Split Complex Lorentzian Dynamics

Part VII: A Forced Product Direction and the Minimal Two-Generator Associative Closure

A formally verified continuation of the dependency audit

Editor: Jan Seeck
AI Assistant: OpenAI ChatGPT (GPT-5.6 Sol)
Lean Agent: Aristotle (Harmonic)

2 September 2026

Abstract

Part VI ended at a sharp negative boundary. Once the already reconstructed longitudinal $1+1$ sectors are required to coexist on the real $3+1$ carrier, no bilinear product closed inside that carrier can be globally associative, and independently no such product can be globally multiplicative for the Lorentz quadratic form. The associativity contradiction already appears for two orthogonal spatial directions. The remaining question was therefore no longer which cross-sector coefficient should be chosen, but whether the requirement of product closure inside the original carrier was itself the obstructing assumption.

This paper reports Task 07, which opens exactly that one assumption and nothing else. The original $3+1$ carrier is injected into an arbitrary real associative unital ambient vector space. Its inherited diagonal square law is retained, but the image is not assumed closed under multiplication. Polarization then recovers the entire previously forced symmetric product after embedding. For two selected orthogonal spatial directions, their images A and B satisfy $A^2 = B^2 = 1$ and $AB + BA = 0$. Only after these relations are derived is the mixed product $P := AB$ introduced.

Associativity forces $P^2 = -1$ together with a complete set of multiplication relations between $1, A, B, P$. The retained old-carrier square law proves that no embedded old vector can square to -1 in this sense; hence P is forced outside the embedded $3+1$ carrier. The four elements $1, A, B, P$ are linearly independent. Their span is multiplicatively closed, has real dimension exactly four, and is minimal among product-closed subspaces containing the two inherited spatial generators and the unit. A fresh four-dimensional witness is constructed only after this table has been derived and is proved directly associative. The resulting minimal closure is unique up to a generator-preserving unital multiplicative linear equivalence.

Only in a final comparison layer is the independently reconstructed algebra identified with 2×2 real matrices, equivalently the standard split-quaternion algebra. This is established mathematics, not a novelty claim. The role of the formal experiment is instead diagnostic: it shows that the Task 06 obstruction was localized at precisely the right assumption for the two-direction subsystem. Associativity does not demand an arbitrary repair; it forces a specific new mixed-product direction, and one such direction suffices to close the two-generator system.

A crucial boundary remains. Task 07 does not construct an associative ambient extension of the entire original `Vec4`. Its structural theorems show what every such extension would have to contain for the chosen pair of directions, while the direct witness establishes only the consistency of that two-generator relation system. The interaction with the third independent spatial direction is intentionally untouched. Task 08 should therefore add exactly one third inherited spatial generator and ask which further mixed products and independent directions are forced. It should stop before norm reconstruction or any conventional target algebra is supplied.

Keywords: Lorentzian geometry; longitudinal split-complex sectors; associative closure; mixed products; real matrix algebra; split quaternions; paravectors; formal verification; Lean 4; conservation of difficulty; dependency analysis.

Contents

1 From a universal obstruction to one opened assumption

Part VI converted a formerly parameter-dependent difficulty into a universal theorem. Let M be any real bilinear product on the original $3+1$ carrier that agrees with every reconstructed longitudinal $1+1$ sector. No choice of its cross-sector alternating defect makes M globally associative, and no choice makes it globally Q_4-multiplicative [9, 1].

For associativity, two orthogonal rest directions already suffice. If their products are denoted schematically by a and b, the inherited sector relations give

$$a^2 = e_0, \qquad b^2 = e_0, \qquad ab + ba = 0. \tag{1}$$

Inside the old carrier, associativity then requires the mixed product $p = ab$ to satisfy

$$p^2 = -e_0, \tag{2}$$

while the inherited square law makes that impossible for every old-carrier element. Part VI therefore ended with a highly localized suspicion:

> Perhaps the failed assumption is not the sector algebra, nor the gluing coefficient, nor isotropy, but the requirement that a mixed product of two old directions must itself remain an old $3+1$ vector.

Task 07 tests precisely this suspicion [2]. It removes only old-carrier product closure. It does not add a third direction, does not introduce a new norm, does not modify the inherited square law, and does not prescribe a known ambient algebra.

