Split Complex Lorentzian Dynamics

Part I: Null-Sector Foundations, Reciprocal Boost Scaling, and an Independent Structural Cross-Check

An exploratory dependency study with formal verification

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Abstract

This paper opens a second experiment on the mathematical structure underlying split-complex Lorentz kinematics. The object under investigation is the *split-complex Lorentzian structure*: a real two-dimensional split algebra with generator $j^2 = 1$, complementary idempotents e_+ and e_-, and the Lorentzian quadratic form that becomes the product of the two idempotent coefficients. The broader program is titled *Split Complex Lorentzian Dynamics*, but Part I deliberately contains no dynamics. It establishes only a carrier, transformation, causal-stratum, and limiting-regime audit.

The study is designed as an independent complement to an earlier dependency experiment. Experiment 1 began with a split-complex Lorentz witness and progressively removed provisional inputs; by its ninth stage it reconstructed a $(1,3)$ Lorentzian spin-factor geometry from a real positive Euclidean three-space, and its final comparison found that the resulting structures coincide with established mathematics while differing mainly in dependency order [10, 11, 12]. Experiment 2 is intentionally isolated from those results at the Lean-agent level. Aristotle receives no theorem, source file, dependency graph, or conclusion from Experiment 1. Its Task 01 prompt instead starts from standard $3+1$ Minkowski mathematics, selects one longitudinal $1+1$ plane, defines the split-complex multiplication locally, and asks for explicit proofs and obstructions without physical ontology [1]. Hence no cross-experiment definitional equality is available to the agent; coherence is assessed only afterward at the editorial level.

Task 01 proves that every longitudinal vector has unique complementary coefficients (a,b), that $Q_2(P(a,b)) = ab$, and that the future-causal coefficient quadrant is partitioned into the origin, two nontrivial null boundary strata, and a timelike interior. Longitudinal boosts act diagonally as

$$(a,b) \longmapsto (e^{\eta}a, e^{-\eta}b),$$

so ab is invariant. A boundary family with one coefficient fixed has a genuine finite vector limit and its quadratic form tends to zero. In contrast, if $ab = I > 0$ is held fixed while one coefficient tends to zero, the other diverges and no finite vector limit exists. Thus "approaching a null coefficient" is not a single limiting operation: boundary convergence and fixed-positive-invariant evolution are mathematically distinct.

The cross-experiment comparison is coherent but not circular. Experiment 1 reconstructs a Lorentz carrier from weaker real data; Experiment 2 independently resolves an already established Lorentz carrier into complementary null sectors and exposes its reciprocal scaling and boundary geometry. The results reinforce the methodological principle called *conservation of difficulty*: decomposition can relocate, expose, or isolate structural burdens, but it cannot erase them. In particular, Part I does not prove that the split-complex multiplication is uniquely forced by Lorentz geometry, does not derive a canonical boundary-to-interior continuation, and does not identify any coefficient with mass, proper time, frequency, phase, or a particle. These unresolved points define the immediate research frontier rather than a predetermined interpretation.

Keywords: split-complex numbers; hyperbolic numbers; Lorentzian geometry; null coordinates; causal cone; rapidity; boundary strata; formal verification; Lean 4; conservation of difficulty; dependency analysis.

Contents

1 Motivation: why take a mature structure apart?

The mathematical structures used in special relativity are mature. Null coordinates, hyperbolic rotations, Lorentz cones, split-complex numbers, Jordan spin factors, Clifford algebras, and their mutual relations all have established literatures [8, 6, 13, 7, 5]. The purpose of the present program is therefore not to rename known mathematics and present it as a new theory. The more modest and, for an exploratory study, more useful question is whether an explicit decomposition of familiar structures can reveal dependencies, equivalences, obstructions, or non-equivalences that are easy to overlook when the final formalism is taken as a single indivisible object.

The initial heuristic motivation was deliberately stronger than the formal input. One may imagine a regular succession of null signals as a timing picture and then wonder whether the difference between null and timelike kinematics is related to the appearance of a complementary degree of freedom. Such imagery is useful only as a generator of questions. It is not evidence, and it is especially dangerous when it silently assigns physical meanings to algebraic coefficients. The formal program therefore strips the heuristic back to a neutral mathematical core before it reaches the theorem prover.

The *object under investigation* is the split-complex Lorentzian structure. In its simplest $1 + 1$ realization one has a real algebra generated by j with

$$j^2 = 1, \tag{1}$$

two complementary idempotents

$$e_+ = \frac{1 + j}{2}, \qquad e_- = \frac{1 - j}{2}, \tag{2}$$

and a decomposition into two real coefficient sectors. Historically, such hyperbolic or split-complex algebras are classical; Cockle's tessarine work already contains a square-$+1$ imaginary direction, and modern treatments emphasize their hyperbolic geometry [2, 3, 13]. Their relation to special-relativistic transformations is also established [6].

What is not mature in the same sense is the particular *dependency experiment*: start from a compact familiar object, remove provisional inputs one by one, reconstruct what can be reconstructed, and then compare independent paths. The scientific value, if any, lies in what survives this process without being inserted by hand.

Methodological principle 1 (Conservation of difficulty)**.** A derivational difficulty is not solved merely because it disappears from the final notation. If a target structure is moved into an earlier definition, representation, convention, or hidden assumption, the difficulty has been relocated rather than discharged. Every claimed derivation must therefore expose which data are input, which are derived, which are equivalent realizations, which require a discrete choice, and which remain open.

This principle was already explicit in the first split-complex witness: once the split carrier and reciprocal action were provided, the Lorentz boost followed, but the reason for selecting that carrier remained open [10]. The later stages of Experiment 1 pushed the same question further, eventually locating a deeper primitive at a real positive Euclidean three-space in the formal chain then available [11]. Version 30 subsequently compared the reconstructed structures against standard mathematics and reported no genuinely different mathematical endpoint among the audited comparisons; the substantive difference was chiefly the ordering and visibility of dependencies [12].

