Split Complex Lorentzian Dynamics

Part XII: Minimal Left Carriers, Axis-Relative Half-Angle Splitting, and the Persistence of State-Selection Freedom

A formally verified continuation of the dependency audit

Editor: Jan Seeck
AI Assistant: OpenAI ChatGPT (GPT-5.6 Sol)
Lean Agent: Aristotle (Harmonic)

3 September 2026

Abstract

Parts X and XI deliberately approached a state-like layer without importing a physical state space. Part X selected one old spatial axis inside the already reconstructed $3+1$ algebra and thereby exposed longitudinal and transverse sectors, full-angle algebra channels, half-angle internal action and two four-real-dimensional eigensectors. Part XI then removed the artificial restriction on the internal lift, proved that the previously found central plane is the exact center of the eight-dimensional associative algebra W, and classified every continuous implementation of the fixed axial automorphism family. The resulting infinitesimal generator splits into a visible noncentral component and an arbitrary central component invisible to conjugation. What remained unresolved was more basic: the entire algebra W was still being used as a provisional carrier. Nothing had shown whether smaller nonzero left-stable carriers exist, whether the Part-X half-angle sectors are such carriers, whether a minimal carrier is unique, or whether selecting one requires new information.

Part XII isolates this carrier problem and removes the remaining target bias. No spinor, particle, inner product, projective quotient, probability interpretation or field equation is permitted in the reconstruction core. A left carrier is defined neutrally as a real subspace $L \subseteq W$ closed under left multiplication by every element of W. For an arbitrary nonzero element $\psi \in W$, the smallest generated left carrier is

$$L_\psi = \{x\psi : x \in W\},$$

implemented formally as the range of right multiplication by ψ. Its dimension is then classified without beginning from the distinguished axis or from the Part-X half-angle sectors.

The first main result is a sharp dimension dichotomy. Every nonzero generated left carrier has real dimension exactly 4 or 8, and both values occur. The distinction is controlled by the central quadratic datum reconstructed in Part IX. If that datum is nonzero, ψ is invertible and $L_\psi = W$, so the generated carrier has dimension 8. If $\psi \neq 0$ lies in the common zero set of the two symmetric Task-IX quadratic branches, then L_ψ has dimension exactly 4. The lower-dimensional case is equivalent to ψ being a zero divisor. Thus the quadratic obstruction that first appeared as a multiplicative-extension problem in Part IX reappears, without new assumptions, as the exact carrier-reduction criterion.

Minimality is even more rigid. Every proper nonzero left carrier is already minimal and has real dimension 4. Equivalently, every left carrier is either 0, all of W, or a minimal four-dimensional carrier; there is no intermediate left-stable dimension and no nontrivial chain of proper carriers. Idempotency is not inserted as a carrier ansatz. Instead, every nonzero null generator is proved to admit a generator of the same carrier satisfying $e^2 = e$, with its unit and central coordinates forced by the algebra. The usual idempotent description therefore emerges as a normal form of the intrinsic null/zero-divisor classification rather than as a conventional construction imported at the start.

The second principal result concerns uniqueness. Minimal carriers do not form a singleton. An explicit real one-parameter family of pairwise distinct minimal subspaces is constructed,

and the full set of minimal carriers is proved infinite. Nevertheless every pair of minimal carriers is related by a right-multiplication map whose restriction is a bijective intertwiner for the left W-action. Hence the current reconstruction yields one intrinsic equivalence class but an infinite family of literal subspaces. The algebra therefore supplies a minimal carrier *type* but does not select one unique carrier. This is a new localization of residual difficulty: reducing W to a minimal state-like carrier does not eliminate selection freedom; it moves the unresolved datum from dimension to choice of representative.

Only after the global classification is complete is the distinguished axis A reintroduced. The result is a clean separation between globally existing carrier structure and axis-relative decomposition. Existence, the $\{4, 8\}$ dimension dichotomy, minimality, the zero-divisor criterion, infinitude, equivalence, central action and internal endomorphism algebra are all axis-independent. The axis contributes instead a pair of distinguished minimal carriers generated by $(1 \pm A)/2$, each fixed by the axial automorphism family, together with the Part-X half-angle sectors H_+ and H_-.

A crucial negative control then overturns the most tempting interpretation of Part X: the four-dimensional half-angle sectors are *not* the minimal left carriers. Neither H_+ nor H_- is stable under left multiplication by all of W; they are right ideals. For H_+, the exact left stabilizer is proved to be a six-dimensional subalgebra, and the sector is reducible under that weaker action because it contains a proper two-dimensional invariant central plane. The analogous exact stabilizer and reducibility theorem for H_- is not formalized in the Task-XII artifact and is recorded here as a genuine residual proof obligation rather than inferred from symmetry.

The relation between the global minimal carriers and the axis-relative half-angle sectors is nevertheless exact. Every minimal left carrier L meets each of H_+ and H_- in real dimension 2, and

$$L = (L \cap H_+) \oplus (L \cap H_-).$$

Thus the direction choice does not create the minimal carrier. A minimal four-dimensional carrier already exists in the global algebra before an axis is selected; the axis subsequently resolves it into two two-dimensional real pieces. This sharpens the reconstruction-order observation of Part X: a distinguished spatial direction reveals an internal decomposition of a pre-existing carrier rather than generating the carrier itself.

Part XII also separates two actions that are easily conflated. Left multiplication by the reference lift, and indeed by every Task-XI full lift, preserves every left carrier simply because a left carrier is stable under all of W. By contrast the algebra automorphism Φ_θ acts on the *family* of minimal carriers: it fixes the two axial members generated by $(1 \pm A)/2$ and moves explicit nonaxial members. The exact fixed locus of this action is not classified. The relation $\Phi_\theta(L) = L\,U(-\theta)$, interpreted as right multiplication by an inverse implementer on the subspace, makes the distinction precise.

Finally the central plane reappears in a third independent structural role. Every minimal carrier is automatically stable under the center, the induced action of S squares to minus the identity, and each minimal carrier contains exactly two independent central directions over the two-real-dimensional central algebra. More strongly, every real-linear endomorphism of a minimal carrier that commutes with the entire left W-action is exactly multiplication by a unique element of the inherited center. The internal endomorphism algebra is therefore two-dimensional over $\mathbb{R}$ and is isomorphic to the exact center previously reconstructed in Part XI. Only in the final comparison layer is this identified with the standard complex-scalar endomorphism algebra of a minimal left ideal of $M_2(\mathbb{C})$.

