

A Spectral Construction of Natural-Number Arithmetic

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ABSTRACT: We construct natural-number arithmetic on an ordered family of correlation modes and retain the expressions by which its elements are generated. Minimal unbiased compositional change determines a rotation plane and a Gaussian law. Local information and total normalized correlation determine a finite window; preservation of constants selects its Neumann closure within the separated homogeneous local boundary class. Modulation and composition of the oriented modes define an ordered Peano semiring with unique factorization. The modes retain quadratic energy and canonical scale transport, while winding degree certifies the arithmetic. Ordered expressions have finite evaluation fibres. Equal root-channel weights and independent subhistory assembly determine a positive history law whose Gaussian marginal is the original mode mass. Conditional readout separates value information from internal history variation by exact variance and entropy identities. Using the same weights, parent–child assembly defines a nonnegative self-adjoint trace-class operator with explicit Gaussian cutoff bounds. Its contraction evolution couples value coefficients to history differences with zero initial readout: an exact matrix element at value four proves this coupling on the complete history space. The coupling persists under small changes of the relevant conditional weights. Eliminating the history differences gives an exact integral equation for the value response. Lean formalizations accompany the arithmetic construction, history-source recursion and assembly results.

KEYWORDS: Construction of natural numbers; Arithmetic expression trees; Conditional expectation; Trace-class operators; Non-Markovian dynamics; Memory kernels

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1 Introduction

The natural numbers may be characterized by successor, represented by sets, specified by an initial property or introduced through inductive constructors [1–4]. We construct arithmetic on modes selected by an analytic correlation law. Ordered expressions record the inputs, operations and brackets by which a mode is assembled; evaluation identifies expressions with the same value. The question is whether the information lost in this identification can affect a subsequent value observation. We construct an operator from the parent–child assembly relations and exhibit a history difference with zero initial value readout and nonzero subsequent response.

The construction takes place in standard analysis and operator theory, where ordinary natural numbers serve as dimensions, finite indices and induction parameters. The constructed carrier is a mode family: correlation response specifies its order and distinguished elements, while function operations define successor, addition and multiplication. Spectral coordinates and the classification of circle endomorphisms then prove its laws; winding degree certifies its arithmetic.

Theorem 5.4 identifies this family as an ordered Peano semiring with unique factorization. Its operations are modulation and cascade. The arithmetic isomorphisms under scale and orientation changes preserve normalized quadratic energy and intertwine the Gaussian responses. Prime residue classes also give the augmentation cotangent basis.

To retain construction data, we resolve each value into a finite fibre of ordered expressions. Equal root-channel weights and independent subhistory assembly specify a positive probability law; phase invariance determines its finite phase extension. Gaussian-weighted summation over histories and phases recovers the original mode mass exactly. Theorem 8.3 gives this source and its conditional readout, including variance and entropy identities that separate root information from internal history variation.

The main analytic result is the assembly evolution of Theorem 11.4. The same source weights on the actual parent-child relations define a nonnegative self-adjoint trace-class operator on the complete counting history space. Its layer traces have an explicit Gaussian tail independent of fibre cardinality. A difference supported on the six histories of value four has a strictly nonzero value response under this infinite operator. Proposition 11.3 preserves the coupling under small changes of the relevant conditional weights. Eliminating the difference component gives an exact integral equation for the value response, so the retained history enters the evolution after the initial value has been read.

An elementary example distinguishes value from history. Six unit leaves joined by additions and the product of additive expressions for three and two have the same value but different leaf counts and local assembly defects. Both histories have positive source probability, so leaf count has positive conditional variance. Their Gaussian responses agree because leaf energy and accumulated defects sum to the same root energy. The history law retains these separate contributions; the assembly operator acts on them.

The analytic source begins with reversible orthogonal change. The rotation plane has the least real carrier dimension among faithful irreducible actions of nontrivial compact connected groups, and orientation fixes its complex structure. Independent composition and exact same-shape self-similarity under unit covariance force square-root scaling and the centered Gaussian law.

Local distinguishability and total normalized correlation mass determine the local and global radii on the displacement line, with ultraviolet and infrared endpoint roles. Preservation of constants selects the Neumann closure within the separated nondegenerate homogeneous local boundary class. Correlation response orders the reflected spectrum before enumeration. Orientation and the rotation generator's nonnegative spectral projection select the analytic modes, on which pointwise modulation and map composition close.

The degree arithmetic of circle endomorphisms is classical [5–7]. Rasekh constructs a natural-number object from the circle's loop space in an elementary higher topos [8]. In the present construction the preceding correlation geometry supplies the circle and its positive modes. It also determines their response order and inner product together with their energy. Arithmetic certification retains their canonical transport. Ordered expression assembly resolves the different constructions identified by arithmetic evaluation. Arithmetic formula trees and their enumeration have an established theory [9]. Here the expression fibres carry a recursive source law with the original Gaussian marginal and a parent-child response using those same weights. Appendix F compares these roles with the standard constructions and with direct circle arithmetic.

Formal differences and fractions give the ordered rational field, and metric completion gives real and complex scalars. Finite rational mode families form a certified dense core

of the correlation Hilbert space. Euclidean division supplies finite residue rings and their orientation-normalized additive characters. The approximation estimates distinguish finite arithmetic certificates from spectral support and from norm control of completed states.

The arithmetic branch of Lean Formalization of Arithmetic and Basic Physical Interfaces [10] accompanies the arithmetic construction, expression normalization and evaluation, history and phase sources, and assembly results. The theorem–manuscript correspondence records the formal declarations; the axiom audit records their dependencies.

Figure 1 displays the passage from the correlation source to arithmetic modes and their history response.

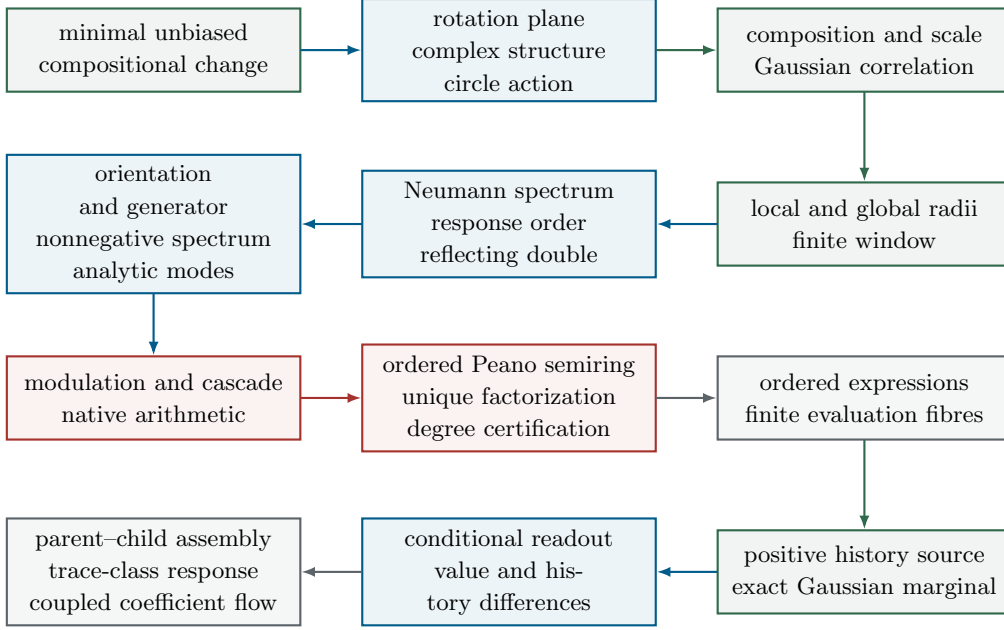


Figure 1. The correlation construction and its history response. The source geometry selects the ordered modes. Modulation and cascade generate arithmetic on them. Degree provides numerical certification. Ordered expressions resolve distinct assemblies before evaluation. Their positive law has the original Gaussian marginal. Conditional readout separates value information from history differences. Parent–child incidence couples these components in the complete assembly response.

2 Minimal change and Gaussian correlation

Minimality selects the least-dimensional real carrier of a faithful, irreducible, nontrivial reversible action. Independent composition and exact self-similarity then determine the accumulation law. These two selection problems give the rotation plane and the Gaussian law, respectively; covariance fixes the dimensional unit.

2.1 The minimal real change carrier

We work with finite-dimensional real inner-product spaces, compact Lie groups, Borel probability measures and Hilbert-space operators. The carrier dimension is minimized among compact connected Lie groups acting faithfully and irreducibly over the reals.

