Split Complex Lorentzian Dynamics

Part II: Intrinsic Reconstruction of the Product, Unit Selection, and a Canonicality Obstruction

A formally verified continuation of the dependency audit

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Abstract

Part I of *Split Complex Lorentzian Dynamics* resolved a fixed longitudinal $1+1$ Lorentz carrier into two complementary null sectors and proved that the Lorentz quadratic form becomes the product of their coefficients. That construction still defined the split-complex multiplication explicitly. Part II removes precisely this remaining input and asks whether the multiplication is forced by weaker Lorentz data, which assumptions are genuinely necessary, and where a non-canonical choice first appears.

The formal Task 02 development is protected by an import firewall. Its reconstruction core imports from Task 01 only the carrier

$$\mathsf{Long} = \mathbb{R}^2$$

and the quadratic form

$$Q_2(t,x) = t^2 - x^2.$$

The Task 01 split-complex product, unit, generator, idempotents, null coordinates, boosts, strata, and limiting theorems are absent from the reconstruction layer and enter only in a final comparison module. Starting from Q_2 alone, Task 02 derives its polarization

$$L_2((t,x),(s,y)) = ts - xy$$

and then quantifies over a genuinely unknown real bilinear product μ on Long. The main admissibility condition assumes only Q_2-multiplicativity and a chosen two-sided unit e; commutativity, associativity, normalization, a square relation, and idempotents are not assumed.

The principal result is an existence-and-uniqueness theorem. An admissible product with unit e exists if and only if $Q_2(e) = 1$, and for each fixed normalized e it is unique. Its independently reconstructed formula is

$$\boxed{\mu_e(X,Y) = L_2(X,e)Y + L_2(Y,e)X - L_2(X,Y)e.}$$

At the coordinate reference unit $u_0 = (1,0)$ this becomes

$$\mu_{u_0}((t,x),(s,y)) = (ts + xy,\, ty + xs),$$

exactly the Task 01 split-complex multiplication. The equality is proved only after the reconstruction, by the uniqueness theorem. Consequently the familiar relation $j^2 = 1$, commutativity, associativity, and the complementary null-idempotent structure are downstream theorems rather than primitive algebraic inputs.

The same development also finds the obstruction at the next dependency level. The Lorentz data do not select a unique normalized future vector. Distinct choices of e yield distinct products on the fixed carrier. Moreover an explicit non-identity, future-preserving Q_2-isometry

$$T_*(t,x) = \left(\frac{5}{4}t + \frac{3}{4}x, \frac{3}{4}t + \frac{5}{4}x\right)$$

has no nonzero fixed vector, and no admissible unital product is equivariant under this fixed-product action. Thus the strongest proved dependency is not "Lorentz form determines the split-complex product", but

$$\boxed{\begin{aligned}(\mathsf{Long}, Q_2) + \text{chosen normalized unit} &\Longrightarrow \text{unique norm-multiplicative bilinear product} \\ &\Longrightarrow j^2 = 1 \text{ and null idempotents.}\end{aligned}}$$

The unresolved burden is the selection of the unit, not the square relation. This is a direct instance of the program's *conservation of difficulty*: successful reconstruction removes several provisional inputs but exposes the next irreducible choice instead of hiding it. Part II introduces no dynamics or physical ontology. In particular, the selected unit is a mathematical datum and is not identified with an observer, clock, particle, velocity, or preferred physical frame.

Keywords: split-complex numbers; Lorentzian geometry; composition algebras; bilinear products; quadratic forms; unit selection; canonicality; formal verification; Lean 4; conservation of difficulty; dependency analysis.

Contents

1 Motivation: removing the last explicitly inserted algebraic operation

Part I began with an established $3 + 1$ Minkowski carrier, selected a fixed longitudinal $1 + 1$ plane, and resolved that plane into complementary null coordinates. The central identities

$$X = a\,e_+ + b\,e_-, \qquad Q_2(X) = ab, \tag{1}$$

and the reciprocal boost action

$$(a, b) \longmapsto (e^{\eta}a, e^{-\eta}b) \tag{2}$$

were formally verified, together with the distinction between a genuine finite null-boundary limit and a fixed-positive-invariant branch with no finite vector limit [9, 1]. One dependency remained deliberately exposed: the multiplication

$$(t, x) \star (s, y) = (ts + xy,\ ty + xs) \tag{3}$$

was defined rather than reconstructed from the Lorentz form.

That is the point at which Part II begins. The new question is not whether (3) is a valid split-complex realization—Part I already established that. The question is whether it is *forced*, and if so by what exact data. This distinction matters methodologically. If a product is declared and then used to derive $j^2 = 1$, idempotents, and null sectors, those later statements may be perfectly correct while the origin of the product remains unexplained. Conversely, if the product can be classified from weaker data, the derivational burden moves downward.

The relevant algebraic background is classical. Bilinear composition laws for quadratic forms go back at least to Hurwitz's systematic study of composition problems [6]; modern composition-algebra treatments organize norm-multiplicative algebras and their split forms in a general structural setting [12]. The present result is not claimed as a new classification theorem in that literature. Its contribution is narrower: an explicit, theorem-by-theorem reconstruction on the concrete Lorentz carrier used by the experiment, with a machine-audited separation between what is input, what is derived, what fails, and what is compared only after the derivation is closed.