Part I asks whether an independent experiment, proceeding in a different direction, reveals a compatible internal anatomy of the Lorentz carrier.

2 Two experiments, one object, no formal cross-feed

The relation between the two experiments is central and must be stated precisely. The experiments are not two files in one cumulative proof. They are two separate interrogations of the same classical mathematical territory.

2.1 Experiment 1: atomize, remove inputs, reconstruct

The first experiment began with an explicitly split-complex Lorentz witness. In its early formulation the algebra

$$A = \mathbb{R}[\epsilon]/(\epsilon^2 - 1) \tag{3}$$

was decomposed into two idempotent sectors and a reciprocal one-parameter action was shown to reproduce $1+1$ Lorentz kinematics [10]. The important methodological admission was immediate: $\epsilon^2 = 1$ selected the split algebra, so the carrier itself had not yet been derived.

Subsequent tasks progressively removed accepted inputs. By Version 9 the construction started formally from

$$V = \mathbb{R}^3, \qquad \langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3, \tag{4}$$

formed the real spin factor

$$\mathcal{S} = \mathbb{R} \oplus V, \qquad (a, u) \circ (b, v) = \big(ab + \langle u, v\rangle, av + bu\big), \tag{5}$$

and derived

$$N_{\mathcal{S}}(a, u) = a^2 - \langle u, u\rangle. \tag{6}$$

The resulting norm, cone, and automorphism structures were then identified with the standard Lorentz structures only after the intrinsic objects existed [11]. Later tasks broadened and hardened this dependency graph; Version 30 served as a comparative null test against established constructions [12].

Thus the direction of Experiment 1 was predominantly

$$\begin{aligned} \text{provisional Lorentz/split structure} &\longrightarrow \text{atomization} \\ &\longrightarrow \mathbb{R}^3 \text{ Euclidean datum} \\ &\longrightarrow \text{reconstruction and comparison.} \end{aligned} \tag{7}$$

2.2 Experiment 2: resolve the already established carrier internally

The second experiment deliberately starts elsewhere. Task 01 is given an ordinary real $3 + 1$ Minkowski carrier and a fixed longitudinal $1 + 1$ plane. It locally defines a split-complex multiplication on that plane and asks what follows. The task prohibits particle terminology, mass, proper time, clocks, periodic phases, fields, interactions, quantum mechanics, projective compactification, and any previous Aristotle task or previous project result [1].

This creates an important information boundary:

> *Aristotle in Experiment 2 is not given the code, theorem inventory, dependency graph, conclusions, or intended cross-experiment comparison of Experiment 1.*

The editorial layer knows that both experiments concern the same broad mathematical object; the formal agent does not receive that relationship as a premise.

This does *not* mean that Task 01 is result-blind in an absolute sense. Its local prompt explicitly asks the agent to test particular algebraic identities, causal classifications, boost actions, and limiting regimes. What is withheld is the cross-experiment target: no theorem from Experiment 1 is imported, no equality between an Experiment 1 object and a Experiment 2 object is available by definition, and no physical interpretation is supplied as a desired endpoint. Negative or obstructive results are explicitly acceptable. This distinction matters because a mathematically

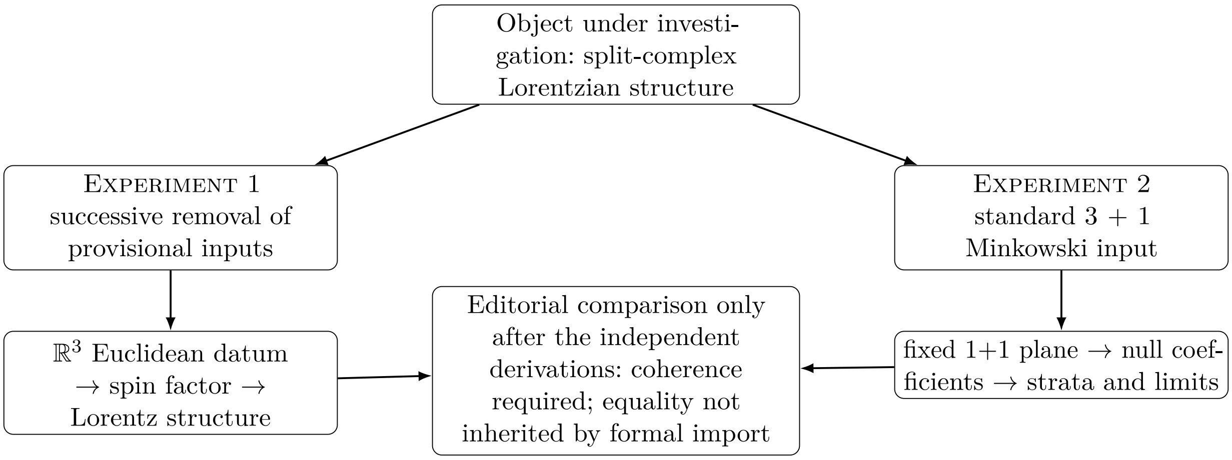

Figure 1: The cross-experiment firewall. The common object is an editorial research target, not a shared formal theorem dependency.

useful blind cross-check should avoid two opposite errors: secretly importing the desired answer, and pretending that a precisely posed theorem-proving task has no target statements at all.

The criterion is therefore not that both experiments must produce identical presentations. They should instead be *coherent where their mathematical domains overlap*, complementary where they expose different dependencies, and free of unresolved contradiction. A contradiction would be scientifically useful: it would trigger an audit of assumptions, definitions, or proofs. Agreement is informative only because the agreement is not obtained by importing one result into the other.

3 Research questions and Part I objectives

The broad program asks whether the split-complex Lorentzian structure can serve as a productive decomposition point from which deeper questions about relativistic dynamics may eventually be posed without smuggling in their answers. Part I is intentionally more restricted.

3.1 Research questions

The first set of questions is structural.

