Split Complex Lorentzian Dynamics

Part XIX: Primitive Bridges, Relation Descent, and the Unresolved Geometry of a Spin-Like Double Structure

From a split-complex Lorentz seed to covering, selector, and extension phenomena without assuming a spin structure

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Abstract

This part records a point at which the reconstruction ceases to be adequately described by a single linear "next obstruction." The experiment began from one of the most elementary and well studied algebraic realizations of Lorentzian geometry: the split-complex plane, with a unit j satisfying $j^2 = +1$, null idempotents $(1 \pm j)/2$, an indefinite quadratic form, and hyperbolic exponentials implementing $1+1$ Lorentz boosts. No spin group, covering-space theorem, bundle, cohomology class, quantum state, or fermionic interpretation was used as input. The subsequent tasks repeatedly enlarged the internal problem while preserving the rule that every new structure must be forced by the inherited algebra and the visible transformation law rather than inserted because a conventional theory suggests it.

By Task 19 that procedure has produced a striking collection of structures. Primitive central bridges are now defined independently of exactness and have a unique two-rate normal form. Global exact existence is equivalent to the existence of a primitive bridge whose reconstruction is exact, and also equivalent to the existence of one continuous directional rate that is simultaneously half-odd and reversal-odd. The three requirements—continuity, half-oddness, and reversal oddness—form an exact minimal unsatisfiable set: every proper subset is realizable, while the full conjunction is not. Independently, the complete visible coincidence relation is decomposed into same-direction, antipodal, and identity-degenerate classes. The first and third are automatic once the local exact rate constraints are imposed; only the antipodal class remains nonautomatic. A relation-complete locally exact open covering system exists if and only if a primitive bridge with globally exact reconstruction exists. Thus the global bridge branch and the relation-completion branch, which appeared independent in Part XVIII, are proved to be two presentations of one obstruction.

The domain branch does not collapse with them. The classification of maximal preconnected admissible domains remains partial. Not every maximal open preconnected admissible domain is an inherited hemisphere-like domain $\mathrm{Dom}(v)$. In the unrestricted admissible class, however, a complete theorem emerges: maximal admissible sets are exactly the unit-direction sets that are dense, antipodally complete, and antipodal-free. Such sets therefore choose exactly one representative from every antipodal pair and are dense in the sphere; explicit examples exist and are nonunique. Ascending unions of admissible sets need not remain admissible, so this maximality theorem is not a routine application of a maximality principle. Disconnected label data are classified by a closure-extension criterion, while pairwise separation of distinct label levels is proved necessary but not sufficient. Finally, the regularity ladder collapses from C^0 through every finite C^k and C^∞, but analyticity separates from the smooth class: an exact analytic plateau is rigid, and arbitrary-domain analytic admissibility remains open.

The resulting picture can be compared, piece by piece, with established mathematics. The visible coincidence relation naturally resembles a kernel-pair/descent problem. Central bridges resemble lift data in a central extension. The affine local labels $\mathbb{Z} + \frac{1}{2}$ with relative differences in $\mathbb{Z}$ resemble a torsor-like structure. Antipodal selection on the direction sphere

points directly to the two-sheeted quotient $S^2 \to \mathbb{RP}^2$. The global three-condition contradiction has a Borsuk–Ulam-type core. The disconnected extension theorem sits inside classical Tietze extension theory, and the smooth hardening belongs to smooth approximation theory. The inherited $2\pi/4\pi$ behavior, central sign, half-angle lift, and double-valued internal refinement are unmistakably spinorial in character.

But these comparisons do not yet fuse into a theorem saying that the reconstructed object *is* a classical spin structure. No principal $\mathrm{Spin}(n)$ bundle has been constructed, no exact sequence with kernel precisely $\mathbb{Z}_2$ has been identified as the total reconstructed carrier, no Stiefel–Whitney obstruction has been computed, and the affine $\mathbb{Z}+\frac{1}{2}$ labels are not automatically the $H^1(-;\mathbb{Z}_2)$ torsor of classical spin structures. The dense maximal selectors and the exact relation-descent theorem also do not appear as immediate restatements of the standard spin-structure existence theorem. The scientifically interesting frontier is therefore no longer that the structure "looks like spin." It is the precise translation problem: which of the reconstructed objects are genuinely equivalent to classical covering, central-extension, descent, projective, or spin objects, which are only analogous, and which residual combinations are not explained by a single established framework.

Part XIX treats this translation gap itself as the new Conservation-of-Difficulty boundary. Task 20 is proposed as a controlled intrinsic-to-classical equivalence audit, with the intrinsic objects frozen first and conventional terminology introduced only in a second comparison layer.

Keywords: split-complex Lorentz structure; primitive bridge; central extension; lift; descent; kernel pair; antipodal relation; real projective plane; maximal selector; Borsuk–Ulam; spin-like double structure; smooth extension; analyticity; formal verification; Lean 4; conservation of difficulty.

Contents

1 Why Part XIX is a different kind of paper

The preceding parts were primarily reconstruction papers. They asked a narrow internal question, let the formal development answer it, and then moved the difficulty to the next exact dependency. Part XIX must do something additional. Task 19 closes enough of the global bridge and relation branches that the next difficulty is no longer simply "prove the next theorem." The formal object now sits at the intersection of several mature mathematical theories, each of which explains one face of the result very well, but none of which has yet been proved to explain the entire cumulative object.

This distinction matters. There are two equally serious ways to overstate the result:

1. to call the emerging structure novel merely because it was reconstructed without using the conventional vocabulary; or

2. to call it "just a spin structure" merely because several of its signatures are spinorial.

Neither statement is currently justified. The first ignores a large body of classical topology, representation theory and extension theory. The second skips precisely the equivalence theorems that would be needed to identify the formal carrier, quotient, transition data, and obstruction class with the conventional objects.

Methodological principle 1 (The explanatory task after Task 19)**.** The correct question is no longer "does this resemble established mathematics?" It plainly does. The correct question is: *which exact reconstructed object corresponds to which established object, under which hypotheses, and where does the correspondence stop?*

The unusual feature is the route by which the question arose. The starting object was not an exotic topological construction. It was the split-complex realization of Lorentzian kinematics, one of the simplest algebraic models of $1+1$ Minkowski geometry. The apparent complexity entered only after asking for internal implementation, extension to arbitrary spatial directions, exact representative independence, joint regularity, local repair, relation completion, and global coherence.

This makes the present boundary scientifically useful. If the eventual answer is "the entire structure is equivalent to a classical construction," the reconstruction will have exposed a dependency path from a very elementary Lorentz algebra to that construction. If the answer is "only part is equivalent," the mismatch itself becomes the object to explain. Either outcome is stronger than an analogy based on vocabulary.

2 The deliberately ordinary starting point: split-complex Lorentz geometry

The split-complex numbers—also called hyperbolic, double, or duplex numbers—form the real algebra

$$\mathbb{D} = \{x + jt : x, t \in \mathbb{R},\ j^2 = +1\}. \tag{1}$$

With conjugation

$$\overline{x + jt} = x - jt, \tag{2}$$

the quadratic product is indefinite:

$$(x + jt)(x - jt) = x^2 - t^2. \tag{3}$$

Up to sign and the choice of which coordinate is called temporal, this is the quadratic form of the $1+1$ Minkowski plane. Hyperbolic exponentiation gives

$$e^{j\eta} = \cosh\eta + j\sinh\eta, \tag{4}$$

and multiplication by $e^{j\eta}$ implements a hyperbolic rotation, equivalently a proper $1+1$ Lorentz boost. This relation between split-complex arithmetic and Minkowski geometry is standard and extensively documented [2, 13].

The algebra also contains the idempotents

$$e_+ = \frac{1+j}{2}, \qquad e_- = \frac{1-j}{2}, \tag{5}$$

which satisfy

$$e_+^2 = e_+, \quad e_-^2 = e_-, \quad e_+e_- = 0, \quad e_+ + e_- = 1. \tag{6}$$

They are null with respect to the indefinite quadratic form and diagonalize the split-complex product. In the Task 1 reconstruction this became the null-coordinate decomposition

$$q_+(t,x) = t + x, \qquad q_-(t,x) = t - x, \tag{7}$$

with the Lorentz quadratic invariant reducing to the product of the two null coefficients. The future-causal sector becomes a positive quadrant, its timelike part the interior, and its two nontrivial null rays the boundary. Boosts rescale the two null coefficients by opposite exponentials while preserving their product.

Nothing in this starting point requires spinors. Nothing requires a covering space. Nothing requires a bundle over a sphere. The split-complex plane is already a complete and conventional description of the longitudinal Lorentzian $1+1$ sector.

Status 1 (Why the starting point matters)**.** The later spin-like signatures should not be attributed to having inserted a spin algebra at the beginning. The initial algebra has $j^2 = +1$ and describes hyperbolic rather than circular rotation. The circular half-angle and $2\pi/4\pi$ phenomena appear only after later internal implementation and spatial-direction reconstruction.

3 How an elementary Lorentz seed accumulated the later structure

A detailed re-derivation of Parts I–XVIII would obscure the present question, but the dependency path matters because the conventional comparisons introduced later must not be read backward as hidden assumptions. The cumulative route can be compressed as follows.

Stage	Relevant structural event
Task 1	Split-complex/null-coordinate realization of the longitudinal Lorentz sector; future-causal quadrant; boost rescaling and invariant product.
Intermediate reconstruction	Enlargement from one longitudinal plane to a $3+1$ directional setting, with spatial generators and an internal algebra reconstructed from required implementation and closure properties.
Task 10	An internal one-axis lift is classified relative to the already derived visible rotation. A half-angle reference implementer appears from the implementation equation; its 2π value is the negative unit and its 4π value the unit.
Tasks 14–15	The one-axis construction extends to arbitrary unit spatial directions. The residual central freedom is classified by directional rates $\alpha(n)$ and $\beta(n)$.
Task 16	Joint regularity forces those rates to be continuous. Exact representative independence globally is impossible. Sign-relaxed global coherence is unique. Proper local exact families exist with half-odd β.

