Split Complex Lorentzian Dynamics

Part III: Lorentz-Covariant Product Transport, Unit Orbits, and the Residual Null-Sector Symmetry

A formally verified continuation of the dependency audit

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Abstract

Part II of *Split Complex Lorentzian Dynamics* reconstructed, rather than assumed, the unique real bilinear product compatible with the longitudinal Lorentz quadratic form once a normalized timelike unit e is chosen. The product is

$$\mu_e(X, Y) = L_2(X, e)Y + L_2(Y, e)X - L_2(X, Y)e,$$

where L_2 is the polarization of $Q_2(t, x) = t^2 - x^2$. The same analysis proved that the Lorentz carrier alone does not select one particular future-normalized unit, apparently leaving a continuous family of products $\{\mu_e\}$. Part III asks whether that family represents genuinely inequivalent algebraic structures or merely different Lorentz-transported representatives of one structure.

The formal Task 03 core imports only the Task 02 general-unit reconstruction. Task 01 split-complex coordinates, idempotents, boosts, rapidity, strata, and limits remain behind an import firewall until the final comparison layer. On the concrete carrier $\mathsf{Long} = \mathbb{R}^2$, Task 03 defines Lorentz linear equivalences solely by preservation of Q_2, derives preservation of L_2 by polarization, and proves the central covariance theorem

$$\boxed{T\big(\mu_e(X, Y)\big) = \mu_{Te}(TX, TY)}$$

for every Lorentz linear equivalence T. Thus the fixed-product obstruction found in Part II is not a failure of the family: a fixed product is invariant exactly when T stabilizes its unit, whereas the complete family is Lorentz-covariant.

For any future-normalized units e and f, Task 03 constructs an explicit transporter directly from their coordinates, without importing a boost or rapidity. The proper future transporter is unique; if spatial orientation is not fixed there are exactly two future-preserving transporters, one with determinant $+1$ and one with determinant -1. Consequently all future-normalized units lie in one proper-future Lorentz orbit, and all reconstructed products lie in one corresponding Lorentz-isomorphism class. The stabilizer of a chosen unit under the full future-preserving Lorentz group has exactly two elements: the identity and an orientation-reversing reflection. This residual $\mathbb{Z}_2$ automorphism exchanges the two complementary null idempotents. Once a spatial orientation is added, the proper stabilizer is trivial and the ordered null pair is unique.

The mathematical content is classical in kind: no novelty is claimed for the $1 + 1$ Lorentz group, its hyperbolic orbits, split-complex realization, or reflection symmetry. The contribution of the experiment is the explicit dependency separation and formal audit. The result sharpens the program's *conservation of difficulty*: the continuous choice of e, which Part II exposed as the next apparent obstruction, is not a continuous family of inequivalent algebras after Lorentz transport is taken into account. What remains in the fixed future-oriented $1 + 1$ sector is a discrete orientation ambiguity in the ordering of the two null directions, together with the still-unopened question of how the longitudinal plane and its time orientation are selected in a larger structure.

A final section translates these statements back into the motivating two-hole-band/laser-pulse picture. That translation is explicitly heuristic. A normalized e may be pictured as a chosen timelike clock or laser direction, while the two null idempotents may be pictured as two counter-propagating null channels. Task 03 then says that changing the chosen timelike direction does not create a new algebra; the entire decomposition is transported covariantly. The labels "left" and "right" remain conventional until a spatial orientation is supplied. No theorem in Part III identifies the algebraic coefficients with material holes, photon densities, wavelengths, clock phases, or a dynamical evolution law.

Keywords: split-complex numbers; Lorentzian geometry; $O(1,1)$; covariance; group orbits; stabilizers; null idempotents; orientation; formal verification; Lean 4; conservation of difficulty; dependency analysis; null-signal heuristic.

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1 Motivation: the obstruction found in Part II was not yet the endpoint

Part I resolved a fixed longitudinal $1+1$ Lorentz carrier into two complementary null sectors, proved that the Lorentz quadratic form becomes the product of their coefficients, and recovered the reciprocal scaling law under boosts [11, 1]. Part II then removed the split-complex multiplication itself from the input. Starting from $\mathsf{Long} = \mathbb{R}^2$, Q_2, an unknown bilinear product, and a chosen two-sided unit, it proved that the unit must satisfy $Q_2(e) = 1$ and that the product is uniquely forced to be [12, 2]

$$\mu_e(X, Y) = L_2(X, e)Y + L_2(Y, e)X - L_2(X, Y)e. \tag{1}$$

At the reference unit $u_0 = (1, 0)$ this is exactly the standard split-complex multiplication. The square relation $j^2 = 1$, commutativity, associativity, and the complementary null idempotents became theorems rather than primitive algebraic assumptions.

Part II also found a genuine canonicality obstruction. The Lorentz data do not select one unique future-normalized vector e, and an explicit nontrivial Lorentz symmetry moves every nonzero vector. Hence no single fixed admissible product can be invariant under every Lorentz map. At that stage the strongest justified statement was therefore

$$(\mathsf{Long}, Q_2) + \text{chosen normalized unit } e \Longrightarrow \text{unique product } \mu_e. \tag{2}$$

There are two mathematically different readings of this endpoint. The pessimistic reading is that the reconstruction has merely replaced one primitive input, the product, by another primitive input, the vector e. The more structural reading is that the family $\{\mu_e\}$ might itself transform naturally under the Lorentz group, so that the distinct products are merely different representatives of one equivariant object. Part II did not decide between these possibilities. Part III does.

