Split Complex Lorentzian Dynamics

Part IV: Gluing Longitudinal Sectors in 3+1, Isotropy, and an Audited Blind-Test Defect

A formally verified continuation of the dependency audit

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Abstract

Parts I–III of *Split Complex Lorentzian Dynamics* isolated and reconstructed the algebraic structure of a single longitudinal $1+1$ Lorentz sector. Part III closed the apparent continuous ambiguity left by the choice of a normalized future unit: the reconstructed products form a Lorentz-covariant family and one proper-future Lorentz orbit. The next dependency question is therefore genuinely higher-dimensional. If every spatial direction through one normalized timelike unit carries its own already reconstructed $1+1$ product, do all such sector products determine a unique bilinear product on the full real $3+1$ carrier?

Task 04 answers this question with an exact local-to-global classification. From the Lorentz quadratic form

$$Q_4(t, \mathbf{x}) = t^2 - \|\mathbf{x}\|^2$$

and a fixed normalized future unit $e_0 = (1, 0, 0, 0)$, the polarization L_4 and the positive Euclidean rest form are derived. Every unit rest direction s defines an isometric longitudinal embedding $\iota_s : \mathbb{R}^{1,1} \to \mathbb{R}^{1,3}$, and the independently reconstructed Task 02 product is transported into that plane. An otherwise unknown real bilinear product M is called sector-compatible when all these longitudinal restrictions agree with the transported product.

The resulting classification is sharp. Sector compatibility alone derives the global two-sided unit e_0, all pure spatial squares, and the entire symmetric part of M. The forced symmetric product is

$$\boxed{\mu_{\mathrm{sym}}(X, Y) = L_4(X, e_0)Y + L_4(Y, e_0)X - L_4(X, Y)e_0.}$$

All remaining freedom is exactly an alternating bilinear defect on the rest space. Conversely, every such defect produces a sector-compatible global product. Thus agreement on every longitudinal sector, including agreement on all pairwise overlaps, does *not* determine multiplication between different spatial directions.

Adding commutativity removes the defect. More significantly, covariance under the full Lorentz stabilizer of e_0 also removes it without assuming commutativity, leaving μ_{sym} uniquely. A late conventional comparison identifies this symmetric branch with the real spin-factor/Jordan product on $\mathbb{R} \oplus \mathbb{R}^3$; its Jordan identity is proved in the core before that name is attached. If only the orientation-preserving stabilizer is required, Task 04 reports a one-real-parameter family μ_λ. Orientation reversal sends $\lambda \mapsto -\lambda$. No real parameter in this family is associative or globally Q_4-multiplicative; each property would force the impossible real equation $-\lambda^2 = 1$, even though every individual longitudinal sector remains associative and norm-multiplicative.

The present paper is deliberately explicit about a methodological defect in this last proper-isotropy result. The Lean theorems are kernel-valid, and Mathlib's conventional `crossProduct` is imported only in the late comparison file. However, the coordinate formula later identified as the three-dimensional vector product was already instantiated in the core as `altBasis` before the proper-isotropy classification established that the equivariant defect space is one-dimensional. A second coordinate copy, `alt3`, is introduced at the beginning of the proper-isotropy module. The audit statement that the basis was chosen only after the dimension statement is therefore not faithful to the actual source order. This does not invalidate the formal classification, but it weakens the intended blind-test evidential claim: the target shape was available to the prover before the classification was complete.

Task 05 is therefore planned primarily as a hardened repetition of the proper-isotropy classification from an abstract unknown defect, with no predeclared coordinate candidate and no conventional target terminology. The space of allowed defects must first be constrained from equivariance alone; only after its dimension and basis values are formally forced may a representative operation be defined and later identified. The same task should also derive the $-\lambda^2 = 1$ obstruction directly from the corresponding hypotheses rather than through an already-proved contradiction. As a narrow forward probe toward Task 06, Task 05 should then ask whether *any* sector-compatible real bilinear extension—without proper-isotropy imposed—can be associative or globally Q_4-multiplicative. This probe is intentionally limited: it should localize whether the obstruction belongs to the isotropic one-parameter branch or to the real four-dimensional gluing problem more generally, without introducing Clifford, complex, spinorial, matrix, or enlarged-carrier structures.

No new mathematical novelty is claimed for the spin factor, Jordan identity, $O(3)/SO(3)$ distinction, or three-dimensional alternating product. The contribution remains the formally audited dependency reconstruction, including an explicit record of where the experimental procedure itself failed to remain fully blind.

Keywords: Lorentzian geometry; 3+1 dimensions; longitudinal sectors; local-to-global gluing; bilinear extensions; spin factors; Jordan algebras; isotropy; orientation; formal verification; Lean 4; conservation of difficulty; dependency analysis; blind-test audit.

Contents

1 Motivation: from one fibre to a gluing problem

The first three parts of this series progressively removed apparent primitive structure inside one longitudinal $1+1$ Lorentz sector. Part I exposed the null decomposition and reciprocal boost scaling. Part II reconstructed the split-complex product from weaker Lorentz data once a normalized timelike unit was selected. Part III showed that changing that unit does not produce a continuous family of inequivalent algebras: the products transform covariantly and lie in one Lorentz orbit [13, 14, 15].

At that point, further analysis inside one fixed $1+1$ plane would largely repeat a fully classified object. The unresolved structural question moved outward. A $3+1$ Lorentz carrier contains infinitely many longitudinal planes through a chosen future unit. Each plane individually inherits the complete Task 02 product. What is not known a priori is how products involving *different* spatial directions are related.