RQ1. Given a fixed longitudinal plane inside standard $3 + 1$ Minkowski space, what exact split-complex and null-coordinate structure can be exhibited and machine-checked?

RQ2. How are the nontrivial null sector and the timelike sector related inside the future-causal coefficient space: are they disconnected constructions, or boundary and interior strata of one carrier?

RQ3. How does a longitudinal Lorentz boost act on the two complementary coefficients, and what invariant is exposed in that basis?

RQ4. Does sending one complementary coefficient to zero necessarily produce a finite null limit, or does the answer depend on what else is held fixed?

RQ5. When compared only after completion, are the results compatible with the independently reconstructed Lorentz structures of EXPERIMENT 1 without requiring cross-experiment imports?

A second set of questions remains deliberately unresolved: whether the split-complex multiplication is forced by weaker Lorentz data, whether any boundary-to-interior continuation is canonical, whether longitudinal decompositions in different spatial directions fit together uniquely, and whether any genuine evolution law follows without additional structure. These are not assumptions of Part I.

3.2 Objectives of Task 01

Task 01 was therefore specified as a *carrier-and-transformation audit*. Its goals were to

(i) restrict the $3+1$ Lorentz form to one transparent $1+1$ longitudinal carrier;

(ii) construct the split-complex idempotents and complementary coordinates explicitly;

(iii) prove the exact quadratic identity $Q_2(P(a,b)) = ab$;

(iv) classify the future-causal coefficient quadrant into zero, two null boundaries, and one interior;

(v) derive the reciprocal boost scaling from the standard hyperbolic boost;

(vi) prove the product invariant independently in two ways; and

(vii) distinguish a finite boundary limit from a fixed-positive-invariant limit that may be obstructed.

No result in this list requires a physical interpretation of a, b, ab, or ε.

4 Mathematical starting data and complete notation

We work in units with $c = 1$. The following table lists every principal mathematical quantity used in Part I and its status in Task 01.

Symbol	Definition / type	Role and status
Vec_4	$\mathbb{R}^4$, coordinates $X = (t, x, y, z)$	Ambient real carrier. INPUT.
Q_4	$t^2 - x^2 - y^2 - z^2$	Ambient Lorentz quadratic form of signature $(+, -, -, -)$. INPUT.
Long	$\mathbb{R}^2$, coordinates (t, x)	Fixed longitudinal plane with $y = z = 0$. INPUT as chosen subspace.
ι	$(t, x) \mapsto (t, x, 0, 0)$	Canonical embedding $\mathsf{Long} \to \mathsf{Vec}_4$. INPUT definition.
Q_2	$t^2 - x^2$	Longitudinal quadratic form; proved to be $Q_4 \circ \iota$. DERIVED.
$\star$	$(t, x) \star (s, y) = (ts + xy, ty + xs)$	Locally defined split-complex multiplication on Long. INPUT definition in Task 01, not derived from Q_2.
1	$(1, 0)$	Multiplicative unit for $\star$. INPUT definition.
j	$(0, 1)$	Split unit; $j \star j = 1$ is proved.
e_+	$\frac{1}{2}(1 + j) = (\frac{1}{2}, \frac{1}{2})$	Plus idempotent / one null direction. DERIVED from definitions.
e_-	$\frac{1}{2}(1 - j) = (\frac{1}{2}, -\frac{1}{2})$	Minus idempotent / complementary null direction. DERIVED from definitions.
q_+	$q_+(t, x) = t + x$	Plus null-coordinate map. INPUT definition.

Symbol	Definition / type	Role and status
q_-	$q_-(t,x) = t - x$	Minus null-coordinate map. INPUT definition.
a, b	real numbers	Neutral complementary coefficients; no physical ontology is assigned.
$P(a,b)$	$ae_+ + be_-$	Assembly map from complementary coefficients to Long.
$\mathrm{invProd}(a,b)$	ab	Neutral shorthand for the coefficient product.
η	real rapidity parameter	Parameter of the standard longitudinal boost; it is not called time.
B_η	$(t,x) \mapsto (\cosh\eta\, t + \sinh\eta\, x, \sinh\eta\, t + \cosh\eta\, x)$	Proper orthochronous longitudinal boost in the chosen sign convention.
a_0, b_0	fixed positive reals	Fixed coefficients in the two elementary boundary families.
ε	nonnegative real parameter	Neutral family parameter. It is not assigned an interpretation.
$F_+(\varepsilon)$	$P(a_0, \varepsilon)$	Plus-boundary family, one coefficient fixed.
$F_-(\varepsilon)$	$P(\varepsilon, b_0)$	Minus-boundary family, symmetric construction.
I	fixed positive real	Value of the invariant product in Regime B; no interpretation beyond a real constant.
$G_I(b)$	$P(I/b, b)$ for the positive branch $b > 0$	Parameterization of $ab = I$. In Lean, division is totalized, so the syntactic value at $b = 0$ is not part of the intended branch.

Proof boundary 1 (No hidden ontology)**.** *None of a, b, I, η, or ε is identified in Part I with mass, energy, frequency, wave number, proper time, clock phase, particle type, field amplitude, or any other physical quantity. The task proves algebraic, topological, and kinematic relations only.*

5 Task 01: formal null-sector foundation

The formal development consists of five substantive Lean modules plus a principal theorem wrapper: `Basic.lean`, `SplitComplex.lean`, `Strata.lean`, `Boost.lean`, `Limits.lean`, and `Task01.lean` [1]. We now state the mathematical results in dependency order.

5.1 Restriction from $3+1$ to the longitudinal $1+1$ plane

Let

$$X = (t,x) \in \mathsf{Long}, \qquad \iota(X) = (t,x,0,0) \in \mathsf{Vec}_4. \tag{8}$$

Then

$$Q_4(\iota X) = t^2 - x^2 = Q_2(X). \tag{9}$$

Proposition 1 (Longitudinal restriction)**.** *For all $X \in \mathsf{Long}$,*

$$\boxed{Q_4(\iota X) = Q_2(X).} \tag{10}$$

This is elementary, but methodologically important: the $1+1$ object is not introduced as an unrelated Lorentz plane. It is the explicit restriction of the ambient $3+1$ form along the chosen longitudinal embedding.