The distinction is essential for the program's methodological principle.

Methodological principle 1 (Conservation of difficulty)**.** When an assumed structure becomes derivable, the remaining explanatory burden must be localized again rather than declared absent. A newly exposed continuous family is not automatically a family of inequivalent structures: one must test whether a symmetry acts transitively on that family, whether the derived structures are transported covariantly, and whether any residual freedom is continuous, discrete, or merely conventional.

This is the exact purpose of Task 03. It is not a search for a new Lorentz group. The $1+1$ Lorentz group and its hyperbolic parametrization are classical [6, 8, 4, 13]. The experiment instead asks which familiar facts remain true when they are reconstructed in a deliberately constrained dependency order.

2 Experimental design and firewall

2.1 Cumulative Task 02 input, Task 01 still withheld

Task 03 is cumulative with Task 02 but remains blind to the Task 01 split-complex realization in its core. The file

```
RequestProject/NullSectorTask03/Core.lean
```

imports exactly

```
RequestProject.NullSectorTask02.GeneralUnit.
```

Through that chain it has access to the carrier Long, the quadratic form Q_2, its polarization L_2, the Task 02 notion of admissibility, the normalized future-unit predicate, and the independently reconstructed product formula. It does *not* import the Task 01 split-complex multiplication, boost, null-coordinate map, idempotents, strata, limits, or the Task 02 comparison layer [3].

Only the final file

```
RequestProject/NullSectorTask03/Task03.lean
```

imports the Task 01 boost module. This allows the independently obtained transporter and null pair to be compared with the previously defined boost and idempotents only after the Task 03 core has compiled.

The distinction between cumulative and circular is therefore explicit. Task 03 is permitted to use the Task 02 theorem

$$\mu_e(X, Y) = L_2(X, e)Y + L_2(Y, e)X - L_2(X, Y)e, \tag{3}$$

because that theorem is the starting point under investigation. It is not permitted to solve the transport problem by importing the standard Task 01 boost action and reading off the answer.

2.2 No high-level $O(1, 1)$ classification imported

Task 03 also forbids solving the central orbit problem merely by importing an existing abstract classification of $O(1, 1)$. The normal form, determinant branches, transporter, stabilizer, and null-sector action are proved by direct coordinate arguments on $\mathbb{R}^2$. This is not because the standard group theory is in doubt; it is because the dependency audit needs to expose which consequences follow directly from preservation of Q_2.

3 Starting data and notation

The carrier and quadratic form remain

$$\mathsf{Long} = \mathbb{R} \times \mathbb{R}, \qquad Q_2(t, x) = t^2 - x^2. \tag{4}$$

The Task 02 polarization is

$$L_2(X, Y) = \frac{Q_2(X + Y) - Q_2(X) - Q_2(Y)}{2}, \tag{5}$$

which in coordinates is

$$L_2((t, x), (s, y)) = ts - xy. \tag{6}$$

A normalized future unit is

$$\mathsf{FutureUnit}(e) \quad :\Longleftrightarrow \quad e_0 > 0, \qquad Q_2(e) = 1. \tag{7}$$

For every normalized e, the notation μ_e denotes the unique Task 02 admissible product given by (1).

A Lorentz linear equivalence is defined without group-theoretic packaging:

$$\mathsf{Lor}(T) \quad :\Longleftrightarrow \quad Q_2(TX) = Q_2(X) \quad \text{for all } X \in \mathsf{Long}. \tag{8}$$

Future preservation is kept separate. A future-timelike vector satisfies

$$X_0 > 0, \qquad Q_2(X) > 0, \tag{9}$$

and T preserves the future component if it maps every such vector to another such vector. Spatial orientation is represented by the determinant in the coordinate basis $u_0 = (1, 0)$, $s_0 = (0, 1)$:

$$\det_L(T) = (Tu_0)_0(Ts_0)_1 - (Ts_0)_0(Tu_0)_1. \tag{10}$$

The proper condition is $\det_L(T) = 1$.

This separation is important. Preserving the quadratic form, preserving the future component, and preserving spatial orientation are three distinct conditions. None is silently folded into another.

4 From quadratic preservation to bilinear preservation

The first Task 03 theorem removes a potential hidden assumption.

Theorem 1 (Polarization is Lorentz-invariant). *If T is real-linear and preserves Q_2, then*

$$L_2(TX, TY) = L_2(X, Y) \tag{11}$$

for all $X, Y \in \mathsf{Long}$.

The proof is immediate in principle but important in the dependency graph: apply the polarization identity to TX and TY, use linearity to replace $TX + TY$ by $T(X + Y)$, and use $Q_2(TZ) = Q_2(Z)$ on the three quadratic terms. Thus preservation of L_2 is DERIVED, not a second defining axiom.

This theorem is what makes the later product covariance nearly inevitable once the Task 02 formula is available. Nevertheless, that consequence is deliberately postponed until the Lorentz maps themselves have been classified concretely.

5 Concrete normal form of Lorentz linear equivalences

Let

$$a = (Tu_0)_0, \qquad c = (Tu_0)_1. \tag{12}$$

Task 03 proves directly from Q_2 preservation that

$$a^2 - c^2 = 1 \tag{13}$$

and that every Lorentz linear equivalence belongs to one of two determinant branches:

$$T(t, x) = (at + cx,\ ct + ax), \qquad \det_L(T) = +1, \tag{14}$$

$$T(t, x) = (at - cx,\ ct - ax), \qquad \det_L(T) = -1. \tag{15}$$

Equation (13) still has two sign branches for a. Consequently the full real $O(1, 1)$ has four connected components, distinguished by time orientation and determinant. This wording is the precise correction to a shorthand phrase in the raw Task 03 audit that described the Lorentz maps as “two components times an $\mathbb{R}$-family”. After the future component is selected, $a > 0$ and only two components remain: the determinant $+1$ and determinant -1 branches.

