Home Assignment 1

1. Consider two identical hedgehogs starting at the vertices A and B of a polygon $ABCDE$. Each minute they simulteneously and independently choose to go clockwise or counter-clockwise in the next vertex.

   The brotherhood of two brave hedgehogs can be in three states: in one vertex, in two adjacent vertices, in two non-adjacent vertices.

   (a) What is the probability that they will be in one vertex after 3 steps?
   (b) Write down the transition matrix of the brotherhood Markov chain.
   (c) What proportion of time the brotherhood will spend in each state in the long run?
   (d) Find the expected time until the hedgehogs meet in one vertex.

2. Each day the Random Restaurant is independently closed with probability p. If the restaurant is open then the number of clients has Poisson distribution with mean μ.

   After N days (working or non-working) the Random Restaurant will permanently close and you are right, N is random and has Poisson distribution with mean n.

   (a) Find the moment generating function of the number of clients during day 1, assuming $N \geq 1$.
   (b) Find the moment generating function of the total number of clients served in the Random Restaurant.

3. Find the probability limit plim X_n, where

$$X_n = \frac{Y_1 + 2Y_2 + 3Y_3 + \ldots + nY_n}{n^2}$$

   and Y_1, Y_2, ... are independent uniform on $[0; 1]$.

   Hint: try to calculate $\mathbb{E}(X_n)$, $\mathbb{V}\text{ar}(X_n)$. You may google the formulas for $1+2+\ldots+n$ and $1^2+2^2+\ldots+n^2$ or ask ChatGPT.

4. Consider the Poisson arrival process X_t with constant rate λ.

   Now let's scale the time in a non-linear fashion, $Y_t = X_{t^2}$.

   (a) Find $\mathbb{E}(Y_t)$, $\mathbb{V}\text{ar}(Y_t)$, $\mathbb{P}(Y_t = 0)$.
   (b) Find $\mathbb{E}(Y_{t+5} \mid Y_t)$ and $\mathbb{V}\text{ar}(Y_{t+5} \mid Y_t)$.

5. Let's toss a dice until the first six appears. Let X be the result of the first toss and Y — the total number of tosses.

   (a) Find $\mathbb{E}(X \mid Y)$, $\mathbb{E}(Y \mid X)$.
   (b) Find $\mathbb{V}\text{ar}(X \mid Y)$, $\mathbb{V}\text{ar}(Y \mid X)$.

6. The joint distribution of X and Y is given in the table

	$X = -1$	$X = 0$	$X = 1$
$Y = 0$	0.1	0.2	0.3
$Y = 1$	0.2	0.1	0.1

   (a) Explicitely find the σ-algebras $\sigma(X)$, $\sigma(Y)$, $\sigma(X \cdot Y)$.
   (b) How many elements are there in $\sigma(X, Y)$, $\sigma(X + Y)$, $\sigma(X, Y, X + Y)$?

Home Assignment 2

Hereinafter (W_t) is a standard Wiener process.

1. Some questions about Wiener process!
   (a) Find $\mathbb{E}(W_7 \mid W_5)$, $\mathbb{V}\text{ar}(W_7 \mid W_5)$, $\mathbb{E}(W_7 W_6 \mid W_5)$.
   (b) Find $\mathbb{E}(W_5 \mid W_7)$, $\mathbb{V}\text{ar}(W_5 \mid W_7)$.
2. Let (W_t) be a standard Wiener process and $Y_t = W_t^3 + t^2 W_t^2$.
   (a) Find $\mathbb{E}(Y_t)$ and $\mathbb{V}\text{ar}(Y_t)$.
   (b) Is Y_t a martingale?
   (c) Find $\mathbb{E}(Y_t \mid W_s)$ for $t \geq s$.
3. Consider two independent Wiener processes A_t and B_t. Check whether these processes are Wiener processes:
   (a) $X_t = (A_t + B_t)/2$.
   (b) $Y_t = (A_t + B_t)/\sqrt{2}$.
4. Using Ito's lemma find dX and the corresponding full form.
   (a) $X_t = W_t^6 \cos t$.
   (b) $X_t = Y_t^3 + t^2 Y_t$ where $dY_t = W_t^2 dW_t + t W_t dt$.
5. Consider $I_t = \int_0^t W_u^2 u^2 du$. Find $\mathbb{E}(I_t)$, $\mathbb{V}\text{ar}(I_t)$ and $\mathbb{C}\text{ov}(I_t, W_t)$.
6. Let $X_t = 42 + t^2 W_t^3 + t W_t^2 + \int_0^t 3 W_u du + \int_0^t W_u^3 dW_u$.
   (a) Find dX_t.
   (b) Is X_t a martingale? Is $Y_t = X_t - \mathbb{E}(X_t)$ a martingale? Provide a short argument for your answer.

Home Assignment 3

1. Consider two-period binomial tree model without dividents. Initial stock price is $S_0 = 200$, in each period the stock price is multiplied by $u = 1.15$ or by $d = 0.75$. One period interest rate is $r = 0.05$.

   (a) Find the risk-neutral probability.

   (b) Price the following binary option: at time $T = 2$ you get 100\$ if $S_1 > 200$ and nothing otherwise.