5.2 Split-complex multiplication and complementary idempotents

On $\mathsf{Long} = \mathbb{R}^2$ define

$$(t, x) \star (s, y) := (ts + xy, ty + xs), \tag{11}$$

with

$$1 = (1, 0), \qquad j = (0, 1). \tag{12}$$

A direct calculation gives

$$j \star j = 1. \tag{13}$$

The two elements

$$e_+ = \frac{1}{2}(1 + j), \qquad e_- = \frac{1}{2}(1 - j) \tag{14}$$

then satisfy

$$e_+ \star e_+ = e_+, \qquad e_- \star e_- = e_-, \tag{15}$$

$$e_+ \star e_- = 0, \qquad e_+ + e_- = 1, \tag{16}$$

$$e_+ - e_- = j. \tag{17}$$

Moreover,

$$Q_2(e_+) = Q_2(e_-) = 0. \tag{18}$$

Thus the algebraic idempotent directions are simultaneously null directions of the longitudinal Lorentz form.

Proof boundary 2 (What is and is not derived here)**.** *The split-complex multiplication $\star$ is a local definition of Task 01. Its compatibility with the Lorentz structure is proved; its uniqueness or inevitability from Q_2 alone is not proved. Consequently, Part I establishes a natural split-complex realization of the longitudinal Lorentz plane, not a uniqueness theorem saying that Lorentz geometry forces one and only one such multiplication.*

5.3 Exact null-coordinate equivalence

For $X = (t, x)$ define

$$q_+(X) = t + x, \qquad q_-(X) = t - x. \tag{19}$$

For arbitrary $a, b \in \mathbb{R}$ define

$$P(a, b) = ae_+ + be_-. \tag{20}$$

Since

$$e_+ = \left(\frac{1}{2}, \frac{1}{2}\right), \qquad e_- = \left(\frac{1}{2}, -\frac{1}{2}\right), \tag{21}$$

one has the derived normal form

$$\boxed{P(a, b) = \left(\frac{a + b}{2}, \frac{a - b}{2}\right).} \tag{22}$$

Therefore

$$q_+(P(a, b)) = a, \qquad q_-(P(a, b)) = b, \tag{23}$$

and conversely

$$P(q_+(X), q_-(X)) = X. \tag{24}$$

Task 01 packages these relations as an explicit $\mathbb{R}$-linear equivalence

$$\boxed{\mathsf{Long} \simeq_{\mathbb{R}} \mathbb{R} \times \mathbb{R}.} \tag{25}$$

The decomposition is unique: if

$$P(a, b) = P(a', b'), \tag{26}$$

then $a = a'$ and $b = b'$.

5.4 The Lorentz form becomes the product of complementary coefficients

Substituting (22) into $Q_2(t,x) = t^2 - x^2$ gives

$$Q_2(P(a,b)) = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 \tag{27}$$
$$= ab. \tag{28}$$

Theorem 1 (Product form of the longitudinal invariant)**.** *For all $a, b \in \mathbb{R}$,*

$$\boxed{Q_2(P(a,b)) = \mathrm{invProd}(a,b) = ab.} \tag{29}$$

Equivalently, for every $X \in \mathsf{Long}$,

$$\boxed{Q_2(X) = q_+(X)q_-(X).} \tag{30}$$

This equality is the algebraic hinge of Part I. It does not introduce a new physical invariant; it exposes the ordinary Lorentz quadratic form in complementary coordinates.

6 Future-causal coefficient space as a stratified quadrant

Restrict to the closed coefficient quadrant

$$a \geq 0, \qquad b \geq 0. \tag{31}$$

For $X = P(a,b)$, (22) gives

$$t = \frac{a+b}{2}, \qquad x = \frac{a-b}{2}. \tag{32}$$

Task 01 proves the exact equivalence

$$\boxed{a \geq 0 \ \wedge \ b \geq 0 \iff t \geq 0 \ \wedge \ Q_2(t,x) \geq 0.} \tag{33}$$

Thus the coefficient quadrant is precisely the future-causal longitudinal cone in these coordinates.

The quadrant admits four mutually exclusive strata:

Stratum	**Coefficient condition**	**Quadratic form**
zero	$a = 0,\ b = 0$	$Q_2 = 0$
plus boundary	$a > 0,\ b = 0$	$Q_2 = 0$
minus boundary	$a = 0,\ b > 0$	$Q_2 = 0$
interior	$a > 0,\ b > 0$	$Q_2 > 0$

Task 01 proves exhaustiveness, pairwise disjointness, and the converses within the future-causal sector:

$$Q_2(P(a,b)) = 0 \iff \text{zero or one of the two boundary strata,} \tag{34}$$

while

$$Q_2(P(a,b)) > 0 \iff \text{interior stratum.} \tag{35}$$

For nonzero future-causal states, therefore,

$$\boxed{\text{nontrivial null sector} = \text{boundary of the positive coefficient quadrant,}} \tag{36}$$

and

$$\boxed{\text{timelike sector} = \text{its open interior.}} \tag{37}$$

This is a topological and algebraic statement. It is not yet a generative law. In particular, saying that the interior has the null rays as its boundary does not imply that a null state physically evolves into a timelike one, or that a unique continuation from boundary to interior exists.

Remark 1 (Orientation dependence of the labels)**.** *The names "plus" and "minus" depend on the orientation of the chosen longitudinal spatial coordinate. Under $x \mapsto -x$,*

$$q_+ \leftrightarrow q_-, \qquad e_+ \leftrightarrow e_-. \tag{38}$$

The robust statement is therefore the existence of two complementary null directions, not an absolute physical distinction between a plus-type and a minus-type object.