Task 03 proves concretely that

$$T \text{ preserves the future component} \quad \Longleftrightarrow \quad a > 0. \tag{16}$$

No rapidity parameter is used in this classification.

6 The unit-orbit problem

The central open question left by Part II can now be stated precisely. Given two normalized future units

$$e = (e_0, e_1), \qquad f = (f_0, f_1), \tag{17}$$

does a future-preserving Lorentz map exist with $Te = f$, and if so, how unique is it?

6.1 The proper transporter constructed from e and f

Define

$$A = e_0 f_0 - e_1 f_1 = L_2(e, f), \qquad C = e_0 f_1 - e_1 f_0. \tag{18}$$

Task 03 proves

$$A^2 - C^2 = Q_2(e)Q_2(f) = 1. \tag{19}$$

It then defines

$$T^+_{e\to f}(t, x) = (At + Cx,\, Ct + Ax). \tag{20}$$

This map preserves Q_2, has determinant $+1$, preserves the future component, and satisfies

$$T^+_{e\to f} e = f. \tag{21}$$

The positivity needed for future preservation is derived algebraically from the future-unit hypotheses.

The construction is noteworthy only in the context of the audit: no boost, no cosh, no sinh, and no rapidity were available as inputs. The usual boost matrix appears later as a comparison.

6.2 The orientation-reversing transporter

There is also a second future-preserving transporter. Define

$$A' = e_0 f_0 + e_1 f_1, \qquad C' = e_0 f_1 + e_1 f_0, \tag{22}$$

and

$$T^-_{e\to f}(t, x) = (A't - C'x,\, C't - A'x). \tag{23}$$

Task 03 proves that this map also sends e to f and preserves the future component, but

$$\det_L(T^-_{e\to f}) = -1. \tag{24}$$

The classification theorem then states that these are the only possibilities.

Theorem 2 (Exactly two future transporters)**.** *For future-normalized e and f, every future-preserving Lorentz linear equivalence satisfying $Te = f$ is either $T^+_{e\to f}$ or $T^-_{e\to f}$. The two maps are distinct.*

Corollary 1 (Unique proper transporter)**.** *If* $\det_L(T) = +1$ *is additionally required, then the transporter from e to f is unique and equals $T^+_{e\to f}$.*

Thus the apparent continuous freedom in the choice of normalized future unit is already one transitive proper-future Lorentz orbit.

7 The central covariance theorem

The Task 02 product was reconstructed as

$$\mu_e(X, Y) = L_2(X, e)Y + L_2(Y, e)X - L_2(X, Y)e. \tag{25}$$

Let T preserve Q_2. By section 4, it also preserves L_2. Therefore

$$\begin{aligned} T\big(\mu_e(X, Y)\big) &= L_2(X, e)TY + L_2(Y, e)TX - L_2(X, Y)Te && (26)\\ &= L_2(TX, Te)TY + L_2(TY, Te)TX - L_2(TX, TY)Te && (27)\\ &= \mu_{Te}(TX, TY). && (28) \end{aligned}$$

This calculation is formalized as the principal Task 03 theorem

$$\boxed{T\big(\mu_e(X,Y)\big) = \mu_{Te}(TX,TY).} \tag{29}$$

Future preservation is not required for (29); preservation of Q_2 alone is sufficient.

The result can be packaged as a genuine unital algebra transport. If

$$\mathcal{A}_e = (\mathsf{Long}, \mu_e, e), \tag{30}$$

then every Lorentz linear equivalence is a multiplicative isomorphism

$$T : \mathcal{A}_e \longrightarrow \mathcal{A}_{Te}. \tag{31}$$

For future-normalized e and f, the proper transporter (20) therefore yields

$$\boxed{\mathcal{A}_e \cong \mathcal{A}_f} \tag{32}$$

by a future-preserving, orientation-preserving Lorentz map.

8 Fixed-fibre invariance is not family covariance

This distinction resolves the apparent tension between Parts II and III.

A fixed product μ_e is invariant under T if

$$T\big(\mu_e(X,Y)\big) = \mu_e(TX,TY) \quad \text{for all } X, Y. \tag{33}$$

Family covariance instead uses the transported unit:

$$T\big(\mu_e(X,Y)\big) = \mu_{Te}(TX,TY). \tag{34}$$

Task 03 proves the exact equivalence

$$\boxed{\mathsf{FixedInvariant}(T,e) \quad\Longleftrightarrow\quad Te = e} \tag{35}$$

for Lorentz T and normalized e. The forward direction ultimately uses the Task 02 uniqueness of the two-sided unit. If a transformed product were still the same fixed product, then its unit would have to remain the same.

Thus the Part II counterexample was correctly interpreted but incomplete as a statement about the full family. A non-stabilizing Lorentz map moves the fibre. It does not destroy covariance.