This makes the task unusually sharp. There are only four logically possible outcomes: the mixed product might still be representable by an old vector; it might leave the old carrier but fail to close after one additional direction; one new direction might suffice; or the retained assumptions might remain inconsistent even in an arbitrary associative ambient space. The formal development resolves this branching without being told which outcome to expect.

2 The retained data: what is not allowed to change

2.1 The old square law

The original real carrier remains

$$\mathsf{Vec4} = \mathbb{R} \oplus \mathbb{R}^3, \qquad Q_4(t, v) = t^2 - \|v\|^2, \qquad e_0 = (1, 0). \tag{3}$$

From the full collection of longitudinal sectors, Parts IV and VI had already forced the symmetric product

$$\mu_{\mathrm{sym}}((t, v), (s, w)) = (ts + v \cdot w,\ tw + sv). \tag{4}$$

Its diagonal is recorded neutrally as

$$\mathsf{oldSq}(t, v) = \mu_{\mathrm{sym}}((t, v), (t, v)) = (t^2 + \|v\|^2,\ 2tv). \tag{5}$$

The decisive sign feature is already visible here:

$$(\mathsf{oldSq} X)^0 = t^2 + \|v\|^2 \geq 0. \tag{6}$$

Thus no old vector has inherited square $-e_0$.

This square law is not a new Task 07 axiom in the explanatory sense. It is retained because Task 06 had proved that every sector-compatible product on the old carrier possesses exactly this diagonal.

2.2 The only relaxed requirement

Task 07 introduces an arbitrary real vector space E with an associative bilinear product $\star$, a two-sided unit 1_E, and an injective real-linear embedding

$$\iota : \mathsf{Vec4} \longrightarrow E, \qquad \iota(e_0) = 1_E. \tag{7}$$

For every old vector X the retained square law is required:

$$\iota(X) \star \iota(X) = \iota(\mathsf{oldSq} X). \tag{8}$$

What is *not* required is

$$\iota(\mathsf{Vec4}) \star \iota(\mathsf{Vec4}) \subseteq \iota(\mathsf{Vec4}). \tag{9}$$

No dimension, basis, grading, involution, norm or conventional algebraic type is assigned to E.

Methodological principle 1 (Single-assumption opening)**.** The old diagonal law is retained exactly; associativity is retained exactly; only closure of mixed products inside the old carrier is removed. Any newly forced object can therefore be attributed to this one change rather than to a wholesale replacement of the prior construction.

3 The old symmetric structure returns automatically

The ambient square condition might at first appear weaker than preserving the complete longitudinally forced product. Bilinearity removes that ambiguity. Applying equation (8) to $X + Y$ and subtracting the two diagonal equations gives the polarization identity

$$\iota(X) \star \iota(Y) + \iota(Y) \star \iota(X) = 2\,\iota(\mu_{\mathrm{sym}}(X, Y))\,. \tag{10}$$

This is the theorem ‘AmbientExt.polarization‘ in the formal development.

The significance is structural. Task 07 has not discarded the old $3+1$ result and invented a new multiplication. Equation (10) says that the entire previously reconstructed symmetric information survives intact. The only room for genuinely new ambient content lies in the antisymmetric part of mixed products.

This is exactly where Task 06 had located its contradiction. The symmetric component remained fixed while no antisymmetric old-carrier correction could rescue associativity. Task 07 now permits that antisymmetric component to leave the old carrier rather than changing its inherited symmetric projection.

4 Two orthogonal directions and the first tension

Choose exactly two orthogonal normalized spatial directions $a, b \in \mathsf{Vec4}$ with

$$L_4(e_0, a) = L_4(e_0, b) = 0, \qquad Q_4(a) = Q_4(b) = -1, \qquad L_4(a, b) = 0. \tag{11}$$

Let

$$A := \iota(a), \qquad B := \iota(b). \tag{12}$$

The retained square law yields

$$A^2 = 1_E, \qquad B^2 = 1_E. \tag{13}$$

Polarization and orthogonality yield

$$AB + BA = 0, \qquad \text{hence} \qquad BA = -AB. \tag{14}$$

No new product object has yet been introduced. These relations arise solely from the old sector data transported into the associative ambient setting.

At this point the tension from Part VI can be stated in its smallest form. The relations (13)–(14) are harmless inside each one-generator longitudinal sector. Their joint associative closure, however, contains information that neither individual sector possesses.

