Split Complex Lorentzian Dynamics

Part IX: The Second Closure Obstruction, Central Quadratic Multiplicativity, and a Discrete Sign Branch

A formally verified continuation of the dependency audit

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Abstract

Part VIII resolved the first of the two universal obstructions isolated in Part VI. The old real $3+1$ carrier could not support the inherited longitudinal square structure together with a globally associative product if all mixed products were required to remain in that carrier. Tasks VII–VIII showed that this was a closure deficit rather than an absolute impossibility: three inherited spatial generators force a minimal eight-dimensional real associative envelope W containing the complete old Vec4, with the old square law and its polarized spin-factor/Jordan product recovered exactly for every embedded old vector. The same reconstruction also forced a central element S satisfying $S^2 = -1$.

This paper reports Task 09, which reopens only the second Part VI obstruction: multiplicativity of the inherited Lorentz quadratic form. The Task VIII product carrier is frozen. No generator or product is changed. The first negative control asks whether a real-valued quadratic map $N_R : W \to \mathbb{R}$ can be both globally multiplicative and equal to Q_4 on the embedded old carrier. It cannot. The contradiction is already visible on $1+S$: homogeneity and multiplicativity force the square of the real number $N_R(1+S)$ to equal -4. Thus enlargement of the product carrier was necessary but not sufficient; the real scalar codomain remains too small.

Only after this no-go is established does Task 09 use the internally derived central plane $Z = \mathrm{span}_{\mathbb{R}}\{1, S\}$. It is proved directly to be two-dimensional, commutative, product-closed and central in W, with multiplication $(a+bS)(c+dS) = (ac-bd)+(ad+bc)S$. An entirely unknown quadratic multiplicative map $N : W \to Z$, required to restrict to Q_4 on old Vec4, is then classified before any explicit candidate is introduced. All eight basis diagonal values and almost the complete polarization table are forced. One residual datum remains, $d = B_N(1, S)$, and multiplicativity implies $d^2 = -4$. Inside Z this has exactly two solutions, $d = \pm 2S$.

Only after rigidity are explicit witnesses constructed. They are globally multiplicative on arbitrary elements of W, reproduce Q_4 on every embedded old vector, and exhaust the solution set. Hence there are exactly two admissible maps, N_+ and N_-, related by the central sign map $S \mapsto -S$. The S component is not optional: every admissible map attains a nonzero S component, while a real-line-valued image is impossible.

The residual sign requires care. The map $a + bS \mapsto a - bS$ is an algebra automorphism of the central plane and exchanges N_+ with N_-, but Task VIII already showed that $S = ABC$ carries the sign of the ordered spatial-generator frame. The full-algebra automorphism used in Task 09 to flip S simultaneously exchanges A and B; a multiplicative map fixing A, B, C pointwise would necessarily fix $S = ABC$. The two branches are therefore equivalent only relative to a data model in which the relevant odd generator relabelling is also regarded as equivalent. They must not be dismissed prematurely as a content-free "complex conjugation" choice.

Only in the final comparison layer is the familiar mathematics named: under the Task VIII isomorphism $W \cong M_2(\mathbb{C})$, N_+ is the determinant and N_- its complex conjugate, while S maps to the central scalar i. Again, the mathematics is standard; the result of the experiment is the audited dependency path. The two Part VI obstructions are now separated sharply: associative product closure required enlargement of the carrier, whereas multiplicative continuation of the Lorentz quadratic form required enlargement of its value space to an internally generated central two-plane.

With these algebraic obstructions resolved, the next task may cautiously probe the boundary between kinematics and dynamics. The proposed Task 10 should fix one nonzero old spatial direction as a momentum-adapted axis, derive the associated longitudinal line and transverse plane inside the already reconstructed $3+1$ geometry, and classify how the intrinsic algebraic symmetries act on them. It should then test, without presupposing particle types, which sectors exhibit vector-like spin-1 behaviour, whether any intrinsic action exhibits the half-angle/double-cover behaviour required for spin 1/2, and whether nontrivial one-parameter structure-preserving evolutions are distinguished without additional dynamical input. The task must explicitly separate propagation direction from spin orientation and must not identify a longitudinal vector mode with a physical massless spin-1 polarization without additional gauge/dynamical structure.