Orthogonality keeps the norm fixed while the response direction changes. Compactness supplies a normalized invariant average over the change group. Connectedness joins each change continuously to the identity. Faithfulness makes distinct group elements distinguishable through their action. Real irreducibility means that no proper nonzero linear subspace is preserved by every change. The plane and its complex structure are obtained within this class before a complex coordinate is introduced.

These analytic arguments use finite dimensions and integer indices together with induction in their usual set-theoretic sense. The generated object is a family of modes equipped with its own order and operations. Those structures are defined on the functions and compared with the standard natural-number structure.

Definition 2.1 (Orthogonal change datum). An orthogonal change datum is a triple (V, G, ρ) in which V is a nonzero finite-dimensional real inner-product space and G is a nontrivial compact connected Lie group. The map

$$\rho : G \longrightarrow O(V) \quad (2.1)$$

is a faithful continuous representation that is irreducible over $\mathbb{R}$. The transformations $\rho(g)$ are the reversible changes. Group composition combines changes. Inversion reverses them. The identity represents no change.

Definition 2.2 (Minimal carrier). An orthogonal change datum is minimal when $\dim_{\mathbb{R}} V$ is least among all data in Definition 2.1. Minimality is applied to the real carrier before a complex coordinate or a phase group has been chosen.

Every orthogonal change datum supplies its Lie algebra $\mathfrak{g}$ with the representation metric

$$\langle X, Y \rangle_{\rho} := -\frac{1}{\dim_{\mathbb{R}} V} \operatorname{Tr}_V(d\rho(X)d\rho(Y)), \quad X, Y \in \mathfrak{g}. \quad (2.2)$$

Faithfulness makes $d\rho$ injective and thus makes this form positive definite.

Theorem 2.3 (Minimal real carrier and its complex structure). (i) *Every minimal orthogonal change datum has*

$$\dim_{\mathbb{R}} V = 2, \quad G \cong SO(2), \quad (2.3)$$

and ρ is orthogonally equivalent to the standard rotation action. For either unit generator X_0 of the one-dimensional Lie algebra set $J := d\rho(X_0)$. Then

$$J^{\top} = -J, \quad J^2 = -I_V, \quad J^{\top}J = I_V. \quad (2.4)$$

(ii) *Choosing one of the two orientations of the minimal carrier selects J . The rule*

$$(a + ic)v := av + cJv, \quad a, c \in \mathbb{R}, \quad (2.5)$$

makes V a one-dimensional complex Hilbert space. In the corresponding complex coordinate the action is

$$\rho(\exp(\theta X_0))v = e^{i\theta}v, \quad SO(2) \cong U(1). \quad (2.6)$$

Reversing the orientation replaces J by $-J$ and complex conjugates this coordinate.

Proof. In real dimension one the group $O(V) \cong O(1) = \{\pm 1\}$ has no nontrivial connected subgroup. Faithfulness therefore excludes $\dim_{\mathbb{R}} V = 1$. The standard rotation action of $SO(2)$ on $\mathbb{R}^2$ is faithful and real irreducible. The minimum is therefore two.

For a minimal datum connectedness places $\rho(G)$ in the identity component $SO(2)$ of $O(2)$. The image is compact and hence closed. A nontrivial connected closed subgroup of the circle $SO(2)$ is the whole circle. Faithfulness now identifies G with $SO(2)$. The resulting two-dimensional action is the standard rotation representation up to an orthogonal change of basis.

The differential image is $\mathfrak{so}(2)$. In an orthonormal basis its two unit generators are $\pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with respect to the metric (2.2). Each is skew-adjoint and squares to $-I_V$. It is therefore orthogonal. This proves (2.4).

Equation (2.4) gives the real-linear complex structure. The compatible Hermitian form is

$$h_J(v, w) := \langle v, w \rangle_V - i \langle Jv, w \rangle_V. \quad (2.7)$$

Throughout this paper Hermitian forms are linear in the first variable and conjugate-linear in the second. This form is Hermitian for the scalar action in (2.5) and satisfies $h_J(v, v) = \langle v, v \rangle_V > 0$ for $v \neq 0$. The exponential identity $e^{\theta J} = \cos \theta I + \sin \theta J$ identifies the rotation action with complex phase multiplication. The two unit generators give the two conjugate orientations. $\square$

2.2 Unbiased compositional change

The rotation action describes an individual reversible change. A probability law on its Lie algebra describes independent accumulation. Let $D_r X := rX$ denote dilation on the real Lie algebra of the minimal carrier.

Axiom 2.4 (Minimal Unbiased Compositional Change). There exists a minimal orthogonal change datum together with Borel probability measures $(\mu_t)_{t>0}$ on its Lie algebra $\mathfrak{g}$ such that

- (C) $\mu_s * \mu_t = \mu_{s+t}$ for $s, t > 0$.
- (S) There is a positive scale factor $a(t)$ such that $\mu_t = (D_{a(t)})_{\#} \mu_1$ for $t > 0$ and $a(1) = 1$.
- (V) μ_1 has finite second moment and covariance $I_{\mathfrak{g}}$.

Clause (C) describes the independent accumulation of changes. Clause (S) requires the accumulated law to keep the same shape after its intensity has been rescaled. The finite second moment in (V) restricts the admissible laws and makes covariance available to measure that intensity. Unit covariance then fixes its dimensional unit. Covariance additivity under (C) combines with (S) and (V) to give

$$a(s+t)^2 = a(s)^2 + a(t)^2, \quad a(1) = 1. \quad (2.8)$$

The nonnegative additive function $a(t)^2$ is therefore linear on $[0, \infty)$. The normalization gives $a(t) = \sqrt{t}$. The compositional variance law thus determines the square-root scale. The

same conditions determine the common shape, including centering and reflection invariance. They also give weak continuity: for every bounded continuous function f one has

$$\int_{\mathfrak{g}} f(X) d\mu_t(X) = \int_{\mathfrak{g}} f(\sqrt{t} X) d\mu_1(X) \longrightarrow f(0) \quad (t \downarrow 0), \quad (2.9)$$

by dominated convergence. The same argument gives weak continuity at every positive time. The term unbiased refers to the reflection symmetry forced by these composition and scaling conditions.

The exact scaling in (S) is stronger than an asymptotic central-limit statement. Together with (C) it requires every time slice to be a rescaled copy of one law. The covariance normalization in (V) removes the remaining change of units. These three requirements uniquely determine the tangent distribution.

Theorem 2.5 (Compositional Gaussian law). *The tangent law in Axiom 2.4 is uniquely*

$$d\mu_t(a) = G_{\sqrt{t}}(a) da, \quad G_{\sqrt{t}}(a) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{a^2}{2t}\right). \quad (2.10)$$

The tangent law is centered and invariant under the full orthogonal group of $(\mathfrak{g}, \langle \cdot, \cdot \rangle_\rho)$. Its covariance is $tI_{\mathfrak{g}}$ at time t .

Proof. Choose a unit coordinate on $\mathfrak{g} \cong \mathbb{R}$ using Theorem 2.3. The scale $a(t)$ in (S) gives $\text{Cov}(\mu_t) = a(t)^2 I$. Covariance adds under convolution. Thus (C) gives (2.8). The function $h(t) := a(t)^2$ is nonnegative and additive on the positive half-line. It is therefore monotone and satisfies $h(t) = th(1) = t$. Positivity of a gives $a(t) = \sqrt{t}$. Finite second moment gives a finite mean $m_t := \int x d\mu_t(x)$. Conditions (C) and (S) respectively imply

$$m_2 = 2m_1, \quad m_2 = \sqrt{2} m_1. \quad (2.11)$$

Thus $m_1 = 0$ and (S) gives $m_t = 0$ for every $t > 0$. Condition (V) gives $\int x^2 d\mu_1(x) = 1$.

Let $\varphi_t(k) := \int e^{ikx} d\mu_t(x)$. The finite second moment permits differentiation twice at the origin and gives

$$\varphi_1(h) = 1 - \frac{h^2}{2} + o(h^2) \quad (h \rightarrow 0). \quad (2.12)$$

For every positive integer N composition and scaling give the exact identity

$$\varphi_t(k) = [\varphi_{t/N}(k)]^N = [\varphi_1(k\sqrt{t/N})]^N. \quad (2.13)$$

For fixed $t > 0$ and $k \neq 0$ the last expression is $[1 - tk^2/(2N) + o(N^{-1})]^N$. This tends to $e^{-tk^2/2}$. The left-hand side is independent of N . The same equality holds at $k = 0$. Hence

$$\varphi_t(k) = e^{-tk^2/2}. \quad (2.14)$$

Uniqueness of characteristic functions proves the assertion.