Only now does the formal development define

$$P := AB. \tag{15}$$

Nothing about P is assumed: not its square, not whether it vanishes, not whether it is old or new, and not whether one such new direction could ever suffice.

5 Associativity forces the missing multiplication channel

Associativity now determines the interaction of P with the generators. From (13)–(15) one obtains

$$AP = B, \qquad PA = -B, \tag{16}$$
$$BP = -A, \qquad PB = A. \tag{17}$$

Most importantly,

$$P^2 = -1_E. \tag{18}$$

For example,

$$P^2 = (AB)(AB) = A(BA)B = -A(AB)B = -(A^2)(B^2) = -1_E. \tag{19}$$

This is no longer a contradiction because Task 07 has stopped demanding that P be an old vector.

The formal source order is methodologically significant:

$$\text{old square law} \longrightarrow \text{polarization} \longrightarrow A^2 = B^2 = 1,\ AB = -BA \longrightarrow P := AB \longrightarrow P^2 = -1. \tag{20}$$

The object with negative square is a theorem-generated product channel, not an inserted repair.

6 Why the new channel cannot be an old vector

Task 07 proves locally, without invoking the global Task 06 no-go, that

$$\mathsf{oldSq}(X) \neq -e_0 \qquad \text{for all } X \in \mathsf{Vec4}. \tag{21}$$

Indeed, the time component of every inherited old square is the nonnegative quantity in (6).

Suppose nevertheless that $P = \iota(X)$ for some old vector X. Then the retained ambient square law and (18) imply

$$\iota(\mathsf{oldSq}X) = P^2 = -1_E = \iota(-e_0). \tag{22}$$

Injectivity gives $\mathsf{oldSq}X = -e_0$, contradicting (21). Hence

$$\boxed{P \notin \text{range}(\iota).} \tag{23}$$

Theorem 1 (Forced new product direction)**.** *For every associative ambient extension satisfying the retained old square law, the mixed product of the two chosen orthogonal spatial generators lies outside the embedded old* $3+1$ *carrier.*

The meaning of the Task 06 obstruction is therefore sharpened. There was no missing choice *inside* the old carrier. The associative product itself demands an output direction that the old carrier does not contain.

7 One new direction is enough

A forced new direction need not, in general, solve a closure problem. Products involving it could require further independent directions, potentially without end. Task 07 therefore does not stop at (23).

First, the formal development proves that

$$1_E,\ A,\ B,\ P \tag{24}$$

are linearly independent over $\mathbb{R}$. The non-membership theorem alone is not used as a shortcut; independence of e_0, a, b in the old carrier is proved separately and combined with injectivity.

Define

$$G := \operatorname{span}_{\mathbb{R}}\{1_E, A, B, P\}. \tag{25}$$

The relations already derived imply that G is closed under multiplication. The complete table is

$\star$	1	A	B	P
1	1	A	B	P
A	A	1	P	B
B	B	$-P$	1	$-A$
P	P	$-B$	A	-1

Every nontrivial entry in this table is collected only after being proved independently from associativity and the generator relations.

For normal forms

$$x = \alpha + \beta A + \gamma B + \delta P, \qquad y = \alpha' + \beta' A + \gamma' B + \delta' P, \tag{26}$$

the derived coefficient law is

$$\begin{aligned} xy = {} & (\alpha\alpha' + \beta\beta' + \gamma\gamma' - \delta\delta') && (27)\\ & + (\alpha\beta' + \beta\alpha' - \gamma\delta' + \delta\gamma')A && (28)\\ & + (\alpha\gamma' + \beta\delta' + \gamma\alpha' - \delta\beta')B && (29)\\ & + (\alpha\delta' + \beta\gamma' - \gamma\beta' + \delta\alpha')P. && (30) \end{aligned}$$

Thus no fifth direction is generated by products within this two-direction subsystem.

Corollary 1 (Exact closure dimension)**.** *The generated associative subspace G has real dimension exactly four.*

The ambient space E remains unrestricted. The number four is an output describing only the smallest closure generated by the two inherited spatial directions.

8 Minimality: the price cannot be made smaller

The closure result shows sufficiency. Minimality asks necessity.