Keywords: Lorentzian geometry; longitudinal sectors; transverse plane; spin factor; associative envelope; central quadratic extension; multiplicativity; determinant; orientation-sensitive sign; spin; helicity; one-parameter automorphisms; formal verification; Lean 4; conservation of difficulty.

Contents

1 After associative closure, one obstruction still survived

Part VI established two independent impossibility statements for products closed inside the original real $3+1$ carrier. First, no sector-compatible bilinear product on that carrier can be globally associative. Second, no such in-carrier product can make the Lorentz quadratic form globally multiplicative [10, 2].

Tasks VII and VIII addressed the first obstruction only. They opened product closure, first for two inherited spatial directions and then for the complete old rest space. The result was a minimal eight-dimensional real associative algebra W generated by

$$1,\ A,\ B,\ C,\ P = AB,\ Q = AC,\ R = BC,\ S = (AB)C, \tag{1}$$

with

$$A^2 = B^2 = C^2 = 1, \qquad AB = -BA, \quad AC = -CA, \quad BC = -CB, \tag{2}$$

and in particular

$$S^2 = -1, \qquad Sx = xS \quad \text{for all } x \in W. \tag{3}$$

The complete old $\mathsf{Vec4}$ embeds linearly and injectively into W, and its inherited square law survives exactly [11, 12, 3].

That success does *not* by itself solve the second Part VI obstruction. The old quadratic form

$$Q_4(t,x,y,z) = t^2 - x^2 - y^2 - z^2 \tag{4}$$

is still a real-valued function defined on the old carrier. Nothing in Task VIII says what value a multiplicative quadratic invariant should assign to a generic element containing P, Q, R or S.

The tension at the opening of Task 09 is therefore deliberately narrow:

> The product carrier is now large enough for associativity. Is the real scalar line still large enough to receive a globally multiplicative quadratic continuation of the old Lorentz form?

Methodological principle 1 (Freeze the successful repair)**.** Task 09 does not enlarge or alter the Task VIII algebra. If the second obstruction persists, the source of failure must be sought elsewhere.

2 The first negative control: keep the codomain real

Let $N_R : W \to \mathbb{R}$ be an unknown map satisfying three requirements:

1. real quadratic homogeneity,
$$N_R(rx) = r^2 N_R(x); \tag{5}$$
2. global multiplicativity,
$$N_R(xy) = N_R(x)N_R(y); \tag{6}$$
3. exact recovery of the inherited Lorentz form on every old vector,
$$N_R(\iota_3 X) = Q_4(X), \qquad X \in \mathsf{Vec4}. \tag{7}$$

Task 09 packages the full quadratic requirement transparently through homogeneous degree two together with bilinearity of the polarization, rather than importing a conventional norm object [6, 4].

The restriction (7) forces

$$N_R(1) = 1, \qquad N_R(A) = N_R(B) = N_R(C) = -1. \tag{8}$$

Multiplicativity and the Task VIII product table then force

$$N_R(P) = N_R(Q) = N_R(R) = 1, \qquad N_R(S) = -1. \tag{9}$$

No freedom remains on the basis diagonals.

The contradiction does not require a global coefficient classification. Since $S^2 = -1$ and S is central,

$$(1+S)^2 = 1 + 2S + S^2 = 2S. \tag{10}$$

Using (5), (6) and (9),

$$N_R(1+S)^2 = N_R\Big((1+S)^2\Big) \tag{11}$$
$$= N_R(2S) \tag{12}$$
$$= 4N_R(S) \tag{13}$$
$$= -4. \tag{14}$$

The left side is the square of a real number and therefore nonnegative. Hence no such map exists.

Theorem 1 (Scalar-codomain obstruction)**.** *There is no real-valued quadratic multiplicative extension of Q_4 from the embedded old* Vec4 *to the Task VIII associative carrier W.*

The formal theorem is stronger in one useful sense: its minimal witness does not use the full polarization axiom. Quadratic homogeneity, multiplicativity and the inherited values on the three spatial generators already suffice to reach the contradiction on $1 + S$ [4].