Methodological principle 1 (Conservation of difficulty)**.** A successful derivation may remove an assumed structure only by transferring the explanatory burden to weaker data or to a newly exposed choice. If $j^2 = 1$, commutativity, associativity, and the idempotents become theorems, the next question is not declared solved; one must identify which remaining datum makes those theorems possible and whether that datum is itself determined.

Part II is therefore a reconstruction-and-obstruction paper. A positive uniqueness result is only half of the target. The other half is to locate exactly where uniqueness stops.

2 Experimental design: continuation without circular import

2.1 Experiment 2 and the Task-02 firewall

The public Experiment 2 repository contains the Part I paper and its Task 01 formal artifact [10]. Task 02 is a cumulative Aristotle development in the same Lean environment, but the classification core was designed to prevent the answer from entering through Task 01 imports [2].

The permitted project-local input is only

$$\mathsf{Long} := \mathbb{R} \times \mathbb{R}, \qquad Q_2(t, x) := t^2 - x^2, \tag{4}$$

from `RequestProject.NullSectorTask01.Basic`. Before the final comparison layer the following Task 01 objects are absent: the multiplication, multiplicative unit, generator j, idempotents, null-coordinate maps, assembly map, boost, strata, invariant-product definition, and limiting families. The six reconstruction files form the dependency chain

$$\begin{aligned}\texttt{Basic} &\to \texttt{LorentzData} \to \texttt{CandidateProduct} \to \texttt{FixedUnit} \\ &\to \{\texttt{Minimality}, \texttt{GeneralUnit}\} \to \texttt{Canonicality}.\end{aligned} \tag{5}$$

Only then does `Comparison.lean` import `Task01.SplitComplex`. A word-boundary token audit of the six core files found no use of the forbidden Task 01 split-complex declarations. Thus the equality found at the end is not a definitional identity available to the reconstruction proof.

Proof boundary 1 (What the firewall establishes)**.** *The firewall does not make the mathematical question assumption-free. Task 02 is explicitly told to investigate bilinear products that multiply the Lorentz quadratic form and to analyse unit laws. What the firewall prevents is importing the previously constructed split-complex multiplication or its characteristic identities as proof data. The agent knows the local classification problem, but not a theorem equating its answer with Task 01.*

2.2 Independence from Experiment 1

The larger methodology still involves two independent experiments. Experiment 1 progressively removed provisional inputs from an initial split-complex Lorentz witness and ultimately reconstructed the standard $(1, 3)$ Lorentz carrier from a real positive Euclidean three-space, with later Clifford and complex structures added downstream [7, 8]. Experiment 2 does not import that theorem graph. It begins instead from standard Lorentz mathematics and asks how much internal algebraic structure can be reconstructed after the Lorentz carrier is already available.

The two directions are therefore complementary:

$$\begin{aligned}\textsc{Experiment 1} &: \text{ weaker real data} \longrightarrow \text{Lorentz structure,} \\ \textsc{Experiment 2} &: \text{ Lorentz structure} \longrightarrow \text{internal null/algebraic resolution.}\end{aligned} \tag{6}$$

Coherence is assessed only after the independent developments exist. Agreement is informative precisely because the equality is not shared as an input; disagreement would expose either a mathematical error, a convention mismatch, or a genuinely different dependency claim that would require resolution.

3 Research questions and Part II objectives

Task 02 was constructed around five questions.

RQ1. Is a real bilinear, Q_2-multiplicative product uniquely determined once a two-sided unit is chosen, or do distinct products remain?

RQ2. Which algebraic properties are genuine assumptions and which are consequences? In particular, must $Q_2(e) = 1$, commutativity, associativity, the relation $j^2 = 1$, and complementary null idempotents be inserted separately?

RQ3. What happens if the right-unit law is removed? Does the classification remain unique, and which of the tested additional axioms eliminate any extra branch?

RQ4. Does the Lorentz quadratic carrier, optionally together with the future component, select a unique unit and hence a unique product on the fixed carrier?

RQ5. Is a fixed admissible product invariant under a nontrivial future-preserving Lorentz symmetry, or is there a concrete obstruction?

The task deliberately does *not* ask for a dynamical evolution law, a physical interpretation of the unit, a preferred inertial frame, a projective completion, or a $3+1$ patching construction. A uniqueness theorem and a no-go theorem are equally successful outcomes if they identify the correct dependency boundary.

4 Starting data and notation

The principal objects used in Part II are listed below. Status refers to the Task 02 reconstruction layer, not to external mathematics.