Stage	Relevant structural event
Task 17	On connected local domains, the half-odd label is constant and local families are exhausted by one integer k. Relative local factors have integer rates. The original hemisphere-like domains are not the whole admissible geometry.
Task 18	A unique central bridge to the global reference is made explicit. Bridge normalization resolves the literal local-to-global normalization mismatch. The remaining global exact obstruction is reduced to continuity + half-oddness + reversal oddness.
Task 19	The bridge is redefined independently of exactness; primitive bridge, global rate, global exact family, and relation-complete local system are linked by genuine equivalences. The domain and regularity branches develop their own independent structure.

The logical direction is therefore important:

$$\begin{aligned}\text{Lorentz implementation constraints} \longrightarrow \text{internal half-angle structure} \longrightarrow \text{central freedom}\\ \longrightarrow \text{global/local coherence problem.}\end{aligned} \tag{8}$$

The experiment did not begin with a double cover and then recover its usual consequences. A double-cover-like pattern became visible only after the internal representative problem was sufficiently constrained.

4 Task XIX verification boundary and provenance

The supplied Aristotle archive reports Task 19 as complete [1]. Its run identifier is

`ca2f0ce1-09a7-4b4d-988f-c0e98f09f2b9`

and the supplied archive file has SHA-256

`36a3484d67d1634334098a1f0750fbd26f757a632daf116c4a4f958c83ec9396`.

The archive records Lean `leanprover/lean4:v4.28.0` and mathlib `v4.28.0` at revision

`8f9d9cff6bd728b17a24e163c9402775d9e6a365`.

The Task 19 audit reports:

- focused Task 19 build: successful;
- whole-project `lake build`: successful, 8238 jobs;
- no warnings in the Task 19 source layer;
- no `sorry`, `admit`, custom `axiom`, `@[implemented_by]`, `native_decide`, `bv_decide`, or `unsafe` in the Task 19 modules;
- all twenty-two principal endpoints report exactly `[propext, Classical.choice, Quot.sound]` under `#print axioms`.

The archive source order is

`SafeBase` → `PrimitiveBridge` → `MinimalObstruction` → `MaximalPreconnected` → `PlainMaximality` → `DisconnectedExtension` → `RelationCompletion` → `RelationBridge` → `HigherRegularity` → `NegativeControls` → `Identification` → `Task19`.

The final `Identification` module is comparison-only and is imported only by the Task 19 endpoint module.

Proof boundary 1 (Build claim versus editorial review)**.** *The successful 8238-job build is an archive-reported claim. The paper-generation environment did not independently rerun the pinned Lean/Lake project. The supplied Task 19 source, theorem statements, archive hash, source order, audit, and prohibited-construct boundary were independently inspected. The distinction remains the same as in Parts XVII and XVIII: a green build verifies the encoded statements, not every stronger interpretation one might attach to their names or prose summaries.*

The user interface again labelled the completed run `REVIEW SUGGESTED`. The archive itself does not call the formal development incomplete. This paper therefore records that UI label as workflow provenance, not as a mathematical status. In this case the suggestion is nevertheless apt: Task 19 contains several refuted intended classifications, a partially closed maximal-domain branch, and results whose relation to classical topology demands careful review.

5 Primitive bridges are now genuinely primitive

Part XVIII contained an important caveat. Its predicate called a "global bridge" already required that multiplying the bridge by the reference family produce a jointly regular globally exact family. The resulting existence equivalence was correct, but partly shaped by the definition itself. Task 19 repairs this by introducing a weaker object.

Definition 1 (Primitive bridge)**.** A primitive bridge $c(n, \theta)$ is constrained only by properties of the central factor itself: centrality on unit directions, normalization at $\theta = 0$, one-parameter multiplicativity, and joint continuity in direction and parameter. Global exactness is not part of the definition.

The exact Lean structure is `IsPrimitiveBridge`. The key consequences are stronger than a mere existence theorem.

Theorem 1 (Primitive bridge normal form)**.** *For every primitive bridge c and every unit direction n, there exist uniquely determined rates*

$$a_c(n), \qquad b_c(n) \tag{9}$$

such that for every real parameter θ,

$$c(n, \theta) = z_{a_c(n), b_c(n)}(\theta). \tag{10}$$

The ordered pair $(a_c(n), b_c(n))$ uniquely determines the bridge on that direction.

Task 19 also proves a negative control that is conceptually important:

$$b_c = b_d \quad \not\Longrightarrow \quad c = d. \tag{11}$$

There are distinct primitive bridges with the same directional b-rate but different a-rates. Thus the rate β that dominated the exactness obstruction is not the whole primitive bridge. The α-direction disappears only after exactness is imposed.

Status 2 (A definition-shaped objection is closed)**.** Primitive bridges exist. Primitive bridges whose reconstructions are locally exact exist. What does not exist is a primitive bridge whose reconstruction is globally exact. The global nonexistence is therefore not an artefact of defining the bridge to be globally exact.

This distinction is one of the most important methodological gains of Task 19. It turns the bridge from a convenient reparameterization into a separately meaningful central one-parameter object that can be studied before exactness is added.

6 Three equivalent formulations of the global exact problem

Task 19 defines the global rate predicate

$$\mathrm{GlobalRate}(b) \quad\Longleftrightarrow\quad \begin{cases} b \text{ is continuous on the unit-direction space,} \\ b(n) \in \mathbb{Z} + \frac{1}{2} \text{ for every unit } n, \\ b(-n) = -b(n) \text{ for every unit } n. \end{cases} \tag{12}$$

It then proves two genuine equivalences.

Theorem 2 (Primitive bridge–rate equivalence)**.** *The following existence statements are equivalent:*

(i) there exists a primitive bridge c whose reconstructed family $\mathrm{Recon}(c)$ *is globally exact;*

(ii) there exists a global rate b satisfying equation (12)*.*

Theorem 3 (Primitive bridge–family equivalence)**.** *The following existence statements are equivalent:*

(i) there exists a primitive bridge with globally exact reconstruction;

(ii) there exists a globally jointly regular exact internal family.

All three existence statements are false.

This is stronger than the Part XVIII formulation for a very precise reason. The primitive bridge in theorems 2 and 3 does not contain exactness in its own definition. Exactness appears as an additional property of its reconstruction and forces the primitive bridge rates into the global rate class.

Methodological caveat 1 (What the equivalence does and does not say)**.** *Task 19 does* not *prove that primitive bridges do not exist. They do exist. It proves that none of them can satisfy the extra global exact compatibility. Any prose statement "there is no global bridge" must therefore specify which bridge property is meant.*

7 The global contradiction is an exact minimal unsatisfiable triple

The three conditions that had emerged gradually by Task 18 are now packaged as a finite logical system:

$$C = \text{continuity}, \qquad H = \text{half-oddness}, \qquad R = \text{reversal oddness}. \tag{13}$$

Task 19 proves not merely that $C \wedge H \wedge R$ is impossible, but that the set $\{C, H, R\}$ is minimal unsatisfiable.

Theorem 4 (Minimal three-condition obstruction)**.** *No directional rate satisfies all three conditions C, H, R. Every proper subset of $\{C, H, R\}$ is satisfiable.*

The explicit witnesses are preserved:

$$C + R : b \equiv 0, \qquad \text{not half-odd,} \tag{14}$$
$$C + H : b \equiv \tfrac{1}{2}, \qquad \text{not reversal-odd,} \tag{15}$$
$$H + R : b = \pm\tfrac{1}{2} \text{ by an antipodal sign choice,} \qquad \text{not continuous.} \tag{16}$$

This result is formally separated at three levels: rates, primitive bridges, and exact families.

At the rate level, the core contradiction can be explained with very little machinery. A continuous odd real-valued function on a connected path from n to $-n$ must pass through zero. Half-oddness excludes zero. The discrete target $\mathbb{Z} + \frac{1}{2}$ strengthens the rigidity further: continuity on a connected set already forces a half-odd rate to be constant. The contradiction can therefore be understood by the intermediate value theorem alone in this particular setting.

Yet this elementary proof does not make the broader topological comparison irrelevant. Borsuk–Ulam says, in one of its equivalent forms, that a continuous antipodal map $S^n \to \mathbb{R}^n$ has a zero; equivalently, there is no continuous antipodal map $S^n \to S^{n-1}$ [9]. The Task 19 obstruction belongs to the same antipodal continuity family, but with a highly constrained one-dimensional codomain and an additional discrete affine lattice.

Proof boundary 2 (Borsuk–Ulam comparison)**.** *The Task 19 Lean proof should not be renamed "the Borsuk–Ulam obstruction." The project proves a more specialized rate contradiction from its own inherited hypotheses. Borsuk–Ulam is an established comparison explaining why antipodal continuity obstructions are natural; an exact theorem equating the Task 19 rate predicate with a particular Borsuk–Ulam formulation has not yet been formalized.*

8 The visible coincidence relation becomes the decisive global object

The visible map

$$(n, \theta) \longmapsto \Phi_n(\theta) \tag{17}$$

is not injective. Exact representative independence is therefore not merely a pointwise requirement over directions. It is a requirement that the internal family take the same value on every pair of parameter points with the same visible image.

Task 19 freezes this as an intrinsic visible relation

$$(n, \theta) \sim (m, \phi) \quad \Longleftrightarrow \quad \Phi_n(\theta) = \Phi_m(\phi). \tag{18}$$

Using the exact Task 16 coincidence classification, every visible coincidence between unit directions falls into one of three classes:

1. **same-direction relations**;
2. **antipodal relations**;
3. **identity-degenerate relations**, where both parameters are integer multiples of 2π and the visible transformation is the identity independently of direction.

For a jointly regular family with the local exact rate constraints $\alpha = 0$ and $\beta \in \mathbb{Z} + \frac{1}{2}$, Task 19 proves:

- the same-direction class is automatic because the complete internal family becomes 2π-periodic;
- the identity-degenerate class is automatic because the internal family takes the required identity value at integral full turns;
- the antipodal class is not automatic.

