Iterative Quantum Phase Estimation: Moving Beyond Traditional QPE

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Introduction

- At this point you've learned a lot about basic quantum algorithms and have begun to start considering the effects of noise.

- Many of these algorithms (Grover's , Shor's, QPE, etc.) require long depth circuits with many nonlocal operators.

- Here we'll discuss a technique to do phase estimation which is more achievable on *today's* quantum computers. Using a *single* auxiliary qubit to store the phase

QPE Review

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For some eigenstate, $|\lambda\rangle$, of U

$$\left.\begin{aligned} U|\lambda\rangle &= \lambda|\lambda\rangle \\ \langle\lambda|U^\dagger &= \langle\lambda|\lambda^* \end{aligned}\right\} \quad \begin{aligned} \langle\lambda|U^\dagger U|\lambda\rangle &= \lambda^*\lambda\langle\lambda|\lambda\rangle \\ &= |\lambda|^2 = 1 \end{aligned}$$

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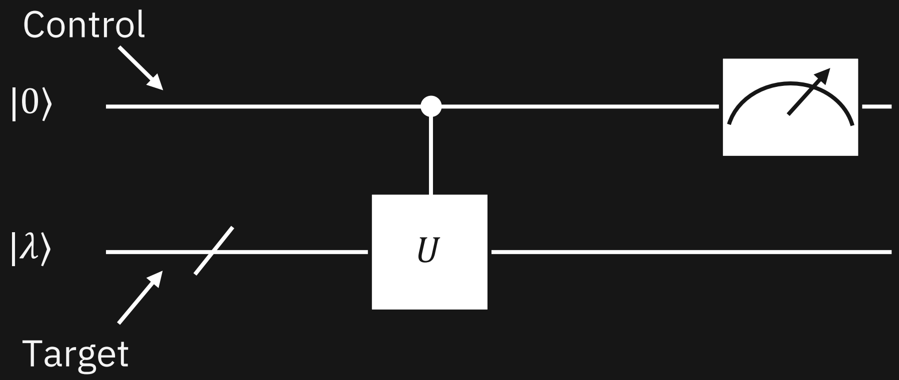
$$\boxed{\lambda = e^{i2\pi\theta}}$$

QPE Review: Phase Kickback



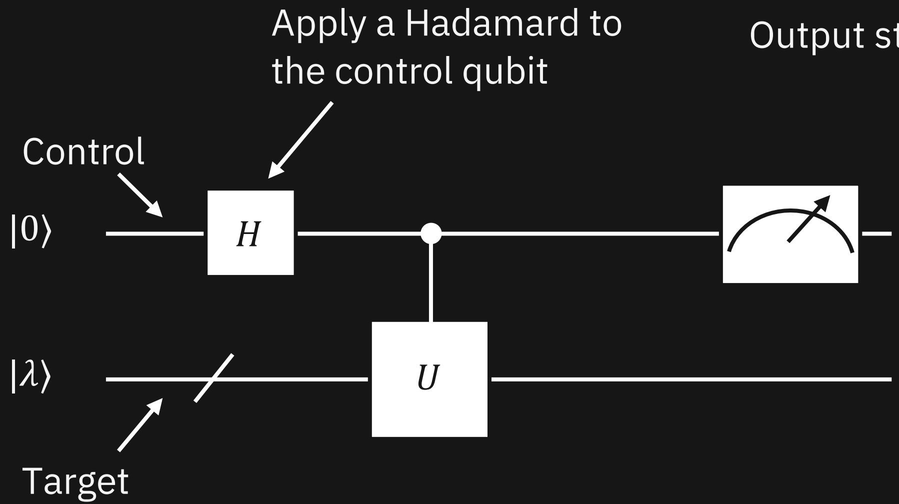
$|0\rangle$; Output State $\rightarrow |0\rangle|\lambda\rangle$

$|1\rangle$; Output State $\rightarrow U|1\rangle|\lambda\rangle$

$$= e^{i2\pi\theta}|1\rangle|\lambda\rangle$$

However, this is difficult to extract. We should try something different

QPE Review: Phase Kickback



Output state becomes:

$$\left[\frac{|0\rangle + e^{i2\pi\theta}|1\rangle}{\sqrt{2}}\right]|\lambda\rangle$$

