Split Complex Lorentzian Dynamics

Part VIII: The Third Direction, Full Associative Ambient Closure, and the Remaining Norm Obstruction

A formally verified continuation of the dependency audit

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Abstract

Part VII resolved the Task 06 associativity obstruction only for two inherited spatial directions. Once mixed products were allowed to leave the original real $3+1$ carrier, two orthogonal generators A and B forced a new product $P = AB$ outside the old carrier, and $\mathrm{span}_{\mathbb{R}}\{1, A, B, P\}$ formed the unique minimal four-dimensional associative closure of that subsystem. A crucial obligation remained deliberately open: whether the complete original Vec4, including its third independent spatial direction, admits a common associative ambient extension preserving the inherited square law for every old vector.

This paper reports Task 08, which adds exactly that third direction and no other new assumption. From the retained old square law and polarization, its image C satisfies $C^2 = 1$ and anticommutes with both A and B. Only after these relations are derived are the mixed products $Q := AC$, $R := BC$ and $S := (AB)C$ introduced. Associativity forces their complete interaction table and the squares $Q^2 = R^2 = S^2 = -1$. Direct membership arguments show that Q, R, S lie outside the embedded old carrier and, together with C, outside the Task 07 closure. The eight elements $1, A, B, C, P, Q, R, S$ are linearly independent in every admissible ambient extension. Their span is multiplicatively closed, has real dimension exactly eight, and is minimal among product-closed subspaces containing the unit and all three inherited spatial generators.

The highest mixed product S is especially diagnostic. It is forced to commute with every generator of the minimal closure and satisfies $S^2 = -1$. This central square root of minus one is not assumed as a scalar, orientation tensor or complex structure. It appears only after the three inherited spatial directions are associatively closed.

After the classification, a fresh eight-dimensional real witness is constructed from the independently derived multiplication table and proved directly associative. The complete original Vec4 is then embedded linearly and injectively into this witness. For every old vector X, its square in the ambient algebra equals the embedded inherited square $\mathsf{oldSq}X$; polarization consequently recovers the complete old symmetric spin-factor/Jordan product. The full-ambient existence obligation left open by Task 07 is therefore resolved positively.

Only in a final comparison layer is the derived algebra identified with $M_2(\mathbb{C})$, equivalently the classical real Clifford algebra $\mathrm{Cl}_{3,0}(\mathbb{R})$. None of this is new mathematics. The formal result is instead a calibrated dependency reconstruction: three inherited spatial directions force precisely the additional product channels required by the standard associative envelope, and the internal central complex structure appears at the exact stage predicted by established Clifford theory.

The Conservation-of-Difficulty analysis consequently changes again. The associativity obstruction from Part VI is now resolved and localized: it was an obstruction to product closure inside the old carrier, not to associative extension itself. The second independent Part VI obstruction remains untouched: global multiplicativity of the Lorentz quadratic form. A narrow Task 09 should therefore ask whether Q_4 admits a multiplicative quadratic extension to the newly derived eight-dimensional associative closure, without importing the late matrix identification, a determinant, conjugation or complex scalars. The already derived central plane $\mathrm{span}_{\mathbb{R}}\{1, S\}$ is the smallest natural internal candidate value space to test, but its conventional interpretation must again be deferred until after the reconstruction.

Keywords: Lorentzian geometry; longitudinal split-complex sectors; spin factor; associative closure; three anticommuting spatial generators; Clifford algebra; complex matrix algebra; central square root of minus one; formal verification; Lean 4; conservation of difficulty; dependency analysis.

Contents

1 The last unresolved direction from Part VII

Part VI produced two logically independent universal obstructions for products closed inside the original real $3+1$ carrier. No sector-compatible bilinear product on that carrier can be globally associative, and independently no such product can make the Lorentz quadratic form globally multiplicative. Task 07 addressed only the first obstruction and only its smallest two-direction witness. It showed that once mixed products may leave $\mathsf{Vec4}$, the relation system generated by two orthogonal inherited spatial directions is consistent and has a minimal four-dimensional associative closure [9, 10, 1].

That result deliberately did *not* establish a full associative ambient extension of the old carrier. A full old vector has three independent spatial components. The third direction could, in principle, have introduced an inconsistency, an infinite cascade of new products, or a finite closure different from the standard structure suggested by recognition after the fact.