The mathematical endpoint is classical: after the reconstruction is complete, W is the familiar real algebra underlying $M_2(\mathbb{C})$, its minimal left carriers correspond to minimal left ideals, the derived idempotent generators correspond to rank-one idempotents, and the center acts as complex scalars. The point of Part XII is not novelty of that algebra but dependency localization. The carrier classification was obtained without using the conventional model to choose the answer, and it separates three previously conflated levels: the global minimal carrier family, the axis-relative half-angle decomposition, and the still-unresolved selection of one member as a physical state carrier.

The proposed Part XIII should therefore remain narrower than a spin-1/2 claim. It should first close the outstanding minus-sector stabilizer/reducibility symmetry, then classify the already-derived axial left action on an *arbitrary* minimal carrier and on its two axis-relative

$2+2$ intersections. It should determine which half-angle properties are carrier-independent, how the two-parameter central lift freedom of Part XI acts on such a carrier, whether the intrinsic equivalences between minimal carriers intertwine the axial action, and exactly which additional datum—if any—is needed to select a representative. Only after that classification should a late conventional comparison ask whether the resulting one-parameter carrier action is the restriction of a standard spinorial representation. Full spatial rotation groups, helicity, particles, field equations and physical dynamics remain beyond Task XIII.

Keywords: minimal left carrier; zero divisor; idempotent normal form; carrier family; half-angle sector; distinguished axis; central action; internal endomorphism; formal verification; Lean 4; conservation of difficulty.

Contents

1 Why Part XII moves from a provisional carrier to a carrier classification **4**

2 Inherited algebra and neutral carrier definitions **4**

3 The exact dimension dichotomy: only four or eight **5**

4 Minimality, properness, and zero divisors **6**

5 Idempotency emerges as a generator normal form **7**

6 An infinite family, but one intrinsic equivalence class **7**

7 Axis audit: what existed before a direction was chosen? **8**

8 The half-angle sectors fail the left-carrier test **9**

9 Every minimal carrier splits $2+2$ relative to the selected axis **9**

10 Left lift action versus automorphism action on the carrier family **10**

11 The center acts internally on every minimal carrier **11**

12 The exact internal endomorphism algebra **11**

13 Late conventional identification **12**

14 Comparison with Experiment 1 **12**

15 Formal verification, source order, and proof status **13**

16 Open obligations after Task XII **14**

17 Outlook: Task XIII should classify the axial action on an arbitrary minimal carrier **15**

18 Conclusion **16**

1 Why Part XII moves from a provisional carrier to a carrier classification

The preceding two parts established enough internal structure to make the next unresolved layer unavoidable. Part X showed that choosing one spatial direction inside the already reconstructed $3+1$ algebra separates longitudinal and transverse transformation channels and admits an internal half-angle implementation. Part XI then hardened the implementation problem on the full eight-dimensional algebra W, obtaining the exact center

$$Z = \mathrm{Center}(W) = \mathrm{span}_{\mathbb{R}}\{1, S\}, \qquad S^2 = -1, \tag{1}$$

and the complete continuous residual lift family [14, 15, 2].

Those results still left a methodological asymmetry. Algebra automorphisms act naturally on W, and internal implementers are themselves elements of W, so using the full regular left carrier was a convenient first probe. Convenience, however, does not imply minimality. In particular, the two Part-X eigensectors

$$H_+ = \{\psi \in W : A\psi = \psi\}, \qquad H_- = \{\psi \in W : A\psi = -\psi\} \tag{2}$$

had real dimension four. Since familiar conventional models also contain four-real-dimensional minimal carriers, there was an obvious danger of identifying the sectors with the desired state carrier before proving the relevant closure property.

Task XII is designed precisely to prevent that shortcut. It forbids conventional module targets in the reconstruction core and asks first for all left-stable real subspaces of the independently reconstructed algebra. Only after that global classification is complete may the axis and half-angle sectors re-enter [3].

Methodological principle 1 (Carrier before representation name)**.** A subspace that has the expected dimension or transformation phase is not a candidate minimal state carrier until the inherited algebra has proved the required stability and minimality. Dimensional coincidence is not a substitute for closure.

This hardening step follows the same Conservation-of-Difficulty logic used throughout the experiment. If a four-dimensional carrier can be derived without the distinguished axis, then the axis cannot be credited with producing that carrier. If the half-angle sector has dimension four but fails left stability, then its dimension does not localize the carrier structure. The purpose is therefore not only to find smaller spaces but to determine which previous ingredients were actually necessary.

2 Inherited algebra and neutral carrier definitions

The carrier algebra is the Task-VIII associative closure with ordered basis

$$1, A, B, C, P = AB, Q = AC, R = BC, S = ABC, \tag{3}$$

and the previously proved multiplication table. Tasks IX–XI supply the central quadratic data, exact center and axial structures but no new product [13, 11, 15].

Task XII introduces only a new classification predicate.

Definition 1 (Left carrier)**.** A real subspace $L \subseteq W$ is a *left carrier* if

$$x\psi \in L \qquad \text{for every } x \in W \text{ and } \psi \in L. \tag{4}$$

It is nonzero if $L \neq 0$, proper if additionally $L \neq W$, and minimal if its only left-stable subspaces are 0 and itself.

This definition deliberately avoids representation-theoretic terminology. It states exactly the closure property needed for a carrier of the left action and nothing more. No inner product, normalization, projective identification or physical interpretation is included.

For arbitrary $\psi \in W$, define the generated carrier

$$L_\psi := \{x\psi : x \in W\}. \tag{5}$$

Formally this is the range of the real-linear map $R_\psi : x \mapsto x\psi$. Associativity immediately gives left stability:

$$y(x\psi) = (yx)\psi \in L_\psi. \tag{6}$$

Since $1\psi = \psi$, the generator itself lies in L_ψ, and every left carrier containing ψ contains L_ψ. Thus equation (5) is the intrinsic smallest left carrier generated by ψ.

The relevant Task-IX quadratic datum is retained in coefficient form. Writing the two real coefficients as $n_{\mathrm{Re}}(\psi)$ and $n_{\mathrm{Sc}}(\psi)$, Task XII defines

$$\mathrm{Null}(\psi) \iff n_{\mathrm{Re}}(\psi) = 0 = n_{\mathrm{Sc}}(\psi). \tag{7}$$

Because the two multiplicative Task-IX branches vanish simultaneously, this definition is branch-symmetric: neither N_+ nor N_- is preferred.