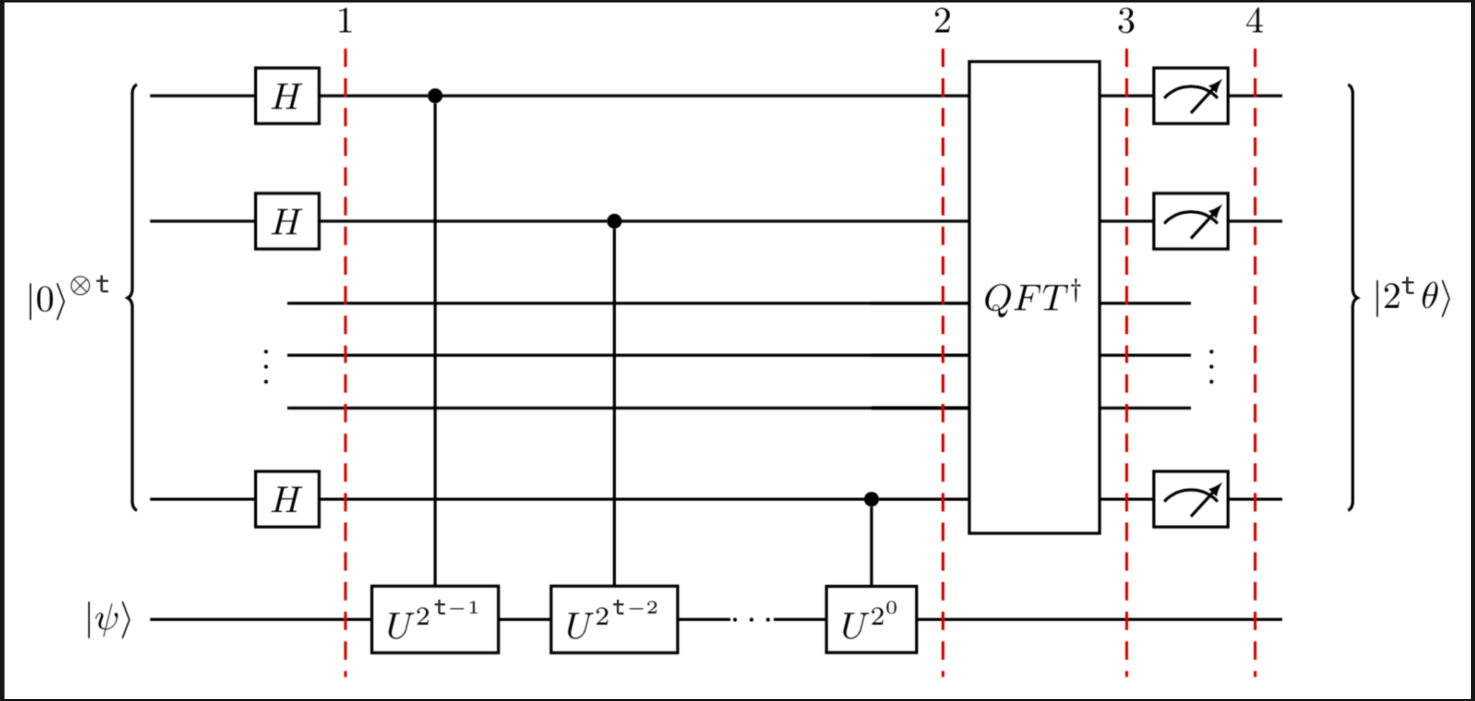
Now the control has a relative phase between basis states determined by θ

Recall also how we store the phase in binary:

$$e^{i2\pi\theta};\ \theta \in (0,1] \qquad \theta = \frac{\theta_1}{2} + \frac{\theta_1}{4} + \cdots + \frac{\theta_m}{2^m} = 0.\theta_1\theta_2 \ldots \theta_m$$

