Week 6

Monday, December 1, 2025 7:47 PM

Week 6

Comparing two populations [ind. & Normal] $X_1, X_2, \dots, X_n \sim N(\mu_x, \sigma_x^2)$, $Y_1, Y_2, \dots, Y_n \sim N(\mu_y, \sigma_y^2)$

- equality of variance
- equality of mean
  - var unknown
    - var. equal
    - var. unequal
  - var known

(1) Equality of two variances $[H_0: \sigma_x^2 = \sigma_y^2]$

$\frac{S_x^2}{S_y^2} \sim F_{n-1, m-1}$ [Under H_0] ⟶ Derivation:

$\frac{(n-1)S_x^2}{\sigma_x^2} \sim \chi^2_{(n-1)}$

$\frac{(m-1)S_y^2}{\sigma_y^2} \sim \chi^2_{(m-1)}$

$\frac{\frac{(n-1)S_x^2}{\sigma_x^2 (n-1)}}{\frac{(m-1)S_y^2}{m-1}} \sim F_{(n-1, m-1)}$

Under H_0, $\sigma_x^2 = \sigma_y^2 \Rightarrow \frac{S_x^2}{S_y^2} \sim F_{(n-1, m-1)}$

Rejection Region: $(-\infty, F_{\frac{\alpha}{2}, (n-1, m-1)}) \cup (F_{1-\frac{\alpha}{2}, (n-1, m-1)}, \infty)$

Constructing a γ-level CI for $\frac{\sigma_y^2}{\sigma_x^2}$

$$k_1 \le \frac{\frac{S_x^2}{\sigma_x^2}}{\frac{S_y^2}{\sigma_y^2}} \le k_2$$

$$\Rightarrow k_1 \le \frac{S_x^2}{S_y^2} \cdot \frac{\sigma_y^2}{\sigma_x^2} \le k_2$$

$$\Rightarrow \frac{k_1}{\frac{S_x^2}{S_y^2}} \le \frac{\sigma_y^2}{\sigma_x^2} \le \frac{k_2}{\frac{S_x^2}{S_y^2}}$$

$[k_1 \to \frac{1-\gamma}{2}$ quantile of F; $k_2 \to \frac{1+\gamma}{2}$ quantile of F$]$

(we check if 1 lies in this interval)

(2) Equality of means [variances known]

$H_0: \mu_x = \mu_y$ [or $\mu_x - \mu_y = 0$]

$\bar{X} \sim N(\mu_x, \frac{\sigma_x^2}{n})$, $\bar{Y} \sim N(\mu_y, \frac{\sigma_y^2}{m})$

$\therefore \bar{X} - \bar{Y} \sim N(\mu_x - \mu_y, \frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m})$

$\Rightarrow$ test stat: $\frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \sim N(0, 1)$

Approach 1: Under H_0: $\frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \sim N(0, 1)$

Approach 2: γ-level CI for $(\mu_x - \mu_y)$: $\bar{X} - \bar{Y} \pm z_{(\frac{1+\gamma}{2})} \sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}$

(3) Equality of two means [variances unknown]

(i) Assuming $\sigma_x = \sigma_y = \sigma$ but unknown.

test-statistic: $\frac{(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)}{S_p \sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{(n+m-2)}$ ⟶ $\frac{(n-1)S_x^2}{\sigma_x^2} \sim \chi^2_{(n-1)}$, $\frac{(m-1)S_y^2}{\sigma_y^2} \sim \chi^2_{(m-1)}$

$\Rightarrow \frac{(n-1)S_x^2}{\sigma_x^2} + \frac{(m-1)S_y^2}{\sigma_y^2} \sim \chi^2_{(n+m-2)}$

★ (1) Under H_0: $\frac{(\bar{X} - \bar{Y})}{S_p \sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{(n+m-2)}$

Rejection region: $(-\infty, t_{\frac{\alpha}{2}, (n+m-2)}, t_{1-\frac{\alpha}{2}, (n+m-2)})$

Our previous test stat: $\frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \sim N(0, 1)$

Now, $\frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sigma \sqrt{\frac{1}{n} + \frac{1}{m}}} \Big/ \frac{1}{\sigma}\sqrt{\frac{(n-1)S_x^2 + (m-1)S_y^2}{n+m-2}} \sim t_{(n+m-2)}$

$\Rightarrow \frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\frac{1}{n} + \frac{1}{m}}} \Big/ \sqrt{\frac{(n-1)S_x^2 + (m-1)S_y^2}{n+m-2}} \sim t_{(n+m-2)}$

$\sqrt{\frac{(n-1)S_x^2 + (m-1)S_y^2}{n+m-2}} \to S_p^2$ (pooled sample variance)

$\Rightarrow \frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{S_p \sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{(n+m-2)}$

(2) Confidence interval

★ ii) if $\sigma_x \neq \sigma_y$ [both unknown], we use different df calculation.