3 The exact dimension dichotomy: only four or eight

The first question is intentionally global: for a completely arbitrary nonzero generator ψ, how large can equation (5) be?

Theorem 1 (Generated-carrier dimension dichotomy)**.** *For every nonzero $\psi \in W$,*

$$\dim_{\mathbb{R}} L_\psi \in \{4, 8\}. \tag{8}$$

Both dimensions occur. More precisely,

$$\neg\,\mathrm{Null}(\psi) \Longrightarrow L_\psi = W, \qquad \dim_{\mathbb{R}} L_\psi = 8, \tag{9}$$

$$\mathrm{Null}(\psi),\ \psi \neq 0 \Longrightarrow \dim_{\mathbb{R}} L_\psi = 4. \tag{10}$$

The non-null branch is conceptually simple but structurally important. The inherited multiplicative quadratic relation supplies an explicit inverse whenever the central quadratic value is nonzero. Hence a non-null element is invertible. If ψ has inverse ψ^{-1}, then every $y \in W$ has the form

$$y = (y\psi^{-1})\psi, \tag{11}$$

so $L_\psi = W$.

The null branch contains the real work. Multiplicativity of the central quadratic datum shows that every element of L_ψ remains null whenever ψ is null. This constrains the generated subspace to a totally isotropic sector of the inherited split quadratic form. The formal proof projects to four coordinates with positive sign in n_{Re} and proves injectivity of that projection on the generated null carrier, giving the upper bound

$$\dim_{\mathbb{R}} L_\psi \leq 4. \tag{12}$$

The matching lower bound is obtained only after a generator normal form is derived; section 5 describes this step.

The resulting theorem is stronger than the original design requirement. There is no two-dimensional generated left carrier and no six-dimensional generated left carrier. The algebra itself enforces a jump directly from four to eight.

Remark 1 (The Part-IX obstruction becomes a carrier criterion)**.** *The same central quadratic datum first appeared because a real-valued multiplicative extension of the Lorentz form failed and a central two-dimensional codomain became necessary. Part XII shows that its zero set also exactly marks the elements whose generated left action fails to fill all of W. The old obstruction therefore reappears as the precise reduction locus for candidate carriers rather than being discarded after Part IX.*

4 Minimality, properness, and zero divisors

The dimension dichotomy leaves open whether a four-dimensional generated carrier could contain a still smaller nonzero left carrier. Task XII closes this possibility completely.

Theorem 2 (Proper equals minimal)**.** *For every left carrier $L \subseteq W$,*

$$L \text{ is proper and nonzero} \quad \Longleftrightarrow \quad L \text{ is minimal.} \tag{13}$$

Every such carrier has real dimension 4*.*

Equivalently, the complete trichotomy is

$$L = 0, \qquad L = W, \qquad \text{or} \qquad L \text{ is minimal with } \dim_{\mathbb{R}} L = 4. \tag{14}$$

There is no longer proper chain to classify.

The proof uses the generated-carrier result rather than a pre-built simple-module theorem. A proper left carrier cannot contain an invertible element, because any invertible $\psi \in L$ would generate all of W. Thus every element of a proper carrier lies on the null locus. Choose any nonzero $\psi \in L$. Then $L_\psi \subseteq L$ and has dimension four. The same isotropy bound constrains L itself to dimension at most four, forcing

$$L = L_\psi. \tag{15}$$

Any nonzero left-stable subspace of L repeats the same argument and must equal L.

The minimal-generator criterion is exact.

Theorem 3 (Minimal generator criterion)**.** *For nonzero $\psi \in W$, the following are equivalent:*

(i) L_ψ *is a minimal left carrier;*

(ii) $\mathrm{Null}(\psi)$*;*

(iii) both Task-IX quadratic branches vanish on ψ*;*

(iv) ψ *is a left zero divisor;*

(v) ψ *is a right zero divisor.*

The zero-divisor formulation is particularly useful because it contains no conventional matrix rank or determinant language. A nonzero ψ is on the null locus exactly when there exists nonzero ϕ with $\psi\phi = 0$, and equivalently when there exists nonzero χ with $\chi\psi = 0$.

Thus the first carrier reduction is not caused by the distinguished axis, a choice of norm, or a hand-selected projector. It is forced exactly at the algebra's intrinsic zero-divisor locus.

5 Idempotency emerges as a generator normal form

A conventional route to minimal left ideals of matrix or Clifford algebras begins by choosing an idempotent and considering the ideal it generates. Such a start would have defeated the purpose of Task XII because it would encode part of the expected answer in the input.

Task XII therefore begins with arbitrary ψ and only later asks whether a useful normalized generator can be derived.

Theorem 4 (Derived self-product generator)**.** *If* $\psi \neq 0$ *and* $\mathrm{Null}(\psi)$*, then there exists* $e \in W$ *such that*

$$e^2 = e, \qquad e \neq 0, \qquad \mathrm{Null}(e), \qquad L_e = L_\psi. \tag{16}$$

Moreover every nonzero self-product element is either the unit or lies on the null locus; in the latter case its unit coordinate is forced to $1/2$ *and its* S*-coordinate to* 0*.*

The idempotent equation is therefore an output of carrier classification. It is used to make the lower bound in equation (10) exact. Right multiplication by e is an idempotent linear map, so its range dimension equals its trace. The formal coordinate trace formula reduces this to the derived unit coordinate $e_0 = 1/2$, producing

$$\dim_{\mathbb{R}} L_e = 4. \tag{17}$$

Methodological principle 2 (Do not normalize before the locus is known)**.** A useful projector-like normal form may emerge after the null/zero-divisor locus has been derived. Beginning with such an element would erase the distinction between the existence of a minimal carrier and the choice of a convenient generator for it.

This is the carrier analogue of the normalization warning from Part XI. A condition that is harmless as a derived normal form can become a Fangschaltung if it is inserted before the residual freedom has been classified.

6 An infinite family, but one intrinsic equivalence class

Finding one minimal carrier would still leave open whether the inherited algebra singles it out. Task XII therefore treats literal equality and structural equivalence separately.

An explicit real one-parameter family $e(u)$ of null self-product generators is constructed, and the map

$$u \longmapsto L_{e(u)} \tag{18}$$

is proved injective. Hence the set of minimal carriers is infinite.