Task 08 therefore makes one addition only: it introduces the missing old spatial direction c. Nothing else is relaxed. The ambient algebra remains real, associative and unital. The embedding of the old carrier remains injective. The inherited old square law remains fixed for all old vectors. No target dimension, grading, conventional algebra, norm, involution or matrix representation is supplied.

The tension is thus sharper than in Part VII. The question is no longer whether one mixed product can escape the carrier. It is whether the *entire* old rest space can coexist inside one associative extension without changing the previously reconstructed $3+1$ square structure.

2 Retained data and the third inherited generator

Let E be an arbitrary real vector space with associative bilinear product $\star$, unit 1_E, and injective linear map

$$\iota : \mathsf{Vec4} \longrightarrow E, \qquad \iota(e_0) = 1_E, \tag{1}$$

satisfying the retained square law

$$\iota(X) \star \iota(X) = \iota(\mathsf{oldSq} X) \qquad \text{for all } X \in \mathsf{Vec4}. \tag{2}$$

As in Task 07, polarization of (2) recovers the complete old symmetric product:

$$\iota(X) \star \iota(Y) + \iota(Y) \star \iota(X) = 2\,\iota(\mu_{\mathrm{sym}}(X, Y)) \,. \tag{3}$$

Choose three mutually orthogonal normalized spatial directions a, b, c in the old rest space and write

$$A := \iota(a), \qquad B := \iota(b), \qquad C := \iota(c). \tag{4}$$

The Task 07 relations are inherited for A and B:

$$A^2 = B^2 = 1_E, \qquad AB = -BA, \qquad P := AB, \qquad P^2 = -1_E. \tag{5}$$

The new direction C is not assigned an algebraic relation by definition. Its square follows from (2):

$$C^2 = 1_E. \tag{6}$$

Its orthogonality to a and b, inserted into (3), gives

$$AC = -CA, \qquad BC = -CB. \tag{7}$$

Thus all three old spatial generators square to $+1$ and pairwise anticommute, but this relation system has been reconstructed from the inherited Lorentz/spin-factor square law rather than supplied as a conventional algebra presentation.

Methodological principle 1 (Third-direction discipline)**.** Task 08 is allowed to know that C is an old spatial vector. It is not allowed to know how many mixed-product directions C will force, whether the resulting span is finite, or which established algebra would realize the relations.

3 Three new products are forced by one old direction

Only after (6)–(7) are established does Task 08 define

$$Q := AC, \qquad R := BC, \qquad S := PC = (AB)C. \tag{8}$$

Associativity immediately removes parenthesization ambiguity. In particular,

$$S = (AB)C = A(BC) = AR = PC = CP, \tag{9}$$

while

$$(AC)B = QB = -S. \tag{10}$$

The channel S is therefore not an independently appended coordinate. It is forced as the product of an already forced Task 07 channel with the third inherited direction.

The three new squares are consequences of the pairwise anticommutation relations:

$$Q^2 = R^2 = S^2 = -1_E. \tag{11}$$

For example,

$$Q^2 = (AC)(AC) = A(CA)C = -A(AC)C = -(A^2)(C^2) = -1_E. \tag{12}$$

The same pattern gives $R^2 = -1_E$. For $S = (AB)C$, the three sign exchanges required to reverse the repeated generators again produce the negative unit.

Task 08 then derives every product of Q, R, S with A, B, C, P rather than importing a multiplication table. A representative subset is

$$\begin{aligned}
AQ &= C, & QA &= -C, && (13)\\
BQ &= -S, & QB &= -S, && (14)\\
CQ &= -A, & QC &= A, && (15)\\
AR &= S, & RA &= S, && (16)\\
BR &= C, & RB &= -C, && (17)\\
CR &= -B, & RC &= B, && (18)\\
PS &= -C, & SP &= -C, && (19)\\
QS &= B, & SQ &= B, && (20)\\
RS &= -A, & SR &= -A. && (21)
\end{aligned}$$

The apparent mixture of commuting and anticommuting behaviour is not chosen. It is the output of associativity acting on the three inherited anticommuting generators.