$\frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \sim t_{(df)}$

$df = \frac{\left(\frac{S_x^2}{n} + \frac{S_y^2}{m}\right)^2}{\frac{(S_x^2/n)^2}{n-1} + \frac{(S_y^2/m)^2}{m-1}}$

★ Paired data: (when obs. of 2 populations are not independent) define $D = X - Y$, sample avg $\bar{D}$, sample std. dev of D, S_d

test stat: $\frac{\bar{D} - \mu_d}{\frac{S_d}{\sqrt{n}}} \sim t_{(n-1)}$

The average amount of time boys and girls aged seven to 11 spend playing sports each day is believed to be the same. A study is done and data are collected, resulting in the data in Table 10.2.1. Each populations has a normal distribution.

Table 10.2.1

	Sample Size		Average Number of Hours Playing Sports Per Day		Sample Standard Deviation	
Girls	9	(n)	2	$(\bar{x})$	0.866	(s_x)
Boys	16	(m)	3.2	$(\bar{y})$	1.00	(s_y)

① Test $H_0: \sigma_x^2 = \sigma_y^2$ v/s $H_a: \sigma_x^2 \neq \sigma_y^2$ at 95% confidence level

② Test $H_0: \mu_x = \mu_y$ v/s $H_a: \mu_x \neq \mu_y$ assuming pop. variances are unknown but equal

③ Test $H_0: \mu_x = \mu_y$ v/s $H_a: \mu_x > \mu_y$ assuming pop. variances are unknown and unequal

④ Test $H_0: \mu_y = \mu_x + 1$ v/s $H_a: \mu_y \neq \mu_x + 1$

[assume population variance is unknown and use your accepted hypothesis from part 1]

① $\gamma = 0.95$ [95% confidence interval]

$\alpha = 0.05 \rightarrow$ level of significance ($\alpha = 1 - \gamma$)

Test statistic: $\frac{S_x^2}{S_y^2} = \frac{(0.866)^2}{1} = 0.75$

Rejection Region: $(-\infty, F_{\frac{0.05}{2}, (8,15)}) \cup (F_{1-\frac{0.05}{2}, (8,15)}, \infty)$

$= (-\infty, 0.244) \cup (3.06, \infty)$

Since 0.75 $\notin$ RR, we fail to reject null hypothesis

Another possible way

95% confidence interval: $\left(\frac{F_{\frac{0.05}{2}, 8, 15}}{0.75}, \frac{F_{1-\frac{0.05}{2}, 8, 15}}{0.75} \right)$

$= \left(\frac{0.244}{0.75}, \frac{3.06}{0.75} \right)$

$= (0.32, 4.08)$

Since 1 lies in our 95% confidence interval, we fail to reject null hypothesis

(2) $$S_p^2 = \frac{(n-1)S_x^2 + (m-1)S_y^2}{n+m-2}$$

$$= \frac{8(0.75) + 15}{9+16-2} = \frac{21}{23} \approx 0.913$$

Under H_0, test statistic: $$\frac{2-3.2}{\sqrt{0.913}\sqrt{\frac{1}{9}+\frac{1}{16}}} = \frac{-1.2}{0.955\sqrt{\frac{25}{144}}}$$

$$= \frac{-1.2}{0.955} \times \frac{12}{5}$$

$$= -3.13$$

Rejection Region: $(-\infty, t_{0.025, 21}) \cup (t_{0.975, 21}, \infty)$

$(-\infty, -2.08) \cup (2.08, \infty)$

Since $-3.13 \in RR$, we reject the null hypothesis

(3) One sided t test

$$v = \frac{\left(\frac{S_x^2}{n} + \frac{S_y^2}{m}\right)^2}{\frac{\left(\frac{S_x^2}{n}\right)^2}{n-1} + \frac{\left(\frac{S_y^2}{m}\right)^2}{m-1}} = \frac{\left(\frac{0.75}{9} + \frac{1}{16}\right)^2}{\frac{\left(\frac{0.75}{9}\right)^2}{8} + \frac{\left(\frac{1}{16}\right)^2}{15}}$$

$$= \frac{(0.083 + 0.0625)^2}{\frac{(0.083)^2}{8} + \frac{(0.0625)^2}{14}}$$

$$= \frac{0.021}{0.001} \approx 21$$

Notice it's $t_{0.95}$ now and not $t_{1-\frac{0.05}{2}}$ [because it's a right tail test now]