Theorem 5 (Non-uniqueness of minimal carriers)**.** *There are infinitely many pairwise distinct minimal left carriers. Every one has real dimension four.*

The next question is whether these are genuinely different carrier types. Task XII defines a neutral equivalence relation on minimal carriers using right multiplication. For minimal carriers L, L', one asks for an element $r \in W$ such that

$$T_r : L \to L', \qquad T_r(\psi) = \psi r, \tag{19}$$

is a bijection. Associativity automatically gives the intertwining law

$$T_r(x\psi) = xT_r(\psi) \qquad (x \in W,\ \psi \in L). \tag{20}$$

The formal artifact proves that such a restricted right-multiplication bijection exists between every pair of minimal carriers. It does *not* prove that the chosen r must itself be an invertible element of all of W; the correct statement concerns the restricted map equation (19). This distinction is retained in the present paper.

Theorem 6 (Single carrier equivalence class)**.** *All minimal carriers are mutually equivalent under bijective right-multiplication intertwiners of the form equation* (19)*. Consequently the reconstruction contains one carrier type but infinitely many literal carrier subspaces.*

The Conservation-of-Difficulty verdict is therefore not “unique minimal state space”. It is

$$\boxed{\text{FAMILY OF EQUIVALENT MINIMAL CARRIERS.}} \tag{21}$$

The algebra has determined dimension, minimality and equivalence class, but not a unique representative. If later physics requires one concrete member, a selection mechanism remains to be explained.

7 Axis audit: what existed before a direction was chosen?

Only after sections 3, 4 and 6 are complete does Task XII re-import the distinguished axis A and the Task-X automorphism family. This source ordering is important because it turns the axis into a controlled negative test rather than an implicit ingredient of the carrier construction.

The axis-independent results are:

- existence of proper nonzero left carriers;
- the exact dimension set $\{4, 8\}$ for generated carriers;
- minimal nonzero dimension four;
- the null/zero-divisor generator criterion;
- infinitude of the minimal-carrier family;
- pairwise carrier equivalence;
- automatic action of the exact center;
- the internal endomorphism classification of section 12.

None requires A.

The axis does, however, produce two particularly transparent null self-product elements,

$$e_+ = \frac{1+A}{2}, \qquad e_- = \frac{1-A}{2}. \tag{22}$$

Their generated left carriers are minimal, distinct, and fixed by the axial automorphism family Φ_θ.

Proposition 1 (What the axis selects)**.** *The distinguished axis does not create the minimal-carrier class. It selects at least two distinguished members of an already existing infinite family, namely L_{e_+} and L_{e_-}, both fixed by every Φ_θ.*

The word “at least” matters. Task XII proves that explicit nonaxial carriers move under Φ_θ, but it does not classify the full fixed locus of the automorphism action. It therefore does not prove that equation (22) are the only globally fixed minimal carriers.

This is a sharper version of the structural observation that motivated Part X. Selecting a direction can make pre-existing structure distinguishable without being responsible for its existence.

8 The half-angle sectors fail the left-carrier test

The strongest negative control in Task XII concerns the Part-X sectors equation (2). Their real dimension is four, exactly the minimal left-carrier dimension. Nevertheless they fail the defining closure property.

Theorem 7 (Half-angle sectors are not left carriers)**.** *Neither H_+ nor H_- is stable under left multiplication by all of W.*

An explicit counterexample uses a transverse generator: multiplying a plus-sector element on the left by B leaves the plus sector. At the same time both sectors are stable under *right* multiplication by arbitrary elements of W. They are therefore right ideals rather than the left carriers classified above.

This resolves a potentially misleading dimensional coincidence:

$$\dim_{\mathbb{R}} H_+ = \dim_{\mathbb{R}} H_- = 4 \qquad \not\Longrightarrow \qquad H_+, H_- \text{ are minimal left carriers.} \tag{23}$$

The exact weaker structure is partially classified. For the plus sector, define the left stabilizer

$$\mathrm{Stab}_L(H_+) = \{x \in W : xH_+ \subseteq H_+\}. \tag{24}$$

Task XII proves that this is a six-dimensional subalgebra, expressible as the kernel of a cross-compression map. It contains $1, S, A$ but not B.

Theorem 8 (Plus-sector reducibility)**.** *Under its exact six-dimensional left stabilizer, H_+ is reducible. In particular*

$$Ze_+ = \mathrm{span}_{\mathbb{R}}\{e_+, Se_+\} \tag{25}$$

is a proper nonzero two-dimensional invariant subspace.

This is another place where the hierarchy becomes clearer. The four-dimensional half-angle sector is not irreducible merely because it was a natural eigenspace of the selected-axis problem.

Methodological caveat 1 (Formal asymmetry remaining in Task XII)**.** *The Task-XII artifact proves the corresponding negative left-carrier statement, right-ideal property and lift stability for H_-, but it does not formalize the exact six-dimensional left stabilizer or an explicit reducibility theorem for the minus sector. The expected $+/-$ symmetry is not promoted to a theorem here. This is the principal small hardening obligation carried forward from Task XII.*

For this reason the most precise status is *core pass with one local symmetry obligation*, rather than an unqualified claim that every requested plus/minus stabilizer theorem has been closed.

9 Every minimal carrier splits $2 + 2$ relative to the selected axis

Although the half-angle sectors are not the global minimal carriers, their intersection with every minimal carrier is rigid.

Theorem 9 (Universal axis-relative splitting)**.** *For every minimal left carrier L,*

$$\dim_{\mathbb{R}}(L \cap H_+) = 2, \qquad \dim_{\mathbb{R}}(L \cap H_-) = 2, \tag{26}$$

and

$$L = (L \cap H_+) \oplus (L \cap H_-). \tag{27}$$

This theorem is conceptually stronger than identifying one special four-dimensional carrier. It applies to the entire infinite family obtained before A was introduced.

The reconstruction order is therefore

$$\begin{array}{c} \text{global associative algebra } W \\ \downarrow \\ \text{infinite family of minimal four-real-dimensional left carriers} \\ \downarrow \quad \text{choose axis } A \\ \text{axis-relative sectors } H_+, H_- \\ \downarrow \\ L = (L \cap H_+) \oplus (L \cap H_-), \quad 4 = 2 + 2. \end{array} \tag{28}$$

The distinguished direction therefore does not produce a state-like carrier. It resolves each pre-existing minimal carrier into two axis-relative pieces.