Remark 1 (A second closure deficit)**.** *Tasks VII–VIII fixed the domain of multiplication. Task 09 now shows that this repair does not fix the codomain of the multiplicative quadratic map. The obstruction has moved, not disappeared.*

3 The smallest internal candidate already exists

Task 09 does not import an external scalar field after the real no-go. Instead it turns to structure that Task VIII had already forced internally. Define

$$Z := \operatorname{span}_{\mathbb{R}}\{1, S\} \subset W. \tag{15}$$

Because $S^2 = -1$, every element has a unique normal form

$$z = a\,1 + b\,S, \qquad a, b \in \mathbb{R}, \tag{16}$$

and direct bilinearity gives

$$(a + bS)(c + dS) = (ac - bd) + (ad + bc)S. \tag{17}$$

Task 09 proves before any conventional naming that

$$\dim_{\mathbb{R}} Z = 2, \qquad ZZ \subseteq Z, \qquad zw = wz \quad (z \in Z,\ w \in W). \tag{18}$$

Thus Z is a two-dimensional commutative central subalgebra of the fixed carrier.

This is the first tested codomain larger than the real unit line. Task 09 proves that it suffices, but it does *not* prove a universal minimality theorem among all conceivable codomains. The exact formal statement is narrower and stronger where it matters: $\mathbb{R}$ fails, while the already-derived internal plane Z succeeds.

Methodological principle 2 (Do not name the answer before it is needed)**.** Before the late comparison layer, Z is only the central plane generated by 1 and S. Its familiar interpretation is withheld from the reconstruction.

4 An unknown central-valued quadratic extension

Now let

$$N : W \longrightarrow Z \tag{19}$$

be entirely unknown. Task 09 requires

$$N(rx) = r^2 N(x), \tag{20}$$
$$N(xy) = N(x)N(y), \tag{21}$$
$$N(\iota_3 X) = Q_4(X)\,1 \qquad (X \in \mathsf{Vec4}), \tag{22}$$

plus real bilinearity of the polarization

$$B_N(x, y) := N(x + y) - N(x) - N(y). \tag{23}$$

No coordinate formula for N is present at this stage. In particular, no determinant, conjugation, matrix representation or complex scalar is available in the reconstruction core [4].

The old restriction again fixes the basis values

x	1	A	B	C	P	Q	R	S
$N(x)$	1	-1	-1	-1	1	1	1	-1

(24)

where each scalar is understood as a multiple of the unit in Z.

This is a useful warning. The new central direction is essential, but it is invisible on all eight basis diagonals. The residual information lives in off-diagonal polarization.

5 Rigidity from the polarization ledger

Task 09 derives the old-Vec4 polarization channels directly from Q_4:

$$B_N(1, A) = B_N(1, B) = B_N(1, C) = 0, \tag{25}$$

$$B_N(A, B) = B_N(A, C) = B_N(B, C) = 0. \tag{26}$$

The mixed channels are then transported by multiplicativity. The key formal identity is a covariance law for polarization under left multiplication. Combined with the Task VIII table and the already fixed diagonal values, it determines the complete 8×8 ledger up to one residual element

$$d := B_N(1, S) \in Z. \tag{27}$$

B_N	1	A	B	C	P	Q	R	S
1	2	0	0	0	0	0	0	d
A	0	-2	0	0	0	0	$-d$	0
B	0	0	-2	0	0	d	0	0
C	0	0	0	-2	$-d$	0	0	0
P	0	0	0	$-d$	2	0	0	0
Q	0	0	d	0	0	2	0	0
R	0	$-d$	0	0	0	0	2	0
S	d	0	0	0	0	0	0	-2

Here ± 2 abbreviates $\pm 2\,1 \in Z$.

Because a real quadratic map is recovered from its polarization by

$$N(x) = \frac{1}{2} B_N(x, x), \tag{28}$$

the ledger immediately gives the rigidity statement:

Theorem 2 (One-datum rigidity)**.** *Every admissible central-valued quadratic multiplicative extension is uniquely determined by the single residual datum $d = B_N(1, S)$.*

No continuous family has yet been ruled out. That happens only after multiplicativity is applied once more to the same diagnostic element that killed the real-valued case.