This is the precise local-to-global question of Task 04:

> If every longitudinal $1+1$ sector is already fixed, what part of a global bilinear product on the full $3+1$ carrier is forced by those sectors, and what additional gluing datum remains invisible to every individual sector?

The question is deliberately stronger than overlap consistency. Two longitudinal planes through the same timelike line agree on their common line almost automatically. A global multiplication law must additionally tell us what to do with a spatial vector from one sector and a spatial vector from another. Task 04 tests whether that cross-sector rule is already encoded in the collection of all longitudinal restrictions.

Methodological principle 1 (Local correctness is not global reconstruction)**.** A family of restrictions may determine every local object and every pairwise overlap while still leaving operations between different local sectors undetermined. The missing information must be classified rather than silently supplied.

This principle is the higher-dimensional form of the conservation-of-difficulty principle used throughout the experiment. The central result of Task 04 is that such a missing datum really appears.

2 Experimental design and formal firewall

2.1 Allowed cumulative input

The Task 04 core uses only the ambient carriers and quadratic forms from Task 01 and the independently reconstructed longitudinal product from Task 02. It does not construct the longitudinal product by importing the explicit Task 01 split-complex multiplication. The relevant chain is

$$\texttt{Task01.Basic} \longrightarrow \texttt{Task02.FixedUnit} \longrightarrow \texttt{Task04 core}. \tag{1}$$

Task 03 is conceptually upstream but no additional Task 03 theorem is needed by the Task 04 core.

The nine core files are

$$\begin{aligned}&\texttt{Lorentz4} \to \texttt{RestSpace} \to \texttt{Sectors} \to \texttt{GlobalExtension} \to \texttt{Classification}\\ &\quad\to \texttt{IdentityAudit} \to \texttt{Isotropy} \to \texttt{ProperIsotropy} \to \texttt{Overlap}.\end{aligned} \tag{2}$$

The late file `Identification.lean` is imported by no core module.

2.2 Experiment-1 firewall

No theorem, source file, formula, terminology target, prompt, or paper from EXPERIMENT 1 is imported into Task 04. In particular, the core does not begin by naming a Jordan algebra, spin factor, Clifford algebra, geometric algebra, quaternionic structure, vector product, bivector, or pseudoscalar. The conventional identifications are postponed until after the core classification.

This firewall matters because EXPERIMENT 1 had already encountered Jordan/spin-factor and Clifford structures on a different reconstruction path. Feeding those results into EXPERIMENT 2 would destroy the intended independence of the comparison. The later agreement is therefore editorial evidence of convergence between two formal paths, not a theorem transferred from one experiment into the other.

2.3 What the firewall did and did not prevent

The import firewall worked in the narrow software sense. Mathlib's `crossProduct` occurs only in the late comparison module, and the Task 01 split-complex files are likewise absent from the core. However, software-import blindness is weaker than conceptual blindness. As documented in section 12, the coordinate formula of the later vector-product comparison was instantiated manually inside the core before the proper-isotropy classification was complete. Part IV treats that fact as an experimental defect rather than hiding it behind the successful import scan.

3 The 3+1 Lorentz data and the derived rest space

The ambient real carrier is

$$\mathsf{Vec4} = \mathbb{R} \oplus \mathbb{R}^3, \tag{3}$$

represented concretely by nested products in Lean. The only quadratic datum imported at this layer is

$$Q_4(t, x, y, z) = t^2 - x^2 - y^2 - z^2. \tag{4}$$

Its polarization is defined by

$$L_4(X, Y) = \frac{Q_4(X + Y) - Q_4(X) - Q_4(Y)}{2}, \tag{5}$$

from which Task 04 derives

$$L_4(X, Y) = tt' - xx' - yy' - zz', \qquad L_4(X, X) = Q_4(X). \tag{6}$$

The uniqueness theorem `L4_unique_of_diagonal` makes the dependency explicit: the bilinear form is not a second primitive independent of the quadratic form.

Fix

$$e_0 = (1, 0, 0, 0), \qquad Q_4(e_0) = 1. \tag{7}$$

The rest space is then derived as

$$\mathsf{Rest} = \{X \in \mathsf{Vec4} : L_4(e_0, X) = 0\}. \tag{8}$$

Writing a rest vector as $(0, v)$ with $v \in \mathbb{R}^3$, define

$$h(v, w) = -L_4((0, v), (0, w)). \tag{9}$$

Task 04 proves

$$h(v, w) = v \cdot w, \qquad h(v, v) \geq 0, \qquad h(v, v) = 0 \Longleftrightarrow v = 0. \tag{10}$$

Thus the positive Euclidean three-space orthogonal to e_0 is DERIVED from the Lorentz form and the selected future unit. It is not supplied as an unrelated metric.

This derived rest-space structure is the first point where Part IV makes direct contact with the endpoint of EXPERIMENT 1: that experiment also exposed a Euclidean $\mathbb{R}^3$ as a structurally important layer. The relationship is discussed only after the Task 04 classification in section 15.

4 All longitudinal sectors

A unit spacelike direction satisfies

$$L_4(e_0, s) = 0, \qquad Q_4(s) = -1. \tag{11}$$

Equivalently, its rest component has Euclidean norm one. For each such s, Task 04 defines

$$\iota_s(a, b) = ae_0 + bs. \tag{12}$$

The formal development proves

$$Q_4(\iota_s(a, b)) = a^2 - b^2 = Q_2(a, b), \tag{13}$$

$$L_4(\iota_s X, \iota_s Y) = L_2(X, Y). \tag{14}$$

Thus every spatial direction through e_0 defines an exact isometric copy of the already reconstructed longitudinal carrier.