Symbol	Definition / type	Role and status
Long	$\mathbb{R} \times \mathbb{R}$	Longitudinal real carrier. INPUT from Task 01 Basic only.
Q_2	$Q_2(t,x) = t^2 - x^2$	Lorentz quadratic form. INPUT.
L_2	$[Q_2(X+Y) - Q_2(X) - Q_2(Y)]/2$	Polarization of Q_2. DERIVED.
μ	real bilinear map $\mathsf{Long} \times \mathsf{Long} \to \mathsf{Long}$	Genuinely unknown candidate product. INPUT variable.
e	vector in Long	Candidate unit; no normalization initially assumed.
u_0	$(1,0)$	Chosen coordinate reference vector. Additional datum, not selected by Q_2.
s_0	$(0,1)$	Second coordinate vector used in the fixed-unit classification.
$\mathrm{LeftUnit}(\mu, e)$	$\mu(e, X) = X$	Candidate left-unit law.
$\mathrm{RightUnit}(\mu, e)$	$\mu(X, e) = X$	Candidate right-unit law.
$\mathrm{Adm}(e, \mu)$	two-sided unit e and $Q_2(\mu(X,Y)) = Q_2(X)Q_2(Y)$	Main admissibility predicate. No commutativity or associativity included.
$\mathrm{LAdm}(e, \mu)$	left unit e and Q_2-multiplicativity	Weakened predicate for the minimality control.
$e^{\perp}$	(e_2, e_1)	Q_2-orthogonal companion used in the general-unit proof; satisfies $Q_2(e^{\perp}) = -Q_2(e)$.
μ_e	reconstructed product at normalized e	DERIVED and unique once e is fixed.
p_+, p_-	$\frac{1}{2}(u_0 \pm s_0)$	Candidate complementary idempotents, defined independently of Task 01.
T_*	rational linear Lorentz symmetry with coefficients $5/4, 3/4$	Explicit canonicality test, constructed without Task 01 boosts.

Proof boundary 2 (No physical meaning assigned to the unit)**.** *The word "unit" means multiplicative identity of a candidate bilinear algebra. Although the normalized future hyperbola is familiar from Lorentzian physics, Task 02 does not identify e with an observer, four-velocity, clock, particle state, or preferred physical frame. Such identifications would require additional semantics absent from the formal development.*

5 Recovering the Lorentz bilinear form from the quadratic carrier

The reconstruction starts without multiplication. Define

$$L_2(X,Y) := \frac{Q_2(X+Y) - Q_2(X) - Q_2(Y)}{2}. \tag{7}$$

For $X = (t,x)$ and $Y = (s,y)$, direct expansion gives

$$\boxed{L_2((t,x),(s,y)) = ts - xy.} \tag{8}$$

Task 02 proves symmetry, real bilinearity in both arguments, subtraction and zero laws, and

$$L_2(X,X) = Q_2(X). \tag{9}$$

It also proves nondegeneracy concretely:

$$[\forall Y \in \mathsf{Long},\ L_2(X,Y) = 0] \Longrightarrow X = 0. \tag{10}$$

No multiplication on Long occurs in this layer. This separation matters because the later product formula will be expressed only in terms of L_2, e, vector addition, and scalar multiplication.

6 The unknown-product problem

Let

$$\mu : \mathsf{Long} \times \mathsf{Long} \to \mathsf{Long} \tag{11}$$

be an arbitrary real bilinear map. In Lean the carrier is `LinearMap.BilinMap ℝ Long Long`, using the standard Mathlib bilinear-map infrastructure [13]. Define

$$\mathrm{LeftUnit}(\mu,e) :\Longleftrightarrow \forall X,\ \mu(e,X) = X, \tag{12}$$

$$\mathrm{RightUnit}(\mu,e) :\Longleftrightarrow \forall X,\ \mu(X,e) = X, \tag{13}$$

$$\mathrm{QMult}(\mu) :\Longleftrightarrow \forall X,Y,\ Q_2(\mu(X,Y)) = Q_2(X)Q_2(Y). \tag{14}$$

The main admissibility condition is

$$\mathrm{Adm}(e,\mu) := \mathrm{LeftUnit}(\mu,e) \wedge \mathrm{RightUnit}(\mu,e) \wedge \mathrm{QMult}(\mu). \tag{15}$$

Commutativity and associativity are intentionally excluded.

The first normalization result is already informative.

Theorem 1 (Normalization is forced)**.** *If μ is bilinear, e is a left unit, and μ is Q_2-multiplicative, then*

$$\boxed{Q_2(e) = 1.} \tag{16}$$

No right-unit law is required for this conclusion.

Proof structure. Take the fixed vector $u_0 = (1,0)$, for which $Q_2(u_0) = 1$. The left-unit law gives $\mu(e,u_0) = u_0$. Applying Q_2-multiplicativity to this product yields

$$1 = Q_2(u_0) = Q_2(\mu(e,u_0)) = Q_2(e)Q_2(u_0) = Q_2(e).$$

The Lean theorem is `Q2_leftUnit_eq_one`. □

The two-sided unit of a fixed product is also unique from the unit laws alone. Thus unit normalization and unit uniqueness are not separate algebraic axioms in the main branch.

7 Fixed-unit reconstruction at $u_0 = (1,0)$

The first complete classification fixes the additional coordinate datum

$$u_0 = (1,0), \qquad s_0 = (0,1), \tag{17}$$

and starts with an arbitrary μ satisfying $\mathrm{Adm}(u_0, \mu)$.

The unit laws determine three entries of the multiplication table:

$$\mu(u_0, u_0) = u_0, \qquad \mu(u_0, s_0) = s_0, \qquad \mu(s_0, u_0) = s_0. \tag{18}$$

Bilinearity therefore leaves only

$$w := \mu(s_0, s_0) \tag{19}$$

unknown. Task 02 applies Q_2-multiplicativity to several explicitly chosen vector pairs and proves

$$\boxed{\mu(s_0, s_0) = u_0.} \tag{20}$$

This is the square relation that, after the late comparison, becomes $j^2 = 1$.