The Lie algebra is one-dimensional. Its orthogonal group consists of the identity and sign reversal. The density in (2.10) is even. Its mean is zero and (S) transports the unit covariance in (V) to $tI_{\mathfrak{g}}$. $\square$

Proposition 2.6 (Independence of the normalized selection conditions). *On the fixed minimal tangent line with its prescribed metric each of the three selection conditions is independent of the other two. Deleting any one of them leaves families other than the standard Gaussian family (2.10).*

Proof. To test independence of (C) take $\mu_t = \frac{1}{2}(\delta_{\sqrt{t}} + \delta_{-\sqrt{t}})$. This family satisfies (S) and (V). The convolution $\mu_1 * \mu_1$ has mass 1/2 at zero whereas μ_2 has none.

To test independence of (S) let μ_t be the law of $P_t^+ - P_t^-$. Here P_t^+ and P_t^- are independent Poisson variables of mean $t/2$. Its characteristic function is

$$\varphi_t(k) = \exp(t(\cos k - 1)). \quad (2.15)$$

The exponent is additive in t and thus proves (C). The variance is t and thus proves (V). Condition (S) would force $a(2) = \sqrt{2}$ by the preceding variance argument. But μ_2 is supported on the integers whereas $(D_{\sqrt{2}})_\# \mu_1$ has positive mass at $\sqrt{2}$.

To test independence of (V) take $\mu_t = N(0, ct)$ with $c > 0$ and $c \neq 1$. Gaussian convolution and scaling give (C) and (S). The unit-time covariance is c instead of one. $\square$

The last example distinguishes the variance unit from the Gaussian shape. The families $N(0, ct)$ differ only by a change of tangent scale. Thus the independence statement concerns the normalized law on the fixed metric line.

The datum

$$\begin{aligned} V = \mathbb{R}^2, \quad G = SO(2), \quad \rho = \rho_{\text{rot}}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \mathfrak{g} = \mathbb{R}, \quad d\mu_t(a) = G_{\sqrt{t}}(a) da \end{aligned} \quad (2.16)$$

satisfies Axiom 2.4. Theorem 2.3 gives minimality and the complex coordinate realizing the $U(1)$ action. Gaussian convolution verifies (C). Quadratic scale covariance verifies (S). Unit-time variance verifies (V).

2.3 Tangent morphisms and correlation states

Identify $\mathfrak{g}$ with its affine line. A tangent change of size a is then the translation automorphism

$$\tau_a(x) := x + a, \quad \tau_a \circ \tau_c = \tau_{a+c}, \quad \tau_a^{-1} = \tau_{-a}. \quad (2.17)$$

The identity τ_0 represents no change. An individual nonzero tangent change is directed whereas the Gaussian law is unbiased under $a \leftrightarrow -a$.

The tangent morphisms have the strongly continuous unitary representation

$$U(\tau_a) : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \quad (U(\tau_a)f)(y) := f(y - a). \quad (2.18)$$

Their ensemble average is

$$C(t) := \int_{\mathbb{R}} U(\tau_a) d\mu_t(a) = e^{-tA/2}, \quad A := -\frac{d^2}{dy^2}. \quad (2.19)$$

The Gaussian convolution identity gives

$$\mu_s * \mu_t = \mu_{s+t}, \quad C(s)C(t) = C(s+t). \quad (2.20)$$

The averaged tangent changes form a Markov semigroup. Each $U(\tau_a)$ is unitary with inverse $U(\tau_{-a})$ whereas $C(t)$ is a positive contraction for $t > 0$. Its Fourier multiplier is $e^{-t\xi^2/2}$. The reciprocal $e^{t\xi^2/2}$ is unbounded. Thus the averaged evolution has no bounded inverse at positive time.

Write

$$t := b^2. \quad (2.21)$$

The unit-mass kernel and unit-peak correlation profile are

$$G_b(d) := \frac{1}{\sqrt{2\pi}b} \exp\left(-\frac{d^2}{2b^2}\right), \quad (2.22)$$

$$K_b(d) := \sqrt{2\pi}b G_b(d) = \exp\left(-\frac{d^2}{2b^2}\right). \quad (2.23)$$

The normalized Hilbert representative centred at x is

$$g_{b,x}(y) := \left(\frac{2}{\pi b^2}\right)^{1/4} \exp\left[-\frac{(y-x)^2}{b^2}\right], \quad \|g_{b,x}\|_2 = 1. \quad (2.24)$$

For the signed displacement $\delta := x - x' \in \mathbb{R}$ and its associated distance $d := |\delta|$ the even profile satisfies

$$\langle g_{b,x}, g_{b,x'} \rangle = K_b(\delta) = K_b(d). \quad (2.25)$$

Hence $g_{b,x}$ is the normalized correlation state associated with the primitive change morphism τ_x .

The normalization and composition laws expressed through the width parameter are

$$\int_{\mathbb{R}} G_b(d) dd = 1, \quad G_{b_1} * G_{b_2} = G_{\sqrt{b_1^2 + b_2^2}}. \quad (2.26)$$

Increasing t corresponds to further composition. The de Bruijn identity relates the entropy derivative along Gaussian convolution to Fisher information. For probability densities of finite Fisher information Shannon entropy is nondecreasing along the semigroup [11, 12].

2.4 Entropy and Fourier characterizations

The compositional Gaussian law also has the classical entropy and Fourier characterizations at its prescribed mean and variance.

Among probability densities on $\mathbb{R}$ with mean zero and variance t the Gaussian density $G_{\sqrt{t}}$ also uniquely maximizes differential entropy [12]. For any such density p relative entropy gives

$$D(p \| G_{\sqrt{t}}) = \frac{1}{2} \log(2\pi et) - h(p) \geq 0. \quad (2.27)$$

Equality holds exactly when $p = G_{\sqrt{t}}$ almost everywhere.

The classical central limit theorem gives the corresponding accumulation statement. Normalized sums of independent identically distributed centered changes with finite nonzero variance converge to the same Gaussian law [13].

For the dimensionless coordinate $x = d/(\sqrt{2\pi}b)$ one has

$$G_b(d) dd = e^{-\pi x^2} dx. \quad (2.28)$$

Under the period-one Fourier convention $\mathcal{F}_{2\pi}f(\xi) = \int f(x)e^{-2\pi i x \xi} dx$ this representative is self-dual. The same convention gives

$$\sigma_x(f) \sigma_\xi(\hat{f}) \geq \frac{1}{4\pi}, \quad (2.29)$$

and equality holds precisely for Gaussians obtained by translation and modulation together with dilation [14]. After centering and removal of modulation the equality family is reduced by Fourier self-duality and unit-mass normalization to $e^{-\pi x^2}$. The factor 2π records the Fourier character convention. These analytic characterizations agree with the compositional law. Theorem 2.3(ii) supplies its compact phase carrier independently of the coordinate used in the Fourier transform.

3 The finite correlation window

The Gaussian law gives a correlation profile on the full displacement line. A finite spectral problem is obtained from its local loss of correlation and its total normalized correlation mass. The first measures the displacement needed to resolve a unit of information. The second records the length carrying the full correlation mass at unit peak. Additive information and linear capacity determine the corresponding local and global radii. We write them as $\ell_{UV}(b)$ and $L_{IR}(b)$ when using them as endpoints of the active spectral carrier. Both calculations retain the complete Gaussian tail.

Preservation of the constant state selects the Neumann closure within the separated nondegenerate homogeneous local boundary class. The selected realization is self-adjoint and has zero endpoint flux. The interval fixes the spectral energy unit and the two response times. Appendix A gives the Gaussian integrals and reflecting-kernel formulas.

3.1 Local information and global correlation mass

The profile K_b has unit peak and is strictly decreasing for $d > 0$. It is integrable. Under the convention $\hat{f}(k) = \int_{\mathbb{R}} f(d)e^{-ikd} dd$ its Fourier transform is

$$\hat{K}_b(k) = \sqrt{2\pi}b \exp\left(-\frac{b^2 k^2}{2}\right) > 0. \quad (3.1)$$

Hence K_b is continuous and positive definite by Bochner's theorem [7].

Let $F : (0, 1] \rightarrow [0, \infty)$ satisfy

$$F(rs) = F(r) + F(s). \quad (3.2)$$

The value $F(1) = 0$ follows by applying this identity at $(r, s) = (1, 1)$. Set $H(x) := F(e^{-x})$ for $x \geq 0$. Equation (3.2) gives $H(x + y) = H(x) + H(y)$. Nonnegativity $H \geq 0$ makes it monotone on the nonnegative line. The standard monotone solution of the Cauchy equation therefore gives $H(x) = cx$ and hence $F(r) = -c \log r$ with $c \geq 0$. Strict detectability gives $c > 0$ while the unit calibration $F(e^{-1}) = 1$ selects $c = 1$. Continuity follows from the composition law and nonnegativity.