The product used inside this copy is not redefined. Task 04 aliases the unique Task 02 reconstruction at $u_0 = (1, 0)$ as μ_{Long} and transports it into the sector:

$$\mathsf{sectorProd}_s(X, Y) = \iota_s\big(\mu_{\text{Long}}(X, Y)\big). \tag{15}$$

Each sector therefore inherits the properties already established in $1 + 1$: a two-sided unit, commutativity, associativity, and multiplicativity of the Lorentz quadratic form within that plane.

The gluing question begins only after this entire family is in place.

5 An unknown global product and sector compatibility

Let

$$M : \mathsf{Vec4} \times \mathsf{Vec4} \to \mathsf{Vec4} \tag{16}$$

be an arbitrary real bilinear map. No commutativity, associativity, global unit, global norm multiplicativity, or covariance is assumed.

Definition 1 (Sector compatibility)**.** M is sector-compatible if, for every unit spacelike s and every $X, Y \in \mathbb{R}^{1,1}$,

$$M(\iota_s X, \iota_s Y) = \iota_s\big(\mu_{\text{Long}}(X, Y)\big). \tag{17}$$

This definition fixes M whenever both arguments belong to the same longitudinal plane. It says nothing explicitly about $M((0, v), (0, w))$ when v and w are linearly independent spatial directions.

The first positive surprise is that sector compatibility already propagates farther than one might expect.

Theorem 1 (Global unit is derived)**.** *Every sector-compatible bilinear product satisfies*

$$M(e_0, X) = X = M(X, e_0) \tag{18}$$

for all $X \in \mathsf{Vec4}$.

The key reason is geometric rather than algebraically imposed: every ambient vector belongs to some longitudinal sector through e_0. Hence the common sector unit becomes the global unit automatically.

6 Exact gluing classification

The central Task 04 theorem identifies exactly what the sectors force and exactly what they do not.

For a pure rest vector $(0, v)$, sector compatibility and bilinearity imply

$$M((0, v), (0, v)) = (v \cdot v)e_0. \tag{19}$$

Polarizing this square relation gives

$$M((0, v), (0, w)) + M((0, w), (0, v)) = 2(v \cdot w)e_0. \tag{20}$$

Thus the complete symmetric part of spatial cross-sector multiplication is fixed.

Combining this with the derived global unit yields the unique symmetric product

$$\boxed{\mu_{\mathrm{sym}}(X, Y) = L_4(X, e_0)Y + L_4(Y, e_0)X - L_4(X, Y)e_0.} \tag{21}$$

In the decomposition $X = (a, v)$ and $Y = (b, w)$,

$$\boxed{\mu_{\mathrm{sym}}((a, v), (b, w)) = (ab + v \cdot w,\ aw + bv).} \tag{22}$$

The antisymmetric part remains free.

Theorem 2 (Task 04 extension classification)**.** *A real bilinear map M is sector-compatible if and only if there exists a unique alternating bilinear map*

$$A : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^{1,3} \tag{23}$$

such that

$$\boxed{M((a, v), (b, w)) = \mu_{\mathrm{sym}}((a, v), (b, w)) + A(v, w),} \tag{24}$$

with

$$A(v, v) = 0 \quad \Longleftrightarrow \quad A(v, w) = -A(w, v). \tag{25}$$

Conversely, every such A produces a sector-compatible product.

This converse is essential. Without it, A might merely represent slack in a proof. The theorem instead shows that the gluing freedom is mathematically real: the entire set of longitudinal sectors is genuinely insensitive to an arbitrary alternating rest-space defect.

The abstract defect space is naturally

$$\mathrm{Hom}(\Lambda^2\mathbb{R}^3, \mathbb{R}^4), \tag{26}$$

which has $3 \times 4 = 12$ real dimensions. The Lean artifact characterizes this datum exactly but does not package the dimension count itself as a separate theorem.

Corollary 1 (Overlap agreement is insufficient)**.** *Pairwise sector products agree on all overlaps, yet cross-sector multiplication remains non-unique.*

For linearly independent spatial directions, two longitudinal planes intersect only on the timelike line generated by e_0. Task 04 proves agreement on that line, and then exhibits two sector-compatible global products that differ on a pair of orthogonal spatial vectors. Local agreement and overlap agreement therefore do not amount to a global gluing law.

7 Commutativity and the full isotropy stabilizer

One obvious way to eliminate an alternating defect is to impose commutativity. Task 04 proves the exact equivalence

$$\mathsf{SectorCompatible}(M) \wedge M(X,Y) = M(Y,X) \iff M = \mu_{\mathrm{sym}}. \tag{27}$$

More interestingly, commutativity need not be assumed if one instead imposes the full spatial isotropy inherited from the Lorentz stabilizer of e_0.

Let R be a real-linear Lorentz equivalence satisfying

$$Re_0 = e_0, \qquad Q_4(RX) = Q_4(X). \tag{28}$$

Task 04 derives that R preserves the rest space and the Euclidean form h. Define full isotropy covariance by

$$R(M(X,Y)) = M(RX,RY) \tag{29}$$

for every such R.

The formal proof uses only explicit coordinate reflections; it does not import a high-level $O(3)$ representation theorem.

Theorem 3 (Full isotropy kills the gluing defect)**.** *If M is sector-compatible and equivariant under the full stabilizer of e_0, then*

$$A = 0 \tag{30}$$

and hence

$$\boxed{M = \mu_{\mathrm{sym}}.} \tag{31}$$

Conversely, μ_{sym} is full-isotropy equivariant.

Thus the symmetric product is not merely the unique commutative extension; it is also the unique extension compatible with the full orientation-allowing spatial isotropy group.