QPE Review

Recall QPE algorithm

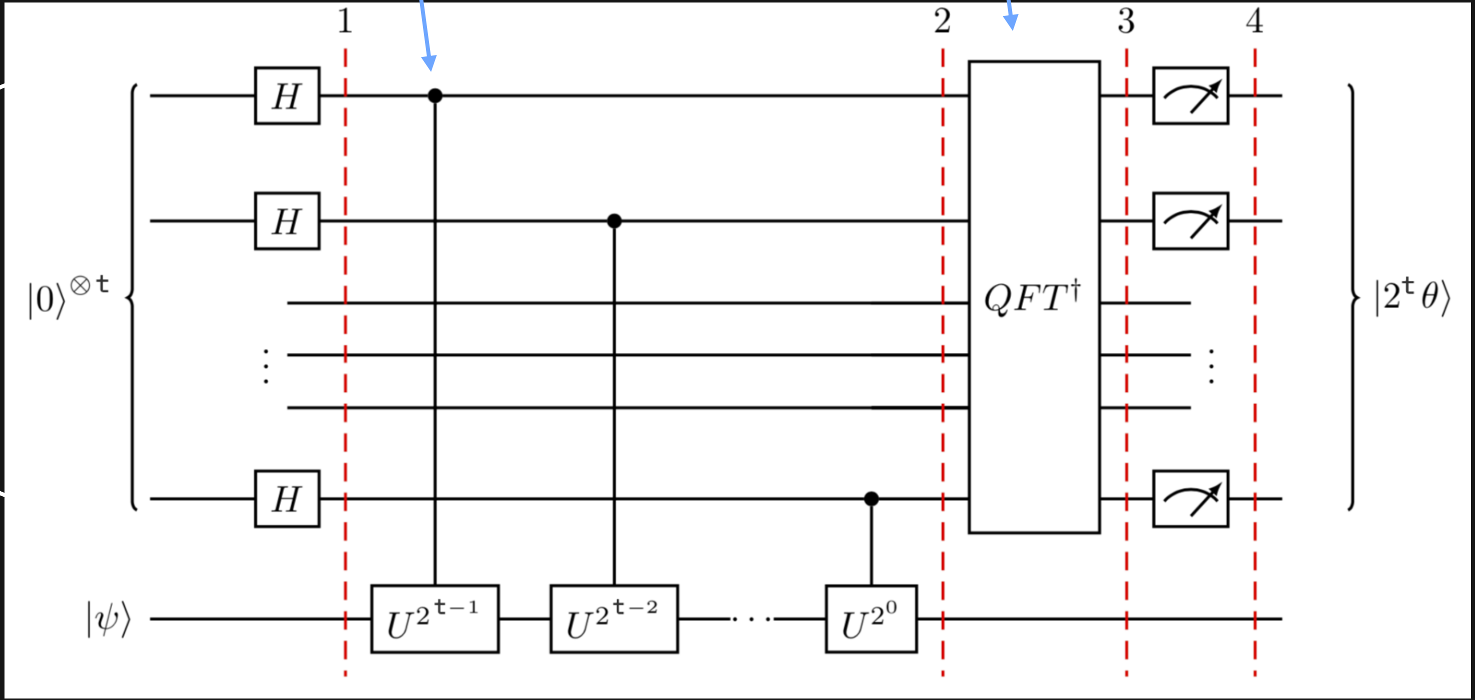


QPE Review

These gates will require *a lot* of SWAPs

Requires QFT (also very expensive!)

We have t qubits to store the phase



Iterative Phase Estimation

Iterative PE

Instead of needing so many auxiliary qubits, let's use one and *iterate* the estimation of the phase.

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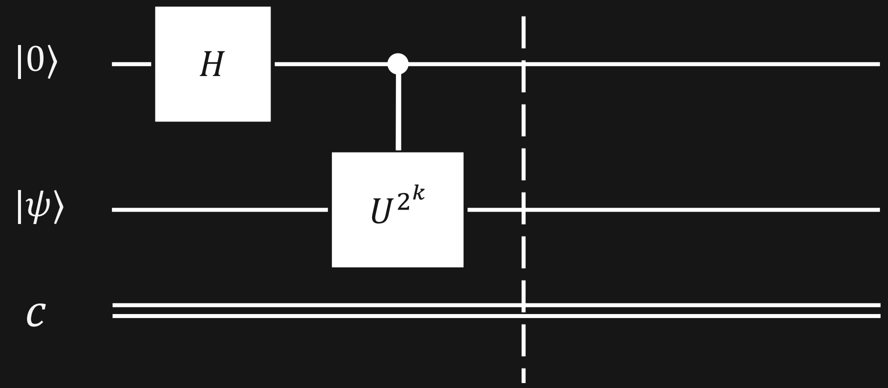