   (c) Price the following chooser option: at $t = 1$ the owner of the option decides whether the option is call or put. The strike price is $K = 200$ and expiry date is $T = 2$.

2. In the framework of Black and Scholes model find the price at $t = 0$ of the following two financial assets, $dS_t = \mu S_t dt + \sigma S_t dW_t$ is the share price equation.

   (a) The asset pays you at time T exactly one dollar if $S_T < K$ where K is a constant specified in the contract.

   (b) The asset pays you at time T exactly S_T^2 dollars.

3. Consider the Vasicek interest rate model,

$$dR_t = 5(0.06 - R_t)\,dt + 3\,dW_t, \quad R_0 = 0.07.$$

   Here R_t is the interest rate.

   (a) Using the substitution $Y_t = e^{at}R_t$ find the solution of the stochastic differential equation. Start by finding dY_t.

   (b) Find $\mathbb{E}(R_t)$ and $\mathbb{V}\text{ar}(R_t)$.

   (c) Which value in this model would you call long-term equilibrium rate and why?

   Hint: you may have integrals in you expression for R_t, but no R_t.

4. Let X_t be the exchange rate measured in roubles per dollar. We suppose that $dX_t = \mu X_t dt + \sigma X_t dW_t$.

   Consider the inverse exchange rate $Y_t = 1/X_t$ measured in dollars per rouble.

   Write the stochastic differential equation for dY_t. The equation may contain Y_t and constants, but not X_t.

5. Solve the stochatic differential equation

$$dY_t = -Y_t dt + dW_t,\ Y_0 = 1$$

   If you are have no clues you may try a substitution $Z_t = f(t)Y_t$. Do not forget that the final answer may contain integrals that can't be calculated explicitely. It's ok.

Home Assignment 4

1. The process (u_t) is a white noise with variance $\mathbb{V}\mathrm{ar}(u_t) = \sigma^2$. Consider the process $b_t = t^2+6t+(1-2L)^2 u_t$.

   (a) Write explicit expression for (b_t) without lag operator L.

   (b) Find $\mathbb{E}(b_t)$ and $\mathbb{V}\mathrm{ar}(b_t)$.

   (c) Find $\mathbb{C}\mathrm{ov}(b_t, b_{t-k})$ and $\mathbb{C}\mathrm{orr}(b_t, b_{t-k})$.

   (d) Is the process (b_t) weakly stationary?

2. Let (u_t) be a white noise process with variance $\mathbb{V}\mathrm{ar}(u_t) = \sigma^2$ and

$$y_t = 1 + u_t + 0.7u_{t-1} + 0.7^2 u_{t-2} + 0.7^3 u_{t-3} + \ldots$$

   (a) Find $\mathbb{E}(y_t)$, $\mathbb{V}\mathrm{ar}(y_t)$.

   (b) Find $\mathbb{C}\mathrm{ov}(y_t, y_{t-k})$.

   (c) Is (y_t) weakly stationary?

   (d) Sketch the autocorrelation function of (y_t) if it is weakly stationary.

3. Provide an example of two dependent processes (a_t) and (b_t) such that each of them is weakly stationary, but their sum is not weakly stationary.

4. Consider three variables (y_1, y_2, y_3) that are jointly normal

$$y \sim \mathcal{N}\left(\begin{pmatrix} 2 \\ 6 \\ 11 \end{pmatrix}; \begin{pmatrix} 16 & 0 & -1 \\ 0 & 4 & 1 \\ -1 & 1 & 4 \end{pmatrix}\right).$$

   Find $\mathbb{C}\mathrm{orr}(y_1, y_2)$ and partial correlation $\mathrm{p}\mathbb{C}\mathrm{orr}(y_1, y_2; y_3)$.

5. Let $y_t = 5 + u_t + u_{t-1} + u_{t-2}$ where (u_t) is a white noise with variance $\mathbb{V}\mathrm{ar}(u_t) = \sigma^2$.

   (a) Is the process (y_t) weakly stationary?

   (b) Find the autocorrelation function ρ_k for this process.

   (c) Find the first two values of the partial autocorrelation function, ϕ_{11} and ϕ_{22}.

6. [bonus] Variables u_1 and u_2 are independent $\mathcal{N}(0; 1)$. Consider the process $y_t = u_1 \cos(\pi t/2) + u_2 \sin(\pi t/2)$.

   (a) Find $\mathbb{E}(y_t)$, $\mathbb{V}\mathrm{ar}(y_t)$, $\gamma_k = \mathbb{C}\mathrm{ov}(y_t, y_{t+k})$.

   (b) Is (y_t) weakly stationary?

   (c) Can (y_t) be represented as $MA(\infty)$ process with respect to *some* white noise, not necessary (u_t)?

   Your know additionally that $y_{100} = 0.2024$.

   (d) What is your best point prediction for y_{104}?

   (e) What is the shortest prediction interval that covers y_{104} with at least 95%-probability?