Let $H \subseteq E$ be any real-linear subspace containing

$$1_E, \qquad A, \qquad B \tag{31}$$

and closed under $\star$. Product closure immediately forces

$$P = AB \in H. \tag{32}$$

Therefore

$$G \subseteq H. \tag{33}$$

The generated G is hence the smallest product-closed linear subspace containing the two inherited generators and the unit.

This gives the Conservation-of-Difficulty statement a quantitative form for the first time in this branch of the experiment. Restoring associativity after opening old-carrier closure is not compatible with an arbitrary infinitesimal adjustment. For the selected pair, one additional independent product direction is necessary, and it is sufficient; the resulting minimal closure has exactly four real dimensions.

The proof does not infer this from a known classification theorem. It follows directly from the forced product P, its independence, and the closure relations.

9 Conditional structure versus unconditional consistency

A methodological distinction is essential here.

Up to this point, Task 07 has proved conditional statements about an arbitrary structure ‘AmbientExt‘: *if* an associative ambient extension of the whole old carrier preserving the inherited square law exists, then its two selected spatial generators necessarily generate the four-dimensional closure G just described.

That does not yet prove that the complete ambient hypothesis set is nonempty. In particular, the formal task intentionally forbids introduction of the third independent old spatial direction.

Task 07 nevertheless checks that the derived two-generator relations themselves are consistent. Only after the complete relation table, dimension and minimality have been established does Aristotle introduce a fresh carrier

$$W \cong \mathbb{R}^4 \tag{34}$$

and define its product using the already derived coefficient law. Bilinearity, a two-sided unit and associativity are then proved directly by calculation. The distinguished generators in W satisfy

$$A^2 = B^2 = 1, \qquad AB = -BA, \qquad P = AB, \qquad P^2 = -1, \tag{35}$$

and $1, A, B, P$ are independent.

Thus the correct formal conclusion is

> Removing old-carrier closure removes the local two-direction contradiction and produces a consistent minimal associative algebra for that two-generator subsystem.

It is *not* yet

> A full associative ambient extension of the entire original `Vec4` has been constructed.

This boundary is recorded explicitly in the Task 07 audit and is retained here.

10 Uniqueness and sign diagnostics

Once the four basis elements and their sixteen products have been forced, bilinearity leaves no further freedom. Task 07 proves that two bilinear products agreeing on these basis pairs are equal. It also constructs, for any two ambient extensions satisfying the Task 07 hypotheses, a generator-preserving equivalence between their respective closures:

$$1 \mapsto 1', \qquad A \mapsto A', \qquad B \mapsto B', \qquad P \mapsto P'. \tag{36}$$

The map is real-linear, unital and multiplicative. Thus the minimal closure is unique up to this generator-preserving algebraic equivalence.

The generator-order audit further shows

$$A \leftrightarrow B \quad \Longrightarrow \quad P \mapsto -P. \tag{37}$$

Negating either one of A or B also negates P, while negating both leaves P unchanged. These are purely algebraic diagnostics in Task 07; no geometric or physical interpretation is attached to them.

The sign behaviour is nevertheless structurally informative. The new direction is not an unrelated appended scalar coordinate. It remembers the ordered pair of generators from which it was produced.

11 Late reflection against established mathematics

Only after the internal classification is complete does the formal development ask what algebra has been reconstructed. It defines an explicit linear map from the direct witness W to 2×2 real matrices by

$$(\alpha, \beta, \gamma, \delta) \longmapsto \begin{pmatrix} \alpha + \beta & \gamma + \delta \\ \gamma - \delta & \alpha - \beta \end{pmatrix}. \tag{38}$$

The map is proved bijective, unital and multiplicative. In particular,

$$A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{39}$$

Therefore

$$\boxed{G \cong M_2(\mathbb{R}).} \tag{40}$$

This real algebra is conventionally described as the split-quaternion algebra and is also the familiar associative algebra generated by two anticommuting square roots of $+1$ [3, 4, 5].

There is no mathematical novelty claim in (40). The tension is precisely the opposite. The formal experiment, after being denied the target object, has reached a standard piece of established algebra at exactly the point where established algebra says it should appear.

In the usual Clifford-algebra language, two independent generators with positive squares and anticommutation generate a four-dimensional associative algebra containing the scalar, the two generators and their mixed product. The mixed product has negative square. Task 07 reconstructs this local pattern without using that language until after the derivation.

