From a Real Euclidean Three-Space to Lorentz Kinematics and Complex Structure

Final dependency closure: spin factor, intrinsic Lorentz cone, real Clifford envelope, and complex structure up to conjugation

Version 9 – Final

Editor: Jan Seeck
AI Assistant: OpenAI ChatGPT (GPT-5.6 Sol)
Lean Agent: Aristotle (Harmonic)

28 August 2026

Abstract

This final version closes a nine-stage dependency audit of split-complex and Lorentz kinematics. The standard Lorentz structures discussed in the literature are not claimed as new. The aim is instead to identify, with explicit formal proof boundaries, which mathematical objects are genuinely required, which are derived, which are only equivalent realizations, and which residual choices are discrete.

The final construction begins with the standard real Euclidean datum

$$V = \mathbb{R}^3, \qquad \langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3,$$

formalized in Lean as `Fin 3 -> R`. From this datum alone one forms the real spin factor

$$\mathcal{S} = \mathbb{R} \oplus V, \qquad (a, u) \circ (b, v) = \big(ab + \langle u, v\rangle,\ av + bu\big).$$

Its intrinsic quadratic norm and polarization are

$$N_{\mathcal{S}}(a, u) = a^2 - \langle u, u\rangle, \qquad B_{\mathcal{S}}((a, u), (b, v)) = ab - \langle u, v\rangle,$$

so the Lorentzian signature $(1, 3)$ is a theorem of the real construction rather than an inserted metric signature. The square cone

$$\mathcal{C}_{\mathcal{S}} = \{x \circ x : x \in \mathcal{S}\}$$

provides the causal cone. The corresponding norm-and-cone automorphism group, with the explicitly necessary real-orientation component condition, is then identified only afterward with the standard proper orthochronous Lorentz group

$$\boxed{\text{JordanConeAut} \cong SO^+(1, 3).}$$

Thus complex scalars, Hermitian matrices, determinant, matrix positivity, and $SO^+(1, 3)$ itself are not prerequisites for the Lorentz carrier.

The same real Euclidean datum admits a canonical Clifford envelope [2]. In

$$\mathrm{Cl}_3(\mathbb{R}) = \mathrm{Cl}(V, q), \qquad q(v) = \langle v, v\rangle,$$

the oriented orthonormal pseudoscalar

$$\omega = e_1e_2e_3$$

is proved intrinsically, using only real Clifford relations, to satisfy

$$\boxed{\omega^2 = -1, \qquad \omega x = x\omega \quad \forall x \in \mathrm{Cl}_3(\mathbb{R}).}$$

Hence a central complex direction is not assumed. It is generated by the real Clifford structure. A faithful Pauli representation then proves the exact centre

$$Z(\mathrm{Cl}_3(\mathbb{R})) = \mathbb{R} \oplus \mathbb{R}\omega \cong \mathbb{C},$$

the real-algebra equivalence

$$\mathrm{Cl}_3(\mathbb{R}) \cong_{\mathbb{R}\text{-alg}} M_2(\mathbb{C}),$$

and the identification $\omega \mapsto iI$. The only central square roots of -1 are $\pm\omega$, while orientation reversal and Clifford reversion send ω to $-\omega$. The resulting scalar complex structure is therefore *derived and canonical up to conjugation*, not primitive.

The final unresolved structural input of the formal chain is consequently not $\mathbb{C}$, i, the Minkowski metric, the Lorentz group, Hermitian matrices, or $SL(2,\mathbb{C})$. It is the three-dimensional positive-definite real Euclidean datum itself. The formal development is carried out on the standard model $\mathbb{R}^3$; a coordinate-free transport theorem to arbitrary three-dimensional Euclidean real vector spaces is not included. Moreover, while $\omega^2 = -1$ and centrality are proved intrinsically in the real Clifford algebra, the current proof that the *entire* centre is exactly $\mathbb{R} \oplus \mathbb{R}\omega$ uses the later faithful Pauli realization. These qualifications are part of the final result, not hidden limitations.

Keywords: Lorentz group; spin factor; Euclidean Jordan algebra; Clifford algebra; complex structure; split-complex numbers; Hermitian matrices; formal verification; Lean 4.

1 Question, scope, and status language

The familiar mathematical objects in this paper – Lorentz transformations, light-cone coordinates, split-complex numbers, Hermitian 2×2 matrices, Pauli matrices, Clifford algebras, and spin representations – are classical [3, 6, 5, 9, 8, 7, 2, 4]. The contribution of the present sequence is not the invention of those structures. It is a formalized dependency analysis of the following question:

> How far can the standard Lorentz construction be decomposed into canonical consequences of weaker mathematical data before a genuinely new input remains?

The final task deliberately uses four distinct statuses.

- **Derived**: a theorem follows from previously available data.
- **Equivalent realization**: a standard object is recognized only after an independently defined object exists.
- **Discrete selection**: the structure exists but a connected component, orientation, or sign must be chosen.
- **Primitive**: the present formal development supplies the object rather than deriving it.

The governing anti-circularity rule is strict: a target may classify the result, but may not define the precursor that is later claimed to generate it. In particular, $SO^+(1,3)$ may not define the quadratic form whose automorphism group is then called Lorentz; det may not define a supposedly determinant-free norm; $\mathbb{C}$ may not occur in the definitions from which its emergence is claimed.

Version 9 reports the cumulative Aristotle Task 9 artifact [1]. The supplied project uses Lean 4.28.0 and mathlib revision `8f9d9cff6bd728b17a24e163c9402775d9e6a365`. The supplied report states that the complete Task 1–9 project builds with `lake build`. Source inspection found no `sorry`, `admit`, project-specific `axiom`, `unsafe`, or `native_decide` escape in the Task-9 modules. The reported `#print axioms` output for every central Task-9 theorem is

`propext`, `Classical.choice`, `Quot.sound`.