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi\theta 2^k}|1\rangle\right)|\psi\rangle$$

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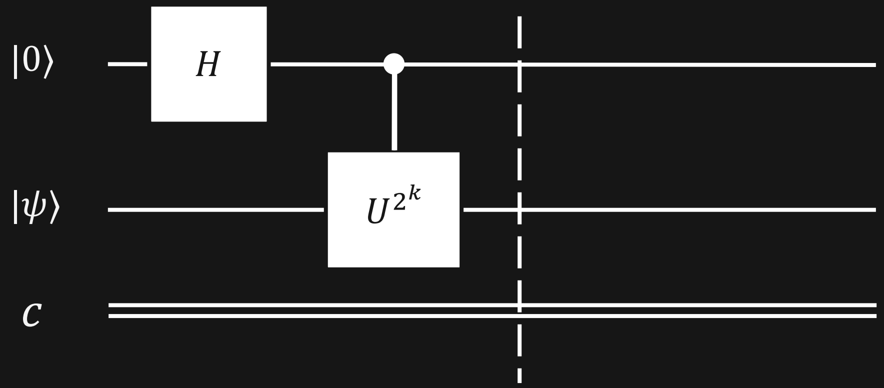

$$k = 1;\ e^{i2\pi\theta 2^0} = e^{i2\pi 0.\theta_1\theta_2\ldots\theta_m}$$

$$k = 2;\ e^{i2\pi\theta 2^1} = e^{i2\pi\theta_1}e^{i2\pi 0.\theta_2\ldots\theta_m}$$

$$k = 3;\ e^{i2\pi\theta 2^2} = e^{i2\pi\theta_1}e^{i2\pi\theta_2}e^{i2\pi 0.\theta_3\ldots\theta_m}$$

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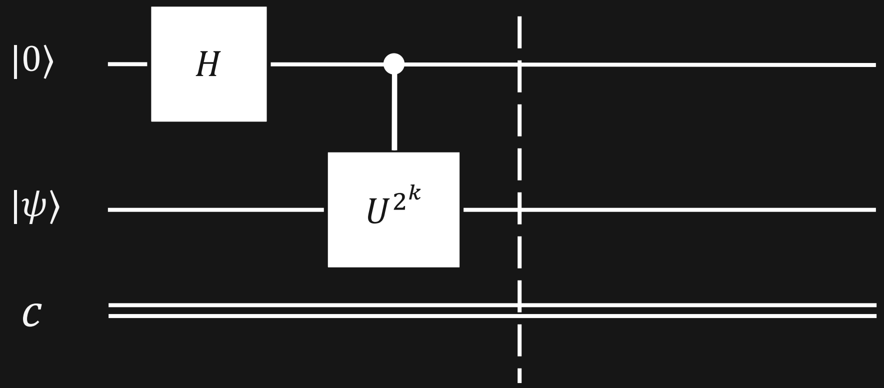

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$k = 1; e^{i2\pi\theta 2^0} = e^{i2\pi 0.\theta_1\theta_2\ldots\theta_m}$

$k = 2; e^{i2\pi\theta 2^1} = \cancel{e^{i2\pi\theta_1}} e^{i2\pi 0.\theta_2\ldots\theta_m}$

$k = 3; e^{i2\pi\theta 2^2} = \cancel{e^{i2\pi\theta_1}} \cancel{e^{i2\pi\theta_2}} e^{i2\pi 0.\theta_3\ldots\theta_m}$