Theorem 2 (Fixed-unit classification)**.** *There exists exactly one bilinear Q_2-multiplicative product with two-sided unit u_0. For all $X = (t, x)$ and $Y = (s, y)$ it is*

$$\boxed{\mu_{u_0}(X, Y) = (ts + xy,\ ty + xs).} \tag{21}$$

The formal proof first derives (20) for a generic product and only then expands bilinearly to (21). Existence is supplied by an explicitly constructed bilinear map `recU`; uniqueness is the theorem `fixedUnit_unique`, bundled as `fixedUnit_existsUnique`.

Two familiar algebraic properties now become corollaries:

$$\mu_{u_0}(X, Y) = \mu_{u_0}(Y, X), \tag{22}$$

and

$$\mu_{u_0}(\mu_{u_0}(X, Y), Z) = \mu_{u_0}(X, \mu_{u_0}(Y, Z)). \tag{23}$$

Neither was included in admissibility.

7.1 Null idempotents are downstream of the reconstruction

Define independently of Task 01

$$p_+ := \frac{1}{2}(u_0 + s_0), \qquad p_- := \frac{1}{2}(u_0 - s_0). \tag{24}$$

For every product admissible at u_0, Task 02 proves

$$\mu(p_+, p_+) = p_+, \qquad \mu(p_-, p_-) = p_-, \tag{25}$$
$$\mu(p_+, p_-) = 0, \qquad \mu(p_-, p_+) = 0, \tag{26}$$

and

$$Q_2(p_+) = Q_2(p_-) = 0. \tag{27}$$

Thus, relative to the Task 02 dependency graph, the complementary null idempotents are not additional algebraic input. They follow after the product and square relation have been reconstructed.

8 Minimality control: dropping the right-unit law

A uniqueness theorem under a strong hypothesis does not identify which hypothesis carries the uniqueness. Task 02 therefore weakens the main branch to

$$\mathrm{LAdm}(u_0, \mu) := \mathrm{LeftUnit}(\mu, u_0) \wedge \mathrm{QMult}(\mu). \tag{28}$$

The result is no longer unique.

Theorem 3 (Complete left-unit dichotomy)**.** *Exactly two coordinate formulas are possible for a bilinear, Q_2-multiplicative product with left unit u_0:*

$$\mu_{u_0}((t, x), (s, y)) = (ts + xy,\ ty + xs), \tag{29}$$

$$\mu_{\mathrm{tw}}((t, x), (s, y)) = (ts - xy,\ ty - xs). \tag{30}$$

Both branches exist and they are distinct.

The second branch is a genuine negative control. It has left unit u_0 and multiplies Q_2, but it fails all three tested stronger properties:

$$\mathrm{RightUnit}(\mu_{\mathrm{tw}}, u_0), \qquad \mathrm{Commutative}(\mu_{\mathrm{tw}}), \qquad \mathrm{Associative}(\mu_{\mathrm{tw}}) \tag{31}$$

are each false.

Conversely, each one of these three conditions separately eliminates the second branch:

Assumptions added to left-unit + Q_2-multiplicative	**Verdict**
right-unit law at u_0	unique: $\mu = \mu_{u_0}$
commutativity	unique: $\mu = \mu_{u_0}$
associativity	unique: $\mu = \mu_{u_0}$

Proof boundary 3 (Precise meaning of the minimality result)**.** *Task 02 proves that each of the three* tested *conditions is sufficient, individually, to restore uniqueness after the right-unit law has been dropped. It does not classify every logically conceivable additional property that might also exclude μ_{tw}. Therefore the formal conclusion is a three-way sufficiency result, not an absolute lattice-theoretic classification of all minimal axiom systems.*

This negative control is important for the dependency argument. The desired product is not an artifact of having represented the unknown multiplication as a ring structure with commutativity or associativity silently pre-installed. Those properties can be absent, and when they are absent an explicit second norm-multiplicative branch survives.

9 Arbitrary normalized unit: the general reconstruction theorem

The fixed reference vector u_0 is convenient for coordinates but should not be confused with an intrinsically selected object. Task 02 therefore repeats the classification for an arbitrary

$$e \in \mathsf{Long}. \tag{32}$$

Define its Q_2-orthogonal companion

$$e^{\perp} := (e_2, e_1). \tag{33}$$

Then

$$L_2(e, e^{\perp}) = 0, \qquad Q_2(e^{\perp}) = -Q_2(e). \tag{34}$$

When $Q_2(e) = 1$, every X admits the explicit decomposition

$$X = L_2(X, e)e - L_2(X, e^{\perp})e^{\perp}. \tag{35}$$

The general classification first proves the corresponding square relation

$$\mu(e^{\perp}, e^{\perp}) = e \tag{36}$$

for every admissible product with unit e. Substitution into the bilinear expansion then yields the central formula.

Theorem 4 (General-unit reconstruction)**.** *Let $e \in \mathsf{Long}$. The following are equivalent:*

(i) there exists a bilinear product μ with two-sided unit e such that

$$Q_2(\mu(X, Y)) = Q_2(X)Q_2(Y)$$

for all X, Y;

(ii) $Q_2(e) = 1$.