Theorem 5 (Only the antipodal class is nonautomatic)**.** *Under joint regularity and the local exact rate constraints,*

$$\textit{complete visible-relation compatibility} \quad \Longleftrightarrow \quad \textit{antipodal compatibility.} \tag{19}$$

There are jointly regular locally exact-rate families for which the antipodal compatibility fails.

This is one of the most important condensations of the cumulative project. The initially large equality relation on all axis–parameter pairs has been reduced to one genuinely nonautomatic class.

9 Relation completeness and the primitive bridge are two presentations of one obstruction

Task 19 then passes from one family to arbitrary indexed local systems. A local system consists of open direction domains covering all unit directions, each carrying a jointly regular locally exact representative. Ordinary overlap coherence means the representatives agree where the domains overlap. Relation completeness is stronger: every visible coincidence must be respected even if the corresponding direction pair is never contained in a common local domain.

The inherited $\{\mathrm{Dom}(v)\}$ system is the ideal negative control. It covers every direction and has coherent ordinary overlaps, but it is not relation complete.

Theorem 6 (Relation-complete system–bridge equivalence)**.** *The following existence statements are equivalent:*

(i) there exists a relation-complete locally exact covering system of open direction domains;

(ii) there exists a primitive bridge whose reconstruction is globally exact.

The forward direction is mathematically substantive. Local β-rates are glued to one global rate. Openness of the domains is used to prove continuity of the glued rate. Relation completeness supplies the cross-domain compatibility needed to make the definition independent of local choice. The resulting rate produces the primitive bridge.

The backward direction restricts the globally reconstructed family to the local domains and inherits every visible relation automatically.

Combining theorems 2 and 6 gives the three-way comparison

$$\begin{aligned}&\exists \text{ global exact primitive bridge}\\&\qquad\Longleftrightarrow \exists \text{ continuous half-odd reversal-odd rate}\\&\qquad\Longleftrightarrow \exists \text{ relation-complete locally exact open covering system,}\end{aligned} \tag{20}$$

and every line is false.

Methodological caveat 2 (A wording boundary inherited from the audit)**.** *The correct statement is not "relation completeness is equivalent to primitive bridge existence." Primitive bridges exist. The correct right-hand side is* primitive bridge with globally exact reconstruction*. Omitting the last clause changes a false existence statement into a true one.*

The global bridge branch and the relation branch of Part XVIII therefore rejoin. They were not independent obstructions after all. One is a rate/factor presentation; the other is a descent/coincidence presentation.

10 The domain branch refuses to collapse

If Task 19 had also reduced maximal-domain geometry to the same global relation theorem, the cumulative picture would have become unexpectedly simple. It does not. The domain problem remains genuinely separate and, in some regimes, becomes more counterintuitive.

10.1 The maximal preconnected classification remains partial

Let `PreconnAdmClass` denote the class of preconnected admissible direction sets. Antipodal completeness means that for every unit direction n, at least one of n or $-n$ belongs to the domain.

Task 19 proves the easy-looking direction that is in fact sufficient:

$$\begin{aligned}&\text{preconnected + admissible + antipodally complete}\\&\qquad\Longrightarrow \text{inclusion-maximal in the preconnected admissible class.}\end{aligned} \tag{21}$$

But the intended converse could not be proved. A maximal preconnected admissible domain is shown to satisfy only a weaker closure-completeness condition: every direction lying in the closure of the domain is represented by one member of its antipodal pair. Full antipodal completeness follows under an additional explicitly named accessibility hypothesis, `AccessibleMissing`.

Status 3 (Maximal preconnected domains)**. Sufficiency of antipodal completeness: PROVED.**
Closure-completeness of maximal domains: PROVED.
Maximality $\Rightarrow$ antipodal completeness without extra accessibility: OPEN.
Converse under `AccessibleMissing`: PROVED.

The obstruction is point-set rather than algebraic. If both n and $-n$ are missing from a maximal preconnected domain, adjoining one of them need not preserve preconnectedness. The straightforward maximality argument therefore breaks exactly where the class constraint matters.

10.2 Maximal open preconnected domains are not all inherited hemispheres

Task 18 had proved that each nonempty inherited domain $\mathrm{Dom}(v)$ is maximal in the class of open preconnected admissible sets. A natural next conjecture was that all maximal members of this class might be of this form. Task 19 refutes it.

Theorem 7 (Non-hemispherical maximal open domain)**.** *There exists a maximal open preconnected admissible domain E containing the previously constructed non-hemispherical domain Y such that*

$$E \neq \mathrm{Dom}(v) \qquad \textit{for every } v. \tag{22}$$

The proof is methodologically significant. It first proves that unions of chains remain inside the *open preconnected admissible* class, and only then invokes a maximality principle. Thus the maximality principle is not used to conceal an unproved closure assumption.

The theorem is existential rather than constructive: it proves that such a maximal domain exists, but does not provide a simple geometric normal form for it.

10.3 Three notions of maximality are genuinely different

Task 19 keeps separate:

- maximal preconnected admissible;
- maximal path-connected admissible;
- maximal open preconnected admissible.

The lexicographic domain lexA is maximal in the first two classes but not open. The inherited $\mathrm{Dom}(e_1)$ is maximal in the third class but not in the first. Hence a statement such as "the maximal local domain" has no invariant meaning until its ambient class is fixed.

11 The unrestricted maximal admissible sets have an exact and unusual classification

Dropping both connectedness and openness changes the problem completely. Here Task 19 obtains a necessity-and-sufficiency theorem.

Define D to be dense if its unit-direction subtype is dense in the full unit-direction space. Antipodal completeness requires at least one point from each pair $\{n, -n\}$; antipodal-freeness requires at most one.

Theorem 8 (Plain maximal admissibility)**.** *For a set D consisting of unit directions,*

$$D \text{ is inclusion-maximal admissible} \iff \begin{array}{l} D \text{ is dense}, \\ D \text{ is antipodally complete}, \\ D \text{ is antipodal-free}. \end{array} \tag{23}$$

The last two conditions say that D contains *exactly one* member of every antipodal pair. The first says that this selector is dense in the sphere.

The role of density follows from the global continuity of the admissible rate. On a dense subset the rate takes values in the discrete affine lattice $\mathbb{Z} + \frac{1}{2}$. Continuity forces the ambient rate into the closure of that lattice everywhere; since the lattice is closed and discrete, the ambient rate becomes globally half-odd and, on the connected direction sphere, constant. If a missing point n is then adjoined, antipodal completeness ensures that $-n$ was already present. Exactness would impose both equality of the constant values and reversal oddness, forcing the value to zero, which is impossible in $\mathbb{Z} + \frac{1}{2}$.

Task 19 constructs an explicit dense antipodal selector `flipSet` and proves it maximal. Negating the selector produces a distinct maximal admissible set. Hence maximal admissible sets exist and are nonunique.

Methodological principle 2 (Noncanonicity strengthens)**.** Part XVIII had shown noncanonicity among maximal preconnected extensions. Task 19 now proves nonuniqueness even in the unrestricted maximal admissible class. Maximality by itself therefore supplies no canonical representative selection.

11.1 Why a naive Zorn argument fails in the unrestricted class

The exact criterion is especially useful because the unrestricted admissible class does *not* have the chain property one might hope for. Task 19 constructs an increasing sequence of admissible sets D_k whose union is not admissible:

$$D_0 \subseteq D_1 \subseteq \cdots, \qquad \mathrm{Adm}(D_k) \text{ for all } k, \qquad \neg\, \mathrm{Adm}\left(\bigcup_k D_k\right). \tag{24}$$

Thus one cannot invoke a maximality principle for the plain admissible class by merely observing that it is partially ordered by inclusion. The obstruction is the incompatibility of the ambient continuous rates carried by the members of the chain.

The existence of maximal admissible sets is therefore stronger than a formal application of Zorn: one is explicitly constructed and checked against theorem 8.

12 Disconnected label data become a precise extension problem

On every connected component of an admissible domain, the half-odd label is constant. Task 18 had already shown that disconnected domains can carry several such values. Task 19 asks when a prescribed integer label function $\ell : D \to \mathbb{Z}$ is realizable by one ambient continuous rate

$$b(n) = \ell(n) + \frac{1}{2} \qquad (n \in D) \tag{25}$$

with the required reversal compatibility.

The exact answer is an extension theorem on the closure.

Theorem 9 (Closure-extension criterion)**.** *For a unit-direction domain D and prescribed integer labels ℓ, the labeling is realizable if and only if the corresponding half-odd function on D extends continuously to the closure of D in the unit-direction space and the extension satisfies the required reversal equation wherever both n and $-n$ belong to D.*

This theorem is sharp enough to refute a tempting simpler classification. If two different labels $j \neq k$ occur, the closures of their level sets must be disjoint. That separation is necessary. It is not sufficient.

Task 19 constructs a sequence of isolated points approaching an accumulation point, with distinct labels assigned along the sequence. Each label level is a singleton, so pairwise level-set closures are disjoint. Nevertheless the prescribed half-odd values cannot extend continuously across the accumulation point. Thus the true datum is not merely pairwise separation; it is continuous extendability of the whole label function.

Proof boundary 3 (What is and is not classified)**.** *The closure-extension theorem is a complete functional criterion. It is not yet a purely combinatorial classification of disconnected component configurations. An open problem is to reformulate continuous extendability using only intrinsic relations among components, closures, accumulation patterns, and antipodal pairing, without existential reference to an extending function.*

This is a recurring theme of Parts XVII–XIX: functional classifications often close before geometric classifications do.

13 The smooth hierarchy collapses; analyticity does not

Task 18 proved that continuous and C^1 admissibility coincide for arbitrary domains. Task 19 inspects the construction rather than extrapolating from it and proves that the staircase used to restore exact half-odd values is actually smooth.

Theorem 10 (Regularity collapse)**.** *For every direction set D and every finite k,*

$$\mathrm{Adm}(D) \quad \Longleftrightarrow \quad \mathrm{Adm}_{C^k}(D), \tag{26}$$

and also

$$\mathrm{Adm}(D) \quad \Longleftrightarrow \quad \mathrm{Adm}_{C^\infty}(D). \tag{27}$$

The construction uses standard extension and smooth-approximation machinery together with a smooth finite staircase restoring the exact discrete target values. This belongs to the established environment of Tietze extension and smooth approximation [15, 14].