The supplied artifact, source, and mechanical dependency audits were inspected for this paper. A separate local kernel rebuild was not available in the editorial environment and is therefore not claimed.

Proof boundary 1 (What the final result does not claim)**.** *The formal chain does not derive the field $\mathbb{R}$, the spatial dimension three, or the positive-definite Euclidean form on $\mathbb{R}^3$. Nor does it claim a physical ontology for these objects, derive dynamics, or establish that the real Clifford algebra is the only conceivable associative envelope of the spin factor. The result is a mathematical dependency closure for the kinematic structures treated here.*

2 The nine-stage atomization in one view

The successive stages can be read as removal of previously accepted construction inputs.

Stage	**Object provisionally accepted**	**What the stage establishes**
v1–v3	two real complementary sectors / split carrier	ordinary multiplication by positive norm-one units realizes exactly $SO_0(1,1)$; the boost need not be separately inserted.
v4	$\mathrm{Herm}_2(\mathbb{C})$ and $SL(2,\mathbb{C})$	determinant gives the $(1,3)$ quadratic form and $SL(2,\mathbb{C})/\{\pm I\} \cong SO^+(1,3)$; the $1+1$ split sector is the exact diagonal restriction.
v5	$GL(2,\mathbb{C})$ and Hermitian congruence	the effective acting group is derived from determinant preservation; $SL(2,\mathbb{C})$ becomes a representative normalization.
v6	$(M_2(\mathbb{C}),\dagger)$	Hermitian carrier, unit group, positive cone and congruence form are largely canonicalized; matrix-unit data are recognized as an equivalent presentation, not a deeper derivation.
v7	real Jordan carrier	trace, quadratic norm and square cone become derivable from Jordan data; several sector relations and normalization facts cease to be primitive.
v8	intrinsic Jordan norm and square cone	determinant, PSD, Minkowski coordinates and $SO^+(1,3)$ are removed from the definition layer; an intrinsic norm-cone automorphism group is defined first.
v9	standard real Euclidean $\mathbb{R}^3$	the Jordan carrier itself is replaced by a purely real spin factor; the real Clifford envelope produces a central $\omega^2=-1$, so the complex scalar structure is reconstructed up to conjugation.

The remainder of the paper presents the final chain in construction order rather than historical order.

3 Primitive real datum

The formal Task-9 construction starts from

$$V = \mathbb{R}^3 = \{u : \mathrm{Fin}\,3 \to \mathbb{R}\} \tag{1}$$

with the standard positive-definite symmetric bilinear form

$$\langle u, v\rangle = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{2}$$

Lean proves symmetry, bilinearity,

$$\langle u, u\rangle \geq 0,$$

and definiteness

$$\langle u, u\rangle = 0 \Longrightarrow u = 0.$$

These are the deepest structural assumptions retained by the present formal development: real scalars, dimension three, and a positive-definite Euclidean form.

Proof boundary 2 (Concrete rather than coordinate-free formalization)**.** *The Lean files work explicitly with* $\mathrm{Fin}\,3 \to \mathbb{R}$ *and the standard dot product. The mathematically standard transport to an arbitrary real three-dimensional Euclidean vector space is not separately formalized. Accordingly, the final theorem is formally a theorem about the standard model* $\mathbb{R}^3$*, although its intended coordinate-free reading is evident.*

4 The purely real spin factor

From (1)–(2) define

$$\mathcal{S} = \mathbb{R} \oplus V \tag{3}$$

and the bilinear product

$$(a, u) \circ (b, v) = (ab + \langle u, v\rangle,\ av + bu)\,. \tag{4}$$

The distinguished unit is

$$1_{\mathcal{S}} = (1, 0).$$

Task 9 proves directly over $\mathbb{R}$ that (4) is commutative, bilinear, unital, and satisfies the Jordan identity

$$(x \circ y) \circ (x \circ x) = x \circ (y \circ (x \circ x)). \tag{5}$$

The exact Lean names include `Task9.sJ_comm`, `sJ_one_left`, `sJ_add_left`, `sJ_smul_left`, and `sJ_jordan`.

This spin factor is a standard example in Euclidean Jordan theory [4]. Here it is important for a narrower reason: it provides a real carrier from which the Lorentz form and causal cone can be built without any occurrence of $\mathbb{C}$.

5 Trace, quadratic norm, and the Lorentzian bilinear form

On $\mathcal{S}$ define

$$\mathrm{tr}_{\mathcal{S}}(a, u) = 2a \tag{6}$$

and

$$N_{\mathcal{S}}(a, u) = a^2 - \langle u, u\rangle. \tag{7}$$

The polarization is

$$B_{\mathcal{S}}(x, y) = \frac{1}{2}\left(N_{\mathcal{S}}(x + y) - N_{\mathcal{S}}(x) - N_{\mathcal{S}}(y)\right). \tag{8}$$

For $x = (a, u)$ and $y = (b, v)$, Lean proves the closed form

$$\boxed{B_{\mathcal{S}}((a, u), (b, v)) = ab - \langle u, v\rangle.} \tag{9}$$

Therefore, relative to the standard basis of $\mathbb{R} \oplus \mathbb{R}^3$, the Gram matrix is

$$\boxed{\operatorname{diag}(1, -1, -1, -1).} \tag{10}$$

The Lorentzian signature $(1, 3)$ is thus a computed property of the construction, not an assumption used to define it.

The same data satisfy the intrinsic rank-two Cayley–Hamilton identity

$$x \circ x - \operatorname{tr}_{\mathcal{S}}(x)x + N_{\mathcal{S}}(x)1_{\mathcal{S}} = 0, \tag{11}$$

formalized as `Task9.sJ_cayleyHamilton`.