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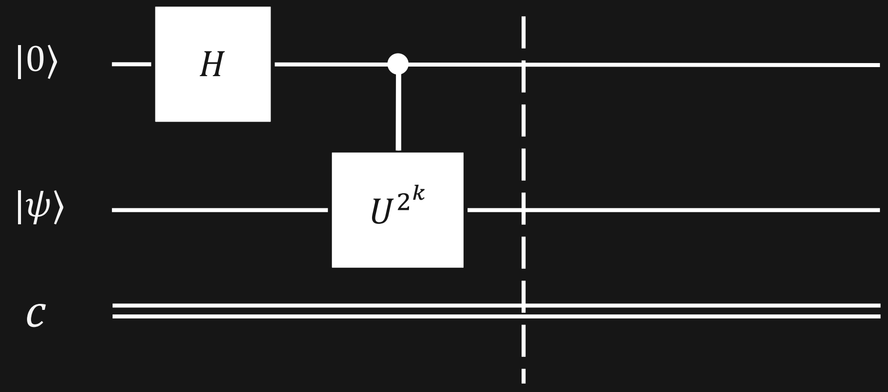

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi\theta 2^k}|1\rangle\right)|\psi\rangle$$

$t = 1; e^{i2\pi\theta 2^0} = e^{i2\pi 0.\theta_1\theta_2\ldots\theta_m}$

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$t = 3; e^{i2\pi\theta 2^2} = e^{i2\pi 0.\theta_3\ldots\theta_m}$

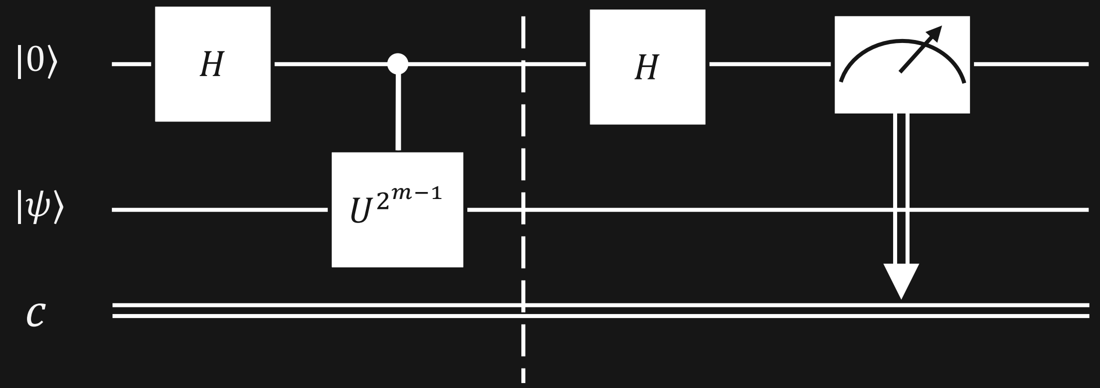
$\vdots$

$k = m - 1;\ e^{i2\pi\theta 2^{m-1}} = e^{i2\pi 0.\theta_m}$

Iterative PE

Instead of needing so many auxiliary qubits, let's use one and *iterate* the estimation of the phase.

What is the first step?



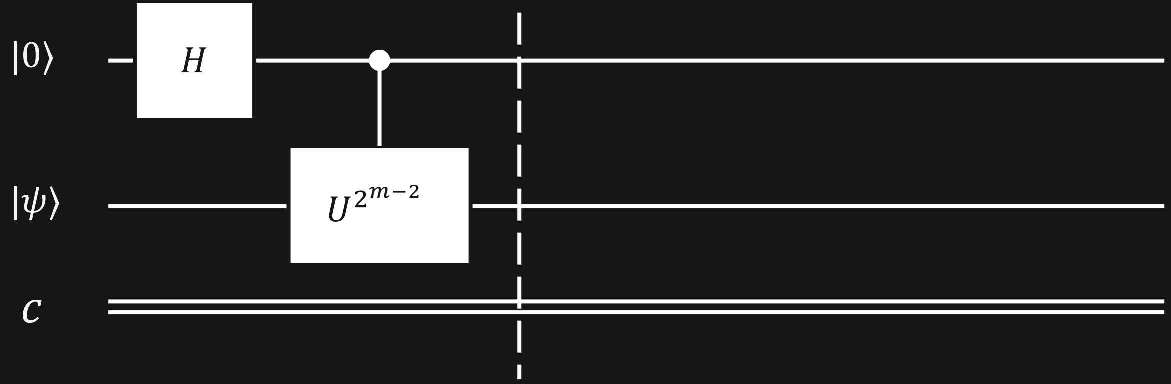
$$\begin{cases} \theta_m = 0; |+\rangle \\ \theta_m = 1; |-\rangle \end{cases}$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + \ e^{i2\pi 0.\theta_m}|1\rangle\right)|\psi\rangle$$

Iterative PE

Instead of needing so many auxiliary qubits, let’s use one and *iterate* the estimation of the phase.

