CDF Method

1. Consider the RV X with the following CDF:

$$F_X(x) = \begin{cases} \frac{1}{2}e^{x} & \text{for } x<0 \\ 1-\frac{1}{2}e^{-x} & \text{for } x\geq 0 \end{cases}$$

Show that $Y=|X|$ (absolute value) follows Exponential(1) distribution

we apply the CDF method
possible values of Y: $Y\geq 0$

$\forall y\geq 0$, $F_Y(y) = P(Y\leq y)$

$= P(|X|\leq y)$ *isolate for X

$= P(-y\leq X\leq y)$

$= P(X\leq y) - P(X<-y)$

$= F_X(y) - F_X(-y)$

$= 1-\frac{1}{2}e^{-y} - \frac{1}{2}e^{-y}$

$= \boxed{1-e^{-y}}$ (and $F_Y(y)=0$ for $y<0$)

Y has the same CDF as Exp(1)
$\Rightarrow Y\sim Exp(1)$

2D PDF Method

2. Let RVs X, Y have the following joint PDF:

$$f_{X,Y}(x,y) = xe^{-x(1+y)} \quad \forall x,y>0 \qquad \text{(and 0 o/w)}$$

Let $W=X$ and $Z=X\cdot Y$. Find the joint PDF of W and Z, and specify their range.

$$\begin{bmatrix} W \\ Z \end{bmatrix} = h(X,Y) = \begin{bmatrix} X \\ X\cdot Y \end{bmatrix} \Rightarrow h^{-1}(W,Z) = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} W \\ Z/W \end{bmatrix}$$

compute $|J(x,y)|$

$$|J(x,y)| = \left|\begin{matrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{matrix}\right| = \left|\begin{matrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial (xy)}{\partial x} & \frac{\partial (xy)}{\partial y} \end{matrix}\right| = \left|\begin{matrix} 1 & 0 \\ y & x \end{matrix}\right| = |x-0| = |x|$$

range of values: $W = X > 0$

$Z = X\cdot Y > 0$

thus $f_{W,Z}(w,z) = \dfrac{f_{X,Y}(h^{-1}(w,z))}{|J(h^{-1}(w,z))|} = \dfrac{f_{X,Y}(w, z/w)}{|J(w, z/w)|} = \dfrac{we^{-w(1+z/w)}}{|w|} = \boxed{e^{-(w+z)}}$ for $z,w>0$

since $w>0$, $|w| = w$

Convolution Method

3. Let X, Y be i.i.d Uniform$(0,1)$ RVs.

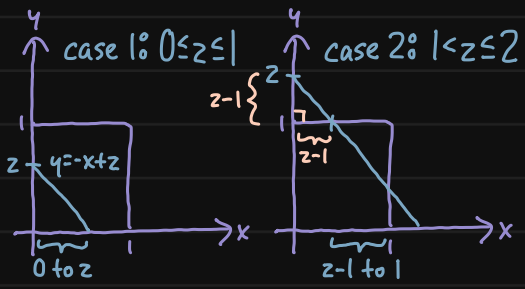
a) Find the joint PDF $f_{X,Y}(x,y)$.

$$X, Y \overset{iid}{\sim} \text{Uniform}(0,1) \Rightarrow f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{o/w} \end{cases} \qquad f_Y(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{o/w} \end{cases}$$

$$\text{by indep, } f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} 1, & 0 \le x, y \le 1 \\ 0, & \text{o/w} \end{cases}$$

$(0 \le Z \le 2)$

b) Find the PDF of $Z = X + Y \Leftrightarrow Y = -X + Z$



by convolution method,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx$$

case 1° $0 \le z \le 1$

$$f_Z(z) = \int_0^z f_{X,Y}(x, z-x)\,dx$$
$$= \int_0^z 1\,dx$$
$$= [x]_{x=0}^{z}$$
$$= z - 0$$
$$= z$$

case 2° $1 < z \le 2$

$$f_Z(z) = \int_{z-1}^{1} f(x, z-x)\,dx$$
$$= \int_{z-1}^{1} 1\,dx$$
$$= [x]_{x=z-1}^{1}$$
$$= 1 - (z-1)$$
$$= 2 - z$$

$$\therefore f_Z(z) = \begin{cases} z, & \text{if } 0 \le z \le 1 \\ 2-z, & \text{if } 1 < z \le 2 \\ 0, & \text{o/w} \end{cases}$$