This distinction is directly relevant to later physical interpretation. If a momentum or relative-motion direction eventually supplies the physical mechanism selecting A, the algebraic carrier need not be generated by that momentum. Rather, the direction could select a decomposition of a carrier already allowed by the global algebra. Part XII does not identify the physical selector; it only separates these logical roles.

10 Left lift action versus automorphism action on the carrier family

Part XI made a sharp distinction between the fixed algebra automorphism Φ_θ and the nonunique internal implementers $U(\theta)$. Task XII now shows that the distinction persists at the carrier level.

Let L be any left carrier. Since $U(\theta) \in W$, left stability immediately gives

$$U(\theta)L \subseteq L. \tag{29}$$

For a full lift the inverse $U(-\theta)$ exists, so equality follows.

Theorem 10 (Full-lift stability)**.** *Every minimal left carrier is preserved by left multiplication with $U_{\mathrm{ref}}(\theta)$ and, more generally, by every Task-XI full lift $U(\theta)$:*

$$U(\theta)L = L. \tag{30}$$

The entire central residual freedom of Part XI is invisible to the question of carrier preservation.

This result is structurally simple once the correct carrier definition is known. Its importance is diagnostic: no additional minimal-carrier selection is hidden in the choice among the Task-XI internal lifts.

The automorphism Φ_θ behaves differently. It maps minimal carriers to minimal carriers,

$$\Phi_\theta(L_\psi) = L_{\Phi_\theta(\psi)}, \tag{31}$$

and hence acts on the whole family. The two axial carriers from equation (22) are fixed, whereas an explicit equatorial carrier is moved at $\theta = \pi/2$.

For any full lift one obtains the useful relation

$$\Phi_\theta(L) = L\,U(-\theta), \tag{32}$$

where the right-hand side denotes the image of L under right multiplication by the inverse implementer. Equation equation (32) explains why left multiplication by U can preserve every carrier while the corresponding algebra automorphism moves a carrier through the family.

Methodological principle 3 (Do not conflate action on elements with action on carrier labels). An internal implementer may preserve a chosen left carrier under left multiplication while the induced algebra automorphism permutes the family of such carriers. These are different actions and answer different structural questions.

The exact orbit decomposition and exact fixed locus of Φ on the full carrier family remain open.

11 The center acts internally on every minimal carrier

The exact center derived in Part XI now acquires a third structural role. Since every minimal carrier is left-stable under all of W, it is automatically stable under

$$Z = \mathrm{span}_{\mathbb{R}}\{1, S\}. \tag{33}$$

Define the induced real-linear map

$$\mathcal{S}_L : L \to L, \qquad \mathcal{S}_L(\psi) = S\psi. \tag{34}$$

Because $S^2 = -1$ in W,

$$\mathcal{S}_L^2 = -\,\mathrm{id}_L\,. \tag{35}$$

No conventional complex scalar is introduced to obtain this relation.

Task XII further proves an intrinsic central-rank statement. For each minimal carrier there exist two elements $\psi_1, \psi_2 \in L$ such that every $\psi \in L$ has a unique expression

$$\psi = (a + bS)\psi_1 + (c + dS)\psi_2, \qquad a, b, c, d \in \mathbb{R}. \tag{36}$$

Thus L has real dimension four but exactly two independent directions over the two-real-dimensional central algebra.

This is a particularly useful reconstruction-order result. The central square-root-of-minus-one structure appeared before any minimal carrier was constructed. Once the carrier is derived, that same center supplies its internal scalar directions automatically; no separate complexification of the carrier is needed.

12 The exact internal endomorphism algebra

The strongest carrier-internal theorem asks for every real-linear map

$$T : L \to L \tag{37}$$

that commutes with the complete left action:

$$T(x\psi) = xT(\psi) \qquad (x \in W,\ \psi \in L). \tag{38}$$

Task XII does not assume that such maps are real scalars, central scalars or anything else.

Theorem 11 (Internal endomorphism rigidity). *Let L be a minimal left carrier. Every real-linear $T : L \to L$ satisfying equation (38) is uniquely of the form*

$$T(\psi) = z\psi \qquad \textit{for a unique } z \in Z. \tag{39}$$

Composition corresponds to multiplication in Z. Consequently

$$\mathrm{End}_W(L) \cong Z, \qquad \dim_{\mathbb{R}} \mathrm{End}_W(L) = 2. \tag{40}$$

The proof uses the derived self-product generator rather than an external commutant theorem. Evaluating T on the generator fixes a candidate central multiplier; compatibility with every left multiplication forces it into the exact center; the relation then extends to the whole generated carrier.

Across Parts IX–XII, the same two-dimensional central structure has therefore appeared in three logically distinct places:

1. as the minimal codomain required for a globally multiplicative quadratic extension;
2. as the exact invisible residual freedom of internal implementations of a fixed algebra automorphism;
3. as the exact internal endomorphism algebra of every minimal left carrier.

These repetitions are theorem-level dependencies inside Experiment 2, not yet a physical interpretation.

13 Late conventional identification

Only after the complete carrier classification compiles does Task XII import the previously established conventional identification of the eight-dimensional real algebra. In this final layer one may recognize

$$W \cong M_2(\mathbb{C}) \tag{41}$$

as real associative algebras [6, 4].

Under that comparison:

intrinsic Experiment-2 result	**standard comparison, only after reconstruction**
L_e for a null self-product generator	a minimal left ideal $M_2(\mathbb{C})P$
$e^2 = e$, $e \neq 0, 1$ derived as normal form	a nontrivial rank-one idempotent
$\dim_{\mathbb{R}} L = 4$	a two-complex-dimensional column-type carrier
$S^2 = -1$ acting on L	multiplication by the conventional complex unit
$\mathrm{End}_W(L) \cong Z$, real dimension 2	complex scalar endomorphisms
infinite equivalent minimal-carrier family	the familiar family of isomorphic minimal left ideals

These are standard algebraic facts once equation (41) is known. They are not claimed as new theorems about matrix algebras. The experimental content is that the left-carrier classification was obtained before using equation (41) as a construction guide.

The conventional literature often introduces spinorial carriers directly through minimal left ideals of Clifford algebras or matrix representations [6, 1]. Task XII deliberately reverses that order: the algebra and center were reconstructed first, the carrier was then classified intrinsically, and only afterward is the familiar language permitted.