This is a useful calibration of the method. A surprising exotic structure at this stage would have been suspicious. Instead, opening one localized obstruction reproduces the standard associative completion of precisely the two-generator relation system that the previous tasks had forced.

12 A short mirror against Experiment 1

The independent relation to Experiment 1 is now sharper than it was in Part IV. That experiment approached Lorentz structure from weaker Euclidean and algebraic data and eventually reached an associative envelope in which products of independent directions need not remain ordinary vectors [7, 8]. Experiment 2 followed the opposite route. It started with Lorentz data, reconstructed the longitudinal split-complex sectors, recovered the spin-factor/Jordan product in $3 + 1$, proved that no in-carrier gluing can restore global associativity, and has now shown for two directions that opening carrier closure forces a new mixed-product direction.

The experiments were not allowed to supply these conclusions to one another. Consequently, the present agreement is not a proof of new mathematics but a nontrivial dependency cross-check. Schematically, the two routes are

$$\textsc{Experiment 1}: \quad \text{weaker real data} \longrightarrow \text{Lorentz/Jordan structure} \longrightarrow \text{associative envelope}, \tag{41}$$

$$\textsc{Experiment 2}: \quad \text{Lorentz data} \longrightarrow \text{longitudinal sectors} \longrightarrow \text{Jordan core}, \tag{42}$$

$$\text{Jordan core} \longrightarrow \text{closure obstruction} \longrightarrow \text{forced mixed direction}. \tag{43}$$

The routes are not identical and should not be made identical retrospectively. Their value lies in the fact that they converge on the same established boundary for different formal reasons.

13 Conservation of Difficulty: the obstruction has changed form

The sequence from Parts IV–VII can now be read as a controlled diagnostic chain.

All longitudinal $1 + 1$ sectors force the symmetric $3 + 1$ product, but leave cross-sector gluing freedom.

Clean-room Task 06: the orientation-sensitive gluing family is real, but no gluing whatsoever restores global associativity inside $\mathsf{Vec4}$.

Localized suspicion: old-carrier product closure itself may be the failed assumption.

Task 07 opens only that assumption. For two orthogonal directions, $P = AB$ is forced and $P \notin \iota(\mathsf{Vec4})$.

One new direction suffices: $\mathrm{span}\{1, A, B, P\}$ is the unique minimal four-dimensional associative closure of the two-generator subsystem.

Remaining difficulty: compatibility with the third inherited spatial direction, and hence nonvacuity of the full ambient hypothesis, is still open.

The important methodological point is that the difficulty has not disappeared merely because the two-generator obstruction has been repaired. It has become more sharply localized.

Before Task 07, the unknown was whether leaving the carrier would help at all. That is now answered positively for the selected pair. After Task 07, the unknown is whether the same ambient logic can accommodate the complete old rest space. The formal programme must not silently extrapolate the two-generator witness to the third direction. Doing so would be another version of the very manoeuvre that the Conservation-of-Difficulty discipline is designed to expose.

Methodological principle 2 (No extrapolation from a solved subsystem)**.** A consistent minimal closure of two generators does not establish a consistent closure of three generators. The unresolved third-direction compatibility is a new obligation, not a detail that may be filled in by recognition of the conventional answer.

14 Formal audit and proof boundary

The audited Task 07 artifact supplied with this work has SHA-256

`5c2cd1a2b0607f1e84f92d642fcd2e50f9604009be9ee9d179f82e5f0ec9e3f1`.

It uses Lean 4.28.0 and the same cumulative mathlib environment as the preceding experiment. Aristotle reports a successful full build of 8050 jobs with no Task 07 warnings or errors [2, 6, 10].

The reconstruction consists of fourteen Task 07 modules. Their essential source order is

$$\begin{aligned}&\texttt{SafeBase} \to \texttt{AmbientExtension} \to \texttt{Polarization} \to \texttt{TwoDirections} \to \texttt{MixedProduct}\\ &\to \texttt{NewDirection} \to \texttt{ForcedRelations} \to \texttt{GeneratedClosure} \to \texttt{Minimality}\\ &\to \texttt{DirectWitness} \to \texttt{Uniqueness} \to \texttt{SignAudit} \to \texttt{Identification} \to \texttt{Task07}.\end{aligned} \tag{44}$$

The conventional identification module is downstream of all reconstruction, closure, dimension, minimality and witness results. A mechanical scan of the reconstruction modules reported no conventional target-structure names.

