Mathematics of Uncertainty

Introduction

Concept of risk

- Unexpected event
- Ignorance of the expected
- Surprise
- Impossibility of knowing the future

- Measurable risks (financial prices, physical events ...)
- Unmeasurable risks (investor psychology and irrationality)

- Known risks
  - Encountered by most or all people
  - Extent and implications can be modelled
  - Can be part of business plans
  - Written in your textbooks
- Unknown risks
  - Encountered by some people, or well described in historical records
  - Extent and implications are unclear

  ...climate change, natural disasters, geopolitics, technological disruptions, wars ...
- **Unknowable risks**
  - **Not seen in known history**

  ...comet striking the earth, Covid-19 ...

Known risks and their management

- **Market risk**

Risk due to changing market prices and quantities

- **Credit risk**

Risk due to default in debt payments

- **Liquidity risk**

Risk of loss in value when trying to buy or sell

- **Operational risk**

Mismanagement, fraud, legal mishaps, operational errors

- **Funding risk** (new!)

Inability to find enough cash to prevent insolvency

- **Diversification**
  - Diversification of balance sheets and portfolios
  - To reduce market risk
- **Hedging**
  - Buying insurance, or hedging to reduce risk
  - To reduce credit and liquidity risk
- **Prudence**
  - Integrity and 'ak akçe kara gün içindir'
  - To reduce operational risk and to prepare for unknown risks

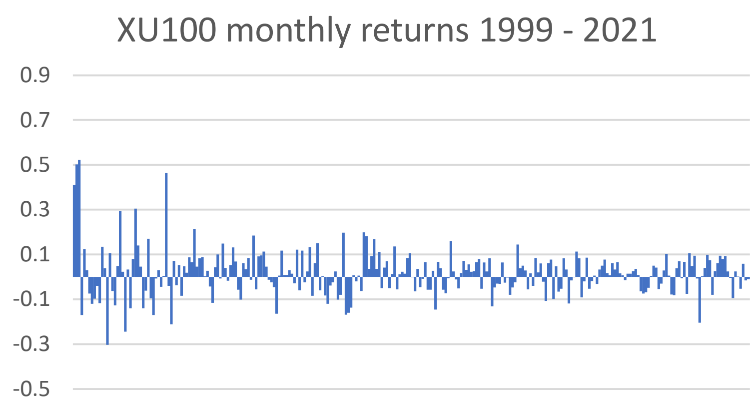
Prices seem to fluctuate randomly

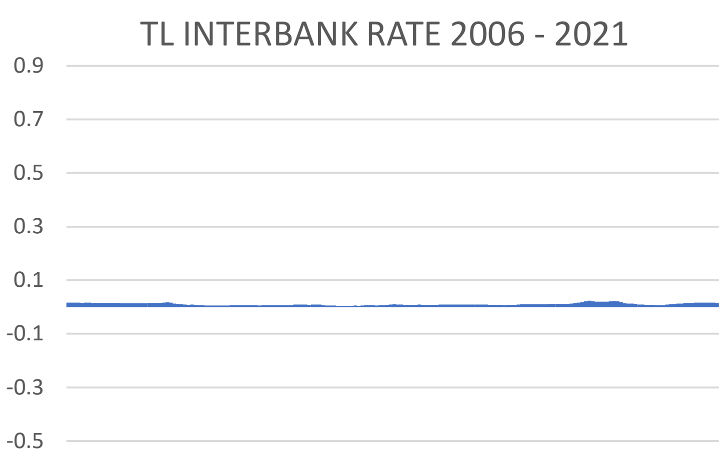
Can we know where the market will be next year?

