A Minimal Split-Complex Witness for $1 + 1$ Lorentz Kinematics

Reciprocal Idempotent Sectors, Invariant Pairings, and a Formally Verified Existence Construction

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Abstract

The Lorentz transformation is usually introduced either from physical postulates or from an already Lorentzian geometric setting. This note isolates a narrower mathematical question: what minimal, explicitly defined algebraic carrier is sufficient to realize $1 + 1$-dimensional Lorentz kinematics with two reciprocal sectors? We use the split-complex algebra

$$A := \mathbb{R}[\varepsilon]/(\varepsilon^2 - 1).$$

Its canonical idempotents $e_+ = (1 + \varepsilon)/2$ and $e_- = (1 - \varepsilon)/2$ decompose every element uniquely into two complementary zero-divisor sectors. A one-parameter action that scales these sectors by e^η and $e^{-\eta}$, respectively, is a group of bijections preserving their product. Under $u = ct + x$ and $v = ct - x$, the preserved product is exactly $c^2t^2 - x^2$, and the induced transformation is the standard $1 + 1$ Lorentz boost, up to the sign convention for rapidity. Contragradiently transformed dual coefficients yield two separately invariant scalar pairings.

The construction is intentionally interpretation-light: c is an input constant, no physical realization of the algebra is asserted, and no $3 + 1$-dimensional extension is claimed. The complete proof obligations have been formalized in Lean 4 and independently reproduced by a clean local build. The central methodological point is a *conservation of difficulty*: the Lorentz structure is derived once the split carrier is given, but the reason for selecting that carrier is not hidden or declared solved.

Keywords: split-complex numbers; hyperbolic numbers; Lorentz transformation; null coordinates; idempotents; rapidity; formal verification; Lean 4.

1 Problem statement: deriving rather than presupposing a Lorentz carrier

There are two distinct mathematical tasks that are easily conflated. The first is to work inside a Lorentzian structure once its metric, null directions, or Lorentz group have already been specified. The second is to identify a weaker algebraic structure from which the characteristic $1 + 1$ Lorentz relations follow. This note addresses only the second task, and only as an existence construction.

The methodological discipline used here may be described as *conservation of difficulty*. A difficult explanatory step is not removed by renaming it or by inserting an interpretation. If a construction begins by assuming the very Lorentz metric it is meant to explain, the derivational burden has merely been concealed. Conversely, if a weaker carrier is posited, one must state exactly which part of the burden has been discharged and which part remains. In the present construction the defining relation

$$\varepsilon^2 = 1$$

is assumed; the Minkowski quadratic form and the reciprocal boost action are then obtained from explicitly stated constructions. Why such a carrier should be selected by any deeper theory is left open.

The target is deliberately modest and exact: construct a real algebra with two complementary sectors, define a one-parameter reciprocal action on them, prove the group law and an invariant product, identify the resulting quadratic form with $1+1$ Minkowski geometry, and prove two invariant scalar pairings under the dual action. No particle model, wave model, clock model, internal motion, or physical ontology is required.

2 Historical and mathematical context

The algebraic ingredients are classical. Cockle's nineteenth-century work on tessarines introduced hyperbolic units whose square is $+1$ [3, 2]. Modern terminology for the two-dimensional real algebra generated by such a unit includes *split-complex*, *hyperbolic*, *duplex*, and *perplex* numbers. Fjelstad used perplex numbers explicitly in a special-relativistic setting [5], and Sobczyk gave a systematic treatment of the hyperbolic number plane [8].

Einstein's 1905 paper derived special-relativistic kinematics from the relativity principle and the light postulate [4]. Minkowski subsequently recast special relativity in a geometric language centered on an invariant spacetime structure and its transformation group [6]. The present note does not offer an alternative empirical foundation for those results. It isolates the algebraic fact that the $1+1$ Lorentz boost becomes diagonal in the canonical idempotent basis of the split-complex algebra.

No novelty is therefore claimed for the connection between split-complex numbers and Lorentz transformations. The contribution here is organizational and formal: the construction is reduced to explicit obligations, the boundary between assumption and consequence is kept visible, and the resulting minimal statement is machine-checked.