The calibrated local information of the Gaussian profile is therefore

$$\mathcal{D}_b(d) := -\log \frac{K_b(d)}{K_b(0)} = \frac{d^2}{2b^2}. \quad (3.3)$$

It is dimensionless and vanishes only at the correlation origin. It is strictly increasing for $d > 0$. Product composition adds its values through

$$-\log(K_1 K_2) = (-\log K_1) + (-\log K_2). \quad (3.4)$$

The calibration fixes the unit in which logarithmic information is measured. The Gaussian profile and its shape are preserved under this calibration.

The global quantity is determined by translation-invariant mass on the additive displacement group. Normalize Haar measure by $m([0, 1]) = 1$. Haar uniqueness on the locally compact additive group identifies m with Lebesgue measure [7]. For a nonzero nonnegative profile $K \in C(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ define

$$\mathcal{C}(K) := \int_{\mathbb{R}} K(d) dm(d), \quad \mathcal{L}(K) := \frac{\mathcal{C}(K)}{\|K\|_{L^\infty(\mathbb{R})}}. \quad (3.5)$$

The first functional is the normalized translation-invariant correlation mass. The second expresses that mass as the width of a constant profile at the same peak height.

The functional $\mathcal{L}$ is invariant under positive amplitude scaling and translation. For $K_a(d) := K(d/a)$ with $a > 0$ it satisfies

$$\mathcal{L}(K_a) = a\mathcal{L}(K). \quad (3.6)$$

The underlying linear mass $\mathcal{C}$ is unique among positive locally finite translation-invariant Borel masses with the stated unit normalization. Amplitude scaling multiplies both terms in the quotient by the same constant. Translation preserves Haar mass and the supremum norm. A change of variables gives $\mathcal{C}(K_a) = a\mathcal{C}(K)$ while the peak remains unchanged. The last assertion is precisely the uniqueness of normalized Haar measure on $(\mathbb{R}, +)$ [7].

Linearity is essential to the meaning of $\mathcal{C}$. It expresses the additivity of correlation content over disjoint regions. Translation invariance then determines Haar mass. Nonlinear L^p widths describe the concentration of a profile and belong to a different class of observables. The uniqueness assertion holds within the additive capacity class.

Applied to the same profile local information fixes its unit local radius and global mass fixes its total correlation radius. The two calculations answer different questions. One compares a translated profile with the original at a specified loss level. The other integrates correlation over the entire displacement line. Both are already lengths on the

native displacement coordinate of K_b and therefore serve directly as endpoints. Each radius is unique within its defining functional class. The local class consists of continuous additive information with the stated unit calibration. The global class consists of translation-invariant correlation mass expressed at fixed peak height.

3.2 The ultraviolet endpoint

The UV endpoint is the shortest separation at which two translated Gaussian states reach one unit of logarithmic distinguishability.

For the profile K_b this distance is

$$\ell_{\text{UV}}(b) := \inf\{d > 0 : \mathcal{D}_b(d) \geq 1\}. \quad (3.7)$$

At information level $h > 0$ the same calculation gives the radius $\sqrt{2h}b$. The endpoint in (3.7) uses one unit of the calibrated information coordinate; changing the level changes the distinguishability being measured.

Proposition 3.1 (Intrinsic UV and IR correlation radii). *The local-information and global-capacity radii are unique within their stated functional classes. The ultraviolet endpoint is*

$$r_b^{\text{geom}} := \ell_{\text{UV}}(b) = \sqrt{2}b. \quad (3.8)$$

At this separation one has

$$\mathcal{D}_b(\ell_{\text{UV}}) = 1, \quad K_b(\ell_{\text{UV}}) = e^{-1}. \quad (3.9)$$

The infrared endpoint is obtained from the same profile by the global capacity rule. The total normalized correlation capacity is

$$\mathcal{C}_b := \int_{\mathbb{R}} K_b(\delta) d\delta = \sqrt{2\pi}b. \quad (3.10)$$

The peak-equivalent width from (3.5) is therefore finite. Define

$$L_{\text{IR}}(b) := \mathcal{L}(K_b), \quad K_b(0)L_{\text{IR}}(b) = \int_{\mathbb{R}} K_b(\delta) d\delta. \quad (3.11)$$

Then

$$R_b^{\text{geom}} := L_{\text{IR}}(b) = \sqrt{2\pi}b, \quad \mathcal{D}_b(L_{\text{IR}}) = \pi, \quad K_b(L_{\text{IR}}) = e^{-\pi}. \quad (3.12)$$

Thus r_b^{geom} and $\ell_{\text{UV}}(b)$ are two notations for the same local correlation radius while R_b^{geom} and $L_{\text{IR}}(b)$ are two notations for the same global correlation radius. The geometric notation records their origin and the endpoint notation records their spectral role.

Proof. Equation (3.3) is continuous and strictly increasing on $(0, \infty)$. The UV equation therefore has the unique solution $\ell_{\text{UV}} = \sqrt{2}b$. The identity $K_b(0) = 1$ makes the capacity equation linear in L_{IR} . Substitution of (3.10) gives $L_{\text{IR}} = \sqrt{2\pi}b$. The remaining infrared identities follow from (3.3). $\square$

3.3 Information-theoretic characterization of the infrared scale

The capacity is the area of a constant profile of height $K_b(0)$ and width $\mathcal{L}(K_b)$. Hence its equivalent width is exactly the global correlation radius $R_b^{\text{geom}} = L_{\text{IR}}(b)$.

We use Rényi's entropy family [15] to give the same width an information-theoretic characterization.

Let $K \geq 0$ be a nonzero element of $L^1(\mathbb{R}, m) \cap L^\infty(\mathbb{R}, m)$ and put

$$p_K := \frac{K}{\mathcal{C}(K)}, \quad H_\alpha^{\text{R}}(p_K) := \frac{1}{1-\alpha} \log \int_{\mathbb{R}} p_K(d)^\alpha dm(d) \quad (\alpha > 1). \quad (3.13)$$

The L^α norms of the probability density $p_K \in L^1 \cap L^\infty$ converge to its essential supremum as $\alpha \rightarrow \infty$. The upper bound $\|p_K\|_\alpha \leq \|p_K\|_\infty^{1-1/\alpha}$ and the sets where $p_K > \|p_K\|_\infty - \epsilon$ give the two inequalities. The identity $H_\alpha^{\text{R}} = \alpha(1-\alpha)^{-1} \log \|p_K\|_\alpha$ together with the fixed Haar normalization $m([0, 1]) = 1$ gives

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} H_\alpha^{\text{R}}(p_K) &= H_\infty^{\text{R}}(p_K) := -\log \|p_K\|_\infty, \\ e^{H_\infty^{\text{R}}(p_K)} &= \frac{\mathcal{C}(K)}{\|K\|_\infty} = \mathcal{L}(K). \end{aligned} \quad (3.14)$$

For the generated Gaussian profile this gives

$$e^{H_\infty^{\text{R}}(p_{K_b})} = \sqrt{2\pi} b = L_{\text{IR}}(b). \quad (3.15)$$

Thus total correlation mass gives the infrared endpoint. The peak-equivalent width and the infinite-order Rényi width give the same quantity.

The interval

$$\mathcal{I}_b := [0, L_{\text{IR}}(b)] \quad (3.16)$$

is the global finite correlation carrier whose radius is the total normalized correlation length $R_b^{\text{geom}} = L_{\text{IR}}(b)$. Unit distinguishability selects the subinterval beginning at ℓ_{UV} as the active window. On any fixed finite carrier the uniform probability density is the unique maximizer of Shannon entropy [12].

3.4 Active window and reflecting closure

The active window generated by K_b is

$$\mathcal{W}_b := [\ell_{\text{UV}}(b), L_{\text{IR}}(b)] = [\sqrt{2}b, \sqrt{2\pi}b]. \quad (3.17)$$

Its length and endpoint ratio are

$$\Delta_b := L_{\text{IR}}(b) - \ell_{\text{UV}}(b) = b(\sqrt{2\pi} - \sqrt{2}), \quad \frac{L_{\text{IR}}(b)}{\ell_{\text{UV}}(b)} = \sqrt{\pi}. \quad (3.18)$$

In the distinguishability coordinate the two radii delimit the scale-independent interval

$$\mathcal{D}_b(\mathcal{W}_b) = [1, \pi], \quad (3.19)$$

while the dilation $u = d/b$ maps every active window to

$$\mathcal{W}_* = [\sqrt{2}, \sqrt{2\pi}]. \quad (3.20)$$

The full Gaussian supplies the correlation law and its scale transport; the local and global radii select the finite spectral carrier without truncating that law.