6 The old contradiction becomes a two-valued constraint

From (24) and (27),

$$N(1 + S) = d, \tag{29}$$

because $N(1) = 1$ and $N(S) = -1$. Using (10), multiplicativity and homogeneity now give

$$d^2 = N((1 + S)^2) = N(2S) = 4N(S) = -4\,1. \tag{30}$$

The same algebraic configuration that produced an impossibility over $\mathbb{R}$ is therefore no longer contradictory in Z; it becomes a rigidity equation.

Write

$$d = p\,1 + q\,S. \tag{31}$$

Then (17) and (30) imply

$$p^2 - q^2 = -4, \qquad 2pq = 0. \tag{32}$$

The real-line possibility $q = 0$ would require $p^2 = -4$ and is impossible. Therefore

$$p = 0, \qquad q = \pm 2, \qquad d = \pm 2S. \tag{33}$$

Corollary 1 (Discrete residual freedom)**.** *The unknown quadratic extension has at most two branches. The residual freedom is a sign, not a continuous parameter.*

This is the precise point at which the second Part VI difficulty stops migrating. It has been compressed to a two-valued central sign.

7 Existence after rigidity

Only after the classification above does Task 09 define explicit candidates. For

$$x = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in W, \tag{34}$$

define two real quadratic coefficients

$$n_{\mathrm{Re}}(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 + x_6^2 - x_7^2, \tag{35}$$
$$n_S(x) = 2(x_0x_7 - x_1x_6 + x_2x_5 - x_3x_4). \tag{36}$$

Then

$$N_\sigma(x) := n_{\mathrm{Re}}(x)\,1 + \sigma\,n_S(x)\,S, \qquad \sigma \in \{+1, -1\}. \tag{37}$$

These formulas are not imported from a known matrix invariant. Their terms are read directly from the forced polarization ledger.

Task 09 proves the coordinate identities

$$n_{\mathrm{Re}}(xy) = n_{\mathrm{Re}}(x)n_{\mathrm{Re}}(y) - n_S(x)n_S(y), \tag{38}$$

$$n_S(xy) = n_{\mathrm{Re}}(x)n_S(y) + n_S(x)n_{\mathrm{Re}}(y), \tag{39}$$

for arbitrary $x, y \in W$. Together with (17), these imply

$$N_\sigma(xy) = N_\sigma(x)N_\sigma(y) \qquad (x, y \in W). \tag{40}$$

Thus multiplicativity is genuinely global; it is not checked only on generators.

For an arbitrary old vector $X = (t, x, y, z)$, the Task VIII embedding has no P, Q, R, S coordinates. Hence

$$n_{\mathrm{Re}}(\iota_3 X) = t^2 - x^2 - y^2 - z^2 = Q_4(X), \qquad n_S(\iota_3 X) = 0, \tag{41}$$

and therefore

$$N_\sigma(\iota_3 X) = Q_4(X)\,1 \qquad \text{for every } X \in \mathsf{Vec4}. \tag{42}$$

Theorem 3 (Exact Task-09 classification)**.** *The admissible quadratic multiplicative extensions are exactly N_+ and N_-. They are distinct and there are no other branches.*

The real scalar line is not recovered accidentally inside the larger codomain. For both branches,

$$N_\sigma(1 + S) = 2\sigma S, \tag{43}$$

so every admissible map actually attains a nonzero central S component.

8 The sign of S is not automatically disposable

The exact two-valued classification invites a tempting shorthand. On the central plane, define

$$\kappa_Z(a + bS) = a - bS. \tag{44}$$

It is real-linear, multiplicative and involutive, and Task 09 proves

$$\kappa_Z \circ N_+ = N_-, \qquad \kappa_Z \circ N_- = N_+. \tag{45}$$

After the late conventional identification this looks exactly like complex conjugation. But that description is potentially misleading if imported backwards into the reconstruction.

Task VIII did not introduce S as an arbitrary square root of -1. It derived

$$S = ABC. \tag{46}$$

Its sign tracks the ordered generator data. Exchanging A and B gives

$$A \leftrightarrow B \quad \Longrightarrow \quad S \mapsto -S, \tag{47}$$

while a cyclic permutation $A \to B \to C \to A$ leaves S unchanged. Task 09 constructs the full-algebra automorphism implementing the transposition $A \leftrightarrow B$ and proves that it exchanges N_+ and N_- in exactly the same way as (44) on the central plane [4].