6 The square cone and time orientation

Define the primary cone entirely within the real Jordan algebra:

$$\mathcal{C}_{\mathcal{S}} = \{x \in \mathcal{S} : \exists y \in \mathcal{S},\ x = y \circ y\}. \tag{12}$$

No PSD condition, Lorentz coordinate inequality, or standard causal terminology enters this definition.

The earlier Task-8 chain proves, after transport to the equivalent Hermitian realization, that the square cone is pointed, contains the Jordan unit, excludes its negative, and is characterized by

$$x \in \mathcal{C}_{\mathcal{S}} \iff \operatorname{tr}_{\mathcal{S}}(x) \geq 0 \quad \text{and} \quad N_{\mathcal{S}}(x) \geq 0. \tag{13}$$

In standard coordinates $x = (T, (X, Y, Z))$, this becomes only afterward

$$T \geq 0, \qquad T^2 - X^2 - Y^2 - Z^2 \geq 0. \tag{14}$$

Thus the standard future causal cone is the comparison image of the square cone, not its definition.

7 The intrinsic Lorentz automorphism group

The correct transformation group is not the Jordan automorphism group. Task 8 proves that a nontrivial Lorentz boost generally fails to preserve the Jordan product itself. What is preserved is the derived quadratic norm and its distinguished square cone.

Let G_{wide} be the group of real-linear self-equivalences of the carrier satisfying

$$N(Fx) = N(x) \tag{15}$$

and

$$x \in \mathcal{C} \iff Fx \in \mathcal{C}. \tag{16}$$

No orientation condition is included yet. Task 9 proves

$$(\det_{\mathbb{R}} F)^2 = 1, \tag{17}$$

so

$$\det_{\mathbb{R}} F \in \{+1, -1\}.$$

The determinant sign defines a surjective homomorphism

$$\epsilon : G_{\text{wide}} \longrightarrow C_2 \tag{18}$$

whose kernel is exactly the proper subgroup G_{proper} with real determinant $+1$. Hence

$$\boxed{G_{\text{wide}}/G_{\text{proper}} \cong C_2.} \tag{19}$$

The real orientation is therefore a discrete component selection, not a continuous structural input. The formal witness for the second component is the Y-reflection from Task 8.

The proper intrinsic group is the previously defined

$$\mathrm{JordanConeAut} = G_{\mathrm{proper}}.$$

Only after this object exists independently does the standard classification theorem enter:

$$\boxed{\mathrm{JordanConeAut} \cong SO^{+}(1,3).} \tag{20}$$

The exact frozen theorem is `Task9.frozen_jordanConeAut_SO13Plus`.

This establishes the principal Lorentz chain already without complex scalars:

$$\boxed{(\mathbb{R}^3, \langle\cdot,\cdot\rangle) \longrightarrow \mathcal{S} = \mathbb{R} \oplus \mathbb{R}^3 \longrightarrow (N_{\mathcal{S}}, B_{\mathcal{S}}, \mathcal{C}_{\mathcal{S}}) \longrightarrow \mathrm{JordanConeAut} \cong SO^{+}(1,3).} \tag{21}$$

8 Hermitian matrices are a later realization, not a prerequisite

Define the familiar real-linear map

$$\Phi(T,(X,Y,Z)) = \begin{pmatrix} T+Z & X-iY \\ X+iY & T-Z \end{pmatrix}. \tag{22}$$

Task 9 proves that Φ is a real-linear equivalence

$$\mathcal{S} \simeq_{\mathbb{R}} \mathrm{Herm}_2(\mathbb{C})$$

and preserves the Jordan product:

$$\Phi(x \circ y) = \frac{1}{2}\big(\Phi(x)\Phi(y) + \Phi(y)\Phi(x)\big). \tag{23}$$

It also transports the complete intrinsic structure:

$$\mathrm{tr}_{\mathcal{J}}(\Phi x) = \mathrm{tr}_{\mathcal{S}}(x), \tag{24}$$

$$N_{\mathrm{intr}}(\Phi x) = N_{\mathcal{S}}(x), \tag{25}$$

$$B_{\mathrm{intr}}(\Phi x, \Phi y) = B_{\mathcal{S}}(x,y), \tag{26}$$

$$\Phi(\mathcal{C}_{\mathcal{S}}) = \mathcal{C}_{\mathcal{J}}. \tag{27}$$

These are respectively `Task9.toHerm_trJ`, `toHerm_Nintr`, `toHerm_Bintr`, and `coneS_image`.

Consequently

$$\boxed{\text{complex scalars are not needed to define the Lorentz Jordan carrier.}} \tag{28}$$

This is formalized as `Task9.complex_scalars_not_needed_for_carrier`.

The classical identity

$$\det \Phi(T,X,Y,Z) = T^2 - X^2 - Y^2 - Z^2$$

now appears only as a comparison theorem between the intrinsic norm and matrix determinant. Likewise matrix PSD is only the later matrix realization of the intrinsic square cone.

9 Why the Jordan carrier itself contains no scalar complex structure

One might hope that a central operator J satisfying

$$J^2 = -I$$

is already present internally in the real spin factor. Task 9 proves the opposite.

Define the Jordan centroid by the condition that a real-linear map T satisfies

$$T(x \circ y) = T(x) \circ y = x \circ T(y) \quad \forall x, y \in \mathcal{S}. \tag{29}$$

A direct basis computation yields

$$\boxed{\mathrm{Cent}(\mathcal{S}) = \mathbb{R}I.} \tag{30}$$

Therefore no centroid element can satisfy $J^2 = -I$:

$$\boxed{\text{the real Jordan carrier contains no internal central complex structure.}} \tag{31}$$

The Lean theorems are `Task9.centroid_eq_scalars` and `Task9.no_complex_structure_in_centroid`.