The second step?



$$\frac{1}{\sqrt{2}}\left(|0\rangle + \ e^{i2\pi 0.\theta_{m-1}\theta_m}|1\rangle\right)|\psi\rangle$$

Iterative PE

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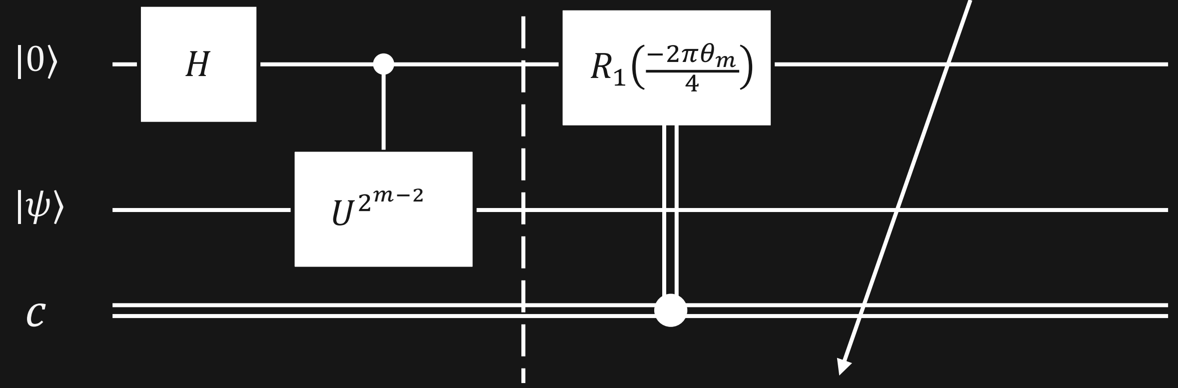
The second step?

Here, we'll isolate θ_{m-1} with a classically controlled rotation



$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi\frac{\theta_{m-1}}{2}} e^{i2\pi\frac{\theta_m}{4}}|1\rangle\right)|\psi\rangle$$

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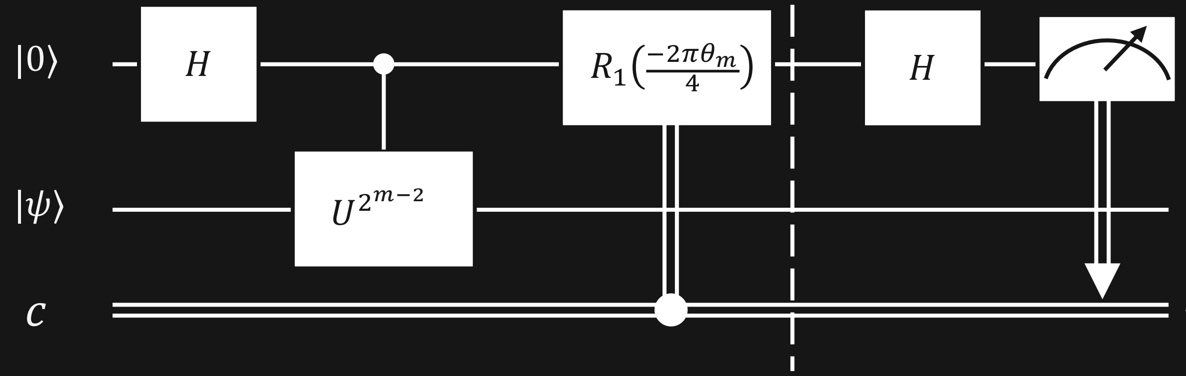
The second step?