Analyticity behaves differently. Task 19 proves an analytic rigidity statement: an analytic real function with a genuine open plateau is globally constant on the connected analytic domain. The smooth staircase therefore cannot simply be declared analytic.

At the same time, antipodal-free domains admit an analytic representative via a constant half-odd rate. Hence Task 19 does *not* prove that analytic admissibility fails. It proves only that the successful arbitrary-domain smooth construction does not transfer by the same plateau mechanism.

Status 4 (Regularity frontier)**.** $C^0, C^1, \ldots, C^k, C^\infty$**: same admissible domains, PROVED.**
Analytic admissibility for antipodal-free domains: PROVED.
Analytic admissibility for every arbitrary admissible domain: OPEN.

14 Established comparison I: this is naturally a descent problem

The first established language that now fits the formal structure is *descent.* This does not mean that Task 19 has constructed a scheme-theoretic descent problem. The comparison is structural and begins with a very elementary observation about the visible map equation (17).

Given any map $f : X \to Y$, its kernel pair is, informally,

$$X \times_Y X = \{(x, x') \in X \times X : f(x) = f(x')\}. \tag{28}$$

In sets this is simply the equivalence relation "have the same image." In category theory the same construction is the pullback of f along itself and is the standard starting point for effective equivalence relations and descent data [12, 11]. The Stacks Project likewise formulates descent data in direct relation to equivalence relations [16].

For the present parameterization, take

$$X = \mathbb{S} \times \mathbb{R}, \qquad f(n, \theta) = \Phi_n(\theta). \tag{29}$$

Then the Task 16/19 visible relation is exactly the set-level kernel pair of f:

$$((n, \theta), (m, \phi)) \in X \times_f X \quad \Longleftrightarrow \quad \Phi_n(\theta) = \Phi_m(\phi). \tag{30}$$

A globally exact internal representative is precisely an internal family whose value is constant on the fibres of f, or equivalently respects all pairs in this relation.

This makes the Task 19 distinction between direction coverage and relation coverage almost inevitable from the established viewpoint. Covering the points of $\mathbb{S}$ only says that every direction appears somewhere. Descent asks whether the local data agree on all identifications induced by the quotient map, including identifications between parameter points that do not lie in one ordinary set intersection.

Proposition 1 (Comparison dictionary: relation layer)**.** *The following established notions provide a natural comparison:*

Reconstructed object	*Established comparison*
visible coincidence relation	*kernel pair / equivalence relation of the visible map*
ordinary overlap coherence	*compatibility on literal intersections*
full relation completeness	*descent compatibility over all equal visible images*
global exact representative	*descended representative depending only on the visible transformation*

Proof boundary 4 (What remains unproved)**.** *Task 19 does not construct the quotient of the parameter space as a topological group, Lie group, groupoid, or categorical coequalizer, nor does it prove that the internal family is an object in a specified fibred category satisfying an established effective-descent theorem. "Descent" is therefore an explanatory comparison whose exact categorical realization remains a Task 20 question.*

The comparison is nevertheless unusually strong because Task 19 independently recovered the one fact that a descent analysis would make central: ordinary overlaps do not suffice; all equivalence relations generated by equal visible images matter.

15 Established comparison II: central bridges and central extensions

The second conventional language is the theory of lifts through central extensions. A typical central extension has the form

$$1 \longrightarrow A \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1, \tag{31}$$

where A lies in the centre of $\widetilde{G}$. If two elements of $\widetilde{G}$ project to the same element of G, their relative factor lies in the central kernel. Projective representations are a standard source of such extensions: lifting a homomorphism into a projective linear group produces a central extension by scalar matrices [4].

The reconstructed pattern is close to this template:

$$\text{visible transformation} \longleftarrow \text{internal implementer}, \tag{32}$$

with relative factors between implementers lying in the exact centre. The reference family U^{ref} supplies one distinguished internal implementation, and the bridge supplies the central relative factor

$$U = \mathrm{Bridge}(U)\, U^{\mathrm{ref}}. \tag{33}$$

Two local exact implementations have relative factor

$$\mathrm{ovl}(U_i, U_j) = \mathrm{Bridge}(U_i)\, \mathrm{Bridge}(U_j)^{-1}. \tag{34}$$

On connected exact domains the absolute bridge rates lie in $\mathbb{Z} + \frac{1}{2}$ and their relative differences lie in $\mathbb{Z}$.

These are exactly the sorts of formulas one expects when comparing different lifts of the same downstairs object.

But an identification with a specific classical central extension has not been proved.

Methodological caveat 3 (The centre is larger than the visible sign)**.** *The reconstructed exact centre carries continuous one-parameter factors with two rates before exactness is imposed. The inherited ± 1 sign defect is only one discrete part of this central structure. A classical spin covering has kernel $\mathbb{Z}_2$. Therefore one cannot identify the entire reconstructed central carrier with the spin-cover kernel merely because a ± 1 sign survives at full turns.*

Task 20 must determine whether there is a naturally defined subgroup of the reconstructed internal units for which the visible projection becomes an exact sequence with kernel precisely $\{\pm 1\}$, whether the continuous central rates live in a larger extension that must first be normalized, or whether the correct established comparison is not a single central extension at all.

16 Established comparison III: the affine half-integer labels form a torsor-like pattern

The local exact labels live in

$$\mathbb{Z} + \frac{1}{2}, \tag{35}$$

while relative labels live in

$$\mathbb{Z}. \tag{36}$$

The latter acts freely and transitively on the former by translation:

$$j \cdot \left(k + \frac{1}{2}\right) = j + k + \frac{1}{2}. \tag{37}$$

Thus $\mathbb{Z} + \frac{1}{2}$ is an affine copy of $\mathbb{Z}$, or in standard language a torsor for the additive group $\mathbb{Z}$: one can compare two points by an integer difference, but there is no preferred zero inside the half-integer set itself.

This comparison explains why the overlap integer is not an independent degree of freedom. It is exactly the difference of two absolute affine labels.

The spin comparison is suggestive but must be handled carefully. In representation theory, double covers of rotation groups naturally admit representations whose weights are half-integral relative to the ordinary rotation parameter, and spin representations do not descend to the underlying special orthogonal group in the same way as integral-weight representations [7, 8, 5]. This makes the appearance of half-integer labels and a $2\pi/4\pi$ internal period unmistakably spinorial in character.

However, classical spin structures on a fixed oriented bundle, when they exist, form an affine space over $H^1(B; \mathbb{Z}_2)$ rather than over $\mathbb{Z}$ [7, 10]. The present $\mathbb{Z} + \frac{1}{2}$ torsor therefore cannot simply be identified with the set of spin structures.

Proof boundary 5 (Two different affine structures)**.** *"Half-integer local labels form an affine $\mathbb{Z}$-torsor" is a direct algebraic statement about the reconstructed rate values. "Spin structures form an $H^1(-;\mathbb{Z}_2)$-torsor" is a classification theorem for lifts of an oriented frame bundle. They may become related after an additional quotient, reduction, or representation-theoretic identification, but that relation has not been proved.*

This mismatch is one of the places where the phrase "it looks like spin" is simultaneously justified and insufficient.

17 Established comparison IV: antipodal direction quotient and selectors

The direction space is the unit sphere S^2 in the inherited three-dimensional spatial carrier. Direction reversal acts freely by

$$n \longmapsto -n. \tag{38}$$

The orbit space is the real projective plane:

$$S^2/\{n \sim -n\} \cong \mathbb{RP}^2, \tag{39}$$

a classical two-sheeted covering $S^2 \to \mathbb{RP}^2$ [6, 17].

Now compare this with the maximal admissible theorem theorem 8. A set $D \subset S^2$ that is antipodal-free and antipodally complete contains exactly one point from each fibre of the quotient map

$$q : S^2 \longrightarrow \mathbb{RP}^2. \tag{40}$$

Set-theoretically, D is therefore a transversal or selector for the fibres of the two-sheeted cover.

Task 19 adds the unexpected condition that a *maximal admissible* selector must be dense in S^2. This density does not come from the definition of a selector. It comes from the interaction with the ambient continuous half-odd rate.

Proposition 2 (Established and project-specific parts of the selector picture)**.**

Property	*Status*
$n \sim -n$ gives $S^2 \to \mathbb{RP}^2$	*classical covering-space topology*
choose exactly one of $n, -n$	*classical set-theoretic transversal/selector*
continuous global selector of the nontrivial cover	*obstructed in classical covering theory*
dense selector	*not required by the cover itself*
dense + exactly-one-per-fibre characterizes maximal admissibility	*specific Task 19 theorem using the rate constraints*

The dense maximal selectors are therefore neither mysterious nor simply identical to a standard spin object. They combine a completely classical quotient with an additional admissibility condition arising from the reconstructed internal rates.

The nonuniqueness theorem also becomes transparent in this language: there are many set-theoretic transversals of a two-sheeted quotient. What is not yet understood intrinsically is which geometric invariants distinguish the maximal admissible ones and whether any additional structure from the reconstruction could select one canonically.

18 Established comparison V: why the global no-go is Borsuk–Ulam-like but not yet a characteristic-class theorem

The global rate obstruction can be read in two established ways of very different strength.

The elementary reading uses only connectedness and the intermediate value theorem. If $b(-n) = -b(n)$ and b is continuous along a path from n to $-n$, the value must cross zero. Half-oddness forbids zero.

The topological reading places this inside the Borsuk–Ulam family. A classical equivalent form of the Borsuk–Ulam theorem says that a continuous antipodal map $S^n \to \mathbb{R}^n$ has a zero; another says there is no continuous antipodal map $S^n \to S^{n-1}$ [9]. A continuous nonzero odd real rate would normalize to a continuous antipodal map $S^2 \to S^0$, which is impossible already by connectedness. The Task 19 half-odd requirement makes the codomain even more rigid than nonvanishing alone.

The tempting next leap is to call this the familiar spin obstruction. That leap is not justified.