The separation-to-scale map

$$t_b(d) := \frac{d^2}{2} = b^2 \mathcal{D}_b(d) \quad (3.21)$$

assigns the two finite endpoint scales

$$t_{\text{UV}} = b^2, \quad t_{\text{IR}} = \pi b^2. \quad (3.22)$$

The capacity definition admits an equivalent dimensionless form in one variable. For a candidate outer correlation length $L > 0$ set

$$c := \left(\frac{L}{\ell_{\text{UV}}} \right)^2, \quad (3.23)$$

In this dimensionless form equality of the local and global capacity readouts is expressed as

$$F_*(c) := \sqrt{c} - \frac{\int_{\mathbb{R}} K_b(\delta) d\delta}{K_b(0)\ell_{\text{UV}}(b)} = \sqrt{c} - \sqrt{\pi} = 0. \quad (3.24)$$

The Gaussian integral and Proposition 3.1 give the second equality in (3.24). The function F_* is strictly increasing on $(0, \infty)$ and changes sign at π . Its unique zero is therefore

$$c_{\text{end}} = \pi. \quad (3.25)$$

Both endpoint lengths scale linearly under $b \mapsto \lambda b$ and $d \mapsto \lambda d$. Their squared ratio is therefore invariant. Equation (3.22) also gives $c_{\text{end}} = t_{\text{IR}}/t_{\text{UV}}$.

Let $R : (0, \infty)^2 \rightarrow (0, \infty)$ be continuous and satisfy common-scale invariance and serial composition through

$$R(\lambda x, \lambda y) = R(x, y), \quad R(x, z) = R(x, y)R(y, z). \quad (3.26)$$

If its logarithmic differential is normalized by

$$\left. \frac{\partial}{\partial \log y} \log R(x, y) \right|_{y=x} = 1, \quad (3.27)$$

then

$$R(x, y) = \frac{y}{x}. \quad (3.28)$$

Consequently the generated endpoint scales have the canonical readout $R(t_{\text{UV}}, t_{\text{IR}}) = \pi$. For uniqueness note that scale invariance gives $R(x, y) = f(y/x)$ with $f(r) := R(1, r)$. Serial composition gives $f(rs) = f(r)f(s)$. Continuity and positivity imply $f(r) = r^q$ for

a real constant q . The differential normalization fixes $q = 1$. Equation (3.22) gives the endpoint value.

The endpoint calculation fixes the interval and its length. A boundary condition is still needed to define evolution on that interval. Reflecting closure preserves the constant state. Within the separated homogeneous comparison class this requirement selects the Neumann condition. The resulting regular domain is self-adjoint. Its vanishing normal derivative gives zero endpoint flux and closes the finite carrier while conserving correlation at its ends. Let $\mathcal{H}_b := L^2(\mathcal{W}_b, dd)$. On this space define the reflecting Laplacian

$$A_b f = -f'', \quad \mathcal{D}(A_b) = \{f \in H^2(\mathcal{W}_b) : f'(\ell_{UV}) = f'(L_{IR}) = 0\}. \quad (3.29)$$

The separated homogeneous boundary conditions for a regular Sturm–Liouville operator form the comparison class [16]. The closure requirements select one member of that class. Self-adjointness is then a consequence of the selected zero-flux domain.

Proposition 3.2 (Selection of the reflecting boundary condition). *Among separated non-degenerate homogeneous local boundary conditions for the Laplacian on $\mathcal{W}_b$ conservation of constants uniquely selects the Neumann conditions in (3.29). The selected domain is self-adjoint. Its endpoint flux vanishes and its zero-energy eigenspace consists exactly of the constant functions.*

Proof. A separated homogeneous boundary condition at an endpoint has the form $af + bf' = 0$ with $(a, b) \neq (0, 0)$. Conservation requires the constant function to belong to the operator domain and thus forces $a = 0$. Nondegeneracy then gives $b \neq 0$. The condition is therefore $f' = 0$. Applying this argument at both endpoints yields the Neumann domain and zero endpoint flux. The regular Neumann realization is self-adjoint by the Sturm–Liouville boundary form. A function in the kernel has $\int_{\mathcal{W}_b} |f'|^2 = 0$ and is therefore constant. $\square$

3.5 Distance from correlation and the Dirichlet form

The tangent line already has the metric used to normalize covariance. The Gaussian profile recovers this metric on the global carrier while the reflecting generator recovers it on the finite window. On the global carrier $\mathcal{I}_b$ define

$$d_{K_b}(x, y) := b \sqrt{2 \log \frac{K_b(0)}{K_b(x - y)}}, \quad x, y \in \mathcal{I}_b. \quad (3.30)$$

The same distance can be recovered independently from the local generator. The Dirichlet form of the reflecting Laplacian is

$$\mathcal{E}_b(f, g) := \int_{\mathcal{W}_b} f'(d) \overline{g'(d)} dd, \quad \mathcal{D}(\mathcal{E}_b) = H^1(\mathcal{W}_b), \quad (3.31)$$

with carré du champ $\Gamma_b(f) = |f'|^2$. Its intrinsic distance is

$$d_{\mathcal{E}_b}(x, y) := \sup \{|f(x) - f(y)| : f \in H^1(\mathcal{W}_b) \cap C(\mathcal{W}_b), \quad \Gamma_b(f) \leq 1 \text{ a.e.}\}. \quad (3.32)$$

The correlation and Dirichlet constructions recover the same distance. The function d_{K_b} is a metric on $\mathcal{I}_b$ and satisfies

$$d_{K_b}(x, y) = |x - y| \quad (x, y \in \mathcal{I}_b). \quad (3.33)$$

On the active window its restriction agrees with the intrinsic Dirichlet metric given by

$$d_{K_b}|_{\mathcal{W}_b \times \mathcal{W}_b} = d_{\mathcal{E}_b}. \quad (3.34)$$

Consequently the two permanent endpoints are intrinsic correlation distances from the correlation origin given by

$$d_{K_b}(0, \ell_{UV}) = r_b^{\text{geom}} = \sqrt{2}b, \quad d_{K_b}(0, L_{IR}) = R_b^{\text{geom}} = \sqrt{2\pi}b, \quad (3.35)$$

and the active-window diameter is $d_{\mathcal{E}_b}(\ell_{UV}, L_{IR}) = \Delta_b$. The Gaussian identity $K_b(x - y) = K_b(0)e^{-|x-y|^2/(2b^2)}$ reduces (3.30) to $|x - y|$. This proves the first assertion and all metric axioms. If $\Gamma_b(f) \leq 1$ almost everywhere the fundamental theorem for Sobolev functions gives $|f(x) - f(y)| \leq |x - y|$ and therefore $d_{\mathcal{E}_b}(x, y) \leq |x - y|$. The affine function $f(d) = d$ belongs to $H^1(\mathcal{W}_b)$ and satisfies $\Gamma_b(f) = 1$. It gives the reverse inequality. The endpoint and diameter formulas now follow from Proposition 3.1.

The Gaussian profile and the reflecting Dirichlet form therefore recover the coordinate distance. Distances are measured in units of b while b^{-1} supplies the associated inverse-length scale.

Theorem 3.3 (Reflecting-double realization). *Let $x = d - \ell_{UV} \in [0, \Delta_b]$ and form the reflected double*

$$\widetilde{\mathcal{W}}_b := \mathbb{R}/(2\Delta_b\mathbb{Z}), \quad \theta := \frac{\pi x}{\Delta_b}, \quad z := e^{i\theta}. \quad (3.36)$$

With normalized Haar measure $d\theta/(2\pi)$ on $\mathbb{T}$ the map

$$(\mathcal{R}_b f)(e^{i\theta}) := \sqrt{\Delta_b} f\left(\ell_{UV} + \frac{\Delta_b}{\pi}|\theta|\right), \quad -\pi \leq \theta \leq \pi, \quad (3.37)$$

is a unitary from $\mathcal{H}_b$ onto the reflection-even subspace $L_{\text{ev}}^2(\mathbb{T})$. It intertwines A_b with $(\pi/\Delta_b)^2(-\partial_\theta^2)$ restricted to that subspace. On the normalized Neumann eigenfunctions it acts by

$$\mathcal{R}_b(\Delta_b^{-1/2}) = 1, \quad \mathcal{R}_b\left[\sqrt{\frac{2}{\Delta_b}} \cos\left(\frac{n\pi(d - \ell_{UV})}{\Delta_b}\right)\right] = \sqrt{2} \cos(n\theta) = \frac{z^n + z^{-n}}{\sqrt{2}} \quad (n \geq 1). \quad (3.38)$$

Proof. Even reflection doubles the integral over $[0, \Delta_b]$. Together with normalized Haar measure the factor $\sqrt{\Delta_b}$ makes (3.37) an isometry onto the even subspace. Even Sobolev H^2 -functions on the double have vanishing normal derivative at the fixed points $\theta = 0, \pi$ while every Neumann function extends evenly. Differentiating under $x = \Delta_b\theta/\pi$ proves the intertwining identity. Substitution of the Neumann eigenfunctions gives (3.38). $\square$

The orientation and the normalization of the angular generator identify the reflected double with the original compact carrier and lift this identification to the covering lines.