There is a simple but important consequence. Suppose a multiplicative endomorphism $\kappa : W \to W$ fixes the three old spatial generators pointwise:

$$\kappa(A) = A, \qquad \kappa(B) = B, \qquad \kappa(C) = C. \tag{48}$$

Then multiplicativity forces

$$\kappa(S) = \kappa(ABC) = ABC = S. \tag{49}$$

Such a map cannot simultaneously send S to $-S$.

Methodological caveat 1 (What "conjugation" means depends on the retained data)**.** *The central sign map is an automorphism of Z. The full Task 09 algebra automorphism that realizes the same sign change does not fix the ordered spatial frame; it exchanges two generators. Therefore N_+ and N_- are equivalent under odd generator relabelling, but they are not identical relative to a pointwise fixed ordered frame.*

This is consistent with established Clifford theory, where several involutions and anti-involutions coexist and are distinguished precisely by their actions on generating vectors and higher products [8, 1]. The late matrix phrase "complex conjugation" names one manifestation of the sign exchange after identification; it should not be promoted to a primitive operation in the reconstruction.

Two interpretations therefore remain logically distinct:

1. if only the unoriented/unordered generator structure is retained, the two branches are related by an allowed relabelling and represent the same orbit of the data;

2. if the ordered generator frame or the sign of $S = ABC$ is part of the distinguished structure, the two branches remain distinct relative to that structure.

Task 09 proves the algebraic exchange. It does not decide which data nature, or a later dynamical theory, should regard as physically fixed.

9 Late comparison with established mathematics

Only after the scalar no-go, central-plane construction, rigidity, existence and exact classification have been completed does Task 09 import the Task VIII conventional identification

$$W \cong M_2(\mathbb{C}). \tag{50}$$

Under this isomorphism,

$$S \longmapsto iI_2, \tag{51}$$

and the central plane Z becomes the scalar centre of the complex matrix algebra. The independently reconstructed branches satisfy

$$N_+(x) \longleftrightarrow \det M_x, \tag{52}$$

$$N_-(x) \longleftrightarrow \overline{\det M_x}. \tag{53}$$

The familiar formulas make the result unsurprising after the fact. They were, however, unavailable to the reconstruction core.

Nothing in this identification is new mathematics. Determinants, quadratic forms, Clifford algebras and their involutions are classical structures [6, 8]. The substantive result of the experiment is the dependency audit:

> A real Lorentz quadratic form that cannot be made globally multiplicative on its original carrier first forces an associative product enlargement; after that enlargement, multiplicativity still fails if the codomain remains real, but succeeds in the central two-plane already generated by the associative closure, leaving only a discrete sign tied to generator ordering.

10 Conservation of Difficulty: the two Part-VI no-gos are now separated

The sequence can now be stated without collapsing distinct difficulties.

Part VI: inside old Vec4, global associativity is impossible and, independently, global real-valued Q_4 multiplicativity is impossible.

$\downarrow$

Tasks VII–VIII: open product closure only. Mixed products leave Vec4 and the three old spatial directions force a minimal eight-dimensional real associative carrier W.

$\downarrow$

Task IX, Phase A: keep W fixed and keep the quadratic codomain real. The obstruction persists; $1 + S$ forces a real square equal to -4.

$\downarrow$

Task IX, Phase B: use only the already derived central plane $Z = \mathrm{span}_{\mathbb{R}}\{1, S\}$. Rigidity leaves $d = B_N(1, S) = \pm 2S$.

$\downarrow$

Exactly two global multiplicative quadratic extensions exist; both recover Q_4 on all old vectors and genuinely use the S direction.

$\downarrow$

The residual sign is exchanged by $S \mapsto -S$ and by the already existing odd generator relabelling $A \leftrightarrow B$.

The two costs are therefore different:

$$\boxed{\text{associativity} \Rightarrow \text{larger product carrier}}, \qquad \boxed{\text{quadratic multiplicativity} \Rightarrow \text{larger value space}}. \tag{54}$$

The second enlargement does not require adding a new external object. The needed two-plane had already appeared as the centre generated by the highest Task VIII product S.