4 The old carrier and the Task VII closure are both too small

The first structural question is whether the new symbols in (8) are actually new. Task 08 proves separately that

$$P, Q, R, S \notin \iota(\mathsf{Vec4}). \tag{22}$$

The old carrier therefore contains the unit and the three inherited spatial vectors, but none of the four mixed-product channels.

The comparison with the Task 07 closure is sharper. Let

$$G_2 := \operatorname{span}_{\mathbb{R}}\{1, A, B, P\}. \tag{23}$$

Task 08 proves

$$C, Q, R, S \notin G_2. \tag{24}$$

These are direct structural membership results rather than deductions from a preassigned target dimension. The third old direction itself is not contained in the two-generator closure, and each of the three products that it generates adds genuinely independent information.

Consequently the natural candidate family is

$$1,\ A,\ B,\ C,\ P,\ Q,\ R,\ S. \tag{25}$$

Task 08 proves that this family is linearly independent in *every* admissible ambient extension satisfying the inherited square law. This is stronger than verifying independence in one convenient representation.

The dimension count is therefore not guessed from established algebra. It emerges from the forced product channels:

$$\text{four inherited channels } (1, A, B, C) \quad + \quad \text{four mixed channels } (P, Q, R, S). \tag{26}$$

5 The eight-channel span closes

Define

$$G_3 := \operatorname{span}_{\mathbb{R}}\{1, A, B, C, P, Q, R, S\}. \tag{27}$$

The full relation ledger derived before this definition is sufficient to prove

$$G_3 \star G_3 \subseteq G_3. \tag{28}$$

No ninth product direction is forced.

Every element of G_3 has a unique normal form

$$x = \alpha 1 + \beta A + \gamma B + \delta C + \varepsilon P + \zeta Q + \eta R + \theta S, \qquad \alpha, \dots, \theta \in \mathbb{R}. \tag{29}$$

The exact real dimension is therefore

$$\boxed{\dim_{\mathbb{R}} G_3 = 8.} \tag{30}$$

Again, the ambient E is not assumed finite-dimensional. Equation (30) concerns only the smallest generated associative subsystem.

The complete multiplication table is collected only after closure has been proved. In the basis order (25) it is

$\star$	1	A	B	C	P	Q	R	S
1	1	A	B	C	P	Q	R	S
A	A	1	P	Q	B	C	S	R
B	B	$-P$	1	R	$-A$	$-S$	C	$-Q$
C	C	$-Q$	$-R$	1	S	$-A$	$-B$	P
P	P	$-B$	A	S	-1	$-R$	Q	$-C$
Q	Q	$-C$	$-S$	A	R	-1	$-P$	B
R	R	S	$-C$	B	$-Q$	P	-1	$-A$
S	S	R	$-Q$	P	$-C$	B	$-A$	-1

The table is a summary of prior theorems, not a defining oracle.

6 Minimality: the third direction costs exactly four channels

Let $H \subseteq E$ be any real-linear subspace containing

$$1,\ A,\ B,\ C \tag{31}$$

and closed under $\star$. Then closure forces

$$P = AB \in H, \qquad Q = AC \in H, \qquad R = BC \in H, \qquad S = (AB)C \in H. \tag{32}$$

Hence

$$G_3 \subseteq H. \tag{33}$$

Together with (28), this proves that G_3 is the unique least product-closed real subspace containing the unit and the three inherited spatial directions.

The cost of the third old direction can therefore be stated exactly. Relative to the Task 07 closure G_2, one adds

$$C \quad \text{and the forced products} \quad Q = AC,\ R = BC,\ S = (AB)C. \tag{34}$$

The inclusion is proper,

$$G_2 \subsetneq G_3, \tag{35}$$

and the real dimension doubles from four to eight. Nothing about this doubling was supplied as a target.

7 The highest product becomes central

Among the four mixed-product directions, $S = (AB)C$ has a special status that emerges only after the three-generator closure has been derived. Task 08 proves

$$S^2 = -1 \tag{36}$$

and

$$SX = XS \qquad \text{for } X \in \{A, B, C, P, Q, R, S\}. \tag{37}$$

By linearity, S commutes with every element of G_3. Thus

$$\boxed{S \in Z(G_3), \qquad S^2 = -1.} \tag{38}$$

This statement must be interpreted with care. No complex scalar field was assumed. No imaginary unit was adjoined. The formal derivation is entirely real. What has been proved is that the minimal real associative closure forced by the three inherited spatial directions contains an internal central element whose square is minus the unit.