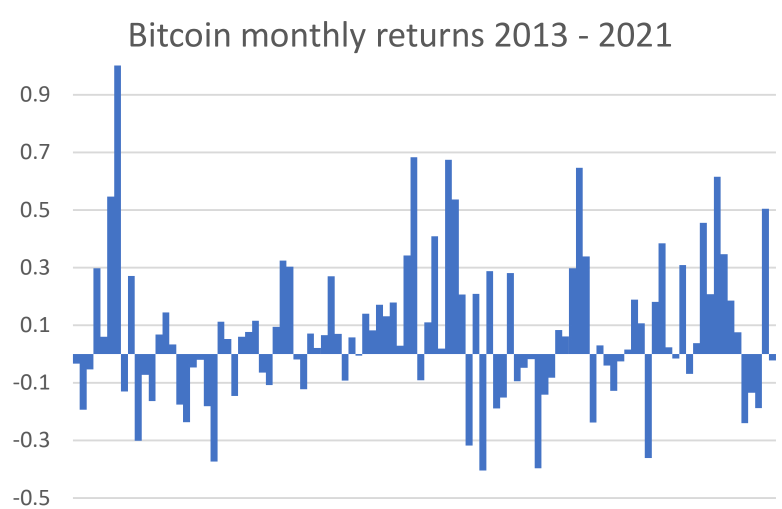
Volatility

Continuous-time rate of return: $R_t = ln(P_t/P_{t-1})$

Discrete-time rate of return: $R_t = (P_t - P_{t-1})/P_{t-1}$



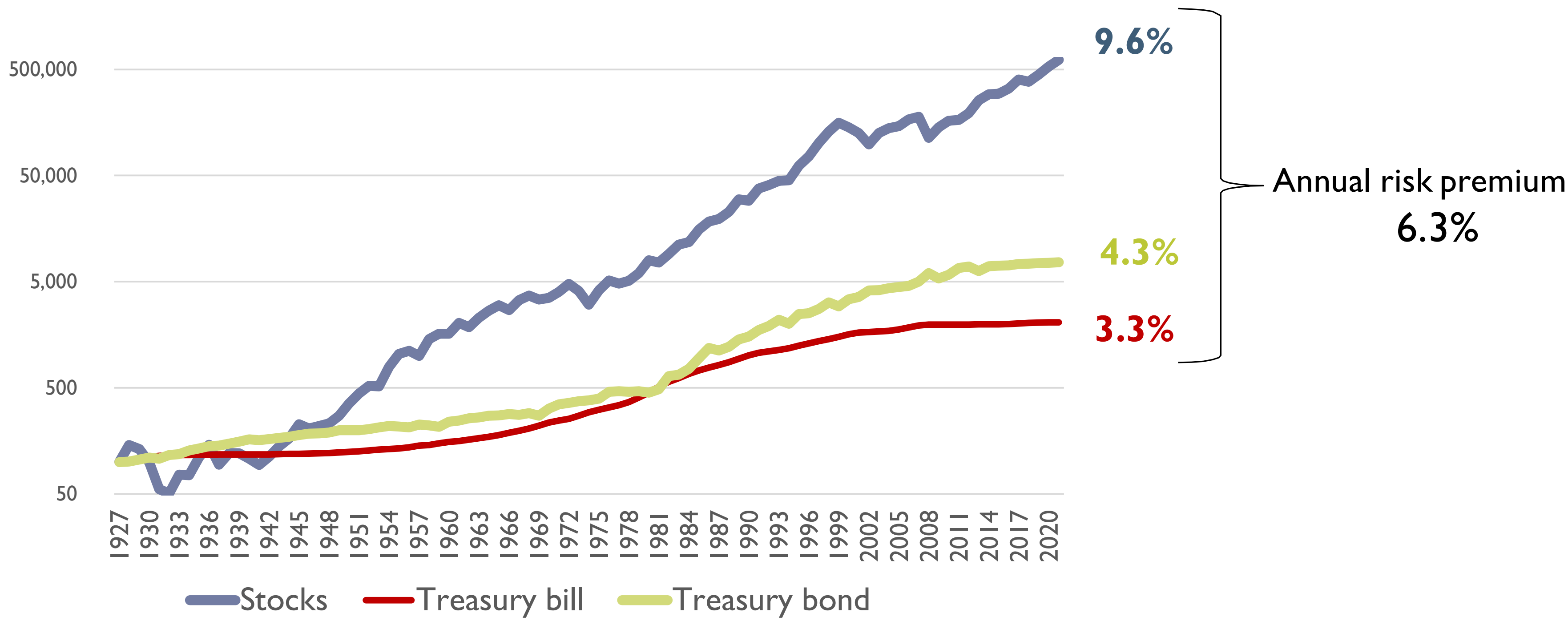




US markets (1927 – 2021)