Classical spin structures concern a lift of an oriented orthonormal frame bundle through the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1, \tag{41}$$

and existence is controlled by the second Stiefel–Whitney class w_2 of the underlying oriented vector bundle [3, 7, 10].

Task 19 has not constructed that frame bundle, has not identified a $\mathrm{Spin}(n)$ principal bundle, and has not computed w_2.

Methodological caveat 4 (Do not identify the no-go with w_2)**.** *The current global contradiction is a theorem about one reconstructed real rate on the direction sphere. The classical w_2 obstruction is a cohomological obstruction to lifting an oriented frame bundle. Similarity of the double-valued behavior does not make these theorems identical. In particular, one must not silently replace the project's direction sphere by the tangent-frame bundle of S^2 or assume that the relevant bundle is the tangent bundle of the sphere.*

This distinction is not pedantry. It is precisely what Task 20 must test: whether the rate obstruction is merely an elementary presentation of a standard covering/lift obstruction in disguise, whether it maps to a characteristic class after an appropriate bundle is identified, or whether the project's exact representative problem is a different descent problem that only shares the same $\mathbb{Z}_2$ geometry.

19 Established comparison VI: disconnected labels are classical extension theory with an extra reversal law

The closure-extension theorem theorem 9 belongs much more directly to established point-set topology. Tietze extension theory says that continuous real-valued functions defined on closed subsets of normal spaces extend continuously to the whole space. The unit sphere is metrizable and hence normal, and Mathlib contains the corresponding extension theorems used in the formal development [15].

The Task 19 problem has two extra features:

1. the prescribed values on the original domain lie in the discrete affine set $\mathbb{Z} + \frac{1}{2}$;

2. where an antipodal pair is present, the prescribed values must satisfy $b(-n) = -b(n)$.

Once the half-odd labeling extends continuously to the closure with the correct reversal relation, the established extension machinery can supply an ambient continuous rate. Conversely, any ambient rate restricts to such an extension.

This explains why pairwise closure separation is insufficient. Continuous extendability is a global condition on accumulation behavior, not a pairwise condition on level sets. The formal counterexample with isolated points converging to one limit is a standard kind of obstruction in extension theory: each level set may be individually harmless while the assigned values fail to converge compatibly at an accumulation point.

Methodological principle 3 (The foreign-looking component structure is not exotic by itself)**.** The disconnected label branch is best understood as a classical continuous-extension problem with a discrete target condition imposed only on the selected domain and an antipodal compatibility law. What remains project-specific is how this extension datum interacts with the internal representative and relation-completion structure.

20 Established comparison VII: smooth approximation explains $C^0 = C^\infty$, while analyticity introduces rigidity

The regularity result theorem 10 is also closely aligned with established analysis. Smooth approximation theorems allow continuous functions on finite-dimensional real manifolds or Euclidean spaces to be approximated by smooth functions while controlling the error; Mathlib provides the corresponding machinery [14]. Task 18 and Task 19 add a finite smooth staircase that maps a sufficiently accurate approximation back onto the exact half-odd values required on the admissible set.

The key point is that smooth functions can have plateaus. Real-analytic functions generally cannot: an analytic function that agrees with a constant on an open interval is forced, by the identity theorem, to agree with it on the connected analytic domain. Task 19 formalizes this plateau rigidity.

Thus the first regularity threshold that genuinely threatens to change the admissible-domain class is not C^1, C^2, or C^∞; it is analyticity.

This is another example of Conservation of Difficulty relocating rather than monotonically increasing. An entire hierarchy of smoothness conditions collapses to one class, and the first new obstruction appears only after passing from smooth to analytic rigidity.

21 Why the cumulative object genuinely looks spinorial

The comparison with spin geometry should now be stated neither apologetically nor triumphantly. Several reconstructed facts are genuinely characteristic of spinorial double structures:

1. the visible spatial transformation is 2π-periodic;
2. the inherited internal reference implementer changes sign after 2π and returns after 4π;
3. the half-angle appears in the internal implementation law rather than being postulated as a spin representation parameter;
4. the discrepancy between equal visible implementations is central;
5. the global sign-relaxed class is unique while exact internal representatives must be treated locally;
6. local exact rates lie in $\mathbb{Z} + \frac{1}{2}$ and their differences lie in $\mathbb{Z}$;
7. antipodal direction reversal is the only nonautomatic visible-relation class after same-direction and identity degeneracies are discharged;
8. the direction quotient by $n \sim -n$ is the standard two-sheeted quotient $S^2 \to \mathbb{RP}^2$.

Classical spin groups are connected two-sheeted covers of special orthogonal groups, constructed via Clifford algebras, and spin representations are genuine representations of the cover that need not descend to the orthogonal group [5, 7, 8]. These facts make the comparison unavoidable.

But a spin *structure* is more specific than spinorial behavior. It is a compatible lift of an oriented orthonormal frame bundle through the double cover equation (41), with existence governed by w_2 and, when it exists, a classification of lifts by an affine $H^1(-;\mathbb{Z}_2)$ action [3, 10]. None of those bundle-level statements is yet a theorem of the project.

Feature	Task-19 status	Classical spin comparison
$2\pi/4\pi$ internal behavior	formally inherited	strongly characteristic of double-cover spin representations
central ± 1 full-turn sign	formally inherited	matches kernel sign in Spin $\to$ SO
half-integer local labels	formally derived	reminiscent of half-integral weights
integer relative labels	formally derived	natural difference lattice, but not yet $\mathbb{Z}_2$ cohomology
antipodal quotient	formally present on directions	exactly $S^2 \to \mathbb{RP}^2$ as a topological quotient
primitive bridge	formally derived	resembles lift/central-extension factor
relation completeness	formally equivalent to global exact bridge existence	resembles effective descent of local lifts
principal Spin bundle	not constructed	essential object in a classical spin structure
w_2 obstruction	not computed	classical existence obstruction
$H^1(-;\mathbb{Z}_2)$ classification	not present	classical classification of spin structures when they exist
physical spin observable	not present	downstream physics, not implied by current algebra

The right conclusion is therefore:

> The reconstructed object has a spinorially characteristic double-valued internal implementation and has entered the same mathematical ecosystem as spin coverings, but the theorem identifying it with a classical spin structure has not been proved.

22 Where the established explanations fail to fuse

At first glance, the preceding comparison sections might suggest that the mystery has disappeared: every individual phenomenon has a familiar name. That would miss the present boundary.

The unresolved issue is that the familiar explanations live in different mathematical categories.

22.1 The descent explanation concerns equality of visible parameterizations

The kernel-pair picture begins with the map from axis–parameter pairs to visible transformations. Its central question is whether internal data descend through an equivalence relation.

22.2 The central-extension explanation concerns relative factors between lifts

Here the central question is whether the internal implementers form a group-like object projecting to the visible transformation group with a central kernel.

22.3 The $S^2 \to \mathbb{RP}^2$ explanation concerns antipodal direction selection

This quotient forgets orientation of an axis. It is a topological statement about the direction sphere, not by itself about the full axis–parameter visible transformation relation.

22.4 The Borsuk–Ulam explanation concerns continuous antipodal maps

This explains why a global odd nonzero rate is impossible, but it does not automatically identify the rate as a characteristic class or the internal algebra as a spin bundle.

22.5 The extension explanation concerns prescribed labels on arbitrary subsets

This is a functional analytic/topological issue. It explains disconnected admissibility but does not by itself explain the $2\pi/4\pi$ internal implementation law.

22.6 The classical spin explanation concerns a lifted frame bundle

This has the right double-cover flavor and central sign, but the relevant bundle and characteristic class have not been identified inside the reconstruction.

The current problem is therefore not a shortage of mathematical analogies. It is an excess of partially matching ones.

Open question 1 (The fusion problem)**.** *Is there one established object or one natural chain of equivalences that simultaneously explains*

(i) the primitive central bridge,

(ii) the exact visible-relation descent theorem,

(iii) the affine half-integer/integer label pair,

(iv) the antipodal selector geometry,

(v) the minimal three-condition global obstruction,

(vi) and the $2\pi/4\pi$ internal behavior?

Or do these remain several coupled structures sharing one reconstructed origin?

No literature comparison examined for this paper establishes such a single identification for the exact cumulative object. This is *not* a novelty claim. It is a statement of the present evidence boundary: the ingredients are classical, while the equivalence of their particular reconstructed combination has not been demonstrated.

23 Why the dense maximal selectors are especially difficult to place

Among the Task 19 results, the theorem theorem 8 is perhaps the least naturally absorbed by the usual spin narrative. A classical double cover already has many set-theoretic selectors, and a nontrivial connected cover does not admit a global continuous section. But the Task 19 theorem selects a very particular class of maximal admissible selectors: they must be dense.

The density is produced by the rate extension requirement. A non-dense selector can leave an open region in which one may add new directions while extending the ambient rate. Density removes this room. Once the selector is also antipodally complete and antipodal-free, any new point creates a full antipodal pair against a rate already forced to be globally constant by density and continuity.

Thus the maximal admissible selector is controlled simultaneously by:

$$\text{two-sheeted antipodal quotient} + \text{continuous ambient extension} + \text{discrete half-odd target}. \quad (42)$$

This three-way combination is not simply the existence or nonexistence of a section of $S^2 \to \mathbb{RP}^2$.

A Task 20 comparison should therefore ask whether these dense selectors have a standard name or classification in the theory of transversals for free group actions, whether the admissibility condition corresponds to a known measurable/Baire/continuous selection problem, and whether the exact maximality theorem can be derived from a classical selector theorem once the rate constraint is translated appropriately.

24 Why the split-complex origin makes the endpoint conceptually sharp

Had this structure been introduced by beginning with Clifford algebras, spin groups, projective representations, or a two-sheeted cover, the later comparison would be almost tautological. The methodological interest lies elsewhere.

The split-complex starting point belongs to a mature and elementary corner of Lorentzian mathematics: a commutative two-real-dimensional algebra reproduces the null decomposition and hyperbolic rotations of the $1+1$ Minkowski plane [2]. The experiment then deliberately asked what additional internal structures are forced when one insists on extending, implementing, composing, and globally identifying transformations in $3+1$ directions.