Whenever these conditions hold, the product is unique and is given by

$$\boxed{\mu_e(X, Y) = L_2(X, e)Y + L_2(Y, e)X - L_2(X, Y)e.} \tag{37}$$

This formula is conceptually important for two reasons. First, it is expressed in terms of the polarized Lorentz form and a chosen normalized unit, rather than in terms of a pre-existing split generator. Second, it is obtained after the existence-and-uniqueness argument; it is not the premise used to make uniqueness true.

At $e = u_0$, using $L_2((t, x), u_0) = t$, (37) reduces immediately to (21). The split-complex coordinate formula is therefore one representative of a general family indexed by normalized units.

10 Where uniqueness stops: the unit is not selected by the Lorentz data

Theorem 4 might tempt an overstatement: perhaps the Lorentz form itself determines the split product. Task 02 proves that this is false on the fixed carrier.

Define neutrally

$$\text{FutureUnit}(e) :\Longleftrightarrow e_1 > 0 \quad \text{and} \quad Q_2(e) = 1. \tag{38}$$

Both

$$u_0 = (1, 0) \tag{39}$$

and

$$e_1 = \left(\frac{5}{4}, \frac{3}{4}\right) \tag{40}$$

are future normalized, and they are distinct. Hence the future Lorentz data do not select one unit.

Because the unit of a fixed product is unique, two products admissible at distinct units must themselves be distinct. Therefore

$$e \neq e' \quad \Longrightarrow \quad \mu_e \neq \mu_{e'} \tag{41}$$

whenever both are defined on the same fixed carrier.

Corollary 1 (Failure of intrinsic product selection)**.** *The data* (Long, Q_2)*, even supplemented by the chosen future component, do not determine a single admissible unital product on the fixed vector space.*

This is the exact new location of the conservation-of-difficulty burden. The algebraic relation $j^2 = 1$ is no longer primitive; the choice of a normalized unit is.

11 Canonicality test: a concrete Lorentz-symmetry obstruction

Non-uniqueness by two examples could still leave open whether some additional symmetry criterion singles out one unit. Task 02 therefore constructs a nontrivial rational linear equivalence independently of Task 01 boosts:

$$\boxed{T_*(t,x) = \left(\frac{5}{4}t + \frac{3}{4}x, \frac{3}{4}t + \frac{5}{4}x\right).} \tag{42}$$

Its inverse is obtained by changing the off-diagonal signs. Direct calculation proves

$$Q_2(T_*X) = Q_2(X), \tag{43}$$

and the map preserves the closed future cone and the future-normalized set. It is not the identity and has no nonzero fixed vector:

$$T_*X = X \Longrightarrow X = 0. \tag{44}$$

For a fixed product μ, define equivariance under a linear equivalence T by

$$T(\mu(X,Y)) = \mu(TX,TY) \qquad \forall X,Y. \tag{45}$$

If e is a two-sided unit and (45) holds, then Te is also a two-sided unit. Uniqueness of the unit forces $Te = e$. Combining this with (44) and $Q_2(e) = 1$ gives a contradiction.

Theorem 5 (Fixed-product canonicality obstruction)**.** *There is no pair (e, μ) such that μ is admissible with unit e and the same fixed product μ is equivariant under T_*:*

$$\boxed{\neg\exists(e,\mu) : \mathrm{Adm}(e,\mu) \wedge \forall X,Y,\ T_*(\mu(X,Y)) = \mu(T_*X,T_*Y).} \tag{46}$$

This theorem must be read carefully. It proves an obstruction to *invariance of one fixed unital product* under the explicit nontrivial Lorentz symmetry. It does not prove that the family $\{\mu_e\}$ lacks a natural Lorentz action.

Open question 1 (Covariance of the reconstructed family)**.** *For a Q_2-preserving linear equivalence T, does the family satisfy*

$$T\big(\mu_e(X,Y)\big) \stackrel{?}{=} \mu_{Te}(TX,TY)? \tag{47}$$

If so, the Lorentz data may determine a canonical orbit or isomorphism class *of products while failing to determine one distinguished representative on the fixed carrier. Task 02 does not formalize this statement.*

The distinction between fixed-product invariance and family covariance is essential. The no-go theorem concerns the former only.

12 Late comparison with Task 01

Only after all preceding theorems are available does `Comparison.lean` import the Task 01 split-complex layer. The Task 01 multiplication is packaged as a generic bilinear map and shown to satisfy Task 02 admissibility at the reference unit u_0. The previously proved fixed-unit uniqueness theorem then yields

$$\boxed{\mu_{\text{Task 01}} = \mu_{u_0}.} \tag{48}$$

This equality is obtained from uniqueness, not merely from a coordinate simplification.

The independently defined Task 02 vectors p_+, p_- are then proved equal to the Task 01 idempotents e_+, e_-. More importantly, several identities that were construction-level facts in Part I become corollaries for an arbitrary Task 02 admissible product at u_0:

$$j^2 = 1, \qquad e_+^2 = e_+, \qquad e_-^2 = e_-, \qquad e_+e_- = 0, \tag{49}$$

together with commutativity, associativity, and nullity of the idempotents.