8 Proper isotropy and the reported one-parameter branch

Task 04 then weakens full isotropy to the orientation-preserving stabilizer. The formal development defines a spatial determinant directly in coordinates and restricts to determinant $+1$ transformations. It reports the classification

$$\mathsf{SectorCompatible}(M) \wedge \mathsf{ProperIsotropyEquivariant}(M) \iff \exists \lambda \in \mathbb{R} : \; M = \mu_\lambda, \tag{32}$$

where

$$\mu_\lambda = \mu_{\mathrm{sym}} + \lambda A_0 \tag{33}$$

for a fixed nonzero alternating defect A_0.

The proof uses explicit half-turns and quarter-turns. These force the values of an equivariant defect on the three spatial basis pairs to lie along the complementary basis directions and then force the three coefficients to agree. The parameterization is proved injective, so the reported freedom is exactly one-dimensional.

Orientation reversal sends

$$\lambda \longmapsto -\lambda. \tag{34}$$

Accordingly, full isotropy fixes only the $\lambda = 0$ branch, while proper isotropy permits a sign-sensitive continuous coefficient.

After the core classification, the late comparison file identifies A_0 with the conventional three-dimensional vector product embedded into the spatial part of $\mathbb{R}^{1,3}$. In conventional notation the family is therefore

$$\mu_\lambda((a,v),(b,w)) = (ab + v \cdot w,\; aw + bv + \lambda\, v \times w). \tag{35}$$

The relation between this familiar formula and the strict blind-test requirement is the subject of section 12.

9 Local associativity does not glue to global associativity

Every longitudinal restriction of every sector-compatible product is associative, because it is literally the transported Task 02 product. The global symmetric extension is not.

Let E_1, E_2 be two orthogonal unit rest directions. For the symmetric branch,

$$\mu_{\mathrm{sym}}(E_1, E_2) = 0, \qquad \mu_{\mathrm{sym}}(E_2, E_2) = e_0, \qquad \mu_{\mathrm{sym}}(E_1, e_0) = E_1. \tag{36}$$

Hence

$$\mu_{\mathrm{sym}}(\mu_{\mathrm{sym}}(E_1, E_2), E_2) = 0, \tag{37}$$
$$\mu_{\mathrm{sym}}(E_1, \mu_{\mathrm{sym}}(E_2, E_2)) = E_1, \tag{38}$$

and associativity fails.

For the reported proper-isotropy family, the same basis test produces the scalar obstruction

$$-\lambda^2 = 1. \tag{39}$$

No real λ satisfies this equation. Consequently Task 04 proves

$$\forall \lambda \in \mathbb{R} : \qquad \neg \mathsf{Associative4}(\mu_\lambda). \tag{40}$$

This is a clean local-versus-global separation:

$$\boxed{\text{every longitudinal sector associative} \not\Rightarrow \text{globally associative gluing.}} \tag{41}$$

The formal theorem establishing the implication to equation (39) is, however, packaged less transparently than the underlying calculation: the wrapper `muFam_associativity_equation` obtains the equation via contradiction from the already established impossibility theorem. The direct derivation exists inside the impossibility proof but is not exposed as the dependency arrow itself. Task 05 should repair that theorem-level provenance; see section 16.

10 Sectorwise norm multiplicativity also fails globally

The same phenomenon occurs for the quadratic norm. Inside every longitudinal sector,

$$Q_4(M(X, Y)) = Q_4(X) Q_4(Y) \tag{42}$$

holds for sector-compatible M when X and Y are in that sector.

Globally, the symmetric branch already fails on two orthogonal spacelike directions:

$$\mu_{\mathrm{sym}}(E_1, E_2) = 0, \tag{43}$$

so

$$Q_4(\mu_{\mathrm{sym}}(E_1, E_2)) = 0 \tag{44}$$

whereas

$$Q_4(E_1) Q_4(E_2) = (-1)(-1) = 1. \tag{45}$$

For the reported proper-isotropy family,

$$Q_4(\mu_\lambda(E_1, E_2)) = -\lambda^2, \tag{46}$$

so global norm multiplicativity again requires

$$-\lambda^2 = 1. \tag{47}$$

Thus

$$\forall \lambda \in \mathbb{R} : \qquad \neg \mathsf{Q4Multiplicative}(\mu_\lambda). \tag{48}$$

The repeated obstruction is structurally important. Two properties that hold perfectly in every $1 + 1$ sector—associativity and multiplicativity of the Lorentz norm—fail when distinct spatial directions must interact through a real bilinear product on the same four-dimensional carrier.

What Task 04 does *not* yet prove is that these properties are impossible for *every* sector-compatible product in the full twelve-parameter defect family. The impossibility has been established only after the proper-isotropy reduction to the reported one-parameter branch. This distinction motivates the narrow forward probe proposed for Task 05.

11 The independently forced symmetric branch

The unique symmetric/full-isotropy branch has several positive identities despite its nonassociativity. Task 04 proves:

- a two-sided unit e_0;
- commutativity;
- flexibility;
- power-associativity in the explicitly tested degrees three and four;
- the Jordan identity.

It refutes left alternativity, full associativity, and global Q_4 multiplicativity by concrete cross-direction counterexamples.

Only in `Identification.lean`, after the core has compiled, is equation (22) compared with the conventional real spin factor / Jordan product on $\mathbb{R} \oplus \mathbb{R}^3$ [7, 6, 16]. The late comparison therefore confirms a standard mathematical identity; it is not used to prove the Task 04 core.

Similarly, the orientation-sensitive alternating operation is compared at the end with the ordinary vector product in three dimensions, a classical structure with the expected $SO(3)$ covariance and orientation sensitivity [5].