The sign/permutation audit reinforces the origin of this element. Exchanging any two of A, B, C sends S to $-S$, whereas the cyclic permutation

$$A \mapsto B, \qquad B \mapsto C, \qquad C \mapsto A \tag{39}$$

leaves it fixed. Negating any one generator also negates S. These facts are derived from the product definitions and anticommutation relations; no orientation tensor is imported.

Methodological principle 2 (Do not upgrade an internal relation into an ontology)**.** The central relation $S^2 = -1$ is a theorem about the derived associative closure. It does not by itself establish that complex numbers, orientation, or any physical degree of freedom are primitive in the underlying model.

8 From conditional ambient algebra to a full Vec4 witness

Up to this point, the structure theorem is conditional: every associative ambient extension preserving the old square law must contain the eight-dimensional G_3. Task 08 then establishes non-vacuity in the strongest form required by the programme.

Only after the full table, independence, dimension and minimality have been derived does it construct a fresh real carrier

$$W \cong \mathbb{R}^8 \tag{40}$$

with basis corresponding to (25). Its bilinear product is transcribed from the already derived coefficient law. The formal development proves directly that this product is unital and associative and satisfies every generator relation above. No matrix algebra or conventional Clifford object is used for this existence proof.

The critical additional step, absent from Task 07, is the full old-carrier embedding

$$\iota_3 : \mathsf{Vec4} \longrightarrow W, \qquad (t,x,y,z) \longmapsto (t,x,y,z,0,0,0,0). \tag{41}$$

The map is linear and injective. For an arbitrary old vector $X = (t,x,y,z)$, the witness product gives

$$\iota_3(X)^2 = (t^2 + x^2 + y^2 + z^2,\ 2tx,\ 2ty,\ 2tz,\ 0,0,0,0). \tag{42}$$

This is exactly

$$\boxed{\iota_3(X)^2 = \iota_3(\mathsf{oldSq}\,X)} \qquad \text{for every } X \in \mathsf{Vec4}. \tag{43}$$

Thus the ambient hypothesis used conditionally in Tasks 07–08 is now explicitly inhabited.

Status 1 (Task-07 open obligation)**.** The existence of a real associative ambient extension of the complete old $\mathsf{Vec4}$ preserving the inherited square law is **resolved positively** by the Task 08 witness.

9 The complete old symmetric product is recovered

Equation (43) is stronger than a basis check. Applying it to $X + Y$ and using bilinearity yields, for arbitrary old vectors,

$$\boxed{\iota_3(X)\iota_3(Y) + \iota_3(Y)\iota_3(X) = 2\,\iota_3(\mu_{\mathrm{sym}}(X,Y))\,.} \tag{44}$$

The whole spin-factor/Jordan product previously reconstructed from the family of longitudinal $1+1$ sectors is therefore the symmetrized restriction of the new associative product to the embedded old carrier.

This resolves a potential ambiguity in the interpretation of Task 08. The eight-dimensional witness is not merely a toy algebra generated by three symbols that happen to satisfy the right basis relations. Its embedded four-dimensional old carrier reproduces the complete inherited diagonal and polarized structure for all linear combinations (t,x,y,z).

Consequently the Task 06 associativity theorem can now be read precisely:

> The old $3+1$ carrier cannot itself be an associative product algebra with the inherited sector rules, but it can sit injectively inside a larger real associative algebra whose symmetrized product restricts to those rules exactly.

The obstruction was therefore a closure obstruction, not an obstruction to associative extension.

10 Late reflection against established mathematics

Only after the internal eight-dimensional reconstruction is complete does Task 08 ask what has been built. The comparison module constructs an explicit real-linear, unital, multiplicative bijection

$$G_3 \longrightarrow M_2(\mathbb{C}). \tag{45}$$

Thus

$$\boxed{G_3 \cong M_2(\mathbb{C})} \tag{46}$$

as real associative algebras.