This is the main dependency shift from Part I to Part II:

Structure	Part I / Task 01	Part II / Task 02
split multiplication	defined	reconstructed after unit selection
$j^2 = 1$	proved from defined product	derived from Lorentz-compatible product classification
commutativity	available for the explicit product	derived from main admissibility
associativity	available for the explicit product	derived from main admissibility
null idempotents	derived from explicit split product	downstream of reconstructed square relation
chosen unit	coordinate definition	proved to be extra data relative to Lorentz symmetry

13 What Part II changes about the reading of Experiment 1

Experiment 1 and Experiment 2 now meet at a sharper interface than they did after Part I. Experiment 1 showed that a standard Lorentz carrier can be reconstructed downstream of weaker real Euclidean and Jordan data, while its final comparative audit found the resulting Lorentz structures to be established objects presented in a different dependency order [8]. Part II asks what algebra is forced once one is already inside a longitudinal Lorentz plane.

The answer is neither "the Lorentz form already contains a unique split multiplication" nor "the split multiplication is arbitrary". The proved statement lies between them:

$$(\mathsf{Long}, Q_2) + \text{chosen normalized unit } e \Longrightarrow \text{unique admissible product } \mu_e. \tag{50}$$

Thus the split product is rigid *relative to a unit choice*, but the unit choice itself is not Lorentz-invariantly selected.

This is coherent with the outcome of Experiment 1. A Lorentzian structure is expected to carry nontrivial symmetry; selecting a specific point on the future unit hyperbola adds data that the symmetric carrier does not itself distinguish. Part II does not attach physical semantics to that fact, but mathematically it prevents a circular claim that the entire algebraic representative was forced by Q_2 alone.

The two experiments therefore continue to satisfy the intended consistency criterion:

- Experiment 1 reconstructs the Lorentz object from a different upstream chain.
- Experiment 2 shows that, within a chosen longitudinal slice, a normalized algebra unit plus the Lorentz form rigidly reconstructs the split product.
- The reconstructed product agrees exactly with Part I only in the late comparison layer.
- The new obstruction—non-selection of the unit by Lorentz data—does not contradict Experiment 1; it refines which data are needed to choose a particular algebra structure on a fixed carrier.

No cross-experiment equality was made available to Aristotle during the Task 02 classification.

14 Conservation of difficulty after Task 02

The methodological gain can now be stated much more sharply than after Part I. At the start of the program one could treat the split algebra as specified by

$$j^2 = 1. \tag{51}$$

Task 02 shows that this relation is not the deepest required input in the present reconstruction. Under the main admissibility conditions it follows from more primitive data. The same is true of several other algebraic properties:

$$\begin{gathered} j^2 = 1 \text{ is no longer an independent input,} \\ \text{commutativity is no longer an independent input,} \\ \text{associativity is no longer an independent input,} \\ \text{null idempotents are no longer independent inputs.} \end{gathered} \tag{52}$$

What remains is not nothing. It is the explicitly isolated selection

$$e \in \mathsf{Long}, \qquad Q_2(e) = 1. \tag{53}$$

The Lorentz data provide an entire normalized hyperbola, not one chosen element of it.

Methodological principle 2 (Updated conservation-of-difficulty ledger)**.** Task 02 reduces the algebraic burden from "choose a split multiplication with $j^2 = 1$" to "choose a normalized unit in the Lorentz carrier". The first choice is then reconstructed uniquely; the second is not removed. The experiment therefore succeeds by relocating the unresolved dependency to a strictly more primitive level and proving that the relocation cannot be skipped under the stated symmetry data.

This is stronger than a stylistic reparameterization. It is a formal dependency result with both a positive theorem and a negative theorem: uniqueness after unit selection, and impossibility of selecting a fixed equivariant unital product under the explicit symmetry T_*.

15 Formal verification and audit boundary

The Task 02 artifact is a Lean 4 development in the same pinned environment as Task 01:

$$\texttt{leanprover/lean4:v4.28.0}, \tag{54}$$

with Mathlib v4.28.0 at commit

$$\texttt{8f9d9cff6bd728b17a24e163c9402775d9e6a365}. \tag{55}$$

The artifact report records a successful `lake build`, no `sorry`, no `admit`, and no project-local axiom. The principal Task 02 theorems report exactly

$$\texttt{propext}, \qquad \texttt{Classical.choice}, \qquad \texttt{Quot.sound} \tag{56}$$

as axiom dependencies, i.e. the standard Lean/Mathlib logical background.

The formal source uses `LinearMap.BilinMap` as the carrier for the unknown product and `LinearMap.mk_2` for explicit witnesses such as the reconstructed and twisted products [13, 14]. The quadratic-form polarization is deliberately implemented directly rather than through Mathlib's higher-level quadratic-map API, keeping the dependency from Q_2 to L_2 explicit in the source.