④ $(t_{0.95, 21}, \infty)$ = RR

RR: $(1.721, \infty)$

$$\frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}} = \frac{2 - 3.2}{\sqrt{\frac{0.75}{9} + \frac{1}{16}}} = \frac{-1.2}{\sqrt{\frac{7}{4 \times 9 \times 3} + \frac{1}{16}}}$$

$$= \frac{-1.2}{\sqrt{\frac{1}{12} + \frac{1}{16}}} = \frac{-1.2}{\sqrt{\frac{4+3}{48}}} = \frac{-1.2}{\sqrt{\frac{7}{3}}} \times 4$$

$$= -\frac{4.8}{1.53} = \boxed{-3.14}$$

$-3.14 \notin$ RR, we fail to reject null hypothesis

⑤ $$\frac{\bar{x} - \bar{y} - \mu_x - \mu_y}{s_p \cdot \sqrt{\frac{1}{n} + \frac{1}{m}}} = \frac{-1.2 - (-1)}{\sqrt{0.913}\sqrt{\frac{1}{9} + \frac{1}{16}}} = \frac{-0.2}{0.955\sqrt{\frac{25}{144}}}$$

$$= \frac{-0.2}{0.955} \times \frac{12}{5}$$

$$= -5.026$$

Rejection Region: $(-\infty, t_{0.025, 21}) \cup (t_{0.975, 21}, \infty)$

$(-\infty, -2.08) \cup (2.08, \infty)$

Since $-5.026 \in$ RR, we reject the null hypothesis

Monday, December 1, 2025 7:52 PM

Week 7

Sufficient Statistic : A statistic $T(X_1, X_2, \dots, X_n)$ is said to be sufficient for θ if conditional dist. of $X_1, X_2, \dots, X_n$ given $T=t$ does not depend on θ.

$X_1, X_2, X_3 \overset{i.i.d}{\sim} \text{Pois}(\lambda)$. Verify $T = \sum_{i=1}^{3} X_i$ is a sufficient statistic for λ.

$$P[X_1, X_2, X_3 \mid \Sigma X_i] = P[X_1 = x_1, X_2 = x_2, X_3 = t - x_1 - x_2 \mid T = t]$$

$$= \frac{P[X_1 = x_1, X_2 = x_2, X_3 = t - x_1 - x_2]}{P[\Sigma X_i = t]}$$

$$= \frac{\frac{e^{-\lambda} \cdot \lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda} \cdot \lambda^{x_2}}{x_2!} \cdot \frac{e^{-\lambda} \cdot \lambda^{t-x_1-x_2}}{(t-x_1-x_2)!}}{P[T=t]} \quad \left[T \sim \text{Pois}(3\lambda)\right]$$

$$= \frac{\frac{e^{-3\lambda} \cdot \lambda^t}{x_1! \cdot x_2! \cdot (t-x_1-x_2)!}}{\frac{e^{-3\lambda} \cdot (3\lambda)^t}{t!}} = \frac{1}{3^{x_1} x_1! \, x_2! \, (t-x_1-x_2)!} \to \text{ind. of } \lambda!$$

Factorization Thm.: factor $L(\theta) = f(x_1, x_2, \dots, x_n \mid \theta)$ → $g(T, \theta) \to$ f^n of T & θ; $h(x_1, x_2, \dots, x_n) \to$ f^n of only obs

$T(X_1, X_2, \dots, X_n)$ is said to be sufficient for θ if the joint probability function factors in the form, $L(\theta) = f(x_1, \dots, x_n \mid \theta) = g[T(x_1, \dots, x_n), \theta] \cdot h(x_1, \dots, x_n)$

Ex 1: $X_1, X_2, \dots, X_n \sim \text{Pois}(\lambda)$

$$L(\lambda) = \frac{e^{-n\lambda} \cdot \lambda^{\Sigma x_i}}{\prod_{i=1}^{n} x_i} = \underbrace{e^{-n\lambda} \cdot \lambda^{\Sigma x_i}}_{g(\Sigma x_i, \lambda)} \cdot \underbrace{\left[\frac{1}{\prod_{i=1}^{n} x_i!}\right]}_{h(x_1, \dots, x_n)}$$