Home Assignment 5

1. Consider the stationary $AR(1)$ process with equation $y_t = 5 + 0.3y_{t-1} + u_t$ where (u_t) is a white noise.
   (a) Find all values of the autocorrelation function ρ_k.
   (b) Find all values of the partial autocorrelation function ϕ_{kk}.
2. Consider the stationary $AR(2)$ process with equation $y_t = 5 + 0.3y_{t-1} - 0.02y_{t-2} + u_t$ where (u_t) is a white noise.
   (a) Find the first two values of the autocorrelation function: ρ_1 and ρ_2.
   (b) Find all values of the partial autocorrelation function ϕ_{kk}.
3. Consider the stationary $ARMA(1,1)$ process with equation $y_t = 1 + 0.5y_{t-1} + u_t - 0.7u_{t-1}$ where (u_t) is a white noise.
   (a) Find μ, δ_1 and δ_2 in the $MA(\infty)$ representation of the process
   $$y_t = \mu + u_t + \delta_1 u_{t-1} + \delta_2 u_{t-2} + \ldots$$
   (b) Assume that $\mathbb{V}\text{ar}(u_t) = 9$, $u_{100} = -2$ and $y_{100} = 3$. Find 95% predictive interval for y_{101} and y_{102}.
4. The semi-annual y_t is modelled by $ETS(AAA)$ process:
   $$\begin{cases} u_t \sim \mathcal{N}(0;4) \\ s_t = s_{t-2} + 0.1u_t \\ b_t = b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + b_{t-1} + s_{t-2} + u_t \end{cases}$$
   Given that $s_{100} = 2$, $s_{99} = -1.9$, $b_{100} = 0.5$, $\ell_{100} = 4$ find 95% predictive interval for y_{102}.
5. Consider the equations (A) $y_t = 4 + 0.6y_{t-1} - 0.08y_{t-2} + u_t$, (B) $y_t = 4 + 0.6y_{t-1} + 1.6y_{t-2} + u_t$ and (C) $y_t = 4 + 0.6y_{t-1} + 0.4y_{t-2} + u_t$. Assume that (u_t) is a white noise.
   (a) How many non-stationary solutions does each equation have?
   (b) How many stationary solutions does each equation have?
   (c) How many stationary solutions that are $MA(\infty)$ with respect to (u_t) does each equation have?
6. (bonus) Consider the process
   $$y_t = \frac{1 - 0.5F}{1 - 0.5L} u_t,$$
   where (u_t) is a white noise and F is the forward operator, $Fu_t = u_{t+1}$.
   (a) Write explicit expression for (y_t) without lag nor forward operator.
   (b) Is (y_t) a white noise?

Bonus Home Assignment

1. The variables $X_1, \ldots, X_n$ are independent identically distributed with $\mathbb{E}(X_i) = 3\alpha + 7$ and $\mathbb{E}(X_i^2) = \alpha^2 + 3\beta$.
   (a) Find the method of moments estimator of α that uses the first moment.
   (b) Find the method of moments estimator of α and β that uses the first two non-central moments.

2. The variables $X_1, \ldots, X_n$ are independent identically distributed with density

$$f(x) = \begin{cases} \lambda \exp(-\lambda(x-\theta)), \text{ if } x \geq \theta \\ 0, \text{ otherwise.} \end{cases}$$

   (a) Find the maximum likelihood estimator of λ for known value $\theta = 1$.
   (b) Find the maximum likelihood estimator of θ for known value $\lambda = 1$.

3. The random variables $X_1, \ldots, X_n$ are independent identically distributed with $\mathbb{P}(X_i = k) = (a-1)^{k-1}/a^k$ for $k \in \{1, 2, \ldots\}$.
   (a) Find $\mathbb{E}(X_i)$.
   (b) Is $\hat{a} = \bar{X}$ an unbiased estimator for a?
   (c) Is $\hat{a} = \bar{X}$ a consistent estimator for a?

4. The random variables $X_1, \ldots, X_n$ are independent identically distributed with $\mathbb{P}(X_i = k) = (a-1)^{k-1}/a^k$ for $k \in \{1, 2, \ldots\}$.
   (a) Find the Fisher information contained in n observations.
   (b) Does $\bar{X}$ attain the Cramer-Rao lower bound? Hint: there is an easy solution that avoids calculation of $\mathbb{E}(\bar{X})$ and $\mathbb{V}\mathrm{ar}(\bar{X})$.
   (c) What is the Cramer-Rao lower bound for an estimator $\hat{a}'$ with $\mathbb{E}(\hat{a}') = 0.9a + 0.20240425$?

5. The random variables $X_1, \ldots, X_n$ are independent identically distributed with $\mathbb{P}(X_i = k) = (a-1)^{k-1}/a^k$ for $k \in \{1, 2, \ldots\}$.

   Find two different sufficient statistics for a.

6. The variables X_1 and X_2 are independent and identically distributed. You would like to test H_0: $X_i \sim U[0;1]$ against alternative H_1: the density function is given by $f(x) = 2x$ for $x \in [0;1]$.
   (a) Using Neyman-Pearson lemma design the most powerful test with 0.1 significance level.
   (b) Do you reject H_0 for $x_1 = 0.7$ and $x_2 = 0.6$ with the test from point (a)?
   (c) What is the power of the test from point (a)?