The principal theorem inventory includes:

Theorem	Formal role
`task07_inherited_square_law`	retained diagonal from longitudinal sectors
`task07_polarization`	recovery of the old symmetric product
`task07_generator_relations`	$A^2 = B^2 = 1$, anticommutation
`task07_forced_relations`	relations involving $P = AB$, including $P^2 = -1$
`task07_new_direction`	P outside the embedded old carrier
`task07_linear_independence`	independence of $1, A, B, P$
`task07_normal_form`	unique four-coefficient representation
`task07_closure`	multiplicative closure of G
`task07_dimension`	$\dim_{\mathbb{R}} G = 4$
`task07_minimality`	least product-closed subspace containing $1, A, B$
`task07_witness`	direct associative four-dimensional model
`task07_uniqueness`	generator-preserving isomorphism class
`task07_sign_audit`	exchange/sign action on P
`task07_identification`	late $M_2(\mathbb{R})$ comparison

For each of the sixteen principal theorem wrappers, the reported axiom set is exactly

$$[\texttt{propext},\ \texttt{Classical.choice},\ \texttt{Quot.sound}], \tag{45}$$

with no ‘sorry‘, ‘admit‘, custom axiom or ‘@[implemented_by]‘ in Task 07.

Methodological caveat 1 (Verification boundary)**.** *The supplied archive reports a green Lean build. The present editorial audit inspects the source structure and proof dependencies, but this paper does not claim an independent second kernel build in a different Lean installation.*

15 What Task 07 proves – and what it does not

The distinction can be summarized compactly.

Established in Task 07	**Not established in Task 07**
The inherited old square law can coexist with associativity for the selected two-generator subsystem once old-carrier closure is removed.	Existence of one associative ambient extension embedding the entire original `Vec4` and preserving the old square law for all old vectors.
The mixed product $P = AB$ is forced outside the old carrier.	Compatibility with the third independent old spatial direction.
One additional independent direction suffices for the two-generator associative closure.	A complete associative closure of all three spatial generators.
The minimal closure has dimension four and is unique up to generator-preserving isomorphism.	Any extension of the Task 06 Lorentz-norm multiplicativity problem.
The late conventional algebra is $M_2(\mathbb{R})$ / split quaternions.	Any claim that the full physical or geometric $3 + 1$ problem has therefore been solved.

This claim boundary is not a weakness to be hidden. It is the next diagnostic location produced by the method.

16 The narrow question for Task 08

Task 07 has answered one question completely:

> For two orthogonal inherited spatial directions, what is the smallest associative closure once mixed products are allowed to leave the old carrier?

The answer is the four-dimensional closure derived above.

TASK 08 should now test only the next unavoidable compatibility question. It should add exactly one third old spatial direction C, orthogonal to both A and B, with the same inherited square law. It should retain the associative ambient setting but supply no conventional target algebra and no expected dimension.

The core questions should be:

1. What relations involving C follow from the inherited old square law and polarization?
2. Are the mixed products AC and BC already contained in the Task 07 closure G, or are new independent product directions forced?
3. After those products are derived, does associativity force an additional product such as $(AB)C$, and is it independent of the previously generated directions?
4. What is the smallest product-closed subspace generated by $1, A, B, C$ under these relations?
5. Can a direct finite-dimensional associative witness for this three-generator subsystem be constructed only after the relation and dimension classification is complete?

The task should stop there. In particular, it should not yet investigate quadratic norms, conjugations, involutions, spinors, matrix representations or arbitrary spatial dimension. It should not be told a conventional algebra name, a target dimension, or how many new mixed-product directions are expected.

This scope is deliberately narrow. Task 07 has shown that recognition of the standard answer is unnecessary for the two-generator case. Task 08 should preserve that discipline and ask only whether adding the third inherited direction forces another enlargement, and exactly how large the minimal associative closure then becomes.

17 Conclusion

Part VI ended with a universal negative statement: no in-carrier bilinear gluing of all longitudinal sectors can be globally associative. Part VII opens exactly the suspected failed condition and obtains a constructive answer for two independent directions.