No novelty is claimed for either conventional identification. The experimental value lies in which assumptions force which part of the product.

12 A methodological blind-test defect

This section records a defect in the experimental procedure openly and precisely.

12.1 What was intended

The Task 04 prompt required the proper-isotropy layer to proceed in the following order:

1. start with an abstract alternating defect A;
2. impose proper-isotropy equivariance;
3. classify the dimension and basis values of the surviving defect space;
4. only after that classification, choose an explicit basis representative;
5. only in a late comparison identify the representative with a conventional object.

The purpose was not mathematical necessity. It was experimental blindness: the prover should not see the expected coordinate shape while proving that the symmetry forces precisely that shape.

12.2 What the source actually does

The formal archive instead defines, already in `Classification.lean`, an explicit nonzero alternating defect

$$\texttt{altBasis}(v,w) = \begin{pmatrix} & v_2w_3 - v_3w_2 \\ 0, & v_3w_1 - v_1w_3 \\ & v_1w_2 - v_2w_1 \end{pmatrix}. \tag{49}$$

This occurs before `ProperIsotropy.lean` proves the one-dimensional equivariant classification. At the beginning of that later module the spatial part is introduced again as `alt3` with the same coordinate pattern. The determinant used for orientation is then defined using that operation.

Only after the classification does the late identification module import Mathlib's actual `crossProduct` and prove equality. Therefore the *software import firewall* is correct, but the stronger conceptual statement

> "the target operation was not available before the classification"

is false.

The audit compounds this issue by stating that "a basis was chosen after the dimension statement was proved." The source order shows the opposite: `altBasis` is already defined in the earlier classification module. The theorems `defect_proper_classification` and `properIsotropy_freedom_one_parameter` occur only later in the proper-isotropy module.

Methodological caveat 1 (Formal validity versus blind-test validity)**.** *The kernel-checked theorems are not invalidated by this source-order defect. The proper-isotropy theorem still proves that every equivariant defect is a scalar multiple of the declared operation, using explicit rotations. What is weakened is the evidential claim that the operation's coordinate shape* emerged without being available to the prover*. Part IV therefore treats the one-parameter proper-isotropy result as mathematically proved but methodologically requiring independent hardening.*

12.3 Why this distinction matters

For ordinary mathematics, there is nothing objectionable about guessing a candidate invariant operation and proving a classification theorem around it. Indeed, that is often an efficient proof strategy. The present experiment asks a different question: can the dependency be reconstructed without smuggling the endpoint into the search space?

The distinction can be summarized as

$$\boxed{\text{theorem correctness} \neq \text{blind reconstruction evidence.}} \tag{50}$$

The first is established by Lean. The second depends on the design and provenance of the formal task.

This is exactly the kind of issue the experiment is intended to expose. A successful formal build is not sufficient if the research question concerns where structural information entered the derivation.

13 Evidence status after the audit

The Task 04 results should therefore be assigned different evidential statuses rather than accepted or rejected as one block.

Result	Formal status	Experimental status
$Q_4 \to L_4$ polarization and rest-space construction	kernel proved	clean
Exact longitudinal embeddings and sector transport	kernel proved	clean
Global unit derived from all sectors	kernel proved	clean
Exact classification $M = \mu_{\text{sym}} + A$ with arbitrary alternating A	kernel proved, converse included	clean
Unique symmetric part μ_{sym}	kernel proved	clean
Commutativity $\Rightarrow A = 0$	kernel proved	clean
Full-isotropy $\Rightarrow A = 0$	kernel proved from reflections	clean
Proper-isotropy $\Rightarrow$ one-parameter family	kernel proved	METHODOLOGICAL CAVEAT: candidate shape available early
Late identification of μ_{sym} as spin factor/Jordan product	kernel comparison	clean as comparison
Late identification of alternating branch with vector product	kernel comparison	identification correct; blind emergence not yet clean
Associativity and norm no-go within reported μ_λ family	kernel proved	dependent on proper-isotropy family; obstruction should be re-exposed directly

This graded status is scientifically preferable to either extreme. It would be incorrect to discard the formal mathematics; it would also be incorrect to claim that the strict blind-test condition was fully met.

14 Conservation of difficulty after Task 04

Ignoring conventional names, the clean part of the dependency graph is now

$$Q_4 \longrightarrow L_4 \longrightarrow (\mathsf{Rest}, h) \longrightarrow \{\text{all longitudinal } 1+1 \text{ sectors}\} \longrightarrow \mu_{\text{sym}} + A, \tag{51}$$

where A is an arbitrary alternating cross-sector defect.

This is the first point in EXPERIMENT 2 where the collection of completely understood local sectors fails to reconstruct the global operation uniquely. The missing information is not visible inside any individual sector and not visible on pairwise overlaps. It lives precisely in products between linearly independent spatial directions.

Two clean selectors are already known:

$$\text{commutativity} \Longrightarrow A = 0, \tag{52}$$

$$\text{full spatial isotropy} \Longrightarrow A = 0. \tag{53}$$

The proper-isotropy reduction to one real parameter is formally proved but must be rerun under stricter blindness before it is used as a dependency result of the experiment.

This distinction also changes the diagnostic interpretation. After Task 03, the remaining freedom looked largely discrete inside one $1+1$ sector. Task 04 reveals that the move to $3+1$ introduces a genuinely new continuous gluing space before global symmetry is imposed. The difficulty has not disappeared; it has moved to the interaction of different longitudinal sectors.

15 Relation to Experiment 1

The cleanest and most significant cross-experiment comparison concerns the symmetric branch.