Theorem 3.4 (Covering compatibility of the generated carrier). *Choose the orientation that selects J in Theorem 2.3(ii). There is a unique orientation-preserving Lie-group isomorphism*

$$\mathfrak{C}_b : G \longrightarrow \widetilde{\mathcal{W}}_b, \quad \mathfrak{C}_b(\exp(\theta J)) := \left[\frac{\Delta_b}{\pi} \theta \right]_{2\Delta_b \mathbb{Z}}, \quad (3.39)$$

whose differential sends the unit angular generator J to Δ_b/π . On the universal covering lines it is induced by the linear map

$$\tilde{\mathfrak{C}}_b : \mathbb{R} \longrightarrow \mathbb{R}, \quad \tilde{\mathfrak{C}}_b(\theta) = \frac{\Delta_b}{\pi} \theta, \quad (3.40)$$

and it intertwines the quotient maps with kernels $2\pi\mathbb{Z}$ and $2\Delta_b\mathbb{Z}$. Thus the tangent line carrying the Gaussian increments and the circle carrying the reflected spectrum are the universal cover and the compact quotient of one normalized change carrier.

Proof. The exponential coordinate of $G \cong SO(2)$ has period 2π . Replacing θ by $\theta + 2\pi m$ changes the right-hand side of (3.39) by $2m\Delta_b$. Thus the map is well defined. It is a continuous group homomorphism. Its lift (3.40) maps the kernel $2\pi\mathbb{Z}$ onto $2\Delta_b\mathbb{Z}$. This proves bijectivity of the circle homomorphism. An orientation-preserving circle homomorphism with the stated differential has this lift and hence is unique. $\square$

In the reflected coordinate $x = 0$ is the translated ultraviolet endpoint $d = \ell_{UV}$. This choice fixes the base point of the compact quotient while keeping the ultraviolet endpoint distinct from the correlation origin $d = 0$ on the original displacement line.

The minimal real action fixes the circle group, while the reflecting window supplies its length scale and generator. Even reflection combines the conjugate modes z^n and z^{-n} into a real mode; orientation distinguishes the two complex representatives.

The windowed change family is

$$C_b(t) := e^{-tA_b/2}, \quad t \geq 0. \quad (3.41)$$

The corresponding unnormalized endpoint correlation operators are

$$T_b^{UV} := e^{-b^2 A_b/2}, \quad T_b^{IR} := e^{-\pi b^2 A_b/2}. \quad (3.42)$$

The Gaussian law on the covering line realizes the window response by averaging the original change action. This connects the source law to the reflecting generator before any arithmetic coordinate is introduced.

Proposition 3.5 (Correlation semigroup and Gaussian averaging). (i) *The family $(C_b(t))_{t \geq 0}$ is a strongly continuous and positive self-adjoint Markov semigroup on $\mathcal{H}_b$. For every $t > 0$ the operator $C_b(t)$ is trace class. The constant function is fixed by every $C_b(t)$.*

(ii) *Let γ_v be the centered Gaussian law of variance $v \geq 0$ on the normalized angular covering line with $\gamma_0 = \delta_0$. On $L^2(\mathbb{T})$ let $T(a)F(e^{i\theta}) = F(e^{i(\theta+a)})$ and set*

$$M(v)F := \int_{\mathbb{R}} T(a)F d\gamma_v(a). \quad (3.43)$$

The integral defines a contraction that preserves the reflection-even subspace and satisfies

$$M(v)\mathcal{R}_b = \mathcal{R}_b C_b(v/E_{b,1}), \quad E_{b,1} := (\pi/\Delta_b)^2. \quad (3.44)$$

The same identity holds after orientation reversal.

Proof. The Neumann Laplacian is nonnegative and self-adjoint. Its quadratic form is $\int_{\mathcal{W}_b} |f'(d)|^2 dd$. Thus the heat semigroup is positivity preserving and conservative. Its eigenvalues on the interval of length Δ_b are 0 and $(n\pi/\Delta_b)^2$ for $n \geq 1$. Hence for every $t > 0$ one has

$$\mathrm{Tr}(e^{-tA_b/2}) = 1 + \sum_{n \geq 1} \exp\left(-\frac{t\pi^2 n^2}{2\Delta_b^2}\right) < \infty.$$

Proposition 3.2 identifies the fixed subspace. This proves part (i).

For part (ii) translations act unitarily and strongly continuously. The Bochner integral therefore exists and is contractive. The characteristic function of the source law gives $M(v)z^n = e^{-vn^2/2}z^n$ for every integer n . The equal multipliers on z^n and z^{-n} preserve the even subspace. Equation (3.37) then identifies these multipliers with those of $C_b(v/E_{b,1})$. Density of Fourier polynomials extends the identity to the Hilbert spaces. Replacing the angular coordinate by its negative leaves γ_v and the multipliers unchanged. $\square$

Thus the Gaussian multiplier on a reflected mode is the mean of the change action that produced the carrier. The normalized angular variance $v = tE_{b,1}$ and the interval parameter t describe this same response in their respective coordinates.

Theorem 3.6 (Finite-window entropy orientation). *Let $\rho_0 \in L^1(\mathcal{W}_b) \cap L^2(\mathcal{W}_b)$ be a nonnegative probability density with respect to dd and put $\rho_t := C_b(t)\rho_0$. Introduce the normalized reference measure and its dimensionless density*

$$dm_b := \frac{dd}{\Delta_b}, \quad \tilde{\rho}_t := \Delta_b \rho_t, \quad \int_{\mathcal{W}_b} \tilde{\rho}_t dm_b = 1. \quad (3.45)$$

For $t > 0$ define

$$H_{\mathrm{pos},b}(t) := - \int_{\mathcal{W}_b} \tilde{\rho}_t \log \tilde{\rho}_t dm_b, \quad \mathcal{I}_{\mathrm{pos},b}(t) := \int_{\mathcal{W}_b} \frac{|\partial_d \rho_t|^2}{\rho_t} dd. \quad (3.46)$$

Then

$$\frac{d}{dt} H_{\mathrm{pos},b}(t) = \frac{1}{2} \mathcal{I}_{\mathrm{pos},b}(t) \geq 0, \quad (3.47)$$

$$\frac{d}{dt} D(\tilde{\rho}_t dm_b \| dm_b) = -\frac{1}{2} \mathcal{I}_{\mathrm{pos},b}(t) \leq 0. \quad (3.48)$$

With $E_{b,1}$ denoting the first positive Neumann eigenvalue one also has

$$\left\| \rho_t - \frac{1}{\Delta_b} \right\|_2 \leq e^{-tE_{b,1}/2} \left\| \rho_0 - \frac{1}{\Delta_b} \right\|_2. \quad (3.49)$$

Thus the uniform density $1/\Delta_b$ is the unique fixed probability density and the unique maximizer of the dimensionless entropy with maximal value zero.

Proof. For every $t > 0$ positivity improvement of the Neumann heat kernel makes ρ_t smooth and strictly positive. The evolution equation $\partial_t \rho_t = \frac{1}{2} \partial_d^2 \rho_t$ and the vanishing Neumann boundary term allow differentiation and integration by parts to give

$$\frac{d}{dt} H_{\text{pos},b}(t) = -\frac{1}{2} \int_{\mathcal{W}_b} (1 + \log(\Delta_b \rho_t)) \partial_d^2 \rho_t dd = \frac{1}{2} \int_{\mathcal{W}_b} \frac{|\partial_d \rho_t|^2}{\rho_t} dd. \quad (3.50)$$

The normalized reference measure gives $D(\tilde{\rho}_t dm_b || dm_b) = -H_{\text{pos},b}(t)$ and thus proves (3.48). The function $\rho_0 - 1/\Delta_b$ is orthogonal to the constant mode. Functional calculus on that orthogonal complement gives (3.49). Equality in the dimensionless entropy bound $H_{\text{pos},b} \leq 0$ holds only for the uniform density. $\square$

The first positive eigenvalue controls convergence to the uniform density. The individual damping rates likewise distinguish the spectral components.