3 The split-complex carrier and its canonical sectors

Let

$$A := \mathbb{R}[\varepsilon]/(\varepsilon^2 - 1), \qquad \varepsilon^2 = 1.$$

Concretely, write $X = a + b\varepsilon$ with $a, b \in \mathbb{R}$, and define multiplication by

$$(a + b\varepsilon)(c + d\varepsilon) = (ac + bd) + (ad + bc)\varepsilon.$$

Define conjugation by

$$\overline{a + b\varepsilon} := a - b\varepsilon.$$

The two canonical idempotents are

$$e_+ := \frac{1+\varepsilon}{2}, \qquad e_- := \frac{1-\varepsilon}{2}.$$

Proposition 1 (Canonical complementary sectors)**.** *The elements e_+ and e_- satisfy*

$$e_+^2 = e_+, \qquad e_-^2 = e_-, \qquad e_+e_- = 0, \qquad e_+ + e_- = 1.$$

Both are nonzero. Every $X = a + b\varepsilon \in A$ has a unique representation

$$X = ue_+ + ve_-$$

with

$$u = a + b, \qquad v = a - b.$$

Proof. Using $\varepsilon^2 = 1$,

$$e_+^2 = \frac{1 + 2\varepsilon + \varepsilon^2}{4} = \frac{1 + \varepsilon}{2} = e_+,$$
$$e_-^2 = \frac{1 - 2\varepsilon + \varepsilon^2}{4} = \frac{1 - \varepsilon}{2} = e_-,$$

and

$$e_+e_- = \frac{1 - \varepsilon^2}{4} = 0, \qquad e_+ + e_- = 1.$$

Moreover,

$$ue_+ + ve_- = \frac{u + v}{2} + \frac{u - v}{2}\varepsilon.$$

Equating coefficients with $a + b\varepsilon$ gives $u + v = 2a$ and $u - v = 2b$, hence $u = a + b$ and $v = a - b$. The linear system is nonsingular over $\mathbb{R}$, so the pair is unique. □

Thus A is canonically decomposed as

$$A = \mathbb{R}e_+ \oplus \mathbb{R}e_-.$$

4 Null coordinates and the quadratic form

Define the sector coordinates

$$u(X) := a + b, \qquad v(X) := a - b.$$

Then

$$X = u(X)e_+ + v(X)e_-.$$

Conjugation exchanges the two sector coordinates, and multiplication by the conjugate gives

$$X\overline{X} = (a^2 - b^2)1 = u(X)v(X)1.$$

The split norm is therefore indefinite. In particular, nonzero elements may have zero norm; the two idempotent sectors are zero-divisor sectors.

Now introduce a fixed positive real constant $c > 0$ and define

$$u := ct + x, \qquad v := ct - x.$$

Then

$$uv = (ct + x)(ct - x) = c^2t^2 - x^2.$$

Proposition 2 (Minkowski form from the sector product)**.** *Under the coordinate identification $u = ct + x$ and $v = ct - x$,*

$$X\overline{X} = (c^2t^2 - x^2)1.$$

Thus the split-conjugation product realizes the $1 + 1$ Minkowski quadratic form with signature $(+, -)$.

Remark 1. *The constant c is not derived. It is an externally supplied positive conversion constant between the two coordinate dimensions. The statement at this stage is structural: the algebraic product of the two canonical sector coordinates has the same quadratic form as the standard $1 + 1$ Minkowski interval.*

5 Reciprocal sector action and the Lorentz boost

For each rapidity parameter $\eta \in \mathbb{R}$, define

$$B_\eta(ue_+ + ve_-) := e^\eta ue_+ + e^{-\eta} ve_-.$$

Proposition 3 (One-parameter reciprocal group)**.** *For all $\eta, \eta_1, \eta_2 \in \mathbb{R}$,*

$$B_0 = \mathrm{id}, \qquad B_{\eta_1} \circ B_{\eta_2} = B_{\eta_1+\eta_2}, \qquad B_\eta^{-1} = B_{-\eta}.$$

Moreover,

$$u(B_\eta X) = e^\eta u(X), \qquad v(B_\eta X) = e^{-\eta} v(X).$$

Proof. The sector formulas are the definition. The group law follows from

$$e^{\eta_1} e^{\eta_2} = e^{\eta_1+\eta_2}$$

in the e_+ sector and

$$e^{-\eta_1} e^{-\eta_2} = e^{-(\eta_1+\eta_2)}$$

in the e_- sector. Replacing η by $-\eta$ gives the inverse. □

5.1 Invariant product

The product of the two sector coordinates is unchanged:

$$u(B_\eta X)v(B_\eta X) = e^\eta e^{-\eta} u(X)v(X) = u(X)v(X).$$

Consequently, with $u = ct + x$ and $v = ct - x$,

$$c^2t'^2 - x'^2 = c^2t^2 - x^2.$$

5.2 Return to (ct, x) coordinates

Recover the original coordinates from the sector variables by

$$ct = \frac{u+v}{2}, \qquad x = \frac{u-v}{2}.$$

After applying B_η,

$$ct' = \frac{e^\eta(ct+x) + e^{-\eta}(ct-x)}{2},$$
$$x' = \frac{e^\eta(ct+x) - e^{-\eta}(ct-x)}{2}.$$

Using

$$\cosh\eta = \frac{e^\eta + e^{-\eta}}{2}, \qquad \sinh\eta = \frac{e^\eta - e^{-\eta}}{2},$$

one obtains

$$ct' = \cosh(\eta)\, ct + \sinh(\eta)\, x,$$
$$x' = \sinh(\eta)\, ct + \cosh(\eta)\, x.$$

This is the standard $1+1$ Lorentz boost up to the sign convention for rapidity. The common convention with negative off-diagonal entries is obtained by $\eta \mapsto -\eta$; equivalently one may reverse the spatial orientation or exchange the two sector labels.