Methodological caveat 2 (No physical state-space or spin claim)**.** *A minimal algebraic left carrier is not yet a physical Hilbert space, one-particle state space, wavefunction space or fermionic state space. No positive inner product, probability rule, projective equivalence, full spatial rotation representation, boost representation or field equation has been derived in Task XII. The late comparison identifies a standard algebraic carrier, not a physical particle.*

14 Comparison with Experiment 1

The independent Experiment 1 remains useful as a blind comparator, but Part XII also marks an important asymmetry between the two reconstruction programs.

Experiment 1 began from positive Euclidean three-space data and reconstructed, through a spin-factor/Jordan stage, Lorentz structure and an associative Clifford envelope. In that route a central pseudoscalar with square -1 emerged, yielding a complex structure up to conjugation [8, 9]. Experiment 2 began from Lorentz-sector data, reconstructed the same spin-factor boundary by gluing longitudinal sectors, encountered the universal associative-closure obstruction, and independently reached the eight-dimensional associative algebra with central $S^2 = -1$ [10, 12, 13].

Parts IX–XI then pushed Experiment 2 beyond the comparison point previously used with Experiment 1: the central plane became the codomain of the multiplicative quadratic extension, the exact center of W, and the exact kernel of invisible implementer-generator freedom. Part XII adds yet another layer by proving that the same center is the complete internal endomorphism algebra of every minimal left carrier.

What should *not* be claimed is equally important. Experiment 1 did not independently perform the Part-XII minimal-carrier classification. Therefore the new statements

$$\dim_{\mathbb{R}} L = 4, \qquad \mathrm{End}_W(L) \cong Z, \qquad L = (L \cap H_+) \oplus (L \cap H_-) \tag{42}$$

are Experiment-2 results, not a second independent convergence theorem.

The comparison nevertheless sharpens the methodological picture:

Experiment 1	**Experiment 2**
Euclidean $\mathbb{R}^3$ datum	Lorentz longitudinal-sector datum
spin-factor/Jordan structure reconstructed	spin-factor/Jordan symmetric product reconstructed by gluing
Lorentz structure derived	Lorentz structure inherited at the start
Clifford/associative envelope reached	associative envelope forced by closure obstruction
central square-root-of-minus-one structure reached	central $S^2 = -1$ reached independently
no carrier classification in the compared route	minimal left carriers now classified intrinsically
no analogue of the Part-X axis-relative carrier split	every minimal carrier splits $2+2$ after axis selection

Thus the two experiments still calibrate one another at the shared algebraic boundary, while Part XII deliberately explores a layer for which the blind comparator has not supplied an independent result. This is preferable to pretending that every later theorem has already been cross-validated.

15 Formal verification, source order, and proof status

The Task-XII Aristotle artifact uses Lean 4.28.0 and Mathlib revision

$$\texttt{8f9d9cff6bd728b17a24e163c9402775d9e6a365}. \tag{43}$$

The supplied archive records thirteen Task-XII modules in the following order [3, 7, 16, 17]:

$$\begin{gathered}\texttt{SafeBase} \to \texttt{LeftCarrier} \to \texttt{GeneratedCarrier}\\ \to \texttt{DimensionClassification} \to \texttt{MinimalCarrier} \to \texttt{CarrierFamily}\\ \to \texttt{AxisRelation} \to \texttt{HalfAngleRelation} \to \texttt{LiftStability}\\ \to \texttt{CentralAction} \to \texttt{CarrierEndomorphisms}\\ \to \texttt{Identification} \to \texttt{Task12}.\end{gathered}$$

The source-order firewall is substantive. The half-angle sectors first enter only after the complete global dimension, minimality and carrier-family classification. Task-XI reference/full-lift structure enters still later. Conventional matrix/complex identification enters only in `Identification`.

The formal build report states

```
lake build RequestProject.NullSectorTask12.Task12
[8102/8102] Built RequestProject.NullSectorTask12.Task12

lake build
Build completed successfully (8146 jobs).
```

A mechanical scan reports no `sorry`, `admit` or custom axiom in the Task-XII modules. The sixteen principal theorems reported by `#print axioms` depend only on

$$\texttt{propext}, \quad \texttt{Classical.choice}, \quad \texttt{Quot.sound}. \tag{44}$$

Proof boundary 1 (Verification boundary for this paper)**.** *The build counts and axiom lists above are taken from the supplied Task-XII audit. The archive, source tree, audit, theorem statements and import ordering were inspected for this paper. The complete pinned Lean/Mathlib build was not rerun in the paper-generation environment. This paper also corrects two prose-level overstatements from the artifact summary: carrier equivalence is stated as a* bijective restricted right-multiplication intertwiner*, not as multiplication by an algebra element proved invertible in all of W; and the half-angle sectors are described as right ideals and axial-lift-stable, not as being preserved* only *by the axial lift.*

The most precise current status is

$$\boxed{\text{TASK XII: CORE PASS, WITH ONE LOCAL } H_- \text{ HARDENING OBLIGATION.}} \tag{45}$$

The core dimension, minimality, family, axis, intersection, lift, center and endomorphism results are formally closed. The missing minus-sector stabilizer/reducibility symmetry is isolated and does not alter those theorems.

16 Open obligations after Task XII

The Task-XII audit records several deliberately unresolved questions. They fall into two classes: one local closure issue that should be handled immediately, and broader classification questions that are not blockers for the main carrier result.

1. **Minus-sector stabilizer and reducibility.** The exact six-dimensional stabilizer and explicit reducibility theorem are formalized for H_+ but not H_-. This is the nearest proof obligation and should be closed before any later theorem relies on a symmetric plus/minus statement.
2. **Global parameterization of the carrier family.** Infinitude, dimension, normal-form generation and one equivalence class are proved, but no exhaustive manifold or homogeneous-space parameterization of all minimal carriers is given.
3. **Exact Φ-fixed locus.** Two axial fixed carriers are proved and moving nonaxial carriers are exhibited, but the complete fixed set is not classified.
4. **Generator nonuniqueness inside one carrier.** Every minimal carrier admits a null self-product generator, but the full set of generators producing the same carrier is not classified.
5. **Carrier-equivalence outside the minimal family.** The intrinsic right-intertwiner relation has the needed properties on minimal carriers; no global equivalence-relation theorem on arbitrary subspaces is attempted.
6. **Selection datum.** The inherited structure determines one carrier equivalence class but no unique representative. No new selection datum is offered.
7. **Physical state structure.** No positivity, norm, projectivization, probability rule, full rotation group, boost action on the minimal carrier, mass relation, Hamiltonian or field equation is part of Task XII.