7 Longitudinal boosts: reciprocal scaling becomes explicit

Define the standard longitudinal boost of rapidity $\eta \in \mathbb{R}$ by

$$B_\eta(t, x) = (\cosh \eta\, t + \sinh \eta\, x, \sinh \eta\, t + \cosh \eta\, x). \tag{39}$$

In $3 + 1$ dimensions the y and z coordinates are left unchanged. Task 01 proves directly that

$$Q_4(B_\eta X) = Q_4(X), \qquad Q_2(B_\eta X) = Q_2(X). \tag{40}$$

The formal proof uses the standard identity

$$\cosh^2 \eta - \sinh^2 \eta = 1, \tag{41}$$

provided in Mathlib as `Real.cosh_sq_sub_sinh_sq` [15].

The key simplification appears in the idempotent basis. From the standard identities

$$\cosh \eta + \sinh \eta = e^\eta, \qquad \cosh \eta - \sinh \eta = e^{-\eta}, \tag{42}$$

formalized respectively as `Real.cosh_add_sinh` and `Real.cosh_sub_sinh` [15], Task 01 derives

$$\boxed{B_\eta(e_+) = e^\eta e_+, \qquad B_\eta(e_-) = e^{-\eta} e_-.} \tag{43}$$

Consequently,

$$\boxed{B_\eta(P(a, b)) = P(e^\eta a, e^{-\eta} b).} \tag{44}$$

Equivalently,

$$q_+(B_\eta X) = e^\eta q_+(X), \qquad q_-(B_\eta X) = e^{-\eta} q_-(X). \tag{45}$$

The familiar hyperbolic mixing in (t, x) coordinates is therefore diagonal reciprocal scaling in the complementary null basis. This is a change in visibility, not a change in mathematics.

7.1 The invariant product and two proof paths

Equation (44) implies

$$(e^\eta a)(e^{-\eta} b) = ab. \tag{46}$$

Task 01 proves this in two dependency-distinct ways.

Direct exponential proof. Using $e^{-\eta} = (e^\eta)^{-1}$ and $e^\eta \neq 0$,

$$(e^\eta a)(e^{-\eta} b) = ab. \tag{47}$$

This proof depends only on exponential identities.

Lorentz-geometric proof. Using

$$Q_2(B_\eta(P(a,b))) = Q_2(P(a,b)), \tag{48}$$

then applying (44) and theorem 1, one obtains exactly (46). Thus the invariant coefficient product is not a new invariant in addition to the Lorentz norm; it is the same invariant written in null coordinates.

This forms a useful consistency triangle:

$$\boxed{\text{Lorentz invariance of } Q_2 \quad \Longleftrightarrow \quad \text{reciprocal coefficient scaling plus } Q_2 = ab.} \tag{49}$$

The logical dependencies are not literally biconditional in the formal library because each direction uses specified prior theorems, but the mathematical consistency is exact.

8 Boundary families: two superficially similar limits that are not the same

The limiting analysis is the most informative negative control in Task 01. It prevents a common but invalid inference: that making one null coefficient small necessarily means approaching a finite null vector.

8.1 Regime A: one coefficient fixed

Fix $a_0 > 0$ and define

$$F_+(\varepsilon) = P(a_0, \varepsilon), \qquad \varepsilon \geq 0. \tag{50}$$

Then

$$\boxed{Q_2(F_+(\varepsilon)) = a_0\varepsilon.} \tag{51}$$

For $\varepsilon > 0$, both coefficients are positive, so $F_+(\varepsilon)$ lies in the interior. At $\varepsilon = 0$,

$$F_+(0) = P(a_0, 0) \tag{52}$$

lies on the plus boundary.

Because P is continuous, Task 01 proves as an actual topological limit in $\mathsf{Long} = \mathbb{R}^2$ that

$$\boxed{F_+(\varepsilon) \longrightarrow P(a_0, 0) \qquad (\varepsilon \to 0^+).} \tag{53}$$

It also proves

$$\boxed{Q_2(F_+(\varepsilon)) \longrightarrow 0.} \tag{54}$$

The symmetric family

$$F_-(\varepsilon) = P(\varepsilon, b_0), \qquad b_0 > 0, \tag{55}$$

obeys the corresponding minus-boundary statements.

Regime A therefore establishes a precise and limited meaning in which an interior family can approach a nonzero null boundary point: one coefficient remains finite and fixed while the complementary coefficient tends to zero, and the Lorentz invariant degenerates to zero with it.

8.2 Regime B: the product is held fixed and positive

Now fix $I > 0$ and consider the level set

$$ab = I. \tag{56}$$

Parameterize its positive branch by

$$G_I(b) = P(I/b, b), \qquad b > 0. \tag{57}$$

For every $b > 0$,

$$\boxed{Q_2(G_I(b)) = I.} \tag{58}$$

However, as $b \to 0^+$,

$$\frac{I}{b} \to +\infty. \tag{59}$$

The ordinary time coordinate is

$$(G_I(b))_t = \frac{I/b + b}{2}, \tag{60}$$

so Task 01 proves

$$\boxed{(G_I(b))_t \longrightarrow +\infty.} \tag{61}$$

From this it proves the stronger vector statement

$$\boxed{\neg\exists\, L \in \mathsf{Long} : G_I(b) \longrightarrow L \quad (b \to 0^+).} \tag{62}$$

A separate theorem records explicit unboundedness arbitrarily close to $b = 0$.

The distinction is therefore exact:

	Regime A	**Regime B**
Constraint	$a = a_0 > 0$, $b \to 0^+$	$ab = I > 0$, $b \to 0^+$
Quadratic form	$Q_2 = a_0 b \to 0$	$Q_2 = I$
Other coefficient	remains finite	$a = I/b \to +\infty$
Vector limit	$P(a_0, 0)$ exists	no finite limit in Long
Geometric outcome	approaches null boundary	escapes every bounded region

Corollary 1 (No finite null boundary at fixed positive invariant)**.** *Within the ordinary vector topology of* Long*, a positive fixed value* $ab = I > 0$ *cannot be maintained while one coefficient tends to zero and the vector simultaneously converges to a finite nonzero null boundary point.*

This negative result is crucial for the broader research program. "One coefficient tends to zero" is not by itself a sufficient specification of a physical or mathematical limiting process. What is constrained simultaneously matters.