The later spin-like structure therefore appears after a long sequence of constraints that were not phrased in spin language. The Conservation-of-Difficulty principle tracked where each attempted closure failed:

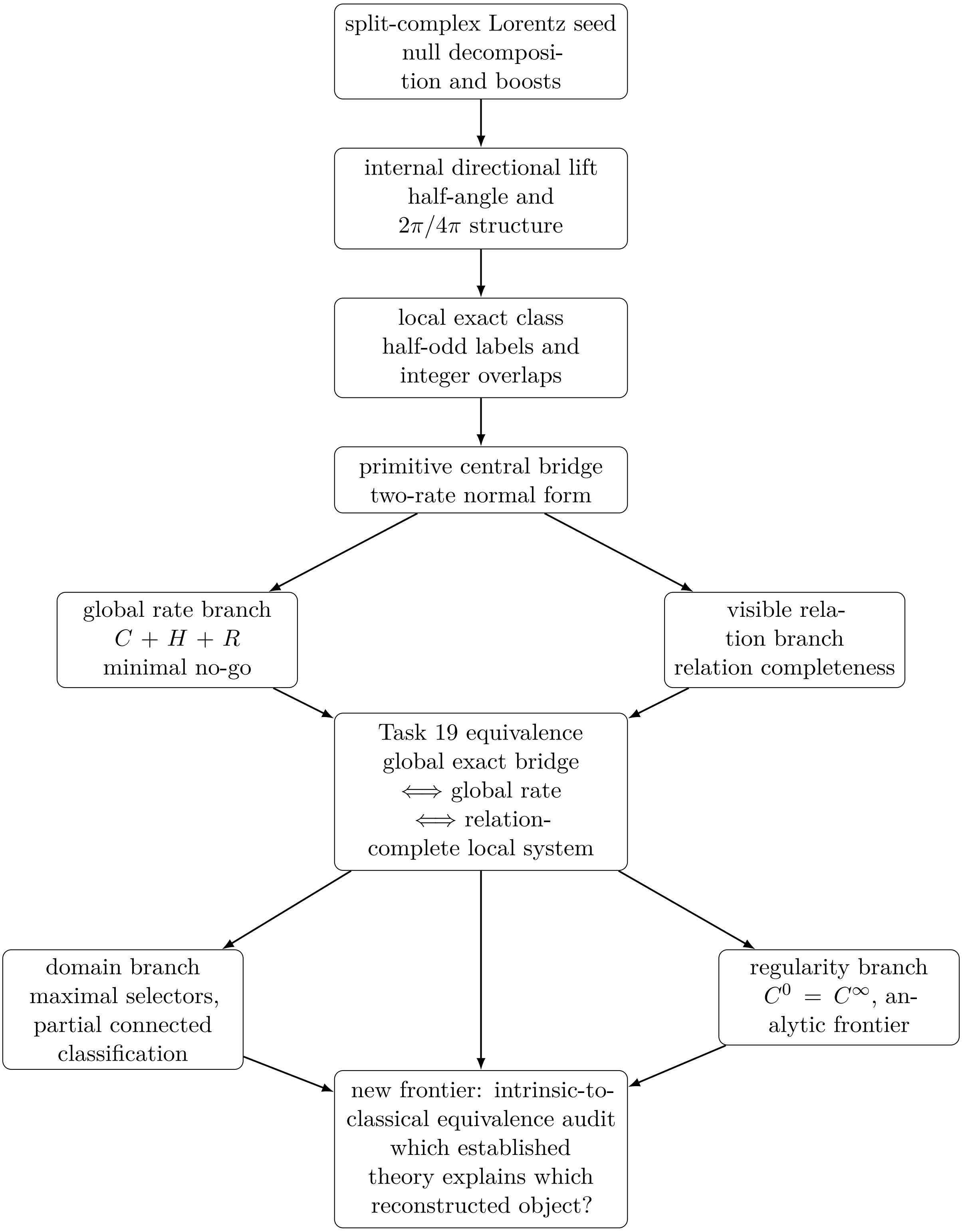
$$\begin{aligned}
\text{split-complex Lorentz seed} &\to \text{spatial extension}\\
&\to \text{internal implementers}\\
&\to \text{half-angle reference}\\
&\to \text{central freedom}\\
&\to \text{joint directional regularity}\\
&\to \text{global exact no-go}\\
&\to \text{local half-odd representatives}\\
&\to \text{integer overlaps}\\
&\to \text{explicit bridge}\\
&\to \text{primitive bridge/rate/relation equivalence.}
\end{aligned} \quad (43)$$

At Task 19 the global obstruction is no longer mysterious internally. It has been reduced to an exact minimal condition set and an exact relation-completeness failure. What is mysterious is the *external explanation*: why did an elementary Lorentz algebraic reconstruction produce, without conventional topological input, a structure that can be read simultaneously through descent, central extensions, antipodal covers, torsors, Borsuk–Ulam, extension theory, and spin-like double behavior?

This is the point at which Conservation of Difficulty stops asking only for another local repair and starts demanding an equivalence theory.

25 Conservation of Difficulty after Task XIX: the problem changes from construction to explanation

Part XVIII ended with a branching diagram. Task 19 now shows that two of those branches rejoin while two others remain genuinely independent.



The global bridge and relation branches have therefore *contracted.* Their difficulty is not preserved by pretending that two equivalent formulations are separate mysteries. The correct conservation statement is that the proof of their equivalence exposes the next problem at a different level.

Methodological principle 4 (Explanatory Conservation of Difficulty)**.** Once an internal obstruction has been reduced to equivalent exact formulations, the remaining difficulty is no longer to reproduce the obstruction in another notation. It is to determine whether those formulations are instances of one established structure and, if so, to prove the equivalence without importing more than the reconstructed hypotheses justify.

Three unresolved families now matter:

1. **Translation/fusion.** Determine whether the primitive bridge, descent relation, antipodal quotient, affine labels, and spinorial double behavior belong to one classical construction or to several coupled ones.
2. **Domain geometry.** The unrestricted maximal class is exactly known, but the maximal preconnected and maximal open preconnected geometry is not structurally classified.
3. **Analytic rigidity.** Smooth regularity adds no restriction; analytic regularity may.

This is a favorable endpoint for the reconstruction. The object has become rigid enough that comparisons can be falsifiable. A claimed identification must now explain not merely $2\pi/4\pi$, but also the exact kernel-pair relation classes, the primitive bridge normal form, the half-integer/integer affine structure, the dense maximal selectors, and the smooth/analytic split.

26 Failed conjectures and changed theorem boundaries

The failed routes in Task 19 are part of the mathematical result because each one rules out a tempting oversimplification.

1. **Unconditional maximal preconnected classification failed.** The intended equivalence "maximal preconnected admissible iff antipodally complete" was not proved. The forward direction was replaced by closure-completeness and a conditional theorem under `AccessibleMissing`.
2. **Pairwise closure separation was not sufficient for disconnected labels.** The accumulating singleton example forced the stronger closure-extension criterion.
3. **Zorn on the unrestricted admissible class was unavailable.** Increasing admissible chains can have nonadmissible unions. An explicit maximal selector was constructed instead.
4. **Maximal open preconnected domains are not exhausted by $\mathrm{Dom}(v)$.** The inherited hemisphere-like family remains important but is not a classification.

These failures make the external comparison more important, not less. They identify the details that any established theory must reproduce. A comparison that explains only the central sign or only the antipodal quotient is incomplete if it cannot also account for the changed maximality classes and the extension obstruction.

27 Formal status ledger after Task XIX

Question	Part-XIX status
Does a primitive bridge exist independently of exactness?	Yes. Centrality, normalization, multiplicativity and joint continuity only.
Is a primitive bridge classified?	Yes, by a unique pair of directional rates on each unit direction.
Does β alone determine a primitive bridge?	No; explicit counterexample with different α.
Primitive exact bridge existence $\leftrightarrow$ global rate?	Yes, genuine equivalence.

Question	Part-XIX status
Primitive exact bridge existence $\leftrightarrow$ global exact family?	Yes, genuine equivalence.
Are continuity, half-oddness and reversal oddness jointly inconsistent?	Yes.
Is the three-condition set minimal unsatisfiable?	Yes; every proper subset has an explicit witness.
Are same-direction visible relations automatic?	Yes under the local exact rate constraints.
Are identity-degenerate relations automatic?	Yes.
Is the antipodal class the only nonautomatic visible relation?	Yes.
Direction coverage $\Rightarrow$ relation completeness?	No; explicitly refuted.
Relation-complete open local system $\leftrightarrow$ primitive bridge with global exact reconstruction?	Yes.
Can the global obstruction be stated as relation incompleteness?	Yes, at the formal existence level.
Maximal preconnected admissible $\leftrightarrow$ antipodally complete?	PARTIAL: sufficiency proved; converse conditional on `AccessibleMissing`; unconditional converse open.
Are all maximal open preconnected admissible domains $\mathrm{Dom}(v)$?	No; refuted.
Are unrestricted maximal admissible domains classified?	Yes: dense + antipodally complete + antipodal-free.
Do unrestricted maximal admissible domains exist?	Yes, explicit `flipSet`.
Are they unique/canonical?	No; explicit nonuniqueness.
Are admissible sets closed under increasing chain unions?	No.
Are disconnected labels classified?	Yes functionally, by continuous extension to the closure plus reversal compatibility.
Is pairwise separation of different label closures sufficient?	No.
Does C^k or C^∞ change admissibility?	No, for arbitrary domains.
Does analyticity change arbitrary-domain admissibility?	OPEN; plateau rigidity proved, antipodal-free analytic existence proved.
Has a principal spin bundle been constructed?	No.
Has the project obstruction been identified with w_2?	No.
Has the affine $\mathbb{Z} + \frac{1}{2}$ structure been identified with classical spin-structure $H^1(-;\mathbb{Z}_2)$ data?	No.
Has physical spin or quantum dynamics been derived?	No.

28 The exact comparison boundary: derived, analogous, and still unidentified

The following table is intentionally stricter than a thematic analogy list.