In standard Clifford terminology, the three relations

$$A^2 = B^2 = C^2 = +1, \qquad AB = -BA, \quad AC = -CA, \quad BC = -CB \tag{47}$$

generate the real Clifford algebra

$$\mathrm{Cl}_{3,0}(\mathbb{R}) \cong M_2(\mathbb{C}), \tag{48}$$

a classical result [5]. The basis

$$1,\ A,\ B,\ C,\ AB,\ AC,\ BC,\ ABC \tag{49}$$

and the central highest product with square -1 are standard features of this algebra.

There is therefore no new algebraic theorem in Task 08. The result is interesting for the experiment precisely because the blind reconstruction lands on established mathematics without being told the standard package. The eight-dimensional basis, the central square root of minus one and the complex matrix identification all appear only after the preceding dependency constraints have forced them.

The spin-factor connection is likewise standard. The symmetrized product

$$X \circ Y := \frac{1}{2}(XY + YX) \tag{50}$$

recovers the old Jordan/spin-factor structure on the paravector subspace [4, 3, 5]. Task 08 does not discover this relationship as new mathematics; it reconstructs why the associative envelope becomes necessary once products of independent spatial directions are required to coexist.

11 A sharper mirror against Experiment 1

The independent convergence with Experiment 1 is now considerably stronger. Experiment 1 began from a different dependency direction and obtained an associative Clifford envelope and a central complex structure only after reconstructing the relevant real geometric data [7, 8]. Experiment 2 was firewalled from those results. It instead followed

$$\begin{aligned}\text{Lorentz data} &\longrightarrow \text{longitudinal sectors} \longrightarrow \text{spin-factor/Jordan core}\\ &\longrightarrow \text{closure obstruction} \longrightarrow \text{associative enlargement}.\end{aligned} \tag{51}$$

Task 08 now reaches, independently, the same standard 3-generator Clifford boundary and the same central square-root-of-minus-one phenomenon.

This agreement should not be overstated. Two reconstruction paths reaching the same known object do not constitute a new proof that nature is fundamentally Clifford-algebraic. They do provide a strong internal calibration of the dependency method: both paths localize the transition from the four-dimensional symmetric carrier to its associative envelope at the products of independent spatial directions.

The convergence is particularly useful because the experiments distribute the explanatory burden differently. Experiment 1 asks how Lorentz and complex structure can emerge from weaker real data. Experiment 2 asks how far one can reconstruct the same algebraic hierarchy when Lorentz structure is accepted at the start but all later multiplicative structure is withheld.

12 Conservation of Difficulty: one obstruction is closed, one remains

The sequence from Parts IV–VIII can now be summarized as a diagnostic chain.

All longitudinal sectors force the symmetric $3 + 1$ spin-factor product but leave cross-sector gluing freedom.

$\downarrow$

Clean-room Task 06: no alternating gluing inside Vec4 can restore global associativity; independently, no in-carrier product can make Q_4 globally multiplicative.

$\downarrow$

Task 07 opens only product closure: two directions force $P = AB$ outside Vec4 and close minimally in dimension 4.

$\downarrow$

Task 08 adds the third old direction: $Q = AC$, $R = BC$, $S = (AB)C$ are forced; the minimal associative closure has dimension 8.

$\downarrow$

A direct 8-dimensional witness embeds the complete old Vec4 and preserves its square law for every old vector. The associativity obstruction is therefore resolved as an old-carrier closure deficit.

$\downarrow$

Remaining independent Part VI obstruction: how, if at all, can the Lorentz quadratic form be extended multiplicatively beyond the old carrier?

The first Task 06 obstruction has therefore been paid for explicitly. The price of associativity is not a free coefficient but a larger product carrier. For the complete old rest space, that carrier is minimally eight-dimensional.

The second Task 06 obstruction must not be silently declared solved by recognition of $M_2(\mathbb{C})$. The old quadratic form Q_4 is defined on Vec4, not on arbitrary elements of G_3. Task 08 constructs no quadratic extension, no determinant, no conjugation and no involution. Consequently the statement

$$Q_4(XY) = Q_4(X)Q_4(Y) \tag{52}$$

is simply not a meaningful global claim on G_3 until a new extension problem has been solved.

This is exactly the next Conservation-of-Difficulty location. The programme should ask whether the norm obstruction was also a closure/codomain issue, rather than importing the standard matrix answer after the identification is recognized.