This negative result is conceptually important. The complex direction that appears later is not hidden in the Jordan carrier itself. It emerges only after passing to an associative envelope generated by the same real quadratic data.

10 The canonical real Clifford envelope

From the same Euclidean $V = \mathbb{R}^3$ define the positive quadratic form

$$q(v) = \langle v, v \rangle \tag{32}$$

and the real Clifford algebra

$$\mathrm{Cl}_3(\mathbb{R}) = \mathrm{Cl}(V, q). \tag{33}$$

The canonical inclusion $\iota : V \to \mathrm{Cl}_3(\mathbb{R})$ satisfies

$$\iota(v)^2 = q(v)1. \tag{34}$$

For an orthonormal basis e_1, e_2, e_3 this gives

$$e_i^2 = 1, \qquad e_i e_j = -e_j e_i \quad (i \neq j). \tag{35}$$

The spin factor embeds by

$$\jmath(a, v) = a1 + \iota(v). \tag{36}$$

Task 9 proves

$$\boxed{\frac{1}{2}(\jmath(x)\jmath(y) + \jmath(y)\jmath(x)) = \jmath(x \circ y).} \tag{37}$$

Thus the spin-factor product is exactly the symmetrized Clifford product on the embedded real carrier. The image of $\jmath$ generates the entire Clifford algebra as a real algebra:

$$\mathrm{Alg}\langle \jmath(\mathcal{S}) \rangle = \mathrm{Cl}_3(\mathbb{R}). \tag{38}$$

The Clifford algebra is canonical from the quadratic data in the precise universal-property sense: if a real algebra A receives a linear map $f : V \to A$ satisfying

$$f(v)^2 = q(v)1,$$

then there exists a unique real-algebra homomorphism

$$F : \mathrm{Cl}_3(\mathbb{R}) \to A$$

extending f. This is packaged as `Task9.clifford_universal`.

Proof boundary 3 (Canonicity of the envelope)**.** *The universal property establishes a canonical associative envelope for representations satisfying the Clifford quadratic relation. It does not prove that every associative algebra whose Jordan symmetrization contains the spin factor must be isomorphic to* $\mathrm{Cl}_3(\mathbb{R})$*. The latter classification remains open and is unnecessary for the Lorentz chain itself.*

11 A central square root of minus one from real data

Let

$$\omega = e_1 e_2 e_3 \tag{39}$$

be the pseudoscalar of an ordered orthonormal basis. Using only (35), one computes

$$\begin{aligned} \omega^2 &= e_1 e_2 e_3 e_1 e_2 e_3 &(40)\\ &= -e_1^2 e_2^2 e_3^2 &(41)\\ &= -1. &(42) \end{aligned}$$

Task 9 proves this intrinsically as

$$\boxed{\omega^2 = -1} \tag{43}$$

with theorem `Task9.omega_sq`.

Because the spatial dimension is odd, the same anticommutation relations imply that ω commutes with each generator and therefore with the entire generated algebra:

$$\boxed{\omega x = x\omega \qquad \forall x \in \mathrm{Cl}_3(\mathbb{R}).} \tag{44}$$

This is `Task9.omega_central`.

The crucial dependency fact is that (39)–(44) are proved in the real Clifford algebra before $\mathbb{C}$, complex matrices, or Lorentz groups appear. Hence the existence of a central square root of -1 is a genuine consequence of the real three-dimensional Euclidean Clifford structure.

It follows already that

$$\mathbb{R}[\omega] = \{a + b\omega : a, b \in \mathbb{R}\} \tag{45}$$

is an internal real algebra with the multiplication law of complex numbers.

12 Orientation and the sign of the complex unit

The sign of ω is not intrinsically selected by an unoriented orthonormal frame. Swapping two basis vectors reverses orientation and Task 9 proves

$$e_2 e_1 e_3 = -\omega. \tag{46}$$

The canonical Clifford reversion also satisfies

$$\widetilde{\omega} = -\omega. \tag{47}$$

Thus the natural residual ambiguity is

$$\boxed{\omega \longleftrightarrow -\omega.} \tag{48}$$

At the level of the emergent complex algebra this is exactly complex conjugation.

This discrete sign ambiguity is distinct from the earlier C_2 choice between orientation-preserving and orientation-reversing Lorentz transformations. The formal development does not identify these two $\mathbb{Z}_2$ structures as the same choice.

13 Pauli realization and exact identification with complex matrices

Only after the real Clifford construction has produced ω does the standard complex matrix realization enter. Define the Pauli representation

$$\rho : \mathrm{Cl}_3(\mathbb{R}) \longrightarrow M_2(\mathbb{C}) \tag{49}$$

by sending the three Euclidean Clifford generators to the three Pauli matrices. Their squares are I and their anticommutators reproduce the Euclidean inner product, so the Clifford universal property yields the algebra map.

Task 9 proves explicitly that ρ is injective and surjective, without relying on an unavailable general finrank theorem for the pinned Clifford API. Hence

$$\boxed{\mathrm{Cl}_3(\mathbb{R}) \cong_{\mathbb{R}\text{-alg}} M_2(\mathbb{C}).} \tag{50}$$

The exact theorem is `Task9.cliffordEquivM2`.

Most importantly,

$$\boxed{\rho(\omega) = iI.} \tag{51}$$

Thus the standard matrix scalar i is identified with a previously constructed real Clifford pseudoscalar. The direction of explanation is

$$\text{real Euclidean data} \to \omega \to \rho(\omega) = iI,$$

not the reverse.

The faithful representation further proves the exact centre

$$\boxed{Z(\mathrm{Cl}_3(\mathbb{R})) = \mathbb{R} \oplus \mathbb{R}\omega \cong \mathbb{C}} \tag{52}$$

and the classification

$$\boxed{x \in Z(\mathrm{Cl}_3(\mathbb{R})),\ x^2 = -1 \Longrightarrow x = \omega \text{ or } x = -\omega.} \tag{53}$$

The latter is `Task9.central_sq_neg_one_eq_pm_omega`.