Reconstructed structure	Established comparison	Exact status of the comparison
split-complex $1+1$ Lorentz algebra	hyperbolic/split-complex Minkowski plane	established identification
visible coincidence relation	kernel pair of parameter-to-visible map	exact at the set-theoretic level; categorical descent object not formalized
relation completeness	descent compatibility	strong structural equivalence; categorical implementation open
primitive central bridge	lift factor in central extension	strong analogy; exact total central extension not identified
$2\pi/4\pi$ reference behavior	double-cover/spinorial behavior	characteristic analogy; no full spin-group identification
local $\mathbb{Z}+1/2$ labels	half-integral weights / affine $\mathbb{Z}$ torsor	exact affine-$\mathbb{Z}$ statement; representation-theoretic identification open
relative $\mathbb{Z}$ labels	differences of affine labels	exact internally; relation to cohomological transition data open
antipodal quotient of directions	$S^2 \to \mathbb{RP}^2$	exact topological identification
maximal dense selectors	transversals of free $\mathbb{Z}_2$ action plus rate constraints	exact selector reading; standard classification correspondence open
global rate no-go	Borsuk–Ulam-type antipodal obstruction	structurally classical; exact equivalence theorem not formalized
spin-structure obstruction	w_2	not identified; must not be claimed
disconnected label criterion	Tietze-style continuous extension	close and partly imported through Mathlib
C^∞ hardening	smooth approximation + smooth staircase	established analytic machinery plus project-specific exact restoration
analytic frontier	identity theorem / analytic continuation rigidity	established rigidity; full admissibility classification open

The table shows why the cumulative endpoint is neither exotic in every component nor reducible to one familiar word.

29 Open explanatory questions created by the spin-like comparison

The following questions now carry more scientific weight than another broad physical extension.

Open question 2 (What is the actual downstairs object?)**.** *Is the image of $(n,\theta) \mapsto \Phi_n(\theta)$ naturally a group or homogeneous subspace for which the Task 19 relation is exactly the kernel pair of a quotient map with the required topological properties? If so, can relation completeness be stated as ordinary effective descent over that quotient?*

Open question 3 (What is the actual upstairs object?)**.** *Do the internal implementers generated by the reconstructed reference and allowed central factors form a natural topological group $\widetilde{G}$*

projecting onto the visible group G? What is the kernel? Is there a distinguished exact subextension with kernel $\{\pm 1\}$, or is the larger continuous centre essential?

Open question 4 (Where exactly does the double cover live?). *The $2\pi/4\pi$ behavior is internal and the direction quotient is $S^2 \to \mathbb{RP}^2$. Are these manifestations of the same two-sheeted structure after an exact identification, or are they two distinct $\mathbb{Z}_2$ phenomena coupled by the implementation law?*

Open question 5 (What is the status of the affine half-integer labels?). *Are the local rates representation weights, choices of logarithm of a central lift, indices of sections, or merely a project-specific parameterization of central one-parameter factors? Why do their differences land in $\mathbb{Z}$, and what, if anything, survives after reducing the central sign to $\mathbb{Z}_2$?*

Open question 6 (Is there a characteristic-class formulation?). *Can one construct from the reconstructed data a bona fide bundle or principal bundle for which the global no-go is represented by a standard characteristic class? If so, which class? If not, can one prove that the current obstruction is categorically different from the w_2 obstruction of a spin structure?*

Open question 7 (What are the maximal selectors geometrically?). *Does the exact criterion "dense + antipodally complete + antipodal-free" correspond to a known class of transversals for free $\mathbb{Z}_2$ actions with extension constraints? Which invariants distinguish the nonunique maximal selectors?*

Open question 8 (Does analyticity produce the next genuine rigidity?). *Can every continuously admissible arbitrary domain still be realized analytically by a construction unlike the smooth staircase, or does analyticity impose a strictly smaller domain class? If smaller, can that class be characterized geometrically?*

The project has reached a point where these are not decorative comparisons. Each has a formal object on the reconstruction side and an established theory on the comparison side. The missing work is to prove or refute the bridge between them.

30 Task XX: freeze the intrinsic object, then test the classical identifications

Task 20 should be the first task whose explicit purpose is not another internal reconstruction step but a controlled *equivalence audit.* That does not mean abandoning the firewall discipline. It means dividing the task into two phases.

Methodological principle 5 (Two-phase comparison protocol). **Phase A** must restate and package the intrinsic Task 19 objects without conventional labels. Only after those invariants are frozen may **Phase B** compare them with covering spaces, central extensions, projective representations, descent, and spin geometry. A conventional theorem may be used to prove an equivalence in Phase B, but may not be used to redefine the intrinsic Phase-A object so that the equivalence becomes tautological.

A suitable title is:

Task XX – Intrinsic-to-Classical Equivalence Audit: Quotient Relations, Central Extension Candidates, Antipodal Selectors, and the Exact Boundary of the Spin Comparison.

30.1 WP1: construct the visible quotient object

Freeze the parameter space of unit directions and real parameters together with the Task 16 visible coincidence relation. Construct, if possible, the quotient by that relation with the strongest topology/algebraic structure justified by the inherited visible transformations. Prove whether the visible map is the quotient/coequalizer of its kernel pair in the chosen category.

Required questions:

- Is the quotient naturally isomorphic to the actual image of the visible transformation family?
- Does it inherit a group law, local group law, homogeneous-space law, or only a set/topological quotient structure?
- Is Task 19 relation completeness exactly descent through this quotient?

If a full quotient theorem is too expensive, isolate the strongest exact fragment and do not replace it with conventional terminology.

30.2 WP2: identify the internal lift object and its centre

Let $\widetilde{G}$ denote only a candidate object until closure is proved. Determine which internal implementers generated by the reference family and primitive bridges are closed under the inherited product and inversion laws. Define the visible projection intrinsically.

Then determine:

- its kernel;
- its exact centre;
- whether the kernel is all of the continuous centre or a discrete subgroup;
- whether there is a distinguished exact subobject whose kernel is precisely $\{\pm 1\}$;
- whether the visible projection is a central extension in the standard algebraic/topological sense.

The task must allow the answer "no single central extension captures the full bridge structure."

30.3 WP3: turn the primitive bridge into a genuine lift/descent datum or refute that identification

Starting from the Phase-A primitive bridge, compare it with the standard notion of relative factor between two lifts through a central extension. Prove whether

$$U = \mathrm{Bridge}(U)U^{\mathrm{ref}} \tag{44}$$

is exactly a lift-factorization theorem for the candidate extension of WP2.

Determine whether the integer overlap law is the standard transition law of those lifts and whether the finite composition identity becomes a conventional descent/cocycle law after the candidate extension is fixed.

Do not call the overlaps a cocycle unless the exact domain, coefficient group, and cocycle equation have been identified.

30.4 WP4: classify the affine half-integer structure

Formalize the intrinsic statement that $\mathbb{Z} + \frac{1}{2}$ is a torsor over $\mathbb{Z}$ and that local relative rates are its differences. Then compare this with:

- weights of double-cover rotation representations;
- logarithmic branches of central one-parameter lifts;
- transition indices of local sections;
- any cohomological data naturally produced by WP2–WP3.

Determine which, if any, is actually equivalent. In particular, test explicitly whether reduction modulo 2 of the integer transition data carries the full global sign information or loses essential continuous/affine data.

30.5 WP5: formalize the antipodal quotient comparison

Construct the quotient of the unit-direction sphere by $n \sim -n$ and identify it, in the comparison layer, with $\mathbb{RP}^2$. Translate:

$$\begin{aligned} \text{antipodal-free} &\leftrightarrow \text{at most one point per quotient fibre}, \\ \text{antipodally complete} &\leftrightarrow \text{at least one point per fibre}, \\ \text{both} &\leftrightarrow \text{set-theoretic selector/transversal}. \end{aligned}$$

Then determine exactly what the Task 19 density condition adds beyond ordinary selector theory.

Test whether maximal admissible selectors correspond to a known standard class of transversals under any additional topological, Baire, measurable, or extension property. If no such standard classification is found, state only that the exact Task 19 combination remains unmatched.

30.6 WP6: compare the minimal rate obstruction with Borsuk–Ulam

Prove the strongest exact implication/equivalence between the Task 19 minimal rate obstruction and standard antipodal-map no-go theorems. Separate at least:

- the elementary path/intermediate-value proof;
- a Borsuk–Ulam formulation;
- nonexistence of a continuous section/selector if that is genuinely equivalent after WP5.

Determine whether Borsuk–Ulam explains the entire obstruction or only the continuity + reversal part, with half-oddness supplying an additional discrete constraint.

30.7 WP7: perform the spin comparison as a theorem, not a resemblance

Only after WP1–WP6 are fixed, compare the candidate internal extension with the standard double cover $\mathrm{Spin}(3) \to \mathrm{SO}(3)$ or any lower-dimensional subgroup actually supported by the inherited visible transformations.

The task must answer separately:

(a) Does the reconstructed reference implementer define a homomorphism into a classical spin group or a representation thereof?

(b) Is the $2\pi/4\pi$ sign exactly the central -1 of that double cover under an explicit isomorphism?

(c) Are the reconstructed spatial generators isomorphic to a Clifford-even algebra used in the spin construction, or merely representation-equivalent on the tested sector?

(d) Does the primitive bridge enlarge the spin cover by an additional central factor?

(e) Does global exactness correspond to a section/lift problem of that classical cover?

A partial identification is an acceptable result. Do not force the full spin group if only a one-parameter or directional substructure is actually proved.

30.8 WP8: audit whether a classical spin structure and w_2 are even the right objects

If WP7 produces a genuine principal-bundle candidate, formulate the corresponding spin-structure lifting problem and compute or characterize its obstruction. Only then test whether the second Stiefel–Whitney class is applicable.

If no appropriate oriented frame bundle has been reconstructed, prove that the current data are insufficient to state a w_2 identification rather than importing one.

Likewise compare the classification of possible lifts with the classical $H^1(-;\mathbb{Z}_2)$ torsor. Explain formally whether and how this differs from the existing affine $\mathbb{Z}+\frac{1}{2}$ local-rate torsor.