Methodological principle 3 (Resolve one obstruction at a time)**.** Task 08 resolves associative carrier closure. It does not resolve the Lorentz-norm problem. The next task should reopen only that second Task 06 obstruction while keeping the now-derived associative algebra fixed.

13 Formal audit and proof boundary

The audited Task 08 artifact supplied with this work has SHA-256

`5bcad4aed7d0784e3333dfc1324f9ae8d60c954d8bdf3ecca9a74848faf34f21`.

It uses Lean 4.28.0 and mathlib revision

`8f9d9cff6bd728b17a24e163c9402775d9e6a365`.

Aristotle reports a successful full build of 8098 jobs with zero errors and zero warnings [2, 6, 11, 12].

The sixteen Task 08 modules form the strict chain

`SafeBase` → `ThirdDirection` → `InheritedRelations` → `NewMixedProducts`
→ `ForcedRelations` → `Membership` → `Independence` → `GeneratedClosure`
→ `Minimality` → `StructuralDiagnostic` → `DirectWitness`
→ `FullVec4Embedding` → `Uniqueness` → `SignPermutationAudit`
→ `Identification` → `Task08`.

The reconstruction core does not import the Task 07 identification layer. A mechanical firewall scan reports zero occurrences of conventional target-structure names in the fourteen reconstruction modules preceding identification.

The principal theorem inventory includes:

Theorem	Formal role
`task08_third_generator_square`	$C^2 = 1$ from inherited square law
`task08_pairwise_anticommutation`	A, B, C pairwise anticommute
`task08_parenthesizations`	consistency of $S = (AB)C = A(BC)$ and related forms
`task08_forced_relations`	complete derived relation ledger
`task08_old_carrier_membership`	mixed channels outside embedded Vec4
`task08_two_generator_closure_membership`	C, Q, R, S outside Task 07 closure
`task08_linear_independence`	independence of $1, A, B, C, P, Q, R, S$
`task08_normal_form`	unique eight-coefficient representation
`task08_closure`	multiplicative closure of G_3
`task08_dimension`	$\dim_{\mathbb{R}} G_3 = 8$
`task08_minimality`	least product-closed subspace containing $1, A, B, C$
`task08_structural_diagnostic`	$S^2 = -1$ and commutation properties
`task08_witness`	fresh direct eight-dimensional associative model
`task08_full_vec4_embedding`	injective full old-carrier embedding and square recovery
`task08_ambient_existence`	nonempty full ambient extension
`task08_uniqueness`	generator-preserving isomorphism class
`task08_sign_permutation_audit`	sign/permutation action on mixed channels
`task08_identification`	late $M_2(\mathbb{C})$ comparison

All twenty-four principal theorem wrappers are reported with exactly

$$[\texttt{propext},\ \texttt{Classical.choice},\ \texttt{Quot.sound}], \tag{53}$$

and the project contains no ‘sorry‘, ‘admit‘, custom axiom or ‘@[implemented_by]‘.

Methodological caveat 1 (Verification boundary)**.** *The supplied archive reports a green Lean build and the editorial audit checks the source order, theorem dependencies and critical algebraic relations. This paper does not claim an independent second kernel build in a separate Lean installation.*

14 What Task 08 proves – and what remains open

Established in Task 08	Not established in Task 08
The third inherited spatial direction is compatible with the Task 07 associative enlargement.	Any multiplicative extension of Q_4 to the new algebra.
Exactly three additional mixed channels Q, R, S are forced by adding C to the Task 07 closure.	Any conjugation, determinant, trace, involution or norm on G_3.
The minimal three-generator associative closure is eight-dimensional.	Any physical interpretation of the new product channels or of the central element S.
S is central in G_3 and satisfies $S^2 = -1$.	A claim that complex numbers are primitive or physically derived from Lorentz geometry alone.
A direct real associative witness embeds the complete old $\mathsf{Vec4}$ and preserves oldSq for every old vector.	Global Q_4 multiplicativity on products outside the old carrier.
The full old symmetric spin-factor product is recovered by polarization.	Spinors, representations, field equations or dynamics.
The late conventional algebra is $M_2(\mathbb{C}) \cong \mathrm{Cl}_{3,0}(\mathbb{R})$.	Any novelty claim about Clifford or matrix algebra.