4 Ordered modes and native arithmetic

The endpoint and boundary calculations have fixed the reflecting generator. Its spectral projections determine the modes and its heat flow orders them by their rates of attenuation. Solving the eigenvalue problem gives coordinates for this ordered family with an energy and a correlation weight attached to each mode.

4.1 The unlabeled spectral carrier

The spectral projections define the carrier independently of any enumeration. Let

$$\mathfrak{E}_b := \{ \mathbf{1}_{\{\lambda\}}(A_b) : \lambda \in \sigma(A_b) \} \quad (4.1)$$

be the family of nonzero spectral projections of the reflecting generator. For $P, Q \in \mathfrak{E}_b$ and $\tau > 0$ define

$$P \prec_b Q \iff \text{Tr}(PC_b(\tau)) > \text{Tr}(QC_b(\tau)). \quad (4.2)$$

This relation compares the responses of the spectral subspaces. The eigenvalue calculation establishes that it is an order independent of τ .

4.2 Neumann spectrum of the active window

Translate the active interval to $[0, \Delta_b]$ by $x = d - \ell_{\text{UV}}(b)$. The Neumann eigenvalue problem

$$-\phi'' = E\phi, \quad \phi'(0) = \phi'(\Delta_b) = 0 \quad (4.3)$$

has the normalized solutions

$$\phi_{b,0}(d) := \Delta_b^{-1/2}, \quad E_{b,0} := 0, \quad (4.4)$$

$$\phi_{b,n}(d) := \sqrt{\frac{2}{\Delta_b}} \cos\left(\frac{n\pi(d - \ell_{\text{UV}})}{\Delta_b}\right), \quad E_{b,n} := \left(\frac{n\pi}{\Delta_b}\right)^2, \quad n \geq 1. \quad (4.5)$$

Here n is the integer coordinate obtained by solving the boundary-value problem. The order on $\mathfrak{E}_b$ is defined by the response comparison in (4.2) independently of enumeration. The Neumann calculation supplies a compatible coordinate for evaluating that order and making the analytic identifications.

Theorem 4.1 (Correlation spectral decomposition). *The family $\{\phi_{b,n} : n \geq 0\}$ is an orthonormal basis of $\mathcal{H}_b$. Every member is an eigenfunction of both A_b and T_b^{UV} satisfying*

$$A_b \phi_{b,n} = E_{b,n} \phi_{b,n}, \quad T_b^{\text{UV}} \phi_{b,n} = \Lambda_{b,n} \phi_{b,n}, \quad (4.6)$$

where

$$\Lambda_{b,n} := \exp\left(-\frac{b^2 E_{b,n}}{2}\right). \quad (4.7)$$

The zero mode is simple and the positive energies satisfy

$$0 < E_{b,1} < E_{b,2} < \cdots, \quad 1 > \Lambda_{b,1} > \Lambda_{b,2} > \cdots > 0. \quad (4.8)$$

Proof. Solving (4.3) gives (4.4)–(4.5). The cosine system is complete in $L^2([0, \Delta_b])$. Functional calculus for the self-adjoint operator A_b gives the eigenvalue formula for T_b^{UV} . Strict monotonicity follows from $E_{b,n} = n^2 E_{b,1}$. $\square$

The integer n records the position of a rank-one projection in the response order. Thus the eigenvalue calculation identifies the carrier defined by the spectral projections, together with its successor relation.

Proposition 4.2 (Unlabeled spectral classification and successor). *The operator A_b is nonnegative and self-adjoint with compact resolvent. Every member of $\mathfrak{E}_b$ has rank one. The family has a unique zero-energy member. The relation (4.2) is independent of τ . It well-orders $\mathfrak{E}_b$ with order type ω . Every member has a unique immediate successor.*

This structure determines a unique order-preserving coordinate $n \mapsto P_{b,n}$ from the metatheoretic nonnegative integers such that

$$P_{b,n} = |\phi_{b,n}\rangle\langle\phi_{b,n}|, \quad E_{b,n} = n^2 E_{b,1}. \quad (4.9)$$

The endpoint-positive unit vector in $\text{Ran } P_{b,n}$ is uniquely $\phi_{b,n}$.

For $P \in \mathfrak{E}_b$ and $\tau > 0$ the response recovers its energy by

$$\mathcal{E}_{b,\tau}(P) := -\frac{2}{\tau} \log \text{Tr}(P C_b(\tau)). \quad (4.10)$$

This value is independent of τ . If

$$\mathcal{E}_b^+(P) := \min(\sigma(A_b) \cap (\mathcal{E}_{b,\tau}(P), \infty)), \quad (4.11)$$

then the response-defined successor is

$$S_b^{\text{sp}}(P) := \mathbf{1}_{\{\mathcal{E}_b^+(P)\}}(A_b). \quad (4.12)$$

It is the unique immediate successor of P in the response order independently of the explicit Neumann coordinate.

Proof. Self-adjointness and nonnegativity follow from the closed Neumann form

$$f \mapsto \int_{\mathcal{W}_b} |f'(d)|^2 dd.$$

The resolvent maps $L^2(\mathcal{W}_b)$ continuously into $H^2(\mathcal{W}_b)$. The latter space embeds compactly into $L^2(\mathcal{W}_b)$ because $\mathcal{W}_b$ is bounded. Thus the resolvent is compact. Solving the scalar Neumann problem gives one-dimensional eigenspaces and a unique constant zero eigenspace. The eigenvalues are strictly increasing as displayed in (4.8). Functional calculus gives

$$\mathrm{Tr}(P_{b,n}C_b(\tau)) = e^{-\tau E_{b,n}/2}.$$

Hence (4.2) is exactly the increasing energy order. The calculation verifies the order type and successor structure of the response relation defined in (4.2). This order has a least member and finite initial segments. Every member has one immediate successor. The resulting order type is ω and gives the unique order coordinate in (4.9). The sign condition at the left endpoint selects the stated unit representative. The same trace formula proves that $\mathcal{E}_{b,\tau}(P)$ is the eigenvalue on $\mathrm{Ran} P$ independently of τ . Compact resolvent together with simplicity and unboundedness gives the least spectral value strictly above it. This determines the asserted successor projection. $\square$

A complete correlation-spectral component is the endpoint-positive unit representative of the range of a projection in $\mathfrak{E}_b$. Proposition 4.2 identifies these representatives with the functions $\phi_{b,n}$ in (4.4)–(4.5). Write

$$\mathcal{E}_b^0 := \{\phi_{b,n} : n \geq 0\}, \quad \mathcal{E}_b^+ := \mathcal{E}_b^0 \setminus \{\phi_{b,0}\}. \quad (4.13)$$

Here a component is the entire single-frequency function $\phi_{b,n}$. Its generator energy $E_{b,n}$ and correlation weight $\Lambda_{b,n}$ are the corresponding spectral data. By Theorem 3.3 every positive component is also the reflection-even combination of the two conjugate circle characters z^n and z^{-n} . The real Neumann spectrum therefore retains the complete frequency component while suppressing orientation. The two orientations are recovered before the arithmetic operations are defined.

Uniform amplitude bound. Every complete component is bounded on the active window with

$$\|\phi_{b,0}\|_\infty = \Delta_b^{-1/2}, \quad \|\phi_{b,n}\|_\infty = \sqrt{2/\Delta_b} \quad (n \geq 1). \quad (4.14)$$

The bound on the positive-mode amplitudes is independent of n whereas the correlation weights strictly decrease with n by (4.7). These formulas follow directly from $|\cos x| \leq 1$ with equality at the left endpoint.

The amplitude bound and $\sum_n \Lambda_{b,n} < \infty$ give a summable majorant independent of $(d, d') \in \mathcal{W}_b^2$. Thus the kernel of T_b^{UV} has the uniformly convergent expansion

$$\mathcal{K}_b(d, d') = \sum_{n=0}^{\infty} \Lambda_{b,n} \phi_{b,n}(d) \phi_{b,n}(d'). \quad (4.15)$$

The Gaussian decay of $\Lambda_{b,n}$ controls the contribution of high components without altering their orthonormal normalization.