EXPERIMENT 1 approached Lorentzian structure from weaker real/euclidean data and encountered a spin-factor/Jordan product as an intermediate reconstruction layer before later

Clifford-related structures were considered [11, 12]. Experiment 2 begins on the opposite side: Lorentz structure is already present, all longitudinal $1+1$ products are reconstructed, and Task 04 asks how they glue in $3+1$.

The independent Task 04 result

$$\mu_{\mathrm{sym}}((a,v),(b,w)) = (ab + v\cdot w, aw + bv) \tag{54}$$

is then identified only afterward as the same classical spin-factor/Jordan product. No Experiment-1 theorem or formula was supplied to Aristotle in Task 04.

Accordingly, the following convergence is experimentally clean:

$$\begin{array}{ccc} \text{Experiment 1} & & \text{Experiment 2} \\ \text{weaker Euclidean data} & & \text{Lorentz data + all } 1+1 \text{ sectors} \\ \downarrow & & \downarrow \\ \text{spin-factor/Jordan layer} & = & \text{spin-factor/Jordan layer.} \end{array} \tag{55}$$

This remains known mathematics. The point is not novelty of the object but independent convergence of two differently ordered dependency audits.

The antisymmetric proper-isotropy branch must be treated more cautiously. Its conventional identification with the three-dimensional vector product is mathematically correct, but because the coordinate pattern was present early in the Task 04 core, its status as an independently *discovered* convergence point is not yet established. Task 05 is designed to decide that question cleanly.

16 Task V: hardening the result without giving Aristotle the target

The primary purpose of Task 05 should be methodological repair, not expansion of scope. The prompt must not ask Aristotle to derive a cross product, a pseudovector, an $SO(3)$ invariant tensor, a Levi-Civita symbol, or a one-parameter family. Those phrases would preannounce the desired endpoint.

Instead, Task 05 should begin from the already clean classification

$$M = \mu_{\mathrm{sym}} + A, \qquad A : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^{1,3} \text{ alternating,} \tag{56}$$

and ask a neutral classification question. The Task 05 prompt itself must not state that Task 04 reported a one-dimensional proper-isotropy branch, must not display the coordinate pattern of equation (49), and must not mention the later conventional identification. Those are editorial facts recorded in Part IV, not inputs to the new prover run. The formal question given to Aristotle should be only:

> Classify all alternating defects A for which the corresponding product is equivariant under the orientation-preserving stabilizer of e_0.

The hardening protocol should enforce the following source order.

16.1 Phase 1: no candidate operation

Before the dimension/classification theorem is complete, the Task 05 core must contain no explicit bilinear coordinate formula with the cyclic determinant pattern of equation (49), no conventional cross-product import, no determinant defined through such an operation, and no predeclared nonzero equivariant basis defect.

Any early nonuniqueness witness needed for regression tests should use a deliberately non-isotropic alternating map, for example one supported on a single ordered basis pair. Its purpose is only to witness the twelve-parameter freedom of sector compatibility, not to resemble the expected proper-isotropic branch.

16.2 Phase 2: derive basis-pair constraints from equivariance

Starting from arbitrary A, explicit proper stabilizer elements may be used to determine the possible values of

$$A(s_1, s_2), \qquad A(s_2, s_3), \qquad A(s_3, s_1). \tag{57}$$

The proof should first establish support directions and coefficient relations abstractly. Only when the allowed values have been forced may the defect space be assigned a dimension.

The important methodological condition is that the theorem statement itself should not contain a prewritten coordinate target. A basis representative can be *defined after* the classification by prescribing the now-proved basis values and extending bilinearly.

16.3 Phase 3: late identification

Only after the independently generated representative exists may a final comparison module import a standard vector-product definition and test equality. If equality holds, the conventional name is attached at that point. If a different operation emerges, that result must be accepted.

16.4 Phase 4: make the obstruction arrows explicit

Task 04 internally calculates the equations needed for the no-go proofs, but the wrapper theorem `muFam_associativity_equation` derives them through contradiction from the impossibility theorems. Task 05 should reverse this packaging:

$$\mathsf{Associative4}(\mu_\lambda) \Longrightarrow \mathcal{E}_{\mathrm{assoc}}(\lambda), \tag{58}$$

$$\mathsf{Q4Multiplicative}(\mu_\lambda) \Longrightarrow \mathcal{E}_{Q}(\lambda), \tag{59}$$

with the equations obtained directly from selected basis evaluations. Only afterward should real algebra be used to decide whether those equations have solutions.

If the hardened classification again produces the same one-dimensional branch and the same equations, the original Task 04 conclusion gains genuinely independent support rather than merely a second proof of a known target.

17 A narrow forward probe toward Task VI

Task 05 should go slightly beyond repair, but only along one tightly controlled axis.

The current no-go theorems establish nonassociativity and failure of global norm multiplicativity for the *proper-isotropic* family. They do not answer the more primitive existence questions

$$\exists M :\ \mathsf{SectorCompatible}(M) \wedge \mathsf{Associative4}(M)? \tag{60}$$

$$\exists M :\ \mathsf{SectorCompatible}(M) \wedge \mathsf{Q4Multiplicative}(M)? \tag{61}$$

when the full twelve-parameter alternating defect is allowed.

These are natural diagnostic probes because they ask whether the obstruction found after isotropy reduction is already inherent in the real four-dimensional gluing problem.

Task 05 should therefore attempt only the following limited extension:

1. decide existence or nonexistence of a sector-compatible associative real bilinear product on the same carrier;
2. separately decide existence or nonexistence of a sector-compatible globally Q_4-multiplicative real bilinear product on the same carrier;
3. if a witness exists, record the minimal basis data needed to distinguish it from the symmetric/proper-isotropic branches, without launching a broad classification unless that classification is elementary;

4. if no witness exists, isolate the smallest explicit algebraic obstruction.