The proof boundary is now unusually clean. Associative ambient existence is no longer hypothetical. The remaining algebraic question inherited from Part VI concerns the quadratic invariant.

15 A narrow Task 09: reopen only the Q4 obstruction

The next task should not add another spatial generator and should not broaden into representations or dynamics. The full old $3+1$ carrier is already embedded in its minimal associative closure. The unresolved Part VI question is now:

> Can the Lorentz quadratic form on the embedded old carrier be extended to a multiplicative quadratic invariant on the derived associative closure without importing the late conventional identification?

A narrow formulation can use only Task 08 reconstruction results preceding the late identification module. In particular, it may use the derived central element S and should first prove internally that

$$Z := \mathrm{span}_{\mathbb{R}}\{1, S\} \tag{54}$$

is a two-dimensional central product-closed subalgebra with $S^2 = -1$. Before any conventional naming, Z is simply the smallest already derived scalar-like internal extension of the real unit line.

The central test should then be restricted to a quadratic map

$$N : G_3 \longrightarrow Z \tag{55}$$

with the following requirements:

1. **Old-carrier recovery:** for every $X \in \mathsf{Vec4}$,

$$N(\iota_3 X) = Q_4(X)\,1. \tag{56}$$

2. **Multiplicativity:**

$$N(xy) = N(x)N(y) \qquad \text{for all } x, y \in G_3. \tag{57}$$

3. **Quadratic homogeneity:**

$$N(rx) = r^2 N(x) \qquad \text{for } r \in \mathbb{R}. \tag{58}$$

The task should *not* be told a formula for N, should not import matrices, and should not use the words determinant, conjugation, adjugate, complex norm or Clifford norm before a late comparison layer.

The diagnostic alternatives should remain open. Task 09 may find:

(a) no such multiplicative quadratic extension exists even with values in Z;

(b) one exists but is non-unique, exposing an additional missing datum;

(c) one is forced uniquely, in which case the second Task 06 obstruction has again been localized as a closure/codomain issue rather than an absolute impossibility.

Only after the existence/classification theorem is complete should a conventional comparison ask whether the result coincides with the standard multiplicative invariant of $M_2(\mathbb{C})$.

This scope is deliberately smaller than "reconstruct the norm theory of a Clifford algebra". It asks one question only: whether the old Lorentz quadratic invariant can survive multiplication once its domain is extended to the associative closure and its values are allowed to use only the central two-plane that Task 08 itself has already forced.

16 Conclusion

Part VIII closes the principal gap left by Part VII. Adding the third inherited spatial direction does not destroy the associative extension and does not lead to an uncontrolled proliferation of products. It forces exactly the additional mixed channels $Q = AC$, $R = BC$ and $S = (AB)C$. Together with $1, A, B, C, P = AB$, these eight directions are linearly independent, form a multiplicatively closed real algebra, and constitute the minimal associative closure of the complete inherited spatial triple.

The highest product S is central and satisfies $S^2 = -1$. This internal relation is derived before any conventional complex language appears. After the abstract classification, a fresh eight-dimensional witness proves the relation system consistent and, critically, embeds the entire old `Vec4`. The old square law holds for every embedded vector and the complete symmetric spin-factor/Jordan product is recovered by polarization. The Task 07 full-ambient existence obligation is therefore genuinely resolved.

The late identification $G_3 \cong M_2(\mathbb{C}) \cong \mathrm{Cl}_{3,0}(\mathbb{R})$ places the result squarely inside established mathematics. That is not a disappointment but a calibration. The blind dependency path has rediscovered the standard associative envelope at the exact point where the old four-dimensional carrier becomes too small.

The Conservation of Difficulty now has a clean split. The associativity obstruction from Part VI is resolved: its price is enlargement of the product carrier. The independent Lorentz-norm obstruction remains. Task 09 should therefore resist the temptation to exploit the recognized matrix algebra and instead ask a single new dependency question: whether Q_4 extends multiplicatively to the derived associative closure, possibly with values in the already forced central plane $\mathrm{span}_{\mathbb{R}}\{1, S\}$, and whether such an extension is forced or requires additional structure.

If that reconstruction succeeds, the programme will have located not only why the old carrier cannot support the full product, but also exactly what additional algebraic information is required for the Lorentz quadratic invariant to coexist with the associative closure. If it fails, the remaining obstruction will be correspondingly sharper. Either outcome is informative; neither should be replaced by importing the standard answer in advance.