Proof boundary 4 (Intrinsic existence versus comparison-verified exactness)**.** *The existence claims $\omega^2 = -1$ and $\omega \in Z(\mathrm{Cl}_3(\mathbb{R}))$ are proved intrinsically using only the real Clifford relations. In the current Lean development, the stronger statement that the* entire *centre is exactly $\mathbb{R} \oplus \mathbb{R}\omega$, and therefore that the central square roots of -1 are exactly $\pm\omega$, is proved through the later faithful Pauli representation into $M_2(\mathbb{C})$. This is logically sound but should be distinguished from a completely internal Clifford proof of the centre theorem.*

The correct final status of the complex scalar structure is therefore

$$\boxed{\textbf{DERIVED, CANONICAL UP TO CONJUGATION.}} \tag{54}$$

14 Reversion, Hermitian adjoint, and definiteness

The real Clifford algebra has a canonical reversion anti-automorphism

$$\widetilde{xy} = \widetilde{y}\,\widetilde{x}, \qquad \widetilde{\iota(v)} = \iota(v). \tag{55}$$

Task 9 proves under the Pauli realization

$$\boxed{\rho(\widetilde{x}) = \rho(x)^\dagger.} \tag{56}$$

Hence the Hermitian adjoint used in the earlier matrix presentation is not an independent structure in this realization; it is the representation of Clifford reversion.

The same faithful representation yields

$$x\widetilde{x} = 0 \Longrightarrow x = 0, \tag{57}$$

formalized as `Task9.star_definite`. This removes the earlier added `StarDefinite` hypothesis inside the Clifford/Pauli realization.

Again the proof-status distinction matters: reversion itself is intrinsic to the real Clifford algebra; the present proof of (56) and the definiteness theorem is comparison/faithfulness based.

15 The old sector data become frame data

The early split and matrix-unit constructions used complementary sectors and a cross channel. Task 9 shows that these can be recovered inside $\mathrm{Cl}_3(\mathbb{R})$ from an orthonormal frame.

Choose the unit vector $n = e_3$ and define

$$p = \frac{1+n}{2}, \qquad q = \frac{1-n}{2}. \tag{58}$$

Since $n^2 = 1$,

$$p^2 = p, \qquad q^2 = q, \qquad p + q = 1, \qquad pq = qp = 0. \tag{59}$$

Using an orthogonal unit vector, concretely e_1, define the cross channel

$$e = e_1 q. \tag{60}$$

Then Task 9 proves

$$pe = e, \qquad eq = e, \qquad e^2 = 0, \tag{61}$$

and, under reversion,

$$e\widetilde{e} = p, \qquad \widetilde{e}e = q. \tag{62}$$

Thus the sector/cross-channel data that earlier appeared as reconstruction inputs are now reclassified as consequences of an orthonormal frame plus normalization. A particular axis is a frame choice, not a canonical direction of the Lorentz carrier.

16 Recovery of the earlier split-complex $1+1$ sector

The original $1+1$ structure is visible directly in the two complementary idempotents. Set

$$\varepsilon = p - q = n. \tag{63}$$

Then

$$\varepsilon^2 = 1. \tag{64}$$

The diagonal real subspace

$$\mathbb{R}p \oplus \mathbb{R}q$$

is therefore the split-complex two-sector carrier. For

$$z = up + vq$$

the rank-two norm reduces to

$$N(z) = uv. \tag{65}$$

Positive norm-one elements act by reciprocal scaling

$$(u, v) \mapsto (e^{\eta} u, e^{-\eta} v), \tag{66}$$

which Version 3 formalized as the exact group equivalence

$$U_1^+ \cong SO_0(1,1). \tag{67}$$

The final construction therefore does not discard the original split-complex route. It places it as a frame-selected longitudinal substructure of the later real spin-factor/Clifford picture.

17 The complete chain to the Lorentz structure

The final dependency graph can now be stated without the historical intermediate choices.

17.1 Kinematic carrier chain

Start with

$$\boxed{V = \mathbb{R}^3, \qquad \langle \cdot, \cdot \rangle > 0.} \tag{68}$$

Then

$$\mathcal{S} = \mathbb{R} \oplus V, \qquad (a, u) \circ (b, v) = (ab + \langle u, v \rangle, av + bu). \tag{69}$$

From this one obtains

$$N_{\mathcal{S}}(a, u) = a^2 - \|u\|^2, \qquad B_{\mathcal{S}}((a, u), (b, v)) = ab - \langle u, v \rangle, \tag{70}$$

with signature $(1, 3)$, together with the square cone $\mathcal{C}_{\mathcal{S}}$. The proper real-linear automorphisms preserving norm and cone form an intrinsic group, and only then

$$\boxed{\mathrm{Aut}^+_{N,\mathcal{C}_{\mathcal{S}}}(\mathcal{S}) \cong SO^+(1,3).} \tag{71}$$

Schematically,

$$\boxed{(\mathbb{R}^3, \langle \cdot, \cdot \rangle) \to \mathbb{R} \oplus \mathbb{R}^3 \to (N, B, \mathcal{C}) \to \mathrm{Aut}^+_{N,\mathcal{C}} \cong SO^+(1,3).} \tag{72}$$

17.2 Associative and complex realization chain

From the same primitive real datum,

$$(\mathbb{R}^3, q) \to \mathrm{Cl}_3(\mathbb{R}) \to \omega = e_1 e_2 e_3, \tag{73}$$

with

$$\omega^2 = -1, \qquad \omega \in Z(\mathrm{Cl}_3(\mathbb{R})). \tag{74}$$

Then

$$\boxed{Z(\mathrm{Cl}_3(\mathbb{R})) = \mathbb{R} \oplus \mathbb{R}\omega \cong \mathbb{C}, \qquad \omega \leftrightarrow -\omega} \tag{75}$$

up to the proof-status qualification already stated. Finally,

$$\boxed{\mathrm{Cl}_3(\mathbb{R}) \cong_{\mathbb{R}\text{-alg}} M_2(\mathbb{C}), \qquad \omega \mapsto iI, \qquad \widetilde{x} \mapsto \rho(x)^\dagger.} \tag{76}$$

The previously established spinorial chain may then be attached:

$$G_{\det}/U(1)_{\mathrm{scalar}} \cong SL(2, \mathbb{C})/\{\pm I\} \cong SO^+(1,3). \tag{77}$$

This chain is now a realization of the already constructed Lorentz carrier, not the foundational definition of that carrier.