Proof boundary 4 (Editorial verification boundary)**.** *The present paper inspects the delivered Lean sources, dependency firewall, theorem statements, audit, toolchain, and manifest. The Aristotle artifact reports a successful build. The editorial environment used for this paper does not independently re-run the Lean kernel build and therefore does not claim an independent compilation of the Lean project. The LaTeX paper itself is compiled and visually checked independently.*

15.1 Principal theorem map

Mathematical statement	Lean theorem(s)
Polarization and nondegeneracy	`L2_apply`, `L2_self`, `L2_nondegenerate`
Unit normalization forced	`Q2_leftUnit_eq_one`, `Q2_unit_eq_one`
Unit uniqueness	`twoSidedUnit_unique`, `unit_unique_mixed`
Fixed-unit square relation	`mu_s0_s0`
Fixed-unit formula and uniqueness	`fixedUnit_formula`, `fixedUnit_unique`, `fixedUnit_existsUnique`
Commutativity and associativity derived	`fixedUnit_commutative`, `fixedUnit_associative`
Independent null idempotents	`mu_pplus_pplus`, `mu_pminus_pminus`, `mu_pplus_pminus`, `Q2_pplus`, `Q2_pminus`
Left-unit dichotomy	`leftAdmissible_dichotomy`, `leftAdmissible_not_unique`
Three sufficiency controls	`testA_unique`, `testB_unique`, `testC_unique`
General existence iff normalization	`exists_admissible_iff`
General product formula and uniqueness	`generalUnit_formula`, `generalUnit_existsUnique`, `eq_recAt`
Future-unit non-uniqueness	`futureUnit_not_unique`, `admissible_product_not_intrinsic`
Explicit symmetry obstruction	`Tsym_Q2`, `Tsym_no_fixed_point`, `no_equivariant_admissible`
Late equality with Task 01	`mulLBil_eq_recU`, `mulL_eq_recU`
Task 01 identities re-derived	`jL_square_derived`, `ep_idem_derived`, `em_idem_derived`, `ep_em_orthogonal_derived`, `mulL_comm_derived`, `mulL_assoc_derived`

16 Why Part II still does not establish dynamics

The program title remains *Split Complex Lorentzian Dynamics*, but no theorem in Part II is an evolution law. The objects e, μ_e, p_+, p_-, and T_* are algebraic or geometric data. The symmetry T_* is a fixed linear equivalence used as a canonicality test; it is not a time step. The family of normalized units is a set of admissible algebraic choices; no path through that set is specified. Consequently Part II does not derive motion, a clock, proper time, phase, frequency, energy, mass, field dynamics, interaction, or a law selecting one unit from another.

This negative statement is scientifically useful. The reconstruction has advanced far enough to show exactly what remains absent. Any later claim of dynamics must introduce or derive structure beyond the present classification, and that new input must be visible in the dependency ledger.

17 Immediate open questions after Task 02

Open question 2 (Lorentz covariance of the product family)**.** *Prove or refute the naturality relation* (47) *for arbitrary Q_2-isometries in the relevant component. This is the most direct next test because it distinguishes failure of a fixed invariant product from possible covariance of the whole reconstructed family.*

Open question 3 (Canonical isomorphism class versus representative)**.** *If all products μ_e for future normalized e are transported into one another by Lorentz symmetries, in what precise* *sense does* (Long, Q_2) *determine the split-complex algebra only up to isomorphism, conjugacy, or torsor-like choice? The answer should be formalized rather than inferred from transitivity heuristics.*

Open question 4 (Transformation of null idempotents under unit change)**.** *For* $e \mapsto e'$*, how do the complementary idempotents associated with* μ_e *transform? Do their null lines coincide with the Task 01 null directions independently of* e*, or does the algebraic idempotent representative depend on the unit while the underlying null cone remains fixed?*

Open question 5 (Compatibility of different longitudinal planes in $3+1$)**.** *Part II remains strictly* $1+1$*. If a product is reconstructed on every longitudinal plane after a corresponding normalized-unit choice, what compatibility data are required to patch these local algebra structures inside the common* $3+1$ *Lorentz carrier?*

Open question 6 (Origin of unit selection)**.** *Does any weaker structure already present upstream of the Lorentz carrier select a normalized unit, or is such a selection necessarily external to the Lorentz-symmetric data? This question is especially relevant to the comparison with* EXPERIMENT *1, but no answer should be imported from that experiment without a new independent derivation.*

Open question 7 (Where can genuine dynamics first enter?)**.** *Only after the static family and its covariance properties are understood should one ask whether a path, section, connection, variational principle, or other new structure can select relations among different units. None of these possibilities is asserted by the current results.*

The order is intentional: covariance and isomorphism-class questions precede any dynamical interpretation because they may remove apparent choices by equivalence without introducing evolution.

18 Discussion: rigidity relative to a choice, obstruction before the choice

Part II yields a particularly clean combination of positive and negative results. The positive result is rigidity: once a normalized two-sided unit is chosen, norm multiplicativity and bilinearity force the product. The negative result is non-selection: the Lorentz data do not provide that unit, and a concrete Lorentz symmetry obstructs invariance of any single fixed unital product.

This combination is more informative than either result alone. If only uniqueness had been proved, it would be easy to hide the unit choice in coordinates and overstate the conclusion. If only non-uniqueness of units had been proved, one might incorrectly infer that the split product itself is arbitrary. Together the results give the exact dependency statement

$$\boxed{\text{Lorentz quadratic data } \not\Rightarrow \text{ one distinguished product,}} \tag{57}$$

but

$$\boxed{\text{Lorentz quadratic data} + \text{chosen normalized unit } \Rightarrow! \text{ admissible product.}} \tag{58}$$

Here $\Rightarrow!$ denotes existence and uniqueness under the stated admissibility conditions, not uniqueness among every conceivable algebraic structure one might place on $\mathbb{R}^2$.