Remark 1. *The distinction between* (33) *and* (29) *is structurally analogous to the distinction between demanding that a tensor have unchanged coordinate components and demanding that it transform covariantly. The analogy is explanatory only; the formal theorem here is the explicit algebraic identity above.*

9 Stabilizer of a chosen unit

Set $f = e$ in the transporter classification. The proper transporter collapses to the identity. The orientation-reversing transporter becomes the reflection

$$R_e(t,x) = \Big((e_0^2 + e_1^2)t - 2e_0e_1x,\ 2e_0e_1t - (e_0^2 + e_1^2)x\Big). \tag{36}$$

It preserves Q_2, preserves the future component, fixes e, and has determinant -1.

Theorem 3 (Future stabilizer)**.** *For a future-normalized unit e, every future-preserving Lorentz map with $Te = e$ is either*

$$\mathrm{id} \qquad \textit{or} \qquad R_e. \tag{37}$$

The two are distinct.

Hence

$$\mathrm{Stab}_{O^+(1,1)}(e) \cong \mathbb{Z}_2. \tag{38}$$

If the determinant $+1$ condition is imposed, the stabilizer becomes trivial.

Both stabilizer elements are automorphisms of the reconstructed algebra $\mathcal{A}_e$. This is not assumed separately; it follows immediately from family covariance together with $Te = e$.

At the reference unit $u_0 = (1, 0)$, the nontrivial stabilizer is simply

$$R_{u_0}(t, x) = (t, -x). \tag{39}$$

This reflection will account exactly for the residual exchange of the two null sectors.

10 Spacelike companions and the complementary null pair

For a normalized unit e, define a normalized spacelike companion by

$$L_2(e, s) = 0, \qquad Q_2(s) = -1. \tag{40}$$

Task 03 proves a complete classification:

$$\boxed{s = (e_1, e_0) \quad \text{or} \quad s = -(e_1, e_0).} \tag{41}$$

The two solutions are distinct. Thus choosing e removes all continuous freedom in the orthogonal normalized spacelike direction; only its sign remains.

For either companion define

$$p_+(e, s) = \frac{1}{2}(e + s), \qquad p_-(e, s) = \frac{1}{2}(e - s). \tag{42}$$

Without importing Task 01 idempotents, Task 03 derives

$$Q_2(p_+) = 0, \qquad Q_2(p_-) = 0, \tag{43}$$
$$\mu_e(p_+, p_+) = p_+, \qquad \mu_e(p_-, p_-) = p_-, \tag{44}$$
$$\mu_e(p_+, p_-) = 0, \qquad p_+ + p_- = e. \tag{45}$$

Moreover,

$$s \mapsto -s \quad \Longrightarrow \quad p_+ \leftrightarrow p_-. \tag{46}$$

This yields a precise canonicality statement.

Theorem 4 (Unordered versus ordered null pair)**.** *For fixed normalized e, the unordered pair*

$$\{p_+, p_-\} \tag{47}$$

is uniquely determined. The ordered pair (p_+, p_-) is not: the nontrivial stabilizer R_e exchanges its entries. Adding a spatial orientation selects one of the two spacelike companions and therefore determines the ordering.

For a future unit e, Task 03 chooses the orientation by requiring the second coordinate of the companion to be positive. Since (e_1, e_0) has second coordinate $e_0 > 0$, this selects the positive-sign companion uniquely.

11 Transport of the null pair

The null-pair transformation law separates the determinant branches cleanly. Linearity alone gives

$$Tp_+(e,s) = p_+(Te,Ts), \qquad Tp_-(e,s) = p_-(Te,Ts). \tag{48}$$

A Lorentz map carries a normalized spacelike companion of e to a normalized spacelike companion of Te.

For the proper transporter from e to f,

$$T^+_{e\to f}(e_1,e_0) = (f_1,f_0), \tag{49}$$

so the two labels are preserved. For the improper transporter,

$$T^-_{e\to f}(e_1,e_0) = -(f_1,f_0), \tag{50}$$

and therefore

$$p_+ \leftrightarrow p_-. \tag{51}$$

Thus the residual $\mathbb{Z}_2$ is not a second algebraic isomorphism class. It is an isotropy symmetry of the same algebraic fibre, acting by exchange of the two null idempotents.

12 What is canonical after Task 03?

The exact answer depends on the level at which "canonical" is asked.

Data / equivalence level	What Task 03 justifies
(Long, Q_2) alone	A Lorentz quadratic carrier; no future component or preferred normalized timelike unit is selected.
(Long, Q_2) plus future component	All future-normalized units form one orbit under the future Lorentz group.
Future component, products modulo proper-future Lorentz transport	One isomorphism class of reconstructed unital products.
Chosen future unit e	One unique product μ_e and one unique unordered complementary null pair.
Chosen e plus spatial orientation	One unique ordered pair (p_+,p_-).

Two qualifications are essential.

First, saying that the products form one orbit does *not* mean that there is one Lorentz-invariant product on the fixed carrier. There is no such fixed representative under the full Lorentz action; (35) says precisely why.

Second, the raw Task 03 audit summarizes the null structure of data sets A and B by saying that the two null lines, and after future selection the two future null rays, are determined. This is the elementary coordinate consequence

$$Q_2(t,x) = 0 \quad \Longleftrightarrow \quad (t-x)(t+x) = 0 \quad \Longleftrightarrow \quad t = x \text{ or } t = -x. \tag{52}$$

However, Task 03 did not package (52) and the positive-ray refinement as standalone Lean classification theorems. Part III therefore treats this A/B statement as standard coordinate mathematics, while the C/D statements concerning the companion classification, unordered pair, non-uniqueness of ordering, and orientation selection are directly covered by the Task 03 formal theorem inventory.