18 Final assumption ledger

Object	Final status	Reason
$\mathbb{R}$	primitive	scalar field of the entire formal development.
$V = \mathbb{R}^3$ and positive Euclidean form	primitive	deepest retained structural datum; concrete standard model is used.
Jordan unit and product on $\mathbb{R} \oplus V$	canonical / derived	explicit formula determined by the Euclidean datum.
Jordan trace	derived	intrinsic spin-factor/Jordan construction.
Quadratic Lorentz norm	derived	$a^2 - \langle u, u \rangle$; determinant only later agrees with it.
Lorentzian bilinear form and signature $(1, 3)$	derived	polarization of the intrinsic norm.
Future causal cone	derived	square cone, later identified with PSD/future cone.
$SO^+(1, 3)$	standard classification target	identified with an already defined intrinsic norm-cone automorphism group.
Real orientation $+1$	discrete component selection	exact quotient by the proper subgroup is C_2.
$\mathrm{Herm}_2(\mathbb{C})$	equivalent realization	real Jordan equivalent to the spin factor.
$\mathrm{Cl}_3(\mathbb{R})$	canonical from quadratic data / realization	universal Clifford envelope; uniqueness among all possible associative envelopes is not claimed.
central $J^2 = -1$	derived	$\omega^2 = -1$ and centrality proved intrinsically.
choice $J \leftrightarrow -J$	canonical up to conjugation	orientation reversal sends ω to $-\omega$; central roots are exactly $\pm\omega$ in the formal comparison proof.
$M_2(\mathbb{C})$	equivalent realization	real-algebra equivalent to $\mathrm{Cl}_3(\mathbb{R})$.
Hermitian adjoint	derived in realization	image of Clifford reversion under the Pauli equivalence.
$SL(2, \mathbb{C})$	realization-only normalization	effective action was already derived from $G_{\det}/U(1)$.
sectors p, q and cross channel e	frame choice / normalization	constructed from an orthonormal frame in $\mathrm{Cl}_3(\mathbb{R})$.

The central revision relative to Version 8 is therefore unequivocal:

$$\boxed{\text{complex scalar structure is no longer in the primitive ledger.}} \tag{78}$$

19 Mechanical dependency audits

Task 8 had already installed a compile-time audit excluding standard determinant, PSD, matrix trace, Lorentz matrices, the earlier determinant-cone group, and Hermitian coordinate maps from the intrinsic Jordan definitions.

Task 9 extends the same method to the real precursor definitions. The audit recursively inspects the project-level dependencies of objects including

`Spin`, `sip`, `sJ`, `sOne`, `strS`, `NS`, `BS`, `ConeS`,

`bil3`, `q3`, `Cl3`, `cle`, `omega`, `spinToCl`, `pSec`, `qSec`, `eCross`

and checks that they do not depend on complex numbers, matrices, matrix determinant or trace, matrix positivity, Hermitian carrier definitions, Minkowski objects, or Task-8 Lorentz groups.

Positive controls are essential. The audit also checks that later comparison theorems *do* expose the expected dependencies: the spin-factor/Hermitian equivalence must mention the Hermitian map; $\rho(\omega) = iI$ must mention i; the Clifford–matrix equivalence must mention matrices. Hence a passing audit is not explained merely by a broken detector.

The dependency result is precisely the one needed for the main claim:

$$\boxed{\text{the definitions that generate } \omega^2 = -1 \text{ are real and complex-free.}} \tag{79}$$

20 Formal theorem map

The main Task-9 theorems are grouped below by logical role.

Lean theorem	**Mathematical content**
`Task9.real_lorentz_carrier_chain`	packages the real spin-factor Jordan carrier, transport of trace/norm/form/cone, and centroid no-go.
`Task9.spin_jordan_equiv_herm`	$\mathcal{S} \cong \mathrm{Herm}_2(\mathbb{C})$ as real unital Jordan carriers.
`Task9.complex_scalars_not_needed_for_carrier`	explicit transport showing that complex scalars are unnecessary for the Lorentz Jordan carrier.
`Task9.orientation_is_a_C2_choice`	norm-and-cone wide group has determinant sign ± 1, kernel equal to the proper group, and quotient of cardinality two.
`Task9.clifford_jordan`	symmetrized Clifford multiplication reproduces the spin-factor Jordan product.
`Task9.adjoin_spinToCl_eq_top`	the embedded spin factor generates $\mathrm{Cl}_3(\mathbb{R})$.
`Task9.omega_sq`	$\omega^2 = -1$ intrinsically.
`Task9.omega_central`	ω commutes with all of $\mathrm{Cl}_3(\mathbb{R})$ intrinsically.
`Task9.center_Cl3`	exact centre $\mathbb{R} \oplus \mathbb{R}\omega$; current proof uses the faithful Pauli realization.
`Task9.center_isomorphic_complex`	$Z(\mathrm{Cl}_3(\mathbb{R})) \cong \mathbb{C}$.
`Task9.central_sq_neg_one_eq_pm_omega`	the only central square roots of -1 are $\pm\omega$.
`Task9.omega_swap01`	swapping two frame vectors sends ω to $-\omega$.
`Task9.reverse_omega`	Clifford reversion sends ω to $-\omega$.
`Task9.cliffordEquivM2`	$\mathrm{Cl}_3(\mathbb{R}) \cong_{\mathbb{R}\text{-alg}} M_2(\mathbb{C})$.
`Task9.rho_omega`	$\rho(\omega) = iI$.
`Task9.rho_reverse`	Clifford reversion corresponds to Hermitian adjoint.
`Task9.star_definite`	$x\widetilde{x} = 0 \Rightarrow x = 0$ in this realization.
`Task9.sectors_from_orthonormal_data`	packages the idempotent sector and normalized cross-channel relations from frame data.
`Task9.clifford_universal`	the universal property of the real Clifford envelope.