Remark 2 (The syntactic value $G_I(0)$ in Lean)**.** *Lean's field division is totalized, so the definition* `G I b := P (I / b) b` *has a syntactic value even at* $b = 0$*. The intended Regime B branch is explicitly* $b > 0$*; all theorems that use the fixed-product identity require* $b \neq 0$*, and the limit is taken from the right. No mathematical endpoint at* $b = 0$ *is inferred from totalized division.*

9 What Task 01 changes about the reading of Experiment 1

Task 01 is not a continuation of the formal dependency chain of Experiment 1. It is more useful than that: it is an independently generated cross-section through the same mature mathematical territory.

9.1 The large Lorentz object acquires an internal resolution

Experiment 1 eventually reconstructed a standard $(1,3)$ Lorentzian carrier from a real spin-factor construction and showed that the endpoint returns to established Lorentz/Jordan/Clifford mathematics [11, 12]. Task 01 now shows that, once such a Lorentz carrier is available, every chosen longitudinal plane admits the explicit complementary resolution

$$X = ae_+ + be_-, \qquad Q_2(X) = ab, \tag{63}$$

and the longitudinal boost becomes

$$(a,b) \mapsto (e^{\eta}a, e^{-\eta}b). \tag{64}$$

Thus the compact object "Lorentz structure" hides at least four mutually linked pieces in this sector:

$$\boxed{\begin{array}{c}\text{two complementary null directions} + \text{ their product}\\ + \text{ reciprocal boost weights} + \text{ boundary/interior stratification}\end{array}} \tag{65}$$

These pieces were always mathematically present. The decomposition changes their visibility, not their truth value.

9.2 The two experiments complement rather than duplicate one another

The two directions can be summarized schematically as

$$\text{Experiment 1}: \qquad \mathbb{R}^3_{\text{Euclidean}} \longrightarrow \text{spin factor / Jordan structure} \longrightarrow 3+1 \text{ Lorentz structure,} \tag{66}$$

and

$$\begin{gathered}\text{Experiment 2 Task 01}:\\ 3+1 \text{ Lorentz structure} \longrightarrow \text{fixed } 1+1 \text{ plane}\\ \longrightarrow \text{complementary null coefficients and strata.}\end{gathered} \tag{67}$$

Read together, these arrows suggest a possible larger dependency loop, but Part I does not close it. The missing link is precise: Task 01 *defines* the split-complex multiplication and proves its compatibility; it does not derive that multiplication uniquely from the Lorentz data produced by Experiment 1. Therefore the current diagram is a coherent pair of constructions, not yet an equivalence theorem for all underlying data.

This is exactly where conservation of difficulty becomes operational. It would be easy to draw a circular figure and call the loop closed. The formal ledger forbids that move: the arrow

$$\text{Lorentz plane} \longrightarrow \text{split-complex multiplication} \tag{68}$$

is presently an Input definition plus compatibility theorem, not a uniqueness derivation.

9.3 A stronger methodological result than "the same mathematics again"

The final comparison of Experiment 1 emphasized that reordering dependencies did not generate a new Lorentz or spin structure [12]. Task 01 clarifies why that null result can still be useful. In ordinary (t,x) coordinates, the boost is a hyperbolic mixing matrix and the invariant is $t^2 - x^2$. In complementary coordinates the same statements become

$$(a,b) \mapsto (e^{\eta}a, e^{-\eta}b), \qquad Q_2 = ab. \tag{69}$$

The latter representation makes the reciprocal structure and the two causal boundary rays manifest.

This is a concrete example of the methodological proposition

$$\boxed{\text{same mathematical endpoint} \not\Rightarrow \text{same visibility of dependencies.}} \tag{70}$$

The proposition is not a mathematical theorem about all formalisms. It is an empirical observation about this controlled decomposition experiment.

9.4 Coherence criteria and possible falsification

The cross-experiment comparison has three possible outcomes.

(a) **Coherence:** overlapping structures agree up to explicit equivalence, sign, orientation, or normalization choices.

(b) **Complementarity:** one experiment reveals dependencies or obstructions not visible in the other without changing the common endpoint.

(c) **Contradiction:** two proved claims cannot simultaneously hold under the asserted identification; this would force an audit before any interpretation is allowed.

Part I currently yields the first two outcomes and no detected contradiction. That is evidence of structural compatibility, not evidence that a new physical theory has been discovered.

10 Formal verification and audit boundary

Task 01 was delivered as a standalone Lean 4 project and accompanying audit [1]. The reported environment is

$$\text{Lean 4.28.0,} \qquad \text{Mathlib } v\text{4.28.0,} \tag{71}$$

with Mathlib commit

$$\texttt{8f9d9cff6bd728b17a24e163c9402775d9e6a365}. \tag{72}$$

The exact revision is independently inspectable in the Mathlib source tree [16]; Lean 4 and Mathlib are described in [9, 14].

The source ledger records, among others, the imported declarations

- `Real.cosh_sq_sub_sinh_sq`,
- `Real.cosh_add_sinh`,
- `Real.cosh_sub_sinh`,
- `Real.exp_neg`,
- topological limit lemmas for inversion and the right-neighbourhood filter,

with the first three verified in the official Mathlib documentation [15].

The artifact reports that `lake build` completes successfully and that no `sorry`, `admit`, custom `axiom`, or `@[implemented_by]` occurs in the development. Its reported `#print axioms` result for the principal theorems is exactly

$$\texttt{propext}, \qquad \texttt{Classical.choice}, \qquad \texttt{Quot.sound}, \tag{73}$$

i.e. standard Lean/Mathlib dependencies.