No carrier enlargement, complexification, exterior algebra, Clifford algebra, matrix realization, spinor construction, or dynamical interpretation should be introduced. Task 05 should stop at the obstruction boundary.

The purpose is to prepare the next question, not answer it. Depending on the result, Task 06 would face two qualitatively different situations:

- If unrestricted sector-compatible products already cannot be associative or norm-multiplicative, then the obstruction belongs to the real $3+1$ carrier/gluing assumptions themselves and Task 06 may ask which assumption must change.
- If such products exist only after sacrificing isotropy, then the next difficulty is a compatibility triangle among sector reconstruction, global algebraic closure, and spatial symmetry.

Part IV deliberately does not decide which branch occurs.

18 Next questions for Part V

The following questions delimit the next formal experiment.

Open question 1 (Blind proper-isotropy classification). *Starting from an arbitrary alternating cross-sector defect and no predeclared coordinate candidate, what is the exact space of defects equivariant under the orientation-preserving stabilizer of* e_0*?*

Open question 2 (Orientation reversal). *After that space has been independently classified, how does an orientation-reversing stabilizer act on it? Is the action discrete, continuous, or mixed?*

Open question 3 (Direct obstruction provenance). *What equations follow* directly *from associativity and from global* Q_4 *multiplicativity on the independently classified family, before any impossibility theorem is invoked?*

Open question 4 (Unrestricted associative gluing). *Does there exist any real bilinear sector-compatible global product on* $\mathbb{R}^{1,3}$ *that is associative if isotropy is not imposed?*

Open question 5 (Unrestricted norm-compatible gluing). *Does there exist any real bilinear sector-compatible global product on* $\mathbb{R}^{1,3}$ *that is globally multiplicative for* Q_4 *if isotropy is not imposed?*

These questions are deliberately narrower than a request to construct a known higher algebra. Their purpose is to establish whether the Task 04 obstruction is a consequence of proper isotropy or a more general failure of real four-dimensional closure.

19 Formal verification and reproducibility

The Task 04 artifact reports the following environment:

Lean	4.28.0
Mathlib	v4.28.0, commit `8f9d9cff6bd728b17a24e163c9402775d9e6a365`
Build	`lake build`, 8054 jobs, successful
Task 04 warning state	no reported warnings
Project-local axioms	none
Task 04 Aristotle run	`8de20cc6-2f0b-4f91-aa3f-bf8807752889`

The archive contains no `sorry`, `admit`, custom `axiom`, or `@[implemented_by]` declaration according to the supplied audit. Principal theorems report only

$$\texttt{propext}, \quad \texttt{Classical.choice}, \quad \texttt{Quot.sound} \tag{62}$$

under `#print axioms` [4, 9, 17].

The formal build status and the methodological blind-test status must not be conflated. The former is green. The latter contains the explicit caveat of section 12.

20 Why Part IV still does not establish dynamics

The word *Dynamics* continues to label the larger research programme, not a result established here. Part IV classifies bilinear extensions, symmetry covariance, algebraic identities, and local-to-global compatibility. It contains no time-evolution law, Hamiltonian, Lagrangian, interaction, field equation, acceleration, backreaction, or state-transition rule.

Likewise, no physical ontology is assigned to the rest-space vectors, the alternating defect, the parameter λ, or the spin-factor product. Such interpretations would require additional assumptions and would obscure the present dependency question.

21 Conclusion

Part IV opens the first genuinely higher-dimensional gluing problem in Experiment 2.

The clean formal result is exact:

$$\boxed{\mathsf{SectorCompatible}(M) \iff M = \mu_{\mathrm{sym}} + A, \qquad A \text{ alternating on } \mathbb{R}^3 \times \mathbb{R}^3.}$$

All longitudinal $1+1$ sectors together derive the global unit and the entire symmetric product, but they do not determine cross-sector multiplication. Pairwise overlap consistency is insufficient. Commutativity or full spatial isotropy removes the defect and yields the unique symmetric branch

$$\boxed{\mu_{\mathrm{sym}}(X,Y) = L_4(X,e_0)Y + L_4(Y,e_0)X - L_4(X,Y)e_0.}$$

Only after this independent derivation is it identified with the classical spin-factor/Jordan product. This provides a clean new convergence point with Experiment 1.

Task 04 further reports that proper spatial isotropy leaves one real parameter and that orientation reversal flips its sign. Within that family neither associativity nor global Lorentz-norm multiplicativity is possible over $\mathbb{R}$; both lead to $-\lambda^2 = 1$. These formal theorems are valid, but the experiment did not satisfy its intended blindness at this stage: the coordinate pattern later identified with the vector product was already available in the core before the proper-isotropy classification.

That methodological miss is not concealed or retroactively reinterpreted. It becomes the primary object of Task 05. The next task must reclassify the proper-isotropy defect space from an entirely unknown alternating map, define any surviving basis operation only after its dimension and basis values have been forced, and perform the conventional identification only at the end. It should also expose the associativity and norm obstructions as direct dependency arrows.

A small forward probe should then ask whether the obstruction persists even when isotropy is removed and the full alternating gluing freedom is allowed. That question is deliberately chosen to prepare, but not prejudge, Task 06. The next conservation-of-difficulty boundary is therefore sharply localized: either the real four-dimensional carrier itself cannot support the desired global properties while retaining all reconstructed sectors, or those properties can be recovered only by introducing additional anisotropic gluing data. Which alternative holds remains open.