5.3 Rapidity and ordinary velocity

Set

$$\beta := \tanh \eta.$$

Then $|\beta| < 1$ and

$$\gamma := \frac{1}{\sqrt{1-\beta^2}} = \cosh \eta, \qquad \gamma\beta = \sinh \eta.$$

Hence the reciprocal sector action reproduces the ordinary $1+1$ Lorentz matrix after the change from rapidity η to the dimensionless velocity parameter β.

6 Two invariant dual pairings

Introduce real dual coefficients $k_+, k_- \in \mathbb{R}$ and define

$$\Phi_+ := k_+ u, \qquad \Phi_- := k_- v.$$

Transform the dual coefficients contragradiently:

$$k'_+ := e^{-\eta} k_+, \qquad k'_- := e^{\eta} k_-.$$

Then

$$\begin{aligned} \Phi'_+ &= k'_+ u' = (e^{-\eta} k_+)(e^{\eta} u) = \Phi_+, \\ \Phi'_- &= k'_- v' = (e^{\eta} k_-)(e^{-\eta} v) = \Phi_-. \end{aligned}$$

Proposition 4 (Sectorwise invariant pairings)**.** *The reciprocal sector representation admits two separately invariant real scalar pairings, one on each sector, under the corresponding dual action. No complex structure is required for this statement.*

If desired, one may subsequently define

$$\psi_+ := e^{i\Phi_+}, \qquad \psi_- := e^{i\Phi_-}.$$

Their invariance is then an immediate corollary of the real equalities $\Phi'_+ = \Phi_+$ and $\Phi'_- = \Phi_-$. The complex exponentials add no content to the Lorentz derivation and are not used elsewhere in this note.

7 Minimal existence theorem

Theorem 1 (Minimal split-complex witness for $1+1$ Lorentz kinematics)**.** *There exists a real commutative algebra A with nonzero complementary idempotents e_+ and e_-, real coordinate maps $u, v : A \to \mathbb{R}$, and a one-parameter family of bijections B_η such that:*

(i) every element has a unique decomposition

$$X = u(X)e_+ + v(X)e_-;$$

(ii) conjugation satisfies

$$X\overline{X} = u(X)v(X)1;$$

(iii)

$$u(B_\eta X) = e^{\eta} u(X), \qquad v(B_\eta X) = e^{-\eta} v(X);$$

(iv)

$$B_0 = \mathrm{id}, \quad B_{\eta_1} B_{\eta_2} = B_{\eta_1+\eta_2}, \quad B_\eta^{-1} = B_{-\eta};$$

(v)

$$u(B_\eta X)v(B_\eta X) = u(X)v(X);$$

(vi) *under $u = ct + x$ and $v = ct - x$ with fixed $c > 0$, the induced transformation is a $1 + 1$ Lorentz boost and preserves $c^2t^2 - x^2$;*

(vii) *for all real k_+, k_-, the pairings k_+u and k_-v are invariant under the reciprocal dual transformation of their coefficients.*

Proof. Take

$$A = \mathbb{R}[\varepsilon]/(\varepsilon^2 - 1), \qquad e_+ = \frac{1+\varepsilon}{2}, \qquad e_- = \frac{1-\varepsilon}{2}.$$

Items (i)–(vii) are exactly the statements established in propositions 1 to 4 and the intervening calculations. □

8 Conservation of difficulty: exact boundary of the result

The strongest conclusion that follows from theorem 1 is *structural sufficiency*: the explicitly defined split-complex carrier is sufficient to realize $1 + 1$ Lorentz kinematics and two reciprocal invariant pairings. A stronger statement—that the carrier itself has been derived from more primitive assumptions—does not follow.

This distinction is the central methodological safeguard. The relation $\varepsilon^2 = 1$ already selects a split algebra rather than an ordinary complex algebra. Once that relation is admitted, the idempotent decomposition follows algebraically. Once the reciprocal exponential action is defined, the Lorentz boost follows by a change of basis. The remaining difficulty is therefore sharply localized: what, if anything, selects this carrier and this action in a deeper construction?