For reproducibility, the Task 01 archive used for the present editorial audit has SHA-256

$$\texttt{4987ce1785fb6bf99483f710748e396f0535d4ba90ac6627de8f8f91d74b816f}. \tag{74}$$

The principal wrapper `Task01.lean` has SHA-256

$$\texttt{0847c91bbcd9a07986b7f59bbdaf353cf44c28b94fcebc0115bb4fff7ddd2da2}. \tag{75}$$

Proof boundary 3 (Editorial verification boundary). *The present paper audits the delivered source, theorem inventory, dependency report, pinned toolchain, and build report. It does not claim that the PDF compilation environment used to typeset this paper independently re-ran Lean itself. The mathematical claims attributed to Aristotle are therefore grounded in the delivered formal artifact and its reported green build, not in a second Lean execution performed during typesetting.*

Two small formal packaging observations do not alter the mathematical results. First, the principal theorem `task01_strata` contains exhaustiveness and the quadratic-form converses, while pairwise disjointness is supplied by the separate theorem `strata_disjoint`; the documentation accurately lists both. Second, the totalized value of $G_I(0)$ in Lean is outside the intended positive branch, as noted in remark 2.

10.1 Principal theorem map

Lean theorem	Mathematical content
`task01_restriction`	$Q_4(\iota X) = Q_2(X)$.
`task01_idempotents`	$j^2 = 1$, complementary idempotents, orthogonality, and nullity of e_+, e_-.
`task01_unique_decomposition`	existence and uniqueness of $X = P(a, b)$.
`task01_Q2_eq_product`	$Q_2(P(a, b)) = ab$.
`task01_strata`	future-causal membership, exhaustive stratum classification, and $Q_2 = 0$ / $Q_2 > 0$ converses.
`strata_disjoint`	pairwise disjointness of the four strata.
`task01_boost_action`	$(a, b) \mapsto (e^\eta a, e^{-\eta} b)$.
`task01_invariant_product`	$(e^\eta a)(e^{-\eta} b) = ab$.
`task01_boundary_family`	finite plus-boundary limit and $Q_2 \to 0$.
`task01_boundary_family_minus`	symmetric minus-boundary limit.
`task01_regimes`	Regime A convergence versus Regime B divergence and non-convergence.

11 Why the title says "Dynamics" although Part I proves none

The title *Split Complex Lorentzian Dynamics* names the research program, not the accomplishment of this first part. This distinction is deliberate. A Lorentz boost is a group action relating frames or coordinates; its rapidity parameter η is not automatically a physical evolution time. Likewise, a one-parameter family such as $F_+(\varepsilon)$ is a curve in coefficient space; without an equation of motion or other dynamical principle, ε is not a time variable. The invariant ab is a quadratic-form value; without further derivation it is not mass, energy, frequency, or proper time.

Part I therefore establishes the kinematic substrate on which a later dynamics question may be asked safely. If future work eventually derives a preferred evolution law, the new input responsible for that law must remain visible under the conservation-of-difficulty rule. If no preferred law can be derived from the present data, that negative result is equally valuable: it identifies where genuine new structure is required.

12 Immediate open questions for Experiment 2, Task 02

Task 01 closes its stated obligations. It also sharpens the next questions. The following are deliberately formulated as tests, not predictions.

Open question 1 (Intrinsic reconstruction and uniqueness of the split product). *Given a real $1+1$ Lorentz vector space, its time orientation, a chosen longitudinal orientation, and possibly* *the proper orthochronous boost subgroup, can one classify all bilinear commutative unital products on the carrier that are compatible with the null directions and Lorentz quadratic form? Is the split-complex product forced up to isomorphism and the exchange $e_+ \leftrightarrow e_-$, or are genuinely different compatible products possible?*

This is the most direct closure question between the experiments. A positive uniqueness theorem would replace the Task 01 multiplication input by a derivation; a negative classification would locate the additional datum precisely.

Open question 2 (Minimal data for null coordinates)**.** *Which subset of the following is sufficient to reconstruct q_+, q_-, and P up to explicit convention: the quadratic form, the future cone, the two null rays, the boost subgroup, and a spatial orientation? Which choices are continuous, and which are only discrete label conventions?*

This question separates genuinely intrinsic data from coordinate conveniences. In particular, it should determine exactly what is lost under $x \mapsto -x$ and whether any ordering of the two null sectors can be canonical without additional input.

Open question 3 (Compatibility across longitudinal directions in $3+1$)**.** *For each unit spatial direction $n \in S^2$, the plane spanned by time and n admits an analogous $1+1$ decomposition. How do these split-complex planes transform and overlap as n varies? Does their compatibility reconstruct only the already known $3+1$ Lorentz structure, or does the patching require additional geometric choices?*

This is a natural bridge to Experiment 1, whose reconstructed carrier is genuinely $3+1$-dimensional. The question must be answered without importing Experiment 1's conclusion into the Task 02 proof environment.

Open question 4 (Classification of boundary-to-interior curves)**.** *Among continuous or differentiable curves $\gamma(s) = P(a(s), b(s))$ beginning at a nonzero null boundary point and entering the interior, does the present structure single out any curve by boost equivariance, homogeneity, variational minimality, or another property already available in the data? Or is every such choice necessarily extra structure?*

Task 01 shows only that the simple fixed-coefficient family exists. It does not establish that this family is canonical. This question is the correct mathematical version of the heuristic phrase "activation of the complementary null degree" before any physical interpretation is attempted.

Open question 5 (The fixed-invariant divergence obstruction)**.** *Does the no-finite-limit result for $ab = I > 0$ admit any completion or normalization that is itself canonically determined by the current Lorentz/split data? If a projective, conformal, or normalized boundary description requires an additional choice, can that non-canonicity be proved before introducing such a completion?*

The point is not to force a compactification. The first obligation is a no-go or uniqueness audit: determine whether the present structure selects one.