$\therefore$ Σx_i sufficient stat for λ

Ex 2: $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2) \to P(x_i = x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$

$$L(\mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \cdot e^{\frac{-1}{2\sigma^2} \cdot \sum_{i=1}^{n}(x_i - \mu)^2}$$

$$= \underbrace{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \cdot e^{\frac{-1}{2\sigma^2} \cdot \left(\sum_{i=1}^{n} x_i^2 - 2\mu \Sigma x_i + n\mu^2\right)}}_{g[T_1, T_2, \mu, \sigma^2]} \cdot 1$$

$\Sigma x_i, \Sigma x_i^2$ are jointly sufficient for μ, σ^2

Ex 3: $X_1, X_2, \dots, X_n \sim \text{Exp}(\lambda)$

$f(x) = \lambda e^{-\lambda x}$

$$L(\lambda) = \underbrace{\lambda^n \cdot e^{-\lambda \cdot \Sigma x_i}}_{g(\Sigma x_i, \lambda)} \cdot \underbrace{1}_{h(x_1, \dots, x_n)}$$

$\hookrightarrow$ Σx_i is the sufficient statistic

$X_1, \dots, X_n \sim \text{Unif}[0, \theta]$

$$L(\theta) = \frac{1}{\theta^n} \cdot \underbrace{I[0 \le x_1, \dots, x_n \le \theta]}_{\max(x_1, \dots, x_n) \le \theta} \times 1$$

$$= \frac{1}{\theta^n} \cdot I[\underbrace{\max(x_1, \dots, x_n)}_{\downarrow} \le \theta] \times 1$$

$\max(x_1, \dots, x_n)$ is the sufficient stat for θ.

Consistent estimator : T_n is said to be consistent in probability if $T_n \xrightarrow{p} \theta$

Ex: $\bar{X} \xrightarrow{p} \mu$ [so $\bar{X}$ is consistent]

[$\bar{X}$ is consistent estimator for $X_i \sim$ Pois(λ) / $N(\mu, \sigma^2)$ / Beo(θ), Exp(β)]

- S^2 is a consistent estimator of σ^2 → Pf: $S^2 = \frac{1}{n-1} \cdot \Sigma(X_i - \bar{X})^2$

$$= \left(\frac{n}{n-1}\right)\left(\frac{1}{n} \cdot \Sigma(X_i - \bar{X})^2\right)$$

$$= \frac{n}{n-1} \cdot \left(\frac{1}{n} \cdot \left[\Sigma X_i^2 - n(\bar{X})^2\right]\right)$$

$$= \underbrace{\frac{n}{n-1}}_{\downarrow 1} \left(\underbrace{\frac{1}{n} \cdot \Sigma X_i^2}_{\downarrow E[X^2]} - \underbrace{(\bar{X})^2}_{\downarrow (E[\bar{X}])^2}\right)$$

$\downarrow \sigma^2$

$\Rightarrow S$ is a consistent estimator of σ (by cont. mapping thm.)

- MSE Consistent: If $MSE(T_n) \to 0$ as $n \to \infty$
- MLE is consistent → Look at Pf again

Score: ① $S(\theta) = \frac{\partial \ell(\theta)}{\partial \theta}$ [derivative of log likelihood] ($S(\theta)$: RV)

② $E[S(\theta \mid X)]\big|_{\theta = \theta_0} = 0$

Fisher Info : $I(\theta_0) = \text{Var}[S(\theta \mid X)|_{\theta = \theta_0}]$

$$= E\left[\left(\frac{\partial}{\partial \theta} \log f(X \mid \theta)\Big|_{\theta = \theta_0}\right)^2\right] = -E\left[\frac{\partial^2}{\partial \theta^2} \log f(X \mid \theta)\Big|_{\theta = \theta_0}\right]$$

$nI(\theta_0) = \text{Var}[S(\theta \mid X_1, \dots, X_n)|_{\theta = \theta_0}]$

Observed Fisher Info: $nI(\hat{\theta}) = -\frac{\partial^2}{\partial \theta^2} \log f(x_1, \dots, x_n \mid \theta)\Big|_{\theta = \hat{\theta}}$ ($\hat{\theta}$: MLE)

Ques $p(x) = \theta a x^{a-1} \exp(-\theta x^a)$, $x > 1$, $\theta > 0$, $a > 0$

$x_1, x_2, \ldots, x_n$ are the sample observations following $p(x)$ density. Assume a is the constant and θ is a parameter. Find the sufficient statistic for θ

Ques Show $E[\, S(\theta \mid x_1, x_2, \ldots, x_n)\,|_{\theta=\theta_0}\,] = 0$ where $x_1, x_2, \ldots, x_n$ follow a bernoulli distribution and θ_0 is the true value of the parameter.