Question	Status
Do complementary idempotent sectors exist in the stated algebra?	Proved
Does reciprocal exponential scaling form a one-parameter group?	Proved
Is the sector product invariant?	Proved
Does the product become $c^2t^2 - x^2$ under $u = ct+x$, $v = ct-x$?	Proved
Does the induced map reproduce the $1 + 1$ Lorentz boost?	Proved
Do two separately invariant dual scalar pairings exist?	Proved
Is c derived by the construction?	No
Is the split-complex carrier physically realized?	Not claimed
Is a $3 + 1$-dimensional Lorentz structure derived?	No
Do Φ_+ and Φ_- have a quantum-mechanical meaning?	Not claimed

Table 1: Exact scope of the construction.

9 Formal verification in Lean 4

The complete obligation set was formalized by the Aristotle Lean agent. The delivered file `RequestProject/SplitComplexLorentz.lean` realizes the algebra concretely as a twocomponent real type with multiplication

$$(a, b)(c, d) = (ac + bd, ad + bc),$$

defines the idempotents and sector coordinates, and proves the decomposition, norm identity, boost group, Minkowski invariance, rapidity relation, and dual-pairing invariance. The bundled final theorem is named

`Split.exists_reciprocal_sectors_with_invariant_phases`.

The formal environment is Lean 4.28.0 with mathlib v4.28.0. The resolved mathlib Git revision in `lake-manifest.json` is

`8f9d9cff6bd728b17a24e163c9402775d9e6a365`.

Lean 4 and mathlib are documented in [7, 9].

The artifact was independently reproduced locally on 27 August 2026. After `lake update` and `lake exec cache get`, `lake build` completed successfully with 8028 jobs. This reproduction is a software-verification fact: it confirms that the supplied project elaborates and builds under its pinned dependency set; it does not enlarge the mathematical conclusion.

Formal proof artifact: `https://aristotle.harmonic.fun/dashboard/requests/bf83645f-493c-47a7-b344-5b1e3d2e04c4` [1]

9.1 Correspondence between mathematical obligations and Lean theorems

Mathematical obligation	Lean theorem(s)
Idempotents and complementarity	`ePlus_sq`; `eMinus_sq`; `ePlus_mul_eMinus`; `ePlus_add_eMinus`
Unique sector decomposition	`existsUnique_idempotent_decomposition`; `eq_uCoord_smul_ePlus_add_vCoord_smul_eMinus`
Split norm and Minkowski form	`mul_conj_eq`; `mul_conj_eq'`; `null_coords_product`; `mul_conj_minkowski`
Reciprocal boost and group law	`boost_sectors`; `boost_zero`; `boost_boost`; `boostEquiv_symm`; `boost_bijective`
Invariant product and Lorentz form	`uCoord_vCoord_boost_invariant`; `norm_boost_invariant`; `boost_ct`; `boost_x`; `minkowski_invariant`; `minkowski_invariant_c`
Rapidity and gamma	`velocity_eq_tanh`; `gamma_eq_cosh`; `tanh_lt_one`
Invariant dual pairings	`phase_plus_invariant`; `phase_minus_invariant`
Bundled result	`exists_reciprocal_sectors_with_invariant_phases`

Table 2: Lean theorem correspondence for the stated obligations.

10 Discussion

In the idempotent basis, the boost is diagonal. The familiar mixing of space and time appears only after returning from sector coordinates (u, v) to (ct, x). This is the most economical reading of the construction: reciprocal scaling is elementary in the sector basis; the hyperbolic Lorentz matrix is the same action expressed in another basis.

The two invariant pairings are equally elementary. They instantiate the standard invariance of a pairing between a representation and its dual: a coordinate of weight $+1$ pairs invariantly with a dual coefficient of weight -1, and conversely. Calling the resulting real scalars *phases* is optional. Exponentiation by i may be appended, but it is not part of the Lorentz structure itself.

The construction is therefore not a physical explanation of special relativity. Its result is narrower: it supplies an exact algebraic witness showing that two complementary reciprocal sectors are sufficient for the $1+1$ Lorentz invariant and its boost group. The unresolved origin of those sectors remains an explicit problem rather than being replaced by imagery.

11 Conclusion

A real split-complex algebra generated by $\varepsilon^2 = 1$ contains two canonical nontrivial idempotents. Their coefficients provide sector coordinates whose product is the split norm. Reciprocal exponential scaling of these coefficients forms a one-parameter group and preserves that product. After the coordinate identification $u = ct + x$ and $v = ct - x$, the invariant becomes $c^2t^2 - x^2$ and the group action becomes the standard $1+1$ Lorentz boost. Dual coefficients transforming with opposite weights provide two separately invariant scalar pairings. The stated obligations have been formalized and machine-checked in Lean 4.

The derivation is complete at the level claimed and deliberately incomplete beyond it. It establishes an existence witness and an exact sufficiency theorem. It does not derive c, select the algebra from deeper principles, establish a $3+1$-dimensional structure, or attach a physical ontology to the two sectors. In the terminology of conservation of difficulty, the remaining difficulty is not removed: it is isolated.

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