$x_1^4 - 4x_1^3$

$f(x_1,x_2) = x_1^3(x_1-4) + 3(x_2-5)^2 + 40$.

$3(x_2^2 - 10x_2 + 25) + 40$.

$3(2x_2 - 10)$

Consider the following function: $f(x_1,x_2) = x_1^3(x_1-4) + 3(x_2-5)^2 + 40$

a) Determine the domain where the function is convex.

a. where it convex. Hessian: $\left.\begin{array}{l} PM_1^i \geq 0 \\ PM_j \geq 0 \end{array}\right\} \Rightarrow$ hessian.

$\frac{\partial f}{\partial x_1} = 4x_1^3 - 12x_1^2 = 4x_1^2(x_1 - 3)$

$\frac{\partial f}{\partial x_2} = 3(2x_2 - 10) = 6(x_2 - 5) = 6x_2 - 30$.

$\frac{\partial^2 f}{\partial x_1^2} = 12x_1^2 - 24x_1 = 12x_1(x_1 - 2)$

$\frac{\partial^2 f}{\partial x_2^2} = 6$

$H(x_1,x_2) = \begin{bmatrix} 12x_1(x_1-2) & 0 \\ 0 & 6 \end{bmatrix}$.

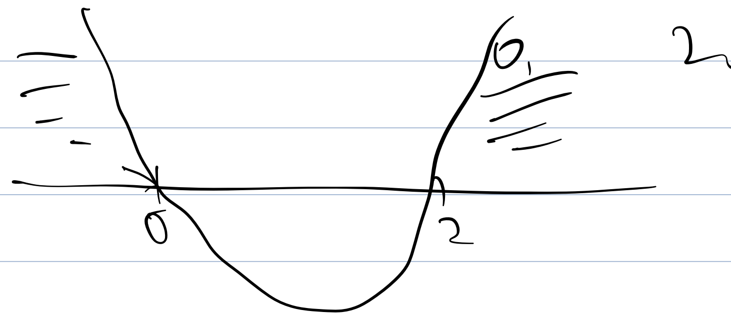
$\frac{\partial^2 f}{\partial x_1 x_2} = 0$

to convex.

$\begin{cases} 12 \times 6\, x_1(x_1-2) \geq 0 \\ 6 \geq 0 \\ 12x_1 \cdot (x_1-2) \geq 0 \end{cases}$.

$x_1(x_1-2) \geq 0$ $\qquad$ $x_1^2-2x_1=0$.

$x_1^2-2x_1 \geq 0$



$x_1 \geq 2$ or $x_1 \leq 0$ convex

$x_1 \in (-\infty, 0] \cup [2, +\infty)$

b. not possible.

b) Determine the domain where the function is concave.

$PM_1^{(2)} > 0$.

c. stationary points:

① $\nabla = 0$,

c) Find the stationary points of the function and determine their type.

$$\begin{cases} 4x_1^2(x_1-3) = 0 \\ 6(x_2-5) = 0 \end{cases} \Rightarrow 4x_1^3 - 12x_1^2 \rightarrow x_1 = 3, \; x_1 = 0.$$

$$= 6x_2 - 30 = 0$$

two stationary points: $(3,5)$ and $(0,5)$.

$D(3,5) = 216 > 0$

$D(0,5) = 0$

$$H(x_1,x_2) = \begin{bmatrix} 12x_1(x_1-2) & 0 \\ 0 & 6 \end{bmatrix}.$$

① $LPM_1 > 0$, $LPM_2 > 0$ $\Rightarrow$ local min

② $LPM_1 < 0$, $LPM_2 > 0$ $\Rightarrow$ local max.

if not 1 or 2 and

(inconclusive).

highest LPM $= 0$ $\Rightarrow$ not conclusive.

highest LPM $\neq 0$ $\Rightarrow$ saddle point

For inconclusive point, we need to check the neighbouring points

for $(3,6)$. $LPM_1 > 0$. $LPM_2 > 0$. $\therefore$ local min.

for $(0,5)$ $LPM_1 = 0$ $LPM_2 = 0$. inconclusive.

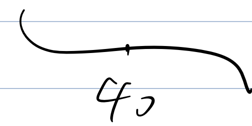
+

$$f = x_1^3(x_1-4) + 3(x_2-5)^2 + 40.$$

$\because$ for $f(0,5) = 40$

$N_1(0.1, 5) < 40$

$N_2(-0.1, 5) > 40.$



$\therefore$ $(0,5)$ saddle point.

<u>QUESTION 2</u> (30 Points)

Consider the function $f(x,y) = x^2 - xy - y^2 - x^3$. Determine the stationary points of $f(x,y)$ and mention whether they are local maximum, local minimum, and saddle points. What can you say about global minimum and maximum points?

① find all stationary points.

$\frac{\partial f}{\partial x} = 2x - y - 3x^2$. ① $\frac{\partial^2 f}{\partial x^2} = 2 - 6x$.