① $L(\theta) = f(x_1, x_2, \ldots, x_n \mid \theta)$

$= \prod_{i=1}^{n} f(x_i \mid \theta)$

$= \prod_{i=1}^{n} \theta a x_i^{a-1} \exp(-\theta x_i^a) \cdot I(\min(x_1, \ldots, x_n) > 1)$

$= (\theta a)^n \cdot \prod_{i=1}^{n} x_i^{a-1} \cdot \exp\left(-\theta \cdot \sum_{i=1}^{n} x_i^a\right) \cdot I(\min(x_1, \ldots, x_n) > 1)$

$= \underbrace{(\theta a)^n \cdot \exp\left(-\theta \cdot \sum_{i=1}^{n} x_i^a\right)}_{g\left(\sum_{i=1}^{n} x_i^a,\ \theta\right)} \cdot \underbrace{\prod_{i=1}^{n} x_i^{a-1} \cdot I(\min(x_1, \ldots, x_n) > 1)}_{g(x_1, \ldots, x_n)}$

$\therefore \sum_{i=1}^{n} x_i^a$ is the sufficient statistic

(a)

$$L(\theta) = f(x_1, x_2, \dots, x_n \mid \theta)$$

$$= \prod_{i=1}^{n} f(x_i \mid \theta)$$

$$= \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$$

$$= \theta^{\sum x_i} (1-\theta)^{\sum 1 - x_i}$$

$$l(\theta) = \log(L(\theta)) = \sum x_i \log(\theta) + (n - \sum x_i) \log(1-\theta)$$

$$S(\theta) = l'(\theta) = \frac{\sum x_i}{\theta} - \frac{n - \sum x_i}{1-\theta}$$

$$= \frac{\sum x_i}{\theta} - \frac{n - \sum x_i}{1-\theta}$$

$$S(\theta \mid x_1, \dots, x_n)\big|_{\theta=\theta_0} = \frac{\sum x_i}{\theta_0} - \frac{n - \sum x_i}{1-\theta_0}$$

$$E\left[S(\theta \mid x_1, x_2, \dots, x_n)\big|_{\theta=\theta_0}\right] = E\left[\frac{\sum x_i}{\theta_0} - \frac{n - \sum x_i}{1-\theta_0}\right]$$

$$= \frac{n\bar{x}}{\theta_0} - \frac{n - n\bar{x}}{1-\theta_0}$$

$$= \frac{n\bar{x}(1-\theta_0) - (n - n\bar{x})\theta_0}{\theta_0(1-\theta_0)}$$

$$= n\bar{x} - \cancel{n\bar{x}\theta_0} - n\theta_0 + \cancel{n\bar{x}\theta_0}$$

Week 8

Distribution of Score $X_1, \dots, X_n \overset{iid}{\sim} \begin{bmatrix} \text{mean} = \mu \\ \text{variance} = \sigma^2 \end{bmatrix}$

$$S(\theta \mid X_1, X_2, \dots, X_n) \xrightarrow{D} N(0, nI(\theta_0))$$

Distribution of MLE: $\dfrac{(\hat\theta - \theta_0)}{\sqrt{\frac{1}{nI(\theta_0)}}} \xrightarrow{D} N(0,1)$ $\quad \left\{ \hat\theta \xrightarrow{D} N\left(\theta_0, \frac{1}{nI(\theta_0)}\right) \right.$

MLE is: (i) asymptotically unbiased (ii) f(n) of sufficient statistic
(iii) consistent (iv) asymptotically efficient.

- these don't work on Unif dist

MLE based CI for θ_0: $\left(\hat\theta - z_{\left(\frac{1+\gamma}{2}\right)} \sqrt{\frac{1}{nI(\theta_0)}} \;,\; \hat\theta + z_{\left(\frac{1+\gamma}{2}\right)} \sqrt{\frac{1}{nI(\theta_0)}} \right)$

can't use [don't know θ_0]

⇓ use Plug-in Fisher Info $I(\hat\theta)$

$$\boxed{\hat\theta \pm z_{\left(\frac{1+\gamma}{2}\right)} \sqrt{\frac{1}{nI(\hat\theta)}}}$$