One qualification remains. Task 09 proves that $\mathbb{R}$ fails and Z succeeds. It does not establish that Z is universally minimal among every conceivable external or internal codomain. That stronger categorical claim is not needed for the present Conservation-of-Difficulty diagnosis.

11 Formal audit and proof boundary

The audited Task 09 archive supplied with this work has SHA-256

`2c4905314a3dcf9b3932485bde147825c1c77ab2bf2d2a20a31876d570ed366f`.

The pinned environment is Lean 4.28.0 with mathlib revision

`8f9d9cff6bd728b17a24e163c9402775d9e6a365`.

Aristotle reports a successful full build of 8108 jobs [4, 9, 13, 14].

The reconstruction chain is deliberately short:

$$\texttt{SafeBase} \to \texttt{RealScalarNoGo} \to \texttt{CentralPlane} \to \texttt{UnknownQuadraticMap}$$
$$\to \texttt{Rigidity} \to \texttt{Existence} \to \texttt{ExactClassification} \to \texttt{CentralSignAudit},$$

followed only then by

$$\texttt{Identification} \to \texttt{Task09}.$$

The reconstruction core imports the Task VIII full-embedding layer but not its conventional identification. The sole apparent pre-identification occurrence of the name 'Matrix' is a Mathlib namespace artefact used to describe the range of a two-element tuple in the proof that $\dim_{\mathbb{R}} Z = 2$; no matrix algebra or determinant enters that proof.

The principal theorem inventory includes:

Theorem wrapper	Formal role
`task09_scalar_no_go`	no real-valued quadratic multiplicative extension
`task09_minimal_scalar_obstruction`	the $1 + S$ contradiction under weaker hypotheses
`task09_central_plane`	closure, commutativity, centrality and dimension of Z
`task09_forced_basis_values`	the eight fixed diagonal values
`task09_old_polarization_constraints`	inherited old-Vec4 channels
`task09_residual_datum`	$d^2 = -4$ and $d = \pm 2S$
`task09_rigidity`	an admissible map is fixed by $B_N(1, S)$
`task09_global_multiplicativity`	multiplicativity for arbitrary $x, y \in W$
`task09_old_Q4_restriction`	exact Q_4 recovery for arbitrary old X
`task09_exact_classification`	exactly the two branches N_+, N_-
`task09_real_line_image_impossible`	the S direction is genuinely required
`task09_central_sign_action`	central sign exchange of the two branches
`task09_generator_sign_tracking`	the same exchange under $A \leftrightarrow B$
`task09_part_vi_verdict`	final codomain-enlargement verdict
`task09_identification`	late determinant comparison

All twenty-one principal theorem wrappers are reported with exactly

$$[\texttt{propext},\ \texttt{Classical.choice},\ \texttt{Quot.sound}], \tag{55}$$

and the project contains no 'sorry', 'admit', custom axiom or '@[implemented_by]'.

Methodological caveat 2 (Verification boundary)**.** *The supplied archive reports a green Lean build, and the editorial audit checks the source order, central algebra and key polynomial identities. This paper does not claim a second independent Lean kernel build in a separately installed toolchain.*

12 What Part IX proves – and what it does not

Established in Task 09	**Not established in Task 09**
No real-valued quadratic multiplicative extension of old Q_4 exists on W.	Universal minimality of Z among all conceivable codomains.
The internal central plane $Z = \mathrm{span}_{\mathbb{R}}\{1, S\}$ is closed, commutative, central and two-dimensional.	Positivity, zero-set geometry, fibres or invertibility properties of N.
Exactly two Z-valued quadratic multiplicative extensions exist.	A primitive or physically preferred choice between the two branches.
Both branches recover Q_4 exactly on every embedded old vector.	A physical interpretation of S, N_+ or N_-.
Every branch genuinely uses the S component away from old Vec4.	Any quantum state space, Hilbert structure or probability rule.
$S \mapsto -S$ exchanges the branches; an odd old-generator relabelling realizes the same exchange on W.	A full-algebra conjugation fixing A, B, C pointwise while sending S to $-S$.
The late conventional comparison gives determinant and conjugate determinant on $M_2(\mathbb{C})$.	Dynamics, field equations, spinor modules or particle identification.