Open question 6 (Where could dynamics first enter?)**.** *Beyond the Lorentz-group action, does the current carrier contain any preferred one-parameter flow that can legitimately be interpreted as evolution? If not, what is the minimal additional mathematical datum required to define one, and which previously proved symmetries constrain it?*

This is the first question deserving the word *dynamics*. It should not be answered by renaming rapidity, ε, or an arbitrary curve parameter as time. A nonexistence result would be a successful outcome because it would locate the missing dynamical input.

These questions are ordered so that Task 02 can remain result-open. The immediate priority is not to add proper time, periodicity, fields, particles, or quantum phases, but to determine whether the algebraic and Lorentzian data already present are sufficient to force any further structure at all.

13 Discussion: what is learned without new physics

Task 01 has a deliberately conservative mathematical content, yet it changes the research landscape in four precise ways.

First, null and timelike sectors are no longer merely juxtaposed. In the future-causal longitudinal coefficient space they are rigorously identified as boundary and interior strata of the same positive quadrant. That is stronger than a heuristic analogy and weaker than a dynamical transition law.

Second, the Lorentz boost is exposed as reciprocal scaling of two complementary null coefficients. This makes the product invariant almost tautological once the null basis is chosen, while the same fact appears as hyperbolic mixing in ordinary coordinates. Again, the gain is structural transparency.

Third, the two limiting regimes demonstrate why conservation of difficulty needs negative controls. A fixed coefficient allows a finite null-boundary limit only because the invariant itself tends to zero. Holding the positive invariant fixed does not reach a finite null state; the carrier coordinate diverges. Any future theory that wants to identify those regimes must supply additional structure capable of explaining the identification.

Fourth, the relation to EXPERIMENT 1 is now more informative. The first experiment shows how a standard Lorentz carrier can be reconstructed from weaker real data along one dependency path. The second shows how the same class of carrier can be resolved into null sectors along another path. Neither path, at present, proves the full inverse of the other. Their coherence is therefore a nontrivial cross-check, while the remaining mismatch in primitive data identifies the next theorem obligations.

Status 1 (Current scientific status)**.** Part I establishes no new empirical prediction and no new physical ontology. It establishes a formally verified decomposition and a set of sharp boundaries on what may be inferred from that decomposition. Its principal value is exploratory: it converts an initially suggestive heuristic into explicit mathematics, proves one nontrivial obstruction, and leaves the next dependency questions exposed rather than resolved by terminology.

14 Conclusion: an intentionally open endpoint

The split-complex Lorentzian structure provides a compact starting object for a controlled decomposition experiment. In a fixed longitudinal plane of $3+1$ Minkowski space, Task 01 proves an exact linear equivalence between ordinary coordinates and complementary coefficients,

$$X = P(a, b) = ae_+ + be_-, \tag{76}$$

with

$$Q_2(X) = ab. \tag{77}$$

The future-causal sector is exactly $a, b \geq 0$; its nonzero null states are the two boundary rays and its timelike states form the interior. A boost acts by reciprocal weights

$$(a, b) \mapsto (e^\eta a, e^{-\eta} b), \tag{78}$$

so ab is invariant. A fixed-coefficient family converges to a finite null boundary point while its invariant tends to zero. A fixed-positive-invariant family does not: one coordinate diverges and no finite vector limit exists.

These facts are coherent with the independent dependency reconstruction of EXPERIMENT 1. They do not complete it in reverse. In particular, Task 01 does not yet derive the split-complex multiplication from Lorentz geometry, does not establish a canonical continuation from null boundary to timelike interior, and does not derive dynamics. Those gaps are not defects to be covered over. They are the present location of the difficulty.

The intended outcome of the program is therefore not a predetermined physical picture. The next step is to ask increasingly restrictive mathematical questions and accept whichever of three outcomes is proved: a canonical derivation, a family of inequivalent possibilities, or an obstruction. Only after those distinctions are formally stable would a physical interpretation be scientifically meaningful.

In this sense, the first part ends where an exploratory study should end: with fewer ambiguities than it began, a stronger negative control than the heuristic supplied, and several sharply posed questions whose answers are not yet known.

A Dependency ledger for Part I

The Task 01 dependency chain can be summarized as

$$\begin{aligned} Q_4 &\longrightarrow (\mathsf{Long}, Q_2) \longrightarrow (\star, e_+, e_-) \longrightarrow (q_+, q_-, P) \\ &\longrightarrow \text{causal strata} \longrightarrow B_\eta \longrightarrow ab \text{ invariant} \\ &\longrightarrow \text{boundary families and obstructions.} \end{aligned} \tag{79}$$

The first arrow is restriction. The second includes a new local definition of $\star$; it is therefore not a derivation of the algebra from the metric. The boost arrow additionally uses standard hyperbolic/exponential identities. The limiting results use the ordinary product topology on $\mathbb{R}^2$.

No dynamical arrow appears in this graph.

B Source and reproducibility ledger

Source class	Material used
Experiment 1 primary sources	Initial split-complex witness [10]; Version 9 dependency closure [11]; Version 30 comparative final [12].
Experiment 2 primary source	Aristotle Task 01 request and delivered Lean/audit artifact [1].
Formal-library primary source	Mathlib v4.28.0 source at commit `8f9d9cff...` [16] and official module documentation for the hyperbolic identities [15].
Historical split-complex context	Cockle's original algebra papers [2, 3]; later hyperbolic-number treatment by Sobczyk [13]; special-relativistic perplex-number formulation by Fjelstad [6].
Lorentzian historical context	Einstein's 1905 kinematics [4] and Minkowski's spacetime formulation [8].
Jordan/spin-factor context	Original Jordan–von Neumann–Wigner algebraic framework [7] and the standard symmetric-cone treatment [5].
Lean and Mathlib	Lean 4 system paper [9]; Mathlib library paper [14].

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