The left-unit negative control further shows that the proof is sensitive to its assumptions. Removing one side of the unit law creates a second explicit norm-multiplicative product, and the three tested conditions remove that branch separately. The reconstruction therefore has a nontrivial falsification surface rather than merely confirming a pre-selected coordinate formula.

Status 1 (Current scientific status)**.** Part II establishes a formally verified dependency theorem for a concrete $1+1$ Lorentz carrier. It does not establish new physical dynamics and does not claim novelty for split-complex or composition-algebra mathematics. Its strongest methodological result is the relocation of the primitive burden from the split multiplication and $j^2 = 1$ to the selection of a normalized algebra unit, together with a proof that the Lorentz-symmetric carrier itself does not make that selection.

19 Conclusion: a sharper open endpoint

Part I exposed the complementary null coordinates of a fixed longitudinal Lorentz plane. Part II goes one layer deeper and removes the split-complex multiplication from the input side of the dependency graph. The resulting classification begins with the bare Lorentz quadratic carrier, derives its polarization, and quantifies over arbitrary bilinear products.

The formal result is exact. A Q_2-multiplicative bilinear product with two-sided unit e exists exactly when $Q_2(e) = 1$. For such a fixed e it is unique and equals

$$\mu_e(X, Y) = L_2(X, e)Y + L_2(Y, e)X - L_2(X, Y)e.$$

The split square relation, commutativity, associativity, and complementary null idempotents follow. At $e = (1, 0)$ the independently reconstructed product agrees with the Task 01 split-complex multiplication by uniqueness.

The derivation does not close the problem by eliminating all choices. It identifies the next one. The future unit hyperbola contains multiple normalized vectors, distinct units induce distinct products on the fixed carrier, and an explicit future-preserving Lorentz symmetry prevents any single admissible unital product from being invariant under that symmetry. Thus the correct endpoint of Part II is

$$\boxed{\text{Lorentz carrier} + \text{unit choice} \longrightarrow \text{unique split product,}} \tag{59}$$

not

$$\text{Lorentz carrier} \longrightarrow \text{unique split product.} \tag{60}$$

The next research frontier is consequently not to attach an interpretation to the selected unit, but to determine the covariance and equivalence structure of the entire family $\{\mu_e\}$. Only after that static question is resolved can the program responsibly ask whether any additional structure deserves to be called dynamical. Part II therefore ends with a stronger reconstruction than Part I and, at the same time, a more sharply localized uncertainty.

A Dependency ledger for Part II

The strongest dependency graph justified by Task 02 is

Long $+$ Q_2

$\downarrow$

L_2 by polarization
symmetric, bilinear, nondegenerate

$\downarrow$ additional datum

choose e with $Q_2(e) = 1$
(not selected by Lorentz data)

$\downarrow$

unique bilinear Q_2-multiplicative product with unit e
$\mu_e(X,Y) = L_2(X,e)Y + L_2(Y,e)X - L_2(X,Y)e$

$\downarrow$

derived square relation
$\mu_e(e^\perp, e^\perp) = e$

$\downarrow$

derived commutativity, associativity,
complementary null idempotents

The arrow from L_2 to the chosen e is not a derivation arrow. It denotes the additional datum that must be supplied. Task 02 proves that replacing it by "the Lorentz data select e" is invalid.

The weakened branch is

$$\begin{array}{c} \text{bilinear} + Q_2\text{-multiplicative} + \text{left unit } u_0 \\ \Downarrow \\ \{\mu_{u_0}, \mu_{\mathrm{tw}}\} \\ \Downarrow + \text{ any one tested condition } \{\text{right unit, commutative, associative}\} \\ \mu_{u_0}. \end{array} \tag{61}$$

B Source and reproducibility ledger

Formal artifact

The audited archive supplied for Task 02 has SHA-256

$$\texttt{927fe7c1b2b7603ae5f75085425974c79c4ea0096483f7b04a0e8118eb518cf8}. \tag{62}$$

The Aristotle summary identifies the Task 02 run as

$$\texttt{d2d4ed8c-0706-4470-a655-bde788333956}. \tag{63}$$

The cumulative request is cited as [2].

Lean and Mathlib

The pinned toolchain is Lean 4.28.0. The Mathlib revision is

$$\texttt{8f9d9cff6bd728b17a24e163c9402775d9e6a365}. \tag{64}$$

The principal imported library facts are standard linear-map infrastructure; the central product classification is proved directly in the project rather than delegated to an external composition-algebra classifier.

Standard mathematical context

Split-complex numbers and their relation to Lorentz transformations are classical [3, 4, 5, 11]. Bilinear composition of quadratic forms has a classical lineage represented by Hurwitz [6], while modern composition-algebra theory provides a broader structural setting [12]. These references provide context only; the exact Task 02 statements are machine-proved in the supplied Lean development.

Experiment repositories

The public EXPERIMENT 2 repository is [10]. The independent earlier dependency experiment is represented here by its Version 9 and final Version 30 reports [7, 8].

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