More volatile markets have yielded higher returns

Required rate of return on risky asset = risk-free rate of return + ***risk premium***



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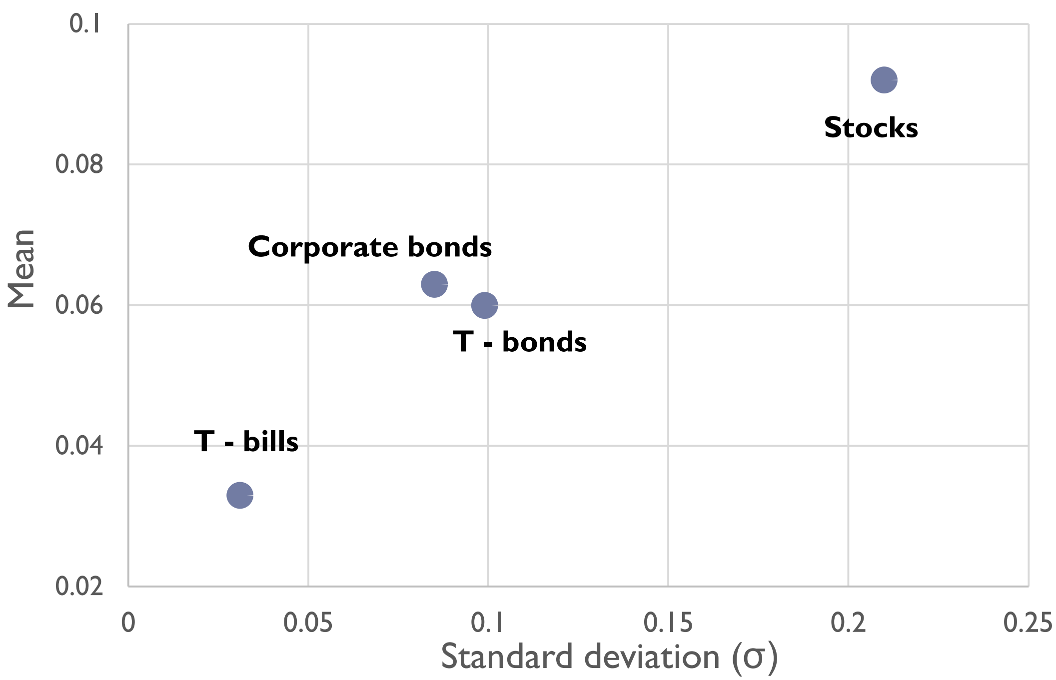


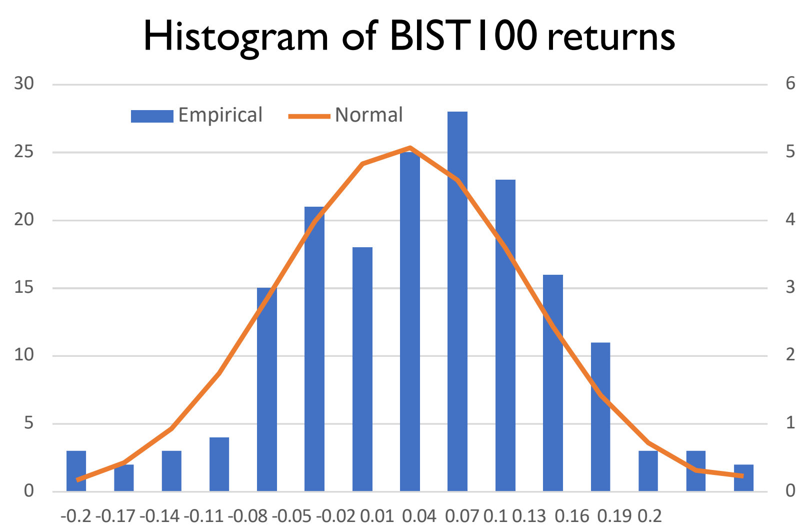
Given $R_1, \ldots, R_T$:

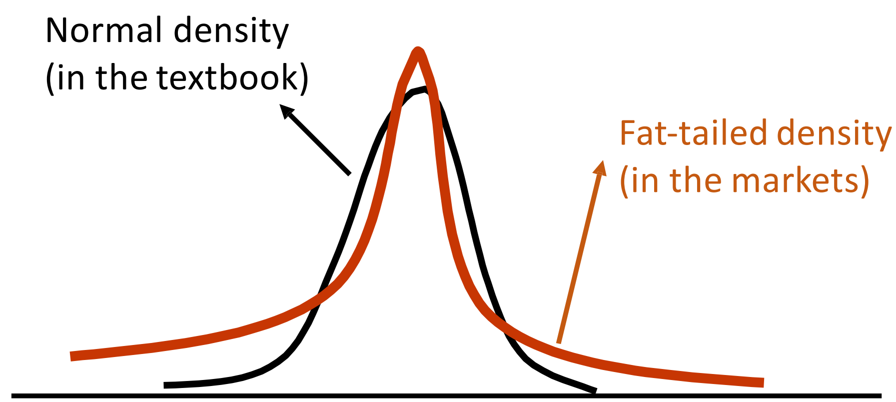
$$\bar{R} = \frac{\left(\sum_{t=1}^{T} R_t\right)}{T}$$