The last row is now the important frontier. The algebraic reconstruction is mature enough to ask whether any part of the familiar state/representation/dynamical hierarchy is forced, but none of it should be imported wholesale.

13 Toward Task X: fix a momentum axis before talking about spin

The first probe beyond the present kinematic algebra should begin with a distinction that is standard in relativistic physics but has not yet been formalized in this experiment: a *state with a distinguished nonzero momentum direction* selects a longitudinal axis inside the old three-dimensional rest space. Relative to a normalized spatial direction n, one has the purely geometric decomposition

$$\mathbb{R}^3 = \mathrm{span}_{\mathbb{R}}\{n\} \oplus n^{\perp}. \tag{56}$$

The associated longitudinal Lorentz plane is

$$L_n := \mathrm{span}_{\mathbb{R}}\{e_0, n\}, \tag{57}$$

while the transverse old spatial sector $n^{\perp}$ is two-dimensional.

This is not yet a particle model. The selected direction may provisionally be called momentum-adapted because that is the physical use one eventually wants to test, but Task 10 should first treat it only as distinguished old spatial data.

13.1 A momentum-adapted symmetry probe

A narrow first phase should classify the subgroup of already reconstructed algebra automorphisms that preserves

$$e_0, \qquad n, \tag{58}$$

and either preserves a chosen pair (S, N_σ) or transports the two sign branches equivariantly. It should then derive, rather than assume, how this stabilizer acts on

$$\mathrm{span}\{n\}, \qquad n^{\perp}, \tag{59}$$

and on the corresponding mixed-product sectors inside W.

The first question is therefore not “where is spin?” but:

> Once one spatial direction is distinguished, which irreducible or invariant longitudinal and transverse transformation sectors are already present in the reconstructed algebra?

13.2 Spin-1 should be a representation test, not a label

For ordinary rotations about a fixed axis, a vector-like three-dimensional object decomposes into one longitudinal component and a two-dimensional transverse plane. In standard relativistic representation theory, the physical interpretation depends crucially on the mass and constraints of the state: a massive spin-1 representation has a longitudinal helicity-zero component as well as two transverse components, whereas a massless spin-1 field has only the two physical transverse helicities after the appropriate gauge/redundancy conditions are imposed [16, 15].

Task 10 should therefore be allowed to identify an *algebraic longitudinal spin-1 candidate sector* and an *algebraic transverse vector sector*, but it must not call the longitudinal mode a photon polarization. Whether it survives as a physical degree of freedom is a later mass-shell/gauge/dynamical question.

This distinction is especially important for the earlier heuristic discussion. A propagation direction is longitudinal; physical polarization need not be. For a massless spin-1 state, propagation along n and polarization in $n^{\perp}$ are compatible statements, not alternatives.

13.3 Spin-1/2 requires a stricter negative control

Half-integer spin is more diagnostic. A spin-1/2 state is not merely a vector pointing transversely or longitudinally. Relative to a momentum axis, one may speak of longitudinal spin/helicity and of transverse spin orientations, but transverse spin states are superpositions of helicity states rather than a second direction of propagation [7, 15].

Task 10 should first ask whether the intrinsic action already available on the algebra W itself exhibits the double-cover/half-angle behaviour characteristic of spin 1/2. A 2π spatial rotation acting trivially on ordinary vectors but as a sign change on a candidate state would be a decisive diagnostic. If no such state exists inside W under the natural automorphism action, that negative result should be recorded rather than repaired immediately.

Only then should a later task consider whether a new *state carrier*, such as a module over the already reconstructed algebra, is required. The algebra should not be enlarged again merely to manufacture half-integer spin. This is precisely the kind of dependency distinction the experiment is designed to expose. The standard Dirac construction provides the familiar comparison target only after such a state-carrier question has been independently localized [5, 8].

13.4 The first controlled step toward dynamics

Even a complete algebra and representation content do not select a physical time evolution. A first dynamics probe should therefore be deliberately weaker than a Hamiltonian or field equation. On whatever state carrier survives the symmetry probe, Task 10 may ask for real one-parameter families of transformations

$$U(\tau), \qquad U(0) = \mathrm{id}, \qquad U(\tau + \sigma) = U(\tau)U(\sigma), \tag{60}$$

that preserve the already reconstructed algebraic and quadratic data appropriate to that carrier.