13 Late comparison with Task 01: the boost reappears only at the end

Only after the core orbit and covariance results are closed does Task 03 import the Task 01 boost. It packages

$$B_\eta(t, x) = (\cosh\eta\, t + \sinh\eta\, x, \sinh\eta\, t + \cosh\eta\, x) \tag{53}$$

as a linear equivalence and proves that it is Lorentz, future-preserving, and proper.

At the reference unit,

$$B_\eta u_0 = (\cosh\eta, \sinh\eta). \tag{54}$$

The comparison theorem then identifies the independently constructed transporter with the Task 01 boost:

$$T^+_{u_0 \to B_\eta u_0} = B_\eta. \tag{55}$$

The Task 03 covariance theorem therefore specializes to

$$B_\eta\big(\mu_e(X, Y)\big) = \mu_{B_\eta e}(B_\eta X, B_\eta Y). \tag{56}$$

The independently reconstructed null pair at u_0 is also proved equal to the Task 01 idempotents:

$$p_+(u_0, s_0) = e_+, \qquad p_-(u_0, s_0) = e_-. \tag{57}$$

Hence the Task 01 reciprocal scaling law returns as a late comparison:

$$B_\eta e_+ = e^\eta e_+, \qquad B_\eta e_- = e^{-\eta} e_-. \tag{58}$$

The reflection $R_{u_0}(t, x) = (t, -x)$ exchanges the two labels.

Rapidity itself is also late. Task 03 proves that every future-normalized unit has the form

$$e = (\cosh\eta, \sinh\eta) \tag{59}$$

for some η, and every proper future Lorentz map is one of these boosts. No such parametrization is used to construct the orbit theorem or the covariance theorem.

14 Heuristic translation: two hole bands and laser pulses

The original motivation for the experiment was not the abstract group orbit. It was a visual timing picture involving two counter-running perforated bands and regularly emitted laser pulses. This section translates the Task 03 result back into that picture. The translation is deliberately separated from the theorem statements because none of the mechanical or optical nouns below occurs in the Lean development.

14.1 The safe dictionary

A useful but non-binding correspondence is:

Formal object	Heuristic picture	Status
Normalized future unit e	A chosen normalized timelike direction of the emitting clock/laser system	HEURISTIC ONLY
Complementary null directions p_+, p_-	Two counter-propagating null signal channels, visualized as the two hole bands	HEURISTIC ONLY
Proper Lorentz transporter T	A change of inertial timelike reference direction while preserving the chosen spatial orientation	kinematic interpretation
Reflection R_e	Reversal of the chosen longitudinal spatial orientation	kinematic interpretation
Null coefficients a, b	Possible marker density, wave-number, or reciprocal-spacing variables only after an extra identification	HEURISTIC ONLY
Product μ_e	Algebraic composition law associated with the selected timelike representative	DERIVED mathematically; physical meaning open

The most important entry is the first. Task 03 does *not* prove that a physical laser selects the unit e. It proves only that once a normalized timelike unit is mathematically selected, changing that choice by a Lorentz transformation transports the corresponding product and null decomposition covariantly.

14.2 What changes when the laser/clock direction changes?

Suppose the heuristic laser emits pulses at a constant intrinsic tick interval and the two signal channels are treated as null. In the reference description choose $e = u_0$. A different inertial motion of the same idealized clock can be represented kinematically by

$$f = B_\eta e. \tag{60}$$

Part II alone could have been read as saying that the new f requires a new product μ_f, leaving an unexplained continuum of algebraic choices. Task 03 supplies the missing relation:

$$B_\eta(\mu_e(X, Y)) = \mu_f(B_\eta X, B_\eta Y). \tag{61}$$

In the heuristic picture this means that changing the timelike laser/clock direction does not require a new underlying split-complex mechanism. The clock direction, null channels, and composition law move together as one covariant family. Holding the old product μ_e fixed while transforming the clock direction to f is exactly the fixed-fibre operation that Task 03 proves to be invariant only when $f = e$.

This gives a useful interpretation of the Part II obstruction. The obstruction was not evidence for a preferred clock. It was evidence that one cannot transform the carrier while simultaneously pretending that the unit defining the chosen algebraic representative has remained absolute.

14.3 The two bands are intrinsically an unordered pair

The two counter-propagating channels are reconstructed as the two complementary null idempotents. But the formal structure does not say which one deserves the name "plus band" and which one deserves the name "minus band". The reflection

$$(t, x) \mapsto (t, -x) \tag{62}$$

leaves the timelike unit u_0 fixed and exchanges

$$e_+ \leftrightarrow e_-. \tag{63}$$

Thus, in the hole-band image, the existence of two opposite null channels is structural, whereas their left/right naming requires a spatial orientation convention. If the longitudinal axis is reversed, the two bands exchange roles without changing the algebraic fibre. This is the precise heuristic meaning of the residual $\mathbb{Z}_2$ stabilizer.

