two - part a

a. Find $E[X^2|Y=2]$.

Using the given formula for $F_{X|Y}(x|y)$, we get

$$F_{X|(Y=2)}(x) = 1 - e^{-x(1+2)} = 1 - e^{-3x}.$$

Notice that $X|(Y=2) \sim \text{Exponential}(\lambda = 3)$.
Consider the definition of variance:

$$V(X|Y=2) = E(X^2|Y=2) - [E(X|Y=2)]^2,$$

$$\implies \frac{1}{\lambda} = E(X^2|Y=2) - (\frac{1}{\lambda})^2.$$

Therefore,

$$\begin{aligned} E[X^2 \mid Y=2] &= \frac{2}{\lambda^2} \\ &= \frac{2}{3^2} \\ &= \frac{2}{9}. \end{aligned}$$

problem two - part b

b. Find the joint PDF of X,Y.

Notice that $(X|(Y = y)) \sim \text{Exponential}(\lambda = y + 1)$. Therefore, by the known PDF of the Exponential distribution,

$$f_{X|Y}(x|y) = (1 + y)e^{-x(1+y)}.$$

Then by multiplication rule for conditional PDFs,

$$\begin{aligned} f_{X,Y}(x, y) &= f_{X|Y}(x|y)f_Y(y) \\ &= (1 + y)e^{-x(1+y)}e^{-y}. \end{aligned}$$

This holds for $x \geq 0, y \geq 0$.