The important point is that these difficulties are now separated. "Find the state space" has become several precise questions: classify the minimal algebraic carrier, classify its axis-relative decomposition, classify its internal automorphisms, and only then ask which additional structures are required for a physical state interpretation.

17 Outlook: Task XIII should classify the axial action on an arbitrary minimal carrier

Part XII changes the appropriate next question. Before this carrier hardening, one might have tried to interpret the Task-X sectors directly as state spaces. That route is now ruled out: H_+ and H_- are not left carriers, whereas every genuine minimal carrier crosses both sectors in a rigid $2+2$ split.

Task XIII should therefore remain an intrinsic transformation-classification task. It should not begin by asking for spin-1/2, a fermion, a Pauli matrix representation or a two-component complex wavefunction. A suitable neutral question is:

> For an arbitrary minimal left carrier L, what transformation structure is forced by the already-derived axial automorphism/lift system on L and on the two real two-dimensional intersections $L \cap H_+$ and $L \cap H_-$?

The task should begin with a short hardening prelude that closes the outstanding H_- stabilizer/reducibility theorem. It should then proceed in a source order such as

$$\begin{array}{c}
\text{arbitrary minimal carrier } L \\
\downarrow \\
\text{complete } H_+/H_- \text{ stabilizer symmetry} \\
\downarrow \\
L = (L \cap H_+) \oplus (L \cap H_-) \\
\downarrow \\
\text{restrict the inherited left lift action to } L \\
\downarrow \\
\text{classify its action on the two 2D real intersections} \\
\downarrow \\
\text{compare different minimal carriers through intrinsic right intertwiners} \\
\downarrow \\
\text{classify the effect of the full central } (\alpha, \beta) \text{ lift freedom} \\
\downarrow \\
\text{late conventional identification only.}
\end{array} \tag{46}$$

Several negative controls are essential.

First, the half-angle behavior found in Part X must be re-derived on an arbitrary minimal carrier rather than inferred from the special regular carrier W. Task XIII should determine whether the two intersections in equation (27) carry universal opposite half-angle factors, whether this depends on the representative L, and whether the right-multiplication equivalences of equation (19) automatically intertwine the restricted axial action. Associativity suggests a strong equivalence result, but it should be proved rather than assumed.

Second, Part XI showed that the internal lift has two central continuous residual parameters before additional quadratic compatibility is imposed. Task XIII should retain the complete family long enough to determine exactly how those central factors act on a minimal carrier and on its two sector intersections. It must not silently choose U_{ref} merely because its $2\pi/4\pi$ behavior is familiar.

Third, the task should distinguish carrier selection from carrier transformation. Even if every minimal carrier carries an equivalent axial action, the existence of an infinite family means a representative remains unselected. Task XIII should state explicitly whether the axial data reduce this family further, whether only the two axial carriers are special, or whether no unique choice emerges.

Only after this intrinsic classification is complete may a final comparison module use the standard language of a minimal left ideal of $M_2(\mathbb{C})$ and ask whether the one-parameter action coincides with the restriction of the familiar spinorial rotation action. Even a positive comparison would establish only an algebraic representation correspondence for one axial subgroup. It would not yet derive a physical fermion, full Spin(3) or Lorentz representation, helicity, a Dirac equation, an inner product or a quantum probability structure [6, 18, 5].

A later task, not Task XIII, could then ask whether the independently reconstructed one-axis action extends canonically to the full family of spatial directions. That would be the appropriate place to test whether the global $3+1$ algebra plus its minimal carriers forces a full rotation/double-cover structure or whether additional orientation and compatibility data reappear. Keeping this extension separate preserves the same blind-reconstruction discipline that made the present $4 \to 2+2$ result informative.

Methodological principle 4 (Task XIII boundary)**.** Derive the axial action on the already-derived minimal carrier before asking for the conventional name of that action. Do not use the known spinorial answer to choose the carrier, normalize the lift, or complete the spatial rotation group.

18 Conclusion

Part XII resolves the most basic state-carrier ambiguity left by Parts X and XI. The full eight-dimensional algebra W is not minimal as a left carrier. Every nonzero element generates either all of W or a four-dimensional minimal carrier, with no other dimension possible. The exact criterion is the vanishing of the inherited central quadratic datum, equivalently the intrinsic zero-divisor condition. Every proper nonzero left carrier is already minimal, so the carrier lattice has a sharp 0/4/8 structure rather than a long hierarchy.

The derivation does not begin from an idempotent. Instead a null self-product generator is recovered as a normal form for every minimal carrier. This reverses the standard construction order and prevents the desired projector from becoming a hidden input.

Minimality does not imply uniqueness. The algebra contains infinitely many distinct minimal carriers, all equivalent under bijective right-multiplication intertwiners of the left action. The reconstruction therefore fixes a carrier equivalence class but leaves a representative-selection freedom. This is the principal new localization of difficulty in Part XII.

The distinguished axis A is then shown to play a more limited but more precise role than one might have inferred from Part X. It is not needed for existence, dimension or minimality of the carriers. It selects two special fixed members of the family and defines the half-angle sectors. Those sectors, despite having the same four-real-dimensional size as a minimal carrier, fail the global left-carrier test. Every genuine minimal carrier instead crosses both sectors and decomposes into two real two-dimensional intersections. The selected direction therefore reveals an internal $2+2$ decomposition of a carrier that already existed globally.

The distinction between the internal lift and the algebra automorphism also survives carrier reduction. Every full lift preserves every minimal carrier under left multiplication, while Φ_θ acts on the family and can move carriers. The central residual freedom from Part XI does not affect carrier preservation.

Finally the exact center reappears as the complete internal scalar algebra of a minimal carrier. Its generator S acts with square -1, each minimal carrier has two independent central directions, and every real-linear endomorphism commuting with the full left action is multiplication by a unique central element. Only after these intrinsic theorems are complete does the conventional $M_2(\mathbb{C})$ comparison identify the result with the standard minimal-left-ideal picture.

The remaining minus-sector stabilizer/reducibility symmetry should be closed explicitly, but it does not alter the core carrier classification. The next substantive task should then restrict the already-derived axial action to an arbitrary minimal carrier and classify the transformation of its two 2-dimensional axis-relative pieces, preserving the full central lift freedom until its effect is understood. Only after that neutral Task XIII should conventional spinorial terminology be allowed to enter as a comparison. Physical state selection, full spatial rotation structure, field equations and dynamics remain later layers.