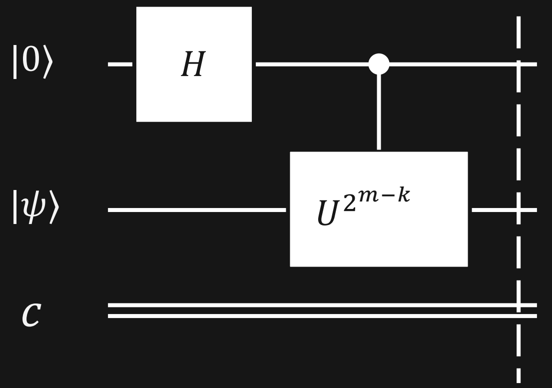


$$\begin{cases} \theta_{m-1} = 0; |+\rangle \\ \theta_{m-1} = 1; |-\rangle \end{cases}$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi\frac{\theta_{m-1}}{2}}|1\rangle\right)|\psi\rangle$$

Iterative PE

Further Steps

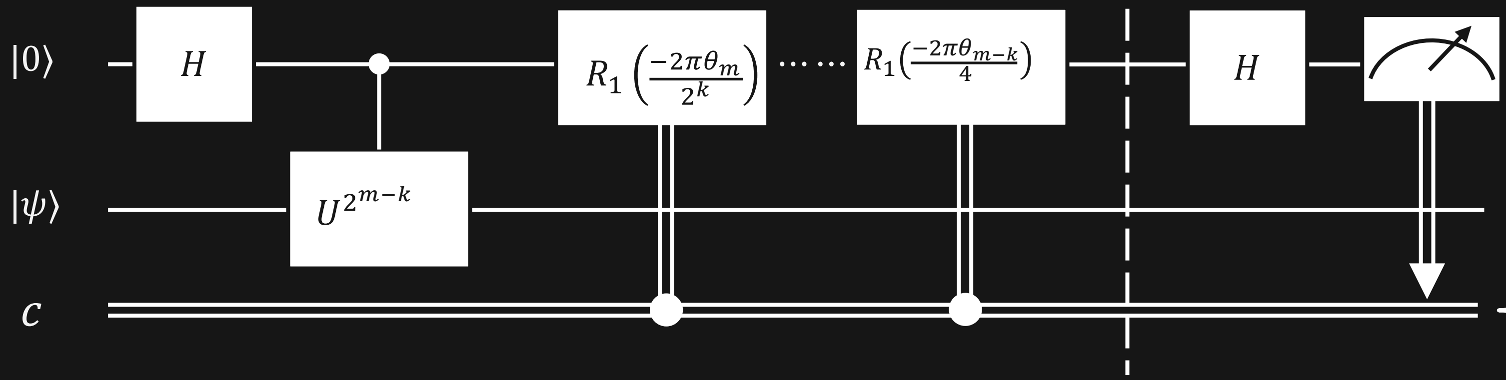


$$\frac{1}{\sqrt{2}}\left(|0\rangle + \ e^{i2\pi 0.\theta_{m-k+1}\ldots\theta_m}|1\rangle\right)|\psi\rangle$$

Iterative PE

Strategy: Iterate this circuit m times by either resetting the aux qubit or use dynamic circuits until the bitstring $\theta = 0.\theta_1\theta_2 \ldots \theta_m$ is found.

Further Steps



$$\begin{cases} \theta_{m-k+1} = 0; |+\rangle \\ \theta_{m-k+1} = 1; |-\rangle \end{cases}$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi\theta_{m-k+1}/2}|1\rangle\right)|\psi\rangle$$

Estimating Observables

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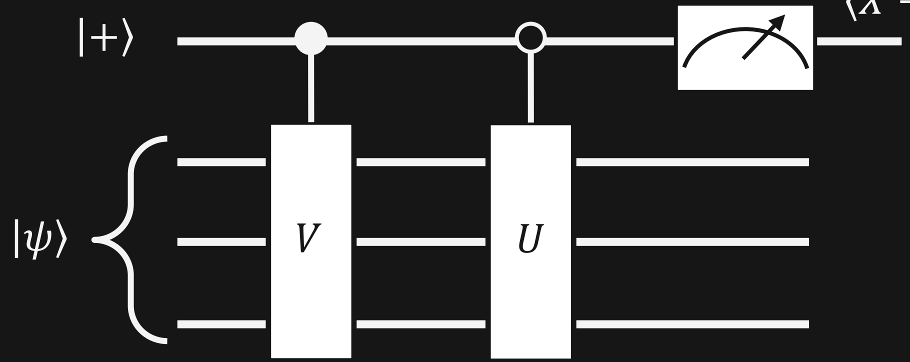
The usage of auxiliary qubits to store phase information ends up being a useful tool!