References

[1] Aristotle (Harmonic). "Null-Sector Foundation – Task 07: Minimal Associative Closure Forced by Two Orthogonal Longitudinal Sectors". Cumulative formal experiment, run cfbefbdb-6c5f-4448-bfb4-711535634ef7; Lean 4.28.0; audited archive SHA-256 5c2cd1a2b0607f1e84f92d642fcd2e50f9604009be9ee9d179f82e5f0ec9e3f1. Sept. 1, 2026.

[2] Aristotle (Harmonic). “Null-Sector Foundation – Task 08: Minimal Associative Closure Forced by the Third Orthogonal Spatial Direction”. Cumulative formal experiment, run f1e4ddfc-ab30-4d3c-87a6-cd2c3a9cac29; Lean 4.28.0; audited archive SHA-256 5bcad4aed7d0784e3333dfc1324f9ae8d60c954d8bdf3ecca9a74848faf34f21. Sept. 2, 2026.

[3] Jacques Faraut and Adam Korányi. *Analysis on Symmetric Cones*. Oxford Mathematical Monographs. Oxford: Oxford University Press, 1994.

[4] Pascual Jordan, John von Neumann, and Eugene Wigner. “On an Algebraic Generalization of the Quantum Mechanical Formalism”. In: *Annals of Mathematics* 35.1 (1934), pp. 29–64. DOI: 10.2307/1968117.

[5] Pertti Lounesto. *Clifford Algebras and Spinors*. 2nd ed. Vol. 286. London Mathematical Society Lecture Note Series. Cambridge University Press, 2001. DOI: 10.1017/CBO9780511526022.

[6] Leonardo de Moura and Sebastian Ullrich. “The Lean 4 Theorem Prover and Programming Language”. In: *Automated Deduction – CADE 28*. Vol. 12699. Lecture Notes in Computer Science. Springer, 2021, pp. 625–635. DOI: 10.1007/978-3-030-79876-5_37.

[7] Jan Seeck, OpenAI ChatGPT, and Aristotle. *From a Real Euclidean Three-Space to Lorentz Kinematics and Complex Structure, Version 9*. Dependency reconstruction in Experiment 1. Aug. 28, 2026. URL: https://github.com/antaris82/Split-Complex-Lorentzian_Kinematics/blob/Intrinsic-Branching-from-a-Derived-Lorentz-Structure/Split_Complex_Lorentzian_Kinematics_v9.tex (visited on 09/01/2026).

[8] Jan Seeck, OpenAI ChatGPT, and Aristotle. *Intrinsic Branching from a Derived Lorentz Structure, Version 30 – Comparative Final*. Final comparative null test of Experiment 1. Aug. 30, 2026. URL: https://github.com/antaris82/Split-Complex-Lorentzian_Kinematics/blob/Intrinsic-Branching-from-a-Derived-Lorentz-Structure/Intrinsic_Branching_from_a_Derived_Lorentz_Structure_v30.tex (visited on 09/01/2026).

[9] Jan Seeck, OpenAI ChatGPT, and Aristotle. “Split Complex Lorentzian Dynamics – Part VI: Clean-Room Replication, Universal 3+1 Obstructions, and the Localization of a Closure Deficit”. Companion paper of Experiment 2, generated from the audited Task-06 artifact. Sept. 1, 2026.

[10] Jan Seeck, OpenAI ChatGPT, and Aristotle. “Split Complex Lorentzian Dynamics – Part VII: A Forced Product Direction and the Minimal Two-Generator Associative Closure”. Companion paper of Experiment 2, generated from the audited Task-07 artifact. Sept. 2, 2026.

[11] The mathlib Community. “The Lean Mathematical Library”. In: *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs*. CPP 2020. ACM, 2020, pp. 367–381. DOI: 10.1145/3372885.3373824.

[12] The mathlib Community. *mathlib4 source at revision 8f9d9cff6bd728b17a24e163c9402775d9e6a365*. 2026. URL: https://github.com/leanprover-community/mathlib4/tree/8f9d9cff6bd728b17a24e163c9402775d9e6a365 (visited on 09/01/2026).