At the infinitesimal level, the question becomes whether there are nonzero generators D satisfying the relevant derivation or invariance conditions. More importantly, Task 10 should ask whether the distinguished direction n, the central element S, and one of the two Task 09 quadratic branches single out any such D *canonically.* Three outcomes are diagnostically distinct:

1. no nontrivial structure-preserving one-parameter evolution exists under the retained requirements;
2. many exist but none is distinguished, locating a genuinely new dynamical datum;
3. the existing structure restricts the generator to a small canonical class.

None of these outcomes should be presupposed.

Methodological principle 3 (Task X should remain a probe)**.** Do not introduce the Dirac equation, Maxwell equations, a Hamiltonian, a Lagrangian, gauge symmetry or a Hilbert-space postulate as input. First classify the momentum-adapted longitudinal/transverse symmetry sectors and the available one-parameter structure-preserving transformations. Let failure itself identify what additional state or dynamical structure is missing.

14 A precise but narrow Task-X question

A suitable Task 10 scope can therefore be summarized in four questions:

1. **Momentum-adapted decomposition.** Fix one nonzero old spatial direction n. Derive the longitudinal line, transverse plane and their induced sectors inside W under the stabilizer of n.

2. **Integer-spin probe.** Determine which internal sectors transform vectorially under rotations about n, separating the longitudinal invariant component from the transverse rotating pair. Do not identify either with a physical photon or massive vector state yet.

3. **Half-integer-spin negative control.** Test whether any intrinsic candidate state inside the present algebra exhibits half-angle/double-cover behaviour. If not, report that a separate state module is required and stop that branch.

4. **First evolution probe.** Classify nontrivial one-parameter transformations preserving the already reconstructed product and appropriate Task 09 quadratic data, and determine whether any generator is selected without a new dynamical assumption.

This is intentionally not yet a theory of dynamics. It is an attempt to locate the first point at which kinematic algebra, representation content and actual evolution cease to be equivalent questions.

15 Conclusion

Part IX completes the algebraic diagnostic opened in Part VI. The associative carrier constructed in Parts VII–VIII is sufficient for multiplication but insufficient, by itself, for a real-valued globally multiplicative continuation of the inherited Lorentz quadratic form. The scalar no-go is elementary and sharp: the internally generated central element S turns the single test element $1 + S$ into a demand for a real square equal to -4.

The obstruction is removed without changing the product carrier. The already derived central two-plane $Z = \mathrm{span}_{\mathbb{R}}\{1, S\}$ is product-closed, commutative and central. For quadratic multiplicative maps into this plane, the entire problem becomes rigid up to the one polarization value $B_N(1, S)$. Multiplicativity forces this value to be $\pm 2S$, and explicit post-rigidity witnesses show that both branches exist globally and recover Q_4 exactly on every old vector. No real-line-valued branch survives.

The residual sign is not an independent continuous parameter. It is exchanged by the central sign map and by the old generator transposition $A \leftrightarrow B$. Nevertheless, the sign cannot be erased without specifying what data are regarded as fixed: any multiplicative map fixing A, B, C pointwise also fixes $S = ABC$. The familiar language of complex conjugation is therefore appropriate only in the late comparison layer, not as an intrinsic primitive of the reconstruction.

After identification, the result is classical: the two maps are determinant and conjugate determinant on $M_2(\mathbb{C})$. As throughout the experiment, the mathematical endpoint is known. The value of the formal reconstruction is the localization of necessity. Part VI contained two different closure deficits. Associativity required a larger product carrier; quadratic multiplicativity required a larger value space. Both enlargements were forced only after the corresponding smaller class had been formally excluded.

This closes the planned algebraic obstruction cycle. The next question is qualitatively different. Once a nonzero old spatial direction is distinguished, the reconstructed 3+1 structure automatically separates longitudinal and transverse directions. Task 10 can therefore begin to test how the full algebra transforms relative to that axis, whether vector-like and half-integer-spin-like sectors arise intrinsically or demand a new state carrier, and whether any nontrivial one-parameter structure-preserving evolution is selected. The hard methodological rule remains unchanged: spin labels, photons, fermions and dynamics are conclusions to be earned, not names to be inserted at the beginning.

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