Lean theorem	Mathematical content
`Task9.frozen_jordanConeAut_SO13Plus`	final classification of the intrinsic proper norm-cone group as $SO^+(1,3)$.

21 Conservation of difficulty

The final reduction is substantial, but it is not a derivation from structure-free data. The Task-9 audit explicitly records where information remains.

21.1 Dimension three is supplied

The formal construction assumes three Euclidean spatial directions. This is precisely what allows a grade-three pseudoscalar and fixes the eventual Lorentz signature to $(1,3)$. The dimension is not derived.

21.2 Euclidean positivity is supplied

The input form on V is positive definite. The Lorentzian minus signs in $N_{\mathcal{S}}$ arise in the spin-factor norm; they are not inserted as a Minkowski metric. Nevertheless, positive Euclidean geometry remains part of the primitive datum.

21.3 Clifford sign convention matters

The formal Clifford relation is

$$v^2 = +q(v)1.$$

For positive q in three dimensions this yields $\omega^2 = -1$. With the opposite Clifford sign convention, the pseudoscalar square changes. The result is therefore a theorem relative to the stated Clifford convention, and the convention is not concealed.

21.4 The sign of the complex unit is not selected

The construction produces the pair $\{\omega, -\omega\}$. Selecting one as $+i$ is equivalent to choosing a complex orientation. The existence of complex structure is derived; its orientation is discrete.

21.5 Associative-envelope uniqueness remains open

The Clifford algebra is canonical through its universal property from the quadratic data. What has not been proved is that every possible associative algebra realizing the same Jordan spin factor must be this Clifford algebra. The Lorentz carrier does not depend on resolving this question.

22 Relation to standard Lorentz and spinor formalisms

The final result can be compared with the standard presentation without changing its dependency direction.

The real spin factor is Jordan equivalent to $\mathrm{Herm}_2(\mathbb{C})$, and the Clifford algebra is real-algebra equivalent to $M_2(\mathbb{C})$. The standard Hermitian determinant then equals the already derived spin-factor norm, and the standard Hermitian congruence action becomes one realization of the already derived Lorentz automorphism group.

Earlier stages proved

$$G_{\det}/U(1)_{\text{scalar}} \cong SL(2,\mathbb{C})/\{\pm I\} \cong SO^+(1,3). \tag{80}$$

Version 9 changes the interpretation of this chain. $SL(2, \mathbb{C})$, the central scalar circle, and $M_2(\mathbb{C})$ are no longer foundational ingredients required to obtain Lorentz kinematics. They belong to an associative/spinorial realization generated from the same real Euclidean datum that already produced the Lorentz Jordan carrier.

This separation also explains why a complex structure need not be present in the Jordan centroid even though a central complex structure appears in the Clifford envelope. These are different algebraic layers and should not be conflated.

23 What the final result establishes – and what it does not

The formally established conclusion is:

(i) a purely real spin factor built from standard Euclidean $\mathbb{R}^3$ carries the quadratic norm, Lorentzian bilinear form, square cone, and intrinsic proper norm-cone group;

(ii) that intrinsic group is subsequently identified with $SO^+(1, 3)$;

(iii) the same real Euclidean data canonically generate a real Clifford envelope in which an intrinsic central pseudoscalar satisfies $\omega^2 = -1$;

(iv) the standard complex matrix realization identifies ω with iI and proves that the central square roots of -1 are exactly $\pm\omega$;

(v) the standard split-complex $1 + 1$ sector reappears as a frame-selected two-idempotent substructure.

The development does *not* establish:

- a physical derivation of spatial dimension three;
- a physical derivation of the Euclidean inner product;
- a dynamical theory or field equations;
- a unique physical selection of future versus past, real orientation, or ω versus $-\omega$;
- uniqueness of the Clifford algebra among every conceivable associative envelope of the spin factor;
- a coordinate-free Lean theorem covering every abstract three-dimensional Euclidean real vector space;
- a completely internal Clifford proof of the exact centre theorem in place of the current Pauli-comparison proof.

These are not omissions hidden behind the final equivalences; they are the exact remaining proof boundaries.

24 Final dependency diagram

The complete construction can be displayed as two branches from one real datum.

$$
\boxed{\begin{array}{cc}
(\mathbb{R}^3, \langle\cdot,\cdot\rangle) & (\mathbb{R}^3, \langle\cdot,\cdot\rangle) \\
\downarrow & \downarrow \\
\mathcal{S} = \mathbb{R} \oplus \mathbb{R}^3 & \mathrm{Cl}_3(\mathbb{R}) \\
\downarrow & \downarrow \\
N_{\mathcal{S}},\ B_{\mathcal{S}},\ \mathcal{C}_{\mathcal{S}} & \omega = e_1e_2e_3, \quad \omega^2 = -1,\ \omega \in Z \\
\downarrow & \downarrow \\
\mathrm{Aut}^+_{N,\mathcal{C}}(\mathcal{S}) & \mathbb{R}[\omega] \cong \mathbb{C} \quad \text{up to } \omega \leftrightarrow -\omega \\
\downarrow \text{ classification} & \downarrow \text{ standard realization} \\
SO^+(1,3) & M_2(\mathbb{C}), \quad \omega \mapsto iI
\end{array}} \tag{81}
$$

The two branches are compatible through the symmetrized Clifford product and through the Jordan equivalence with $\mathrm{Herm}_2(\mathbb{C})$. They should nevertheless be distinguished: Lorentz kinematics is already present on the real Jordan branch, whereas the scalar complex structure appears in the associative Clifford branch.