14.4 Reciprocal scaling and the marker-spacing interpretation

At the Task 01 reference unit any longitudinal vector can be decomposed as

$$X = ae_+ + be_-. \tag{64}$$

Under a proper boost,

$$(a, b) \longmapsto (e^{\eta}a, e^{-\eta}b). \tag{65}$$

The invariant product is

$$ab = Q_2(X). \tag{66}$$

One possible heuristic realization, called here the *fixed-null-transport interpretation*, keeps the propagation speeds of the two signal channels at $\pm c$ and assigns the changing coefficients to a marker wave number or density. With one convention,

$$k_+ = k_0 e^{\eta}, \qquad k_- = k_0 e^{-\eta}, \tag{67}$$

$$\lambda_+ = \lambda_0 e^{-\eta}, \qquad \lambda_- = \lambda_0 e^{\eta}, \tag{68}$$

$$\rho_+ = \rho_0 e^{\eta}, \qquad \rho_- = \rho_0 e^{-\eta}. \tag{69}$$

Reversing the spatial orientation exchanges the + and − rows. The formal content behind (69) is only the reciprocal coefficient scaling (65); identifying a coefficient with k, ρ, or $1/\lambda$ is an additional modeling step.

In the extreme limit $\eta \to +\infty$, this interpretation gives

$$\lambda_+ \to 0, \quad \rho_+ \to \infty, \qquad \lambda_- \to \infty, \quad \rho_- \to 0. \tag{70}$$

If one says colloquially that the holes on one band "coincide", the mathematically defensible meaning is that the separation of any fixed pair of corresponding markers tends to zero in the chosen parametrization. It is not a theorem that an infinite material band literally collapses to one point.

14.5 Why the bands should not be interpreted as material objects whose speed changes

A different heuristic would hold marker spacing fixed and vary the transport speed of the bands. If the two bands are meant to represent the formal null directions, this is not the same Lorentz interpretation. Null propagation remains null under Lorentz transformations; its local invariant speed is not continuously slowed from c to zero by a boost. Such a variable-speed band model would therefore add new structure beyond the Task 03 Lorentz carrier.

A third possibility is to vary both marker spacing and transport rate. Then a synchronization condition alone supplies only one relation between two changing quantities. The microscopic decomposition is underdetermined unless an additional law is supplied. Task 03 does not remove that underdetermination because it classifies the Lorentz transformation of the resulting null coefficients, not a hidden mechanical mechanism that produces them.

The three possibilities can therefore be summarized as follows:

Model	Heuristic assignment	Relation to Task 03
A	Fixed null speeds $\pm c$; variable marker density/wave number	Structurally compatible with the reciprocal null-coefficient scaling, after an extra coefficient-to-density identification.
B	Fixed marker spacing; variable propagation speed	Not a standard null interpretation if the channels are meant to remain Lorentz-null.
C	Both spacing and transport rate vary	Underdetermined by the Lorentz/null-coordinate relations alone; needs an additional microscopic law.

14.6 What Task 03 does not say about the laser

Even with the heuristic dictionary, the following claims are *not* established:

- that a physical laser is the mechanism that selects e;
- that the null idempotents are photons or particle species;
- that the coefficients a, b are frequencies, energies, masses, or particle numbers;
- that the marker density law (69) is a microscopic physical law;
- that η is an evolution time;
- that the orbit parameter represents acceleration or a dynamical process;
- that one null sector dynamically creates the timelike interior.

The laser-pulse picture remains a visualization of a kinematic decomposition and of its covariance properties.

15 Conservation of difficulty after Task 03

The dependency burden has moved again, but it has not vanished.

After Task 01 one might have suspected that the relation $j^2 = 1$ or the split-complex multiplication was the essential input. Task 02 showed that, after choosing a normalized timelike unit, both are forced by weaker Lorentz data. The apparent remaining difficulty then became the continuous choice of e. Task 03 shows that this continuous family is one proper-future Lorentz orbit and that the corresponding products are all transported into one another by (29).

The new dependency picture is therefore

$$
\begin{aligned}
(\mathsf{Long}, Q_2) &\Longrightarrow L_2 \\
&\Longrightarrow \text{Lorentz transport law} \\
&\xRightarrow{\text{choose future component}} \text{one orbit of normalized future units} \\
&\Longrightarrow \text{one orbit of reconstructed products} \\
&\xRightarrow{\text{choose a representative } e} \text{one fixed product and one unordered null pair} \\
&\xRightarrow{\text{choose spatial orientation}} \text{one ordered null pair.}
\end{aligned}
$$

Several subtleties matter.

First, the choice of a *specific* representative e has not become canonical. What has disappeared is its status as a continuous family of inequivalent algebraic structures. If a calculation is performed in one fixed fibre, a representative still has to be chosen.

Second, the future component itself is additional structure relative to the bare quadratic form. The map $T = -\mathrm{id}$ preserves Q_2 but reverses time orientation. Task 03 explicitly contains this negative control. Thus one must not compress

$$(\mathsf{Long}, Q_2) \tag{71}$$

into

$$\text{future-oriented Lorentz carrier} \tag{72}$$

without recording the added sign choice.

Third, after a future component and a unit representative are selected, the residual ambiguity is exactly the spatial reflection that exchanges the two null sectors. Under the proper future Lorentz subgroup the stabilizer is trivial. The $\mathbb{Z}_2$ is therefore an isotropy symmetry of the full future-preserving group, not a second algebraic isomorphism class.

In the engineering language motivating the conservation-of-difficulty principle, Task 03 has ruled out another candidate "fault location". The continuous unit choice looked like a possible missing structural part after Task 02; the orbit and covariance theorems show that, at the level of Lorentz isomorphism, it is instead symmetry-related redundancy. The unresolved burden moves outward: to time orientation, spatial orientation when ordered labels matter, and ultimately to the selection and compatibility of the longitudinal plane itself in a larger-dimensional structure.