$\frac{\partial f}{\partial y} = -x - 2y.$ ② $\frac{\partial^2 f}{\partial y^2} = -2.$ $\frac{\partial^2 f}{\partial x \partial y} = -1.$

$$\begin{cases} 2x - y - 3x^2 = 0 \\ -x - 2y = 0 \end{cases} \Rightarrow 2y = -x. \quad y = -\frac{x}{2}.$$

$y = 2x - 3x^2.$ $2x - 3x^2 = -\frac{x}{2}.$

$$-3x^2 + 2x + \frac{x}{2} = 0.$$

$$-3x^2 + x\left(2 + \frac{1}{2}\right) = 0$$

$x_1 = \frac{5}{6},$ $x_2 = 0.$

$\therefore$ candidate points: $(0,0)$ and $(\frac{5}{6}, -\frac{5}{12})$

to cheen local max, local min etc.

we need H.Matrix.

$\therefore H(x,y) = \begin{bmatrix} 2-6x & -1 \\ -1 & -2 \end{bmatrix}$

① $LPM_1 > 0$, $LPM_2 > 0$ local min,

② $LPM_1 < 0$, $LPM_2 > 0$ local max.

not ①/② and $LPM_2 = 0$ not in, $LPM_2 \neq 0$ saddle

for point $(0,0)$ $LPM_1 > 0$, $LPM_2 < 0$

$\therefore (0,0)$ saddle point

for point $(\frac{5}{6}, -\frac{5}{12})$ $LPM_1 = -3 < 0$

$LPM_2 = -3x-2 - 1\times 1$

$= 6 > 0.$

$\therefore \left(\frac{5}{6}, -\frac{5}{12}\right)$ local max.

To say about global max or min:

for $f(x,y) = x^2 - xy - y^2 - x^3$.

is the highest

if $x \to +\infty$, $f(x,y) \to -\infty$,

if $x \to -\infty$, $f(x,y) \to +\infty$

$\Rightarrow$ there is not any global max or min for this.

QUESTION 3 **(40 Points)**

Consider the following NLP where ln denotes the natural logarithm:

$$\max\ f(x,y) = \ln(1+x) - y^2$$

s.t.

$$x + 2y \le 3$$

$$y \ge 0$$

a) Write down the KKT conditions to show that (0,0) is not a local maximum solution. (20)

KKT: ① write lambda.

② write KKT conditions, ① $\frac{\partial f}{\partial x} = 0$

② original

③ complementary

④ λ (or μ) ≥ 0.

$\max f = \ln(1+x) - y^2$.

s.t. $x + 2y \leq 3$.

$-y \leq 0$

$[\ln(1+x)]' = \frac{1}{1+x}$.

$$L = \ln(1+x) - y^2 + \mu_1(3-x-2y) + \mu_2 \cdot y$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\frac{\partial L}{\partial x} = \frac{1}{1+x} - \mu_1 = 0 \quad ①$$

$$\frac{\partial L}{\partial y} = -2y + \mu_2 - 2\mu_1 = 0 \quad ②$$

$$x+2y \leq 3 \quad ③$$

$$y \geq 0 \quad ④$$

$$\mu_1(3-x-2y) = 0 \quad ⑤$$

$$\mu_2 y = 0 \quad ⑥$$

$$\mu_1 \geq 0 \quad ⑦$$

$$\mu_2 \geq 0 \quad ⑧$$

①

a) Write down the KKT conditions to show that (0,0) is not a local maximum solution. (20)

show (0,0) is not local max.

$\Downarrow$

show that (0,0) does not satisfy KKT conditions

from ① $(x=0, y=0)$: $1-\mu_1=0. \Rightarrow \mu_1=1$.

from ② $\mu_2=2\mu_1$

from ③ $3\mu_1=0$.

①,②,③ does not satisfy at the time.

$\therefore$ (0,0) is not local max.

b) Find a local optimal solution and mention why it is also a global optimal solution. (20)

We need to show: $\frac{\partial^2 f}{\partial x} = -\frac{1}{(1+x)^2}$

$$\frac{\partial^2 f}{\partial y} = -2.$$

$$H = \begin{bmatrix} -\frac{1}{(1+x)^2} & 0 \\ 0 & -2 \end{bmatrix}$$

$PM_1^1 = -\frac{1}{(1+x)^2} < 0$

$PM_1^2 < 0$.

$PM_2 > 0$ $\therefore$ f is concave.

and feasible region is convex.

$\therefore$ any local optimal is global optimum.

Because we have 2 of μs. μ_1 and μ_2.

we have 4 possibilities

case ① $\mu_1 = 0$. $\mu_2 = 0$. do not satisfy KKT ✗.

case ② $\mu_1 = 0$, $\mu_2 > 0$. from ① $\frac{1}{1+x} \neq 0$. Not possible.

case ③ $\mu_1 > 0$, $\mu_2 = 0$. ② $-2y - 2\mu_1 = 0$.

$\mu_1 = -y$.

④ $y \geq 0$. $\Rightarrow \mu_1 < 0$. Not possible.

case ④ $\mu_1 > 0$, $\mu_2 > 0$.

$\mu_1 = 0 < \begin{matrix} \mu_2 = 0 \\ \mu_2 > 0 \end{matrix}$

$\mu_1 > 0 < \begin{matrix} \mu_2 = 0 \\ \mu_2 > 0 \end{matrix}$

⑤ from ⑤ $\begin{cases} x+2y=3 \\ y=0 \end{cases} \Rightarrow \begin{cases} x=3 \\ y=0 \end{cases}$.

$\therefore$ $(3,0)$ is a local max and also a global max

$\mu_1 = \frac{1}{4}$.

$\mu_2 = \frac{1}{2}$