25 Final conclusion

The nine-stage analysis began with a chosen split-complex $1+1$ carrier and asked what part of Lorentz kinematics was genuinely being inserted. Successive rounds removed the boost as an independent input, reclassified $SL(2,\mathbb{C})$ as a normalization of the effective action, replaced determinant and PSD by intrinsic Jordan constructions, and finally replaced the complex Hermitian carrier itself by a purely real spin factor.

The final kinematic chain is therefore

$$
\boxed{\begin{array}{c}
(\mathbb{R}^3, \langle\cdot,\cdot\rangle) \longrightarrow \mathbb{R} \oplus \mathbb{R}^3 \longrightarrow N = a^2 - \|u\|^2, \\
B \text{ of signature } (1,3),\ \mathcal{C} \longrightarrow \mathrm{Aut}^+_{N,\mathcal{C}} \cong SO^+(1,3).
\end{array}} \tag{82}
$$

No complex scalar is required in this chain.

The same real Euclidean datum generates the associative Clifford branch

$$
\boxed{(\mathbb{R}^3, q) \longrightarrow \mathrm{Cl}_3(\mathbb{R}) \longrightarrow \omega = e_1e_2e_3, \qquad \omega^2 = -1, \qquad \omega \in Z(\mathrm{Cl}_3(\mathbb{R})).} \tag{83}
$$

The standard Pauli realization then identifies

$$
\boxed{Z(\mathrm{Cl}_3(\mathbb{R})) = \mathbb{R} \oplus \mathbb{R}\omega \cong \mathbb{C}, \qquad \mathrm{Cl}_3(\mathbb{R}) \cong_{\mathbb{R}-\mathrm{alg}} M_2(\mathbb{C}), \qquad \omega \mapsto iI.} \tag{84}
$$

The central complex unit is unique up to

$$
\boxed{\omega \leftrightarrow -\omega,}
$$

i.e. up to complex conjugation, with the stated qualification that exact centre uniqueness is presently comparison-verified through the faithful Pauli representation.

The deepest retained non-derived structure is therefore not i, $\mathbb{C}$, the Minkowski metric, the Lorentz group, Hermitian matrices, or $SL(2,\mathbb{C})$. In the formal model it is the standard three-dimensional real Euclidean datum itself. Everything downstream in the chain above is either derived, a later standard identification, a realization choice, or a discrete component/orientation selection.

This is the closure point of the present series. Any further foundational reduction would no longer be an atomization of the Lorentz formalism treated here; it would have to explain why the primitive real Euclidean three-dimensional datum itself is available.

Acknowledgements

The editor thanks Thomas Klimpel ("Jakito") for discussions that sharpened the distinction between resolving a mathematical difficulty and relocating it into an earlier assumption. That distinction became the organizing audit criterion of the series. The final Task-9 falsification branch was deliberately allowed to reject the anticipated conclusion that complex structure might remain primitive; the formal result instead reconstructs a central square root of -1 from the real Clifford envelope, up to conjugation.

References

[1] Aristotle (Harmonic). *Formal proof artifact: Final Closure Audit – Is Complex Structure the Last Genuine Input?* Lean 4.28.0; mathlib revision 8f9d9cff6bd728b17a24e163c9402775d9e6a365; supplied cumulative Task-9 artifact. 2026. URL: `https://aristotle.harmonic.fun/dashboard/requests/bf83645f-493c-47a7-b344-5b1e3d2e04c4` (visited on 08/28/2026).

[2] Claude Chevalley. *The Algebraic Theory of Spinors and Clifford Algebras.* Ed. by Pierre Cartier and Catherine Chevalley. Vol. 2. Collected Works. Springer, 1997. ISBN: 9780387570631.

[3] Albert Einstein. "Zur Elektrodynamik bewegter K"orper". In: *Annalen der Physik* 322.10 (1905), pp. 891–921. DOI: `10.1002/andp.19053221004`.

[4] Jacques Faraut and Adam Korányi. *Analysis on Symmetric Cones.* Oxford Mathematical Monographs. Oxford University Press, 1994. ISBN: 9780198534778. DOI: `10.1093/oso/9780198534778.001.0001`.

[5] Paul Fjelstad. "Extending Special Relativity via the Perplex Numbers". In: *American Journal of Physics* 54.5 (1986), pp. 416–422. DOI: `10.1119/1.14605`.

[6] Hermann Minkowski. "Raum und Zeit". In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* 18 (1909). Lecture delivered 21 September 1908, pp. 75–88.

[7] Gregory L. Naber. *The Geometry of Minkowski Spacetime: An Introduction to the Mathematics of the Special Theory of Relativity.* 2nd ed. Vol. 92. Applied Mathematical Sciences. Springer, 2012. DOI: `10.1007/978-1-4419-7838-7`.

[8] Roger Penrose and Wolfgang Rindler. *Spinors and Space-Time, Volume 1: Two-Spinor Calculus and Relativistic Fields.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1984. DOI: `10.1017/CBO9780511564048`.

[9] Garret Sobczyk. "The Hyperbolic Number Plane". In: *The College Mathematics Journal* 26.4 (1995), pp. 268–280. DOI: `10.1080/07468342.1995.11973712`.