16 Relation to Experiment 1 and mathematical novelty

The two experiments continue to approach the same classical mathematical territory from different directions. Experiment 1 progressively atomized a Lorentz/split-complex starting point and reconstructed Lorentzian and complex structures from weaker real data, eventually comparing the result with standard Jordan, Clifford, and Lorentz constructions [9, 10]. Experiment 2 instead begins with an already established Lorentz carrier and asks how much of its split-complex internal structure is forced and how the resulting family transforms.

Task 03 receives no theorem from Experiment 1. The present comparison is editorial, after the formal result. The compatibility is nevertheless notable: the independent Task 03 path does not generate an exotic alternative algebra or an anomalous transformation law. It recovers the standard $1+1$ Lorentz orbit structure and split-complex null decomposition expected from classical mathematics [4, 13, 8].

No mathematical novelty is claimed for any individual theorem in the $O(1,1)$ classification, the transitivity of the future unit hyperbola, the determinant branches, the null-direction exchange, or the split-complex realization. The contribution is methodological and formal:

1. the product is reconstructed before it is identified with the standard split-complex product;
2. the family covariance theorem is proved before boosts are imported;
3. the transporter is constructed algebraically before rapidity is introduced;
4. continuous and discrete non-uniqueness are separated by explicit negative controls;
5. the import boundary is machine-auditable.

For a mature structure such as Lorentz geometry, recovering known mathematics is the expected calibration result. A genuinely unfamiliar result at this stage would demand more skepticism, not less.

17 Formal verification and audit boundary

The Task 03 artifact reports the following environment:

Lean	4.28.0
Mathlib	v4.28.0, commit `8f9d9cff6bd728b17a24e163c9402775d9e6a365`
Build	`lake build`, 8043 jobs, successful
Task 03 linter state	zero errors, zero linter warnings
Project-local axioms	none

No `sorry`, `admit`, or custom `axiom` declaration occurs anywhere in the project artifact. The principal Task 03 theorems report only the standard Lean/Mathlib foundations

$$\texttt{propext}, \quad \texttt{Classical.choice}, \quad \texttt{Quot.sound} \tag{73}$$

under `#print axioms` [3, 7, 15].

The Task 03 core imports only `Task02.GeneralUnit`. The late comparison file imports the Task 01 boost after the core is complete. A word-boundary firewall scan found no forbidden Task 01 declaration in the core; the two occurrences of the isolated token `B` were comment labels ("Case B" and "data sets A and B") and were documented individually rather than suppressing the scan.

The verified negative-control ledger closes every requested row:

Question	Formal outcome
Does Q_2 preservation imply L_2 preservation?	PROVED
Do future-preserving Lorentz maps act on future units?	PROVED; false without future preservation
Is the product family covariant?	PROVED
Are all future-normalized units in one orbit?	PROVED
Is transport unique without orientation?	PROVED EXACTLY TWO
Is proper-future transport unique?	PROVED UNIQUE
Is one fixed product invariant under all Lorentz maps?	PROVED IMPOSSIBLE
Is fixed invariance exactly unit stabilization?	PROVED
Is a normalized spacelike companion unique?	PROVED EXACTLY TWO
Is the ordered null pair intrinsic?	PROVED NON-UNIQUE
Is the unordered pair intrinsic once e is chosen?	PROVED
Are all μ_e Lorentz-isomorphic in the future sector?	PROVED

No requested Task 03 mathematical obligation remains unresolved within the fixed $1+1$ scope.

18 Why Part III still does not establish dynamics

The title *Split Complex Lorentzian Dynamics* continues to name the research program, not a theorem proved in Part III. The Task 03 results concern group actions, algebra transport, orbit classification, stabilizers, and null-sector labeling. A path

$$\eta \longmapsto B_\eta e \tag{74}$$

is a parametrized orbit in the space of inertial directions; it is not automatically a physical time evolution.

No equation of motion, Hamiltonian, Lagrangian, interaction law, field equation, acceleration law, backreaction term, or dynamical selection rule appears. Likewise, the hole-band/laser section does not turn the orbit parameter into proper time or the null coefficients into observable pulse frequencies. Such identifications would require new assumptions and new negative controls.

19 Open boundary after Task 03

The fixed future-oriented longitudinal 1+1 carrier is now highly constrained. Further decomposition of the same fibre is unlikely to reveal a new continuous algebraic freedom: Task 03 has classified the unit orbit and the residual stabilizer completely. The next structurally distinct questions therefore lie outside the hard boundary of Task 03.

Open question 1 (Selection of the longitudinal plane)**.** *If the ambient carrier is* $3+1$ *dimensional and no spatial direction is selected in advance, what data determine a longitudinal* $1+1$ *plane, and how do the reconstructed products associated with different planes compare?*

Open question 2 (Compatibility across planes)**.** *Do products reconstructed independently on intersecting longitudinal planes admit a coherent global compatibility law, or does the patching require additional structure such as a spatial orientation, rotation structure, Clifford multiplication, or something weaker?*

Open question 3 (Time orientation)**.** *Can the choice of future component be derived from a larger construction, or must it remain a discrete external datum at the relevant level of description?*

Open question 4 (Heuristic underdetermination)**.** *If the null coefficients are to be interpreted microscopically in the hole-band/laser picture, what additional law distinguishes variable marker density from variable transport rate or a coupled variation of both?*

These are questions, not conclusions. Part III stops before introducing any $3+1$ patching theorem or dynamical mechanism.

20 Conclusion

Part III closes the canonicality question left open by Part II at the level appropriate to the fixed $1+1$ Lorentz carrier.