The old sector information survives through the inherited square law and polarization. Two orthogonal embedded spatial generators therefore square to the unit and anticommute. Their mixed product is not optional: associativity forces it. Its square is negative, so the old carrier cannot contain it. The new product direction is linearly independent, but it does not trigger an uncontrolled proliferation. Together with the unit and the two old generators it forms an exactly four-dimensional, minimal, multiplicatively closed associative subsystem, unique up to generator-preserving isomorphism. Only after that derivation is complete does the standard identification $G \cong M_2(\mathbb{R})$ appear.

The result is established mathematics reached by an intentionally indirect route. That is the desired calibration. The formal search has not manufactured a novel structure to escape the Task 06 obstruction; it has reproduced the standard minimal associative response to two anticommuting square-$+1$ generators.

The Conservation of Difficulty has therefore moved again, but it has not vanished. The two-direction problem is closed. The full old $3 + 1$ embedding is not. The next unresolved connection is the third independent spatial direction. Task 08 should expose that connection with the same discipline: add one generator, derive rather than recognize, classify the minimal closure, and stop before the next layer of established structure is supplied as an answer.

References

[1] Aristotle (Harmonic). "Null-Sector Foundation – Task 06: Clean-Room Reconstruction of Orientation-Sensitive Gluing and Universal 3+1 Obstruction Audit". Cumulative formal experiment; Lean 4.28.0, mathlib revision 8f9d9cff6bd728b17a24e163c9402775d9e6a365; audited artifact supplied with this work. Sept. 1, 2026.

[2] Aristotle (Harmonic). "Null-Sector Foundation – Task 07: Minimal Associative Closure Forced by Two Orthogonal Longitudinal Sectors". Cumulative formal experiment, run cfbefbdb-6c5f-4448-bfb4-711535634ef7; Lean 4.28.0; audited archive SHA-256 5c2cd1a2b0607f1e84f92d642fcd2e50f9604009be9ee9d179f82e5f0ec9e3f1. Sept. 1, 2026.

[3] James Cockle. "On Certain Functions Resembling Quaternions, and on a New Imaginary in Algebra". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science.* 3rd ser. 33 (1848), pp. 435–439. URL: `https://zenodo.org/records/1431091/files/article.pdf`.

[4] James Cockle. "On a New Imaginary in Algebra". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science.* 3rd ser. 34 (1849), pp. 37–47. URL: `https://zenodo.org/record/1431095/files/article.pdf`.

[5] Pertti Lounesto. *Clifford Algebras and Spinors.* 2nd ed. Vol. 286. London Mathematical Society Lecture Note Series. Cambridge University Press, 2001. DOI: `10.1017/CBO9780511526022`.

[6] Leonardo de Moura and Sebastian Ullrich. "The Lean 4 Theorem Prover and Programming Language". In: *Automated Deduction – CADE 28.* Vol. 12699. Lecture Notes in Computer Science. Springer, 2021, pp. 625–635. DOI: `10.1007/978-3-030-79876-5_37`.

[7] Jan Seeck, OpenAI ChatGPT, and Aristotle. *From a Real Euclidean Three-Space to Lorentz Kinematics and Complex Structure, Version 9.* Dependency reconstruction in Experiment 1. Aug. 28, 2026. URL: `https://github.com/antaris82/Split-Complex-Lorentzian_Kinematics/blob/Intrinsic-Branching-from-a-Derived-Lorentz-Structure/Split_Complex_Lorentzian_Kinematics_v9.tex` (visited on 09/01/2026).

[8] Jan Seeck, OpenAI ChatGPT, and Aristotle. *Intrinsic Branching from a Derived Lorentz Structure, Version 30 – Comparative Final.* Final comparative null test of Experiment 1. Aug. 30, 2026. URL: `https://github.com/antaris82/Split-Complex-Lorentzian_Kinematics/blob/Intrinsic-Branching-from-a-Derived-Lorentz-Structure/Intrinsic_Branching_from_a_Derived_Lorentz_Structure_v30.tex` (visited on 09/01/2026).

[9] Jan Seeck, OpenAI ChatGPT, and Aristotle. "Split Complex Lorentzian Dynamics – Part VI: Clean-Room Replication, Universal 3+1 Obstructions, and the Localization of a Closure Deficit". Companion paper of Experiment 2, generated from the audited Task-06 artifact. Sept. 1, 2026.

[10] The mathlib Community. "The Lean Mathematical Library". In: *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs.* CPP 2020. ACM, 2020, pp. 367–381. DOI: `10.1145/3372885.3373824`.