30.9 WP9: explain relation completeness in established descent language

Assuming WP1 provides an appropriate quotient and WP2–WP3 provide an internal lift object, prove whether Task 19's

$$\text{relation-complete local system} \Longleftrightarrow \text{globally exact primitive lift}$$

is an instance of an established effective-descent theorem.

If it is, identify all hypotheses actually used: openness, coverage, centrality, continuity, and the exact relation classes. If it is not, isolate the property in which the Task 19 system differs from standard descent.

30.10 WP10: continue the two unresolved independent branches

The comparison task should not erase the unresolved internal questions. In parallel:

- test the unconditional maximal-preconnected $\Rightarrow$ antipodally-complete implication, or preserve it as open with a sharper counterexample/criterion;
- investigate arbitrary-domain analytic admissibility and decide whether analyticity produces a genuinely smaller admissible class.

These should remain secondary to the equivalence audit; they are included because either may reveal which established comparison is actually correct.

30.11 Task-XX decision table

The final Task 20 report should distinguish at least:

Question	Required verdict
Visible relation is the kernel pair of a quotient in the chosen category?	PROVED / PARTIAL / REFUTED
Visible quotient identified with a standard transformation group/object?	PROVED / PARTIAL / OPEN

Question	Required verdict
Internal implementers form a central extension of that visible object?	PROVED / PARTIAL / REFUTED
Kernel of the relevant exact subextension is precisely $\{\pm 1\}$?	PROVED / REFUTED / NOT APPLICABLE
Primitive bridge is exactly a lift-relative central factor?	PROVED / PARTIAL / REFUTED
$\mathbb{Z} + \frac{1}{2}$ labels have a standard representation-theoretic/torsor interpretation?	PROVED / PARTIAL / OPEN
Antipodal selector theorem identified with standard $S^2 \to \mathbb{RP}^2$ selection theory?	PROVED / PARTIAL
Task-19 rate no-go equivalent to a Borsuk–Ulam formulation?	PROVED / PARTIAL / REFUTED
Reference $2\pi/4\pi$ lift identified with a classical spin double cover?	PROVED / PARTIAL / REFUTED
A genuine spin structure has been constructed?	PROVED / NOT YET / NOT THE RIGHT OBJECT
w_2 is the project's global obstruction?	PROVED / REFUTED / NOT APPLICABLE / OPEN
Classical $H^1(-;\mathbb{Z}_2)$ lift torsor explains the project labels?	PROVED / PARTIAL / REFUTED
Task-19 relation theorem is an instance of standard effective descent?	PROVED / PARTIAL / REFUTED
Maximal preconnected classification closed?	PROVED / REFUTED / OPEN
Arbitrary-domain analytic admissibility classified?	PROVED / PARTIAL / OPEN
Overall cumulative object fully identified with one established construction?	YES / PARTIAL / NO

The desired endpoint is not "Spin confirmed." It is a map of exact equivalences and exact failures of equivalence.

31 Why Task XX should precede any broader dynamics

The reconstruction has now reached the point at which moving immediately into interactions, Hamiltonians, or quantum-state dynamics would create an avoidable ambiguity. If the current internal structure is already equivalent to a known classical lift/descent construction, then that equivalence tells us which additional dynamical structures are canonical and which are coordinate artifacts. If it is not equivalent, then the mismatch is potentially more informative than any dynamics built on top of an unidentified carrier.

In particular, a physical claim about spin would require far more than the present $2\pi/4\pi$ behavior. One would need a state space, representation action, observable structure, and an interpretation connecting the formal internal implementers to physical states. None of that follows merely from a double cover. The next scientifically disciplined move is therefore mathematical translation, not physical promotion.

32 Conclusion

Task 19 closes the global internal obstruction more sharply than any preceding stage. Primitive bridges are defined without exactness, classified by two rates, and shown to exist. Global exactness then selects a narrower class and is equivalent both to one continuous half-odd reversal-odd directional rate and to a relation-complete locally exact open covering system. Those conditions are impossible. The global bridge branch and the relation branch have therefore converged to one exact obstruction rather than remaining separate difficulties.

The obstruction itself is minimal: continuity, half-oddness and reversal oddness are jointly inconsistent, while every proper subset is realizable. Same-direction and identity-degenerate visible coincidences are automatic under the local exact constraints, leaving the antipodal class as the only nonautomatic relation. Ordinary overlaps are consequently proven insufficient in the strongest possible sense: they may all be coherent while the full visible relation still fails.

The domain branch behaves differently. Its connected maximal classification remains partial, maximal open connected domains extend beyond the inherited $\mathrm{Dom}(v)$ family, and unrestricted maximal admissible sets acquire the exact but unusual form of dense antipodal selectors. The unrestricted class is not closed under ascending chain unions, so maximality is not a routine set-theoretic consequence. Disconnected labels reduce to a continuous extension problem on the closure, and smooth regularity all the way to C^∞ adds no new restriction. Analyticity is the first regularity regime at which the construction itself encounters a genuine rigidity boundary.

Every one of these results has a recognizable established analogue. Split-complex numbers already encode $1+1$ Lorentz geometry. The visible relation resembles a kernel pair and the local-to-global theorem resembles descent. Central bridges resemble lift factors in a central extension. The local half-integer labels form an affine $\mathbb{Z}$-torsor and recall half-integral weights. Antipodal direction selection is literally selection over the double cover $S^2 \to \mathbb{RP}^2$. The global no-go belongs to the Borsuk–Ulam family of antipodal continuity obstructions. The disconnected and smooth branches sit inside classical extension and approximation theory. The inherited $2\pi/4\pi$ sign behavior is unmistakably spinorial.

Yet the established explanations do not presently fuse. The project has not constructed a principal spin bundle, has not identified its total internal carrier with $\mathrm{Spin}(3)$ or another specific spin group, has not reduced the full central structure to a $\mathbb{Z}_2$ kernel, has not computed w_2, and has not identified the affine $\mathbb{Z}+\frac{1}{2}$ labels with the classical $H^1(-;\mathbb{Z}_2)$ torsor of spin structures. Conversely, the classical spin-structure theorem does not immediately explain the Task 19 dense maximal selectors, the exact closure-extension criterion, or the particular relation-completeness equivalence without additional identifications.

This is the present Conservation-of-Difficulty frontier. The difficulty is no longer hidden in an algebraic calculation. It is exposed as an explanatory problem:

> Starting from a conventional split-complex Lorentz structure, the reconstruction has generated a formally rigid network of lift, descent, antipodal, affine-label, and double-cover phenomena.
>
> The next task is to determine which of these are genuinely the same established object, which are only parallel manifestations, and which residual combinations require a separate explanation.

That is an unusually favorable place for a reconstruction experiment to arrive. The mathematics has become familiar enough to compare rigorously, but not yet so identified that the comparison is tautological. Task 20 should therefore perform the translation explicitly and allow the answer to be full equivalence, partial equivalence, or genuine mismatch.

References

[1] Aristotle (Harmonic). *Task 19: Intrinsic Global-Obstruction Decomposition.* Run ca2f0ce1-09a7-4b4d-988f-c0e98f09f2b9; Lean archive supplied by the editor; archive SHA-256 36a3484d67d1634334098a1f0750fbd26f757a632daf116c4a4f958c83ec9396. 2026.

[2] Francesco Catoni, Dino Boccaletti, Roberto Cannata, Vincenzo Catoni, Enrico Nichelatti, and Paolo Zampetti. *The Mathematics of Minkowski Space-Time: With an Introduction to Commutative Hypercomplex Numbers.* Birkhäuser Basel, 2008. DOI: 10.1007/978-3-7643-8614-6.

[3] Encyclopedia of Mathematics. *Characteristic class.* Includes the spin lifting sequence and the second Stiefel–Whitney obstruction; accessed 4 September 2026. URL: https://encyclopediaofmath.org/wiki/Characteristic_class.

[4] Encyclopedia of Mathematics. *Projective representation.* Central extension associated with a projective representation; accessed 4 September 2026. URL: https://encyclopediaofmath.org/index.php?title=Projective_representation.

[5] Encyclopedia of Mathematics. *Spinor group.* Accessed 4 September 2026. URL: https://encyclopediaofmath.org/wiki/Spinor_group.

[6] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002. URL: https://pi.math.cornell.edu/~hatcher/AT/AT.pdf.

[7] H. Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry.* Princeton University Press, 1989.

[8] Pertti Lounesto. *Clifford Algebras and Spinors.* 2nd ed. Cambridge University Press, 2001.

[9] Jiri Matousek. *Using the Borsuk–Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry.* Universitext. Springer, 2003.

[10] John W. Milnor and James D. Stasheff. *Characteristic Classes.* Princeton University Press, 1974.

[11] nLab contributors. *Descent morphism.* Accessed 4 September 2026. URL: https://ncatlab.org/nlab/show/descent+morphism.

[12] nLab contributors. *Kernel pair.* Accessed 4 September 2026. URL: https://ncatlab.org/nlab/show/kernel+pair.

[13] J. Rooney. "On the Relationship between Rotations and Lorentz Transformations in Two, Three, and Four Dimensions". In: *Environment and Planning B* 6.4 (1979), pp. 413–439. DOI: 10.1068/b060413.

[14] The mathlib Community. *Mathlib.Analysis.Calculus.BumpFunction.SmoothApprox.* Smooth approximation results used in the regularity branch. URL: https://leanprover-community.github.io/mathlib4_docs/Mathlib/Analysis/Calculus/BumpFunction/SmoothApprox.html.

[15] The mathlib Community. *Mathlib.Topology.TietzeExtension.* Documentation for the Tietze extension results used in the Task-19 dependency chain. URL: https://leanprover-community.github.io/mathlib4_docs/Mathlib/Topology/TietzeExtension.html.

[16] The Stacks Project Authors. *Descent data give equivalence relations.* Stacks Project, Tag 024D. Accessed 4 September 2026. URL: https://stacks.math.columbia.edu/tag/024D.

[17] Eric W. Weisstein. *Real Projective Plane.* Wolfram MathWorld. Accessed 4 September 2026. URL: https://mathworld.wolfram.com/RealProjectivePlane.html.