References

[1] Rafal Ablamowicz and Bertfried Fauser. “On the Transposition Anti-Involution in Real Clifford Algebras III: The Automorphism Group of the Transposition Scalar Product on Spinor Spaces”. In: *arXiv preprint arXiv:1102.3304* (2011). arXiv: `1102.3304 [math-ph]`. URL: `https://arxiv.org/abs/1102.3304`.

[2] Aristotle (Harmonic). “Null-Sector Foundation – Task 11: Hardening the Axial Lift: Full-Carrier Classification, Central Freedom, Continuity, and Infinitesimal Separation”. Cumulative formal Experiment-2 artifact; Lean 4.28.0; mathlib revision 8f9d9cff6bd728b17a24e163c9402775d9e6a365; full build reported 8133 jobs; supplied archive SHA-256 4c208650cdaa95493425a8785fba96557b74f74fac9cf0d6b470bcb4fff8b198. Sept. 2, 2026.

[3] Aristotle (Harmonic). “Null-Sector Foundation – Task 12: Hardening the Candidate State Carrier: Invariant Left Carriers, Reducibility, Minimality, and Axial Stability”. Cumulative formal Experiment-2 artifact, run 25283ae2-f202-40c9-bf0e-f839b8c14593; Lean 4.28.0; mathlib revision 8f9d9cff6bd728b17a24e163c9402775d9e6a365; full build reported 8146 jobs; supplied archive SHA-256 386071a08299c28515e9f05a779e53a7e9f4b56b08bfc1e8ca0c689e8d70462c. Sept. 3, 2026.

[4] Tsit-Yuen Lam. *Introduction to Quadratic Forms over Fields.* Vol. 67. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2005. DOI: `10.1090/gsm/067`.

[5] Elliot Leader. *Spin in Particle Physics.* Vol. 15. Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology. Cambridge University Press, 2001. DOI: `10.1017/CBO9780511524455`.

[6] Pertti Lounesto. *Clifford Algebras and Spinors.* 2nd ed. Vol. 286. London Mathematical Society Lecture Note Series. Cambridge University Press, 2001. DOI: `10.1017/CBO9780511526022`.

[7] Leonardo de Moura and Sebastian Ullrich. “The Lean 4 Theorem Prover and Programming Language”. In: *Automated Deduction – CADE 28.* Vol. 12699. Lecture Notes in Computer Science. Springer, 2021, pp. 625–635. DOI: `10.1007/978-3-030-79876-5_37`.

[8] Jan Seeck, OpenAI ChatGPT, and Aristotle. *From a Real Euclidean Three-Space to Lorentz Kinematics and Complex Structure, Version 9.* Dependency reconstruction in Experiment 1. Aug. 28, 2026. URL: `https://github.com/antaris82/Split-Complex-Lorentzian_Kinematics/blob/Intrinsic-Branching-from-a-Derived-Lorentz-Structure/Split_Complex_Lorentzian_Kinematics_v9.tex` (visited on 09/01/2026).

[9] Jan Seeck, OpenAI ChatGPT, and Aristotle. *Intrinsic Branching from a Derived Lorentz Structure, Version 30 – Comparative Final.* Final comparative null test of Experiment 1. Aug. 30, 2026. URL: `https://github.com/antaris82/Split-Complex-Lorentzian_Kinematics/blob/Intrinsic-Branching-from-a-Derived-Lorentz-Structure/Intrinsic_Branching_from_a_Derived_Lorentz_Structure_v30.tex` (visited on 09/01/2026).

[10] Jan Seeck, OpenAI ChatGPT, and Aristotle. “Split Complex Lorentzian Dynamics – Part IV: Gluing Longitudinal Sectors in 3+1, Isotropy, and an Audited Blind-Test Defect”. Companion paper of Experiment 2, generated from the audited Task-04 artifact. Sept. 1, 2026.

[11] Jan Seeck, OpenAI ChatGPT, and Aristotle. “Split Complex Lorentzian Dynamics – Part IX: The Second Closure Obstruction, Central Quadratic Multiplicativity, and a Discrete Sign Branch”. Companion paper of Experiment 2, generated from the audited Task-09 artifact. Sept. 2, 2026.

[12] Jan Seeck, OpenAI ChatGPT, and Aristotle. “Split Complex Lorentzian Dynamics – Part VI: Clean-Room Replication, Universal 3+1 Obstructions, and the Localization of a Closure Deficit”. Companion paper of Experiment 2, generated from the audited Task-06 artifact. Sept. 1, 2026.

[13] Jan Seeck, OpenAI ChatGPT, and Aristotle. “Split Complex Lorentzian Dynamics – Part VIII: The Third Direction, Full Associative Ambient Closure, and the Remaining Norm Obstruction”. Companion paper of Experiment 2, generated from the audited Task-08 artifact. Sept. 2, 2026.

[14] Jan Seeck, OpenAI ChatGPT, and Aristotle. “Split Complex Lorentzian Dynamics – Part X: A Distinguished Axis, Transverse Channels, Half-Angle State Action, and the First Dynamics Boundary”. Companion paper of Experiment 2, generated from the audited Task-10 artifact. Sept. 2, 2026.

[15] Jan Seeck, OpenAI ChatGPT, and Aristotle. “Split Complex Lorentzian Dynamics – Part XI: Exact Central Freedom, Full-Carrier Lift Rigidity, and the Separation of Observable and State Generators”. Companion paper of Experiment 2, generated from the audited Task-11 artifact. Sept. 2, 2026.

[16] The mathlib Community. “The Lean Mathematical Library”. In: *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs*. CPP 2020. ACM, 2020, pp. 367–381. DOI: `10.1145/3372885.3373824`.

[17] The mathlib Community. *mathlib4 source at revision 8f9d9cff6bd728b17a24e163c9402775d9e6a365*. 2026. URL: `https://github.com/leanprover-community/mathlib4/tree/8f9d9cff6bd728b17a24e163c9402775d9e6a365` (visited on 09/01/2026).

[18] Eugene P. Wigner. “On Unitary Representations of the Inhomogeneous Lorentz Group”. In: *Annals of Mathematics*. 2nd ser. 40.1 (1939), pp. 149–204. DOI: `10.2307/1968551`.