The main result is the family covariance law

$$\boxed{T(\mu_e(X,Y)) = \mu_{Te}(TX,TY).}$$

All normalized future units form one proper-future Lorentz orbit, and the associated unital products form one Lorentz-isomorphism class. Between two fixed future units there are exactly two future-preserving transporters; choosing spatial orientation selects the unique determinant-+1 transporter. The stabilizer of a fixed unit under the full future-preserving group is exactly $\mathbb{Z}_2$, and its nontrivial element exchanges the two complementary null idempotents. The proper stabilizer is trivial.

Accordingly, the continuous unit freedom found in Part II is not a continuous family of inequivalent split-complex algebras. It is the freedom to choose a representative in a transitive Lorentz orbit. What remains after fixing the future sector and one representative is a discrete orientation symmetry in the ordering of the null pair.

In the two-hole-band/laser-pulse heuristic, the result says that changing the normalized timelike laser/clock direction should be pictured as transporting the entire algebraic decomposition, not as selecting a different underlying algebra. The two counter-propagating null channels are structurally complementary, while their $+/-$, left/right naming is orientation-dependent. Reciprocal coefficient scaling may be visualized as reciprocal marker density or spacing only after an additional modeling identification; it is not itself a microscopic physical law.

The conservation-of-difficulty principle therefore moves the diagnostic boundary again. The split-complex product was not the irreducible input; the chosen unit was not a continuous inequivalent input after covariance was tested; the residual ordering ambiguity is discrete. The next significant dependency question is no longer internal to one fixed $1+1$ plane. It concerns the selection and mutual compatibility of such planes inside the larger structure.

A Principal theorem map

Task 03 declaration	Role
`L2_of_isLorentz`	Derives preservation of L_2 from Q_2 preservation.
`lorentz_normal_form`	Concrete two-branch coordinate normal form for Q_2-preserving linear equivalences.
`preservesFuture_iff`	Future preservation iff the timelike coefficient a is positive.
`detL_eq_one_or_neg_one`	Determinant branch is exactly ± 1.
`transport`, `transportFlip`	Explicit proper and improper transporters between normalized units.
`futureUnit_orbit_transitive`	One future-unit orbit.
`transporter_classification`	Exactly two future-preserving transporters between fixed units.
`proper_transporter_unique`	Unique determinant-$+1$ future transporter.
`mu_covariant`	$T(\mu_e XY) = \mu_{Te}(TX)(TY)$.
`product_orbit`	All future reconstructed products form one proper-future Lorentz orbit.
`fixedInvariant_iff`	Fixed-product invariance iff $Te = e$.
`stabilizer_classification`	Future stabilizer of e is identity or reflection.
`proper_stabilizer_trivial`	Proper future stabilizer is identity only.
`companion_classification`	Normalized spacelike companions are exactly $\pm(e_1, e_0)$.
`nullPair_unordered`	Companion sign changes only swap the null pair.
`nullPair_ordered_not_determined`	Ordered pair is not intrinsic without orientation.
`proper_transport_nullPair`	Proper transport preserves null labels.
`improper_transport_nullPair`	Improper transport exchanges null labels.
`reflectAt_swaps_nullPair`	Nontrivial stabilizer acts by the null-sector swap.
`transport_u0_eq_boost`	Late comparison: independent transporter equals Task 01 boost.
`futureUnit_rapidity`	Late comparison: every future unit admits hyperbolic parametrization.
`proper_future_lorentz_is_boost`	Late comparison: every proper future Lorentz map is a standard boost.

B Dependency ledger for Part III

(Long, Q_2)

↓

L_2 and Q_2-preserving linear maps

↓

Concrete Lorentz normal form: determinant ± 1 and time-orientation branches

↓ extra discrete datum: time orientation

Choose future component: future-normalized units

↓

One proper-future orbit of units; exactly two future transporters without spatial orientation

↓

Task 02 μ_e plus Task 03 covariance $T\mu_e = \mu_{Te}(T\cdot, T\cdot)$

↓

One Lorentz-isomorphism class of reconstructed products

↓ choose a fibre representative

Choose representative e: stabilizer $\{\mathrm{id}, R_e\} \cong \mathbb{Z}_2$

↓

Unique unordered null pair; ordered pair exchanged by R_e

↓ extra discrete datum: spatial orientation

Add spatial orientation: unique ordered null pair and unique proper transporter

C Source and reproducibility ledger

Formal artifact

The cumulative Aristotle archive audited for Part III is

`bac68dbb-f023-46c2-ad78-b77f44254031-aristotle.tar(1).gz`

with local SHA-256

`164b49f3104aa6c94e74dceb212422b5587b098921a2cf53eedf6becae150fa7`.

The Task 03 Aristotle run recorded in the supplied summary is

`e65666e4-8611-42a9-9e02-9cab6d32b0d5`.

The formal source and audit are cited as [3].

Lean and Mathlib

The project pins Lean 4.28.0 and Mathlib v4.28.0 at commit

`8f9d9cff6bd728b17a24e163c9402775d9e6a365`.

The formalization context is described in [7, 15, 16].

Standard mathematical context

The classical background for Lorentz geometry and the $1 + 1$ hyperbolic/split-complex realization is represented here by [6, 8, 4, 13]. Composition-algebra context used in Part II is represented by [5, 14]. These sources are context and comparison references; the Task 03 transporter and normal-form proofs do not import their theorems.

Experiment repositories and companion papers

The relation to the independent reconstruction experiment is documented by [9, 10]. The earlier stages of the present experiment are [11, 12, 1, 2].

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