Fisher Information estimate, $nI(\hat\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \log f(x_1, \dots, x_n \mid \theta)\big|_{\theta=\hat\theta}\right] = -\frac{\partial^2}{\partial\theta^2} \log\left(f(x_1, \dots, x_n \mid \theta)\big|_{\theta=\theta_0}\right)$

plug-in fisher info

Ex: $X_1, X_2, \dots, X_n \sim \text{Pois}(\lambda)$

$L(\lambda) = \dfrac{e^{-n\lambda} \cdot \lambda^{\sum x_i}}{\prod_{i=1}^n x_i!}$, $\ell(\lambda) = -n\lambda + \sum x_i \log\lambda + \text{const}$

$\ell'(\lambda) = -n + \dfrac{\sum x_i}{\lambda}$ $\qquad \ell''(\lambda) = -\dfrac{\sum x_i}{\lambda^2}$

By CLT, $\bar X \xrightarrow{D} N\left(\lambda, \frac{\lambda}{n}\right)$

$nI(\lambda) = -E[\ell''(\lambda)] = -E\left[-\frac{\sum x_i}{\lambda^2}\right] = \frac{1}{\lambda^2} \cdot E[n\bar X] = \frac{n\bar X}{\lambda^2}$

we didn't even need fisher info when MLE = $\bar X$

MLE for λ

$nI(\hat\lambda) = nI(\bar X) = \dfrac{n\bar X}{(\bar X)^2} = \dfrac{n}{\bar X} \Rightarrow$ $\boxed{CI: \bar X \pm z_{\frac{1+\gamma}{2}} \sqrt{\frac{\bar X}{n}}}$

Efficiency: $\text{eff}(T_1, T_2) = \dfrac{\text{Var}[T_2]}{\text{Var}[T_1]}$

When T_1, T_2 are unbiased or have same bias: $\text{eff}(T_1, T_2) > 1 \Rightarrow \text{Var}[T_2] > \text{Var}[T_1] \Rightarrow T_1$ is more efficient

Cramer Rao lower Bound (CRLB): $\text{Var}[T] \geq \dfrac{1}{nI(\theta_0)}$ → fisher info

Pf: $Z = S(\theta \mid x_1, \dots, x_n)\big|_{\theta=\theta_0}$

$-1 \leq \text{Corr}[T, Z] \leq 1$

$\Rightarrow (\text{Cor}[T, Z])^2 \leq 1$

$\Rightarrow \left(\dfrac{\text{Cov}[T, Z]}{\sqrt{V[T]\,V[Z]}}\right)^2 \leq 1$

$\Rightarrow \dfrac{\text{Cov}[T, Z]}{V[T] \cdot V[Z]} \leq 1$

$\Rightarrow \dfrac{|\text{Cov}[T, Z]|}{nI(\theta_0)} \leq V[T]$

$\Rightarrow \dfrac{1}{nI(\theta_0)} \leq V[T]$

Ques Let $X_1, X_2, \ldots, X_n$ follow Pois(λ) distribution. Find γ level MLE based confidence interval for λ.

(ii) Calculate CRLB

Ex: $X_1, X_2, \ldots, X_n \sim \text{Pois}(\lambda)$

$$L(\lambda) = \frac{e^{-n\lambda} \cdot \lambda^{\sum x_i}}{\prod_{i=1}^{n} x_i!}, \quad \ell(\lambda) = -n\lambda + \sum x_i \log\lambda + \text{const}$$

$$\ell'(\lambda) = -n + \frac{\sum x_i}{\lambda} \qquad \ell''(\lambda) = -\frac{\sum x_i}{\lambda^2}$$

$$n\mathcal{I}(\lambda) = -E[\ell''(\lambda)] = -E\left[-\frac{\sum x_i}{\lambda^2}\right] = \frac{1}{\lambda^2} \cdot E[n\bar{X}] = \frac{n\bar{x}}{\lambda^2}$$

MLE for λ

$$n\mathcal{I}(\hat{\lambda}) = n\mathcal{I}(\bar{X}) = \frac{n\bar{x}}{(\bar{x})^2} = \frac{n}{\bar{x}} \Rightarrow \Rightarrow \text{CI: } \bar{X} \pm z_{\frac{1-\gamma}{2}} \sqrt{\frac{\bar{x}}{n}}$$

$$\text{CRLB: } \frac{1}{n\mathcal{I}(\lambda_0)} = \frac{1}{\frac{n\bar{x}}{\lambda_0^2}} = \frac{\lambda_0^2}{n\bar{x}}$$