A Principal Task 04 theorem map

Task 04 declaration	Role
`L4_unique_of_diagonal`	Shows that Q_4 determines the polarized bilinear form L_4.
`task04_rest_space`	Packages the e_0-orthogonal rest space and its positive form.
`Q4_iotaS_Q2`, `L4_iotaS`	Every unit rest direction gives an exact $1+1$ Lorentz sector.
`muLong_unique`	Records that the sector product is the Task 02 reconstructed product, not a new explicit definition.
`M_globalUnit`	Derives the global two-sided unit from sector compatibility.
`M_rest_sq`, `M_rest_symm`	Forces pure spatial squares and the symmetric cross-sector combination.
`mu4sym_coordfree`	Gives the unique symmetric product in Lorentz-invariant form.
`sectorCompatible_iff`	Exact classification $M = \mu_{\mathrm{sym}} + A$ with alternating defect.
`defect_unique`	Uniqueness of the defect datum.
`muOf_sectorCompatible`	Converse: every alternating defect produces a compatible global product.
`sectorCompatible_not_unique`	Explicit nonuniqueness witness.
`sectorCompatible_comm_unique`	Commutativity selects μ_{sym} uniquely.
`sectorCompatible_isotropy_unique`	Full stabilizer covariance selects μ_{sym} uniquely.
`defect_proper_classification`	Reported proper-isotropy classification relative to the predeclared `altBasis`; mathematically proved, methodologically caveated.
`sectorCompatible_properIsotropy_classification`	Reported one-parameter proper-isotropy family.
`muFam_orientation_reversal`	Orientation reversal sends λ to $-\lambda$.
`mu4sym_jordan`	Proves the Jordan identity before conventional identification.
`mu4sym_not_associative`	Cross-direction counterexample to global associativity.
`mu4sym_not_Q4Multiplicative`	Cross-direction counterexample to global norm multiplicativity.
`muFam_not_associative`	No real parameter in the reported proper-isotropy family is associative.
`muFam_not_Q4Multiplicative`	No real parameter in that family is globally norm-multiplicative.
`sector_overlap_agreement`	Sector products agree on all overlaps.
`overlap_does_not_determine_cross_sector`	Overlap agreement does not determine gluing.
`mu4sym_eq_spinFactorProduct`	Late comparison with the conventional spin-factor/Jordan product.
`alt3_eq_crossProduct`	Late comparison with Mathlib's conventional three-dimensional vector product.

B Dependency ledger for Part IV

Q_4 on $\mathbb{R}^{1,3}$

$\downarrow$

L_4 by polarization; choose normalized future unit e_0

$\downarrow$

Derived Euclidean rest space (Rest, h)

$\downarrow$

All unit rest directions s and all longitudinal embeddings ι_s

$\downarrow$

Transport the independently reconstructed Task 02 product into every sector

$\downarrow$

$\mathsf{SectorCompatible}(M)$

$\downarrow$

Exact clean classification: $M = \mu_{\mathrm{sym}} + A$, A arbitrary alternating rest-space defect

$\downarrow$

Add commutativity or full isotropy: $A = 0$, unique μ_{sym}

$\downarrow$

Add proper isotropy only: reported one-parameter branch; theorem valid, blind reconstruction requires Task 05 hardening

$\downarrow$

Within reported branch: associativity and global Q_4 multiplicativity each imply the real obstruction $-\lambda^2 = 1$

C Methodological provenance ledger

Provenance item	Assessment
Task 01 explicit split-complex product	Withheld from the Task 04 core; longitudinal product imported from Task 02 reconstruction.
Experiment 1 Jordan/Clifford results	Not imported or used in Task 04.
Mathlib `crossProduct`	Imported only in `Identification.lean`; not used by core theorem proofs.
Core `altBasis`	Explicit cyclic coordinate formula defined in `Classification.lean` before the proper-isotropy classification. This violates the intended source-order blindness.

Provenance item	Assessment
Core `alt3`	Same spatial cyclic formula reintroduced at the start of `ProperIsotropy.lean` and used to define the hand-written determinant.
Audit claim about basis timing	The statement that the basis was chosen after the dimension result is inconsistent with the actual module/source order.
Formal theorem validity	Preserved. Lean proves the stated classifications from the supplied definitions and hypotheses.
Blind-test evidential strength	Weakened for the proper-isotropy branch; requires Task 05 reconstruction from an abstract defect.
Symmetric spin-factor/Jordan convergence	Clean: μ_{sym} is independently forced before conventional naming and agrees with the Experiment 1 intermediate structure.

D Source and reproducibility ledger

Formal artifact

The cumulative Aristotle archive audited for Part IV is

`bac68dbb-f023-46c2-ad78-b77f44254031-aristotle(1).tar(1).gz`

with local SHA-256

`8dbfb640ea218cec28a1b0a8d9d052280a33ce131404a34892220886754f0afa`.

The Task 04 run recorded in the supplied summary is

`8de20cc6-2f0b-4f91-aa3f-bf8807752889`.

The formal source and audit are cited as [4].

Lean and Mathlib

The project pins Lean 4.28.0 and Mathlib v4.28.0 at commit

`8f9d9cff6bd728b17a24e163c9402775d9e6a365`.

The formalization context is described in [9, 17, 18].

Standard mathematical context

The Lorentz background is represented by [8, 10]. The classical spin-factor/Jordan context is represented by [7, 6, 16]. Three-dimensional vector-product context is represented by [5]. These references provide conventional context only; the Task 04 clean symmetric classification does not import their theorems.

Experiment repositories and companion papers

The independent reconstruction path is documented by [11, 12]. Earlier stages of the present experiment are [13, 14, 15, 1, 2, 3].

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