We can use this technique of phase kickback to directly measure expectations values of a pair of observables

$$\langle X + iY \rangle = \langle U^\dagger V \rangle$$

V or $U^\dagger$ can be *any* unitary operator, including the identity: I

Estimating Observables

But how can this be the case?

We know that: $\langle X + iY \rangle = \langle HZH \rangle + \langle ZHZH \rangle$

And: $|\Psi\rangle = \left[\frac{|0\rangle U|\psi\rangle + |1\rangle V|\psi\rangle}{\sqrt{2}}\right]$

$$\langle HZH \rangle = \langle \Psi | HZH | \Psi \rangle$$

$$= \tfrac{1}{2}\left[\langle 0|U^\dagger\langle\psi| + \langle 1|V^\dagger\langle\psi|\right] HZH \left[|0\rangle U|\psi\rangle + |1\rangle V|\psi\rangle\right]$$

$$= \tfrac{1}{2}\langle\psi|U^\dagger V|\psi\rangle + \tfrac{1}{2}\langle\psi|V^\dagger U|\psi\rangle$$

$$\langle ZHZH \rangle = \langle \Psi | ZHZH | \Psi \rangle$$

$$= \tfrac{1}{2}\left[\langle 0|U^\dagger\langle\psi| + \langle 1|V^\dagger\langle\psi|\right] ZHZH \left[|0\rangle U|\psi\rangle + |1\rangle V|\psi\rangle\right]$$

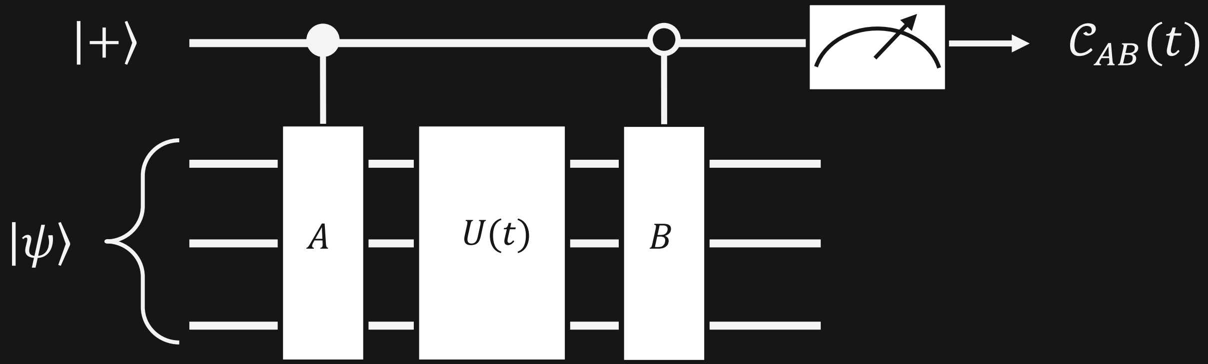
$$= \tfrac{1}{2}\langle\psi|U^\dagger V|\psi\rangle - \tfrac{1}{2}\langle\psi|V^\dagger U|\psi\rangle$$

$$\boxed{\langle X + iY \rangle = \langle\psi|U^\dagger V|\psi\rangle = \langle U^\dagger V\rangle}$$

Estimating Observables

We can also utilize time evolution to characterize physical systems

$$\langle X + iY \rangle = \langle e^{iHt} B^{\dagger} e^{-iHt} A \rangle$$



Review

- “Traditional” QPE is too expensive to achieve given the current hardware available. Both in terms of the circuit depth and the required number of qubits.
- Using an auxiliary qubit to measure individual bits in the estimation string is much more achievable.
    - This allows us to estimate the phase by instead running more circuits which is generally cheaper to do
- This technique of utilizing an auxiliary qubit to store and measure phase information can be extended to characterize simulations of physical systems. Very useful indeed!

Thank you