$$\sigma^2 = \frac{\left(\sum_{t=1}^{T} (R_t - \bar{R})^2\right)}{T}$$

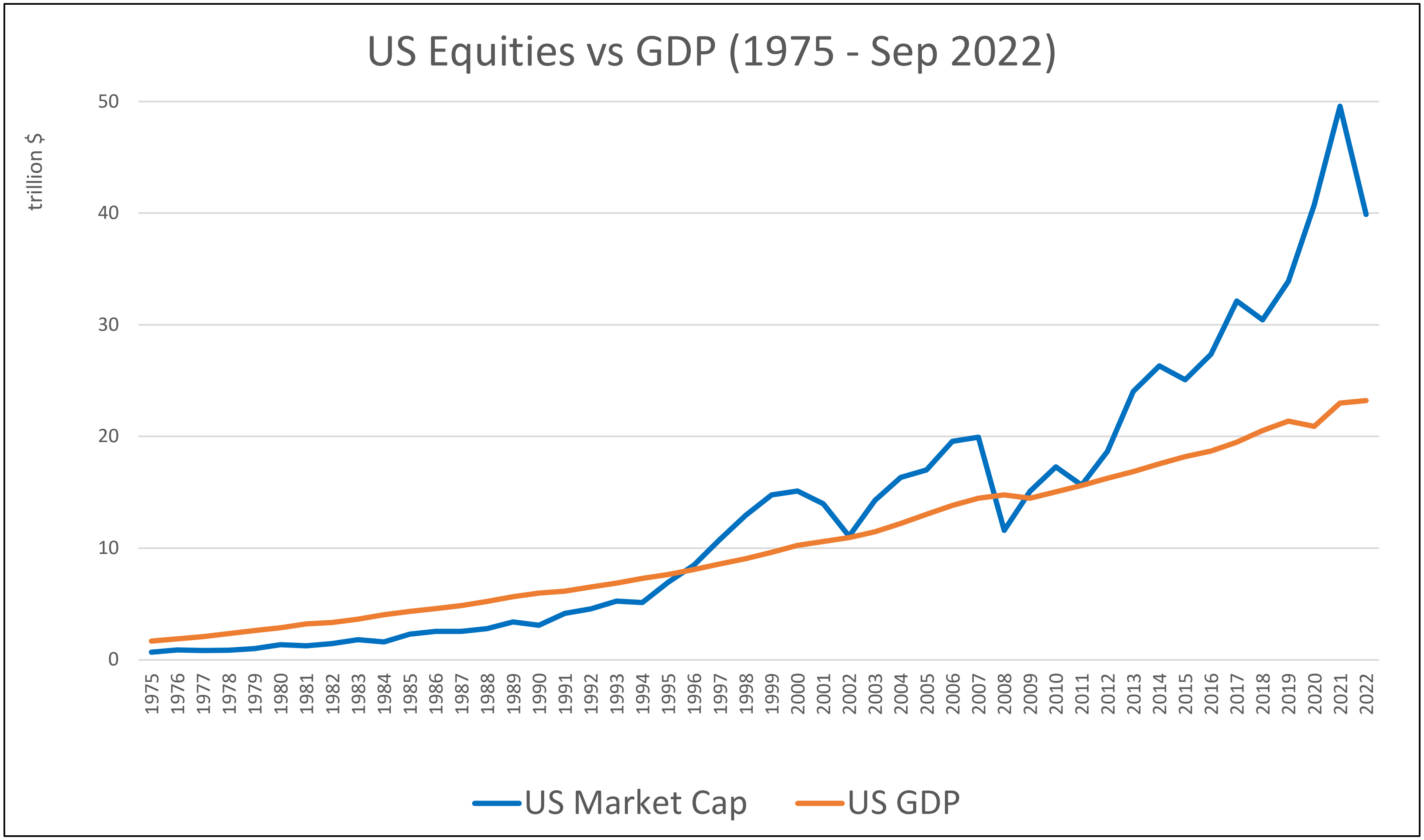
$$\sigma = \sqrt[+]{\sigma^2}$$

Are rates of return normally distributed?





Price = Value ?


US Equities vs GDP (1975 - Sep 2022)
trillion $
50
40
30
20
10
0
1975 1976 1977 1978 1979 1980 1981 1982 1983 1984 1985 1986 1987 1988 1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011 2012 2013 2014 2015 2016 2017 2018 2019 2020 2021 2022
US Market Cap
US GDP

The idea of probability

Objective interpretation: Counting of outcomes based on experiments or empirical data

> **Example:** Toss a fair coin 10 times to get 2 heads and 8 tails, 100 times to get 40 tails and 60 heads, 1,000 times to get 450 tails and 550 heads, 1,000,000 times to get 499,999 tails and 500,001 heads and so on ... $\Pr(H) = \Pr(T) = 0.5$

➔Objective probabilities attempt to describe frequencies and other regularities in financial markets.

Subjective interpretation: ("belief") Subjective (speculative) estimate of the likelihood of a future event

> **Examples:** Pr(*stock price will increase by X % in a year*), Pr(*rain tomorrow*), Pr(*winning the election*)..., which are not based on any experiment or empirical data.

➔Subjective probabilites describe personal preferences in risk taking in financial markets

Just watch youtube.com or www.twitter.com

The idea of probability

Frequency plus belief: Empirical counting modified with subjective believes (statistics + analysis)

Example: The annual rate of return on a common stock has been positive 30 times and negative 20 times during the last 50 years. Therefore, empirically,

$$\Pr(R > 0) = 0.60$$

However, this year, the firm has made a new investment, which is expected to double its profits. Therefore, the final assessment of probability may be

$$\Pr(R > 0 \mid \text{new info}) > 0.60$$

Ideas from finance

Uncertainty about future values of financial assets (stock prices, interest rates, exchange rates, commodity prices etc) spans the full range from certainty to impossibility

[Almost certain] [.................. Probable ....] [Almost impossible]

Example: The payment of the nominal principal of a Treasury bill maturing tomorrow is a certainty.

Example: The rate of return on any stock in BIST exceeding 50% in one day is impossible (because of daily price limits set by the Exchange, $\pm 10\%$ / session, or $\pm 21\%$ / day)

Example: The bankruptcy of the most profitable firm in the country in a week is possible but not probable (that is, $\Pr(event) = 0$)

Example: Before 1990's, the bankruptcy of the most reputable firm in the country in a month was possible but not probable ($\Pr(event) = 0$). After the GFC of 2008, we now know that $\Pr(event) > 0$. This is a new type of an "**extreme**" or "**tail**" event.

The unpredictability of tomorrow's prices

Random walk: If tomorrow's price were predictable today, it would become unpredictable through trading today. Trading today would continue until tomorrow's price is no longer predictable.

- Today's final price discovered as such is called the ***equilibrium price*.**
- A market which makes possible such a price discovery in a timely fashion is an ***efficient market****.*

Word of caution: Unlike Keynes's implication that stock markets are gambling casinos where prices are randomly set, our approach is:

➔ Probability models of financial prices are meaningful ex ante and not necessarily ex post.

➔ The challenge is the formulation of useful probability models of the uncertainty in future prices for:

- Assessing the riskiness of investments
- Ranking investments in order of riskiness
- Developing pricing / valuation relations