4.3 Order from correlation response

For $\tau > 0$ and $\phi \in \mathcal{E}_b^0$ define its survival response under an additional change by

$$R_{b,\tau}(\phi) := \langle \phi, C_b(\tau)\phi \rangle_{\mathcal{H}_b}. \quad (4.16)$$

The eigenvalue relation $A_b\phi = E_\phi\phi$ gives

$$R_{b,\tau}(\phi) = e^{-\tau E_\phi/2}. \quad (4.17)$$

For distinct $\phi, \psi \in \mathcal{E}_b^0$ set

$$\phi \prec_b \psi \iff R_{b,\tau}(\phi) > R_{b,\tau}(\psi) \text{ for one, equivalently every, } \tau > 0. \quad (4.18)$$

On the real components the response order of Proposition 4.2 takes the form

$$\phi \prec_b \psi \iff E_\phi < E_\psi. \quad (4.19)$$

For every ordered pair $\phi \prec_b \psi$ the logarithm of the ratio of their survival responses increases along the flow according to

$$\frac{d}{d\tau} \log \frac{R_{b,\tau}(\phi)}{R_{b,\tau}(\psi)} = \frac{1}{2}(E_\psi - E_\phi) > 0. \quad (4.20)$$

The ordered family $(\mathcal{E}_b^0, \prec_b)$ is a well-order of type ω . It has a least component and every component has a unique immediate successor. Equation (4.17) proves the equivalence and the independence of τ . Differentiating its ratio gives (4.20). Thus increasing the response parameter separates two distinct energy levels more strongly while preserving their order.

The resulting order is

$$\phi_{b,0} \prec_b \phi_{b,1} \prec_b \phi_{b,2} \prec_b \cdots. \quad (4.21)$$

The index in this display enumerates the response order. It is used to write eigenfunctions and prove completeness as well as evaluate traces. The zero mode and the first positive mode are distinguished by the response itself. We retain this order on the oriented analytic representatives of the real components used to define arithmetic operations.

The real cosine modes identify opposite frequencies and their pointwise product generally contains both a sum frequency and a difference frequency. An operation returning a single mode is obtained from oriented representatives on the reflected circle. Products of these representatives remain single characters. They are also circle endomorphisms and can be composed. These two operations define addition and multiplication on the mode family. The unitary lift retains the correlation generator and its flow.

4.4 The oriented analytic lift

The orientation from the UV endpoint towards the IR endpoint fixes a normalized rotation flow on the reflected double. Its generator selects the analytic channel before a character or an integer degree is assigned. Let V_b be the positively oriented invariant vector field

whose flow has least positive period 2π under the normalization of Theorem 3.4. On the complex Hilbert space of the double equipped with normalized Haar measure define

$$D_b := -iV_b, \quad \mathcal{D}(D_b) = H^1(\widetilde{\mathcal{W}}_b), \quad \mathbf{P}_{b,\pm}^{\text{an}} := \mathbf{1}_{[0,\infty)}(\pm D_b). \quad (4.22)$$

Here the Sobolev domain is periodic on the double. In the arclength coordinate x and its normalized angle the same operator is

$$D_b = -i \frac{\Delta_b}{\pi} \partial_x = -i \partial_\theta. \quad (4.23)$$

The reflection $r_b(x) = -x$ induces the complex-linear unitary pullback

$$(\mathcal{J}f)(x) := f(r_b x), \quad (\mathcal{J}f)(z) = f(\bar{z}). \quad (4.24)$$

Thus $\mathcal{J}(cf) = c\mathcal{J}f$ for every $c \in \mathbb{C}$.

Theorem 4.3 (Selection by the oriented rotation generator). *The oriented generator D_b is self-adjoint and has compact resolvent. The spaces*

$$H_{b,\pm}^2 := \text{Ran } \mathbf{P}_{b,\pm}^{\text{an}} \quad (4.25)$$

are determined by the oriented rotation flow and its spectral measure. Their intersection is the constant subspace. Their orthogonal complements to constants give a direct-sum decomposition of the zero-mean circle space. They satisfy

$$\mathcal{J}D_b\mathcal{J} = -D_b, \quad \mathcal{J}\mathbf{P}_{b,+}^{\text{an}}\mathcal{J} = \mathbf{P}_{b,-}^{\text{an}}. \quad (4.26)$$

An orientation-preserving normalized carrier isomorphism transports these projections by unitary conjugacy. Orientation reversal exchanges them.

Proof. Integration by parts on the double makes $-iV_b$ symmetric on the periodic H^1 domain. A vector in its adjoint domain has a weak first derivative in L^2 . The boundary term against all periodic test functions then imposes the same periodic trace condition and proves self-adjointness. The graph norm is equivalent to the H^1 norm. The Sobolev embedding in L^2 on the compact circle is compact. The resolvent is therefore compact. The kernel consists of constants. The spectral theorem partitions its orthogonal complement into the positive and negative spectral subspaces. This gives the asserted decomposition. Reflection reverses V_b and preserves its domain. Functional calculus therefore gives (4.26). A normalized oriented carrier isomorphism intertwines the rotation flows and hence their generators and spectral projections. $\square$

Put

$$\theta_b(d) := \frac{\pi(d - \ell_{\text{UV}})}{\Delta_b}, \quad 0 \leq \theta_b(d) \leq \pi. \quad (4.27)$$

Theorem 3.3 identifies this angular coordinate with one half of the reflected circle. On the full circle put

$$\chi_k(z) := z^k, \quad k \in \mathbb{Z}. \quad (4.28)$$

This integer is the degree invariant for circle maps [5]. The group-endomorphism law fixes the representative in each degree class.

Every continuous group endomorphism of the minimal scalar carrier $\mathbb{T}$ is exactly one character χ_k . Thus

$$\text{End}_{\text{cts}}(\mathbb{T}) = \{\chi_k : k \in \mathbb{Z}\}, \quad \deg(\chi_k) = k. \quad (4.29)$$

For each nonzero degree magnitude the two frequency orientations are exchanged by the complex-linear reflection pullback $\mathcal{J}\chi_k = \chi_{-k}$. For a continuous homomorphism $u : \mathbb{T} \rightarrow \mathbb{T}$ choose the continuous lift $\tilde{u} : \mathbb{R} \rightarrow \mathbb{R}$ of $u(e^{i\theta})$ with $\tilde{u}(0) = 0$. For $\theta, \varphi \in \mathbb{R}$ the defect

$$\tilde{u}(\theta + \varphi) - \tilde{u}(\theta) - \tilde{u}(\varphi)$$

takes values in $2\pi\mathbb{Z}$. It is continuous and vanishes at $(0, 0)$. It therefore vanishes everywhere. The lift is a continuous additive map and has the form $\tilde{u}(\theta) = a\theta$ for some $a \in \mathbb{R}$. The identity $e^{i(\theta+2\pi)} = e^{i\theta}$ gives $2\pi a \in 2\pi\mathbb{Z}$. Write $a = k \in \mathbb{Z}$. Thus $u(z) = z^k$. The winding integral gives degree k and reflection changes its sign.

The same compact carrier now has an analytic and a topological description. Its reflection-even Laplacian has the compact resolvent and square spectrum of Theorem 4.1 while its endomorphisms have the degree lattice just obtained. The reflecting identification (3.38) pairs χ_n with χ_{-n} and gives

$$n = \sqrt{\frac{E_{b,n}}{E_{b,1}}} = |\deg \chi_{\pm n}|. \quad (4.30)$$

The Gaussian endpoint width Δ_b fixes the dimensional energy unit $E_{b,1}$. A directed traversal of the active window fixes one of the two conjugate orientations. A change of b rescales all energies together while preserving the dimensionless integer in (4.30). The Gaussian data also determine the response weights $e^{-tE_{b,n}/2}$ and the endpoint times at which they are evaluated. The analytic lift retains these response weights and evaluation times.

Solving $D_b \chi_k = k \chi_k$ now gives the coordinate description of the intrinsically selected spaces. Under the normalized circle identification write $H_{\pm}^2(\mathbb{T}) = H_{b,\pm}^2$. Then

$$H_+^2(\mathbb{T}) = \overline{\text{span}}\{z^n : n \geq 0\}, \quad H_-^2(\mathbb{T}) = \overline{\text{span}}\{z^{-n} : n \geq 0\}, \quad (4.31)$$

$$u_{b,n}^+(z) := z^n, \quad u_{b,n}^-(z) := z^{-n}. \quad (4.32)$$

The notation $H_{\pm}^2$ follows the Hardy-space convention for square-integrable boundary values. The sign records the selected frequency orientation. Write $H_{\pm,0}^2(\mathbb{T}) := H_{\pm}^2(\mathbb{T}) \ominus \mathbb{C}1$. The zero-mean circle space splits as

$$L^2(\mathbb{T}) \ominus \mathbb{C}1 = H_{+,0}^2(\mathbb{T}) \oplus H_{-,0}^2(\mathbb{T}), \quad (4.33)$$

and $\mathcal{J}$ exchanges the two summands. The positive representative agrees with the UV-to-IR orientation by Theorem 4.3. The characters $z^{\pm n}$ describe its spectral subspaces in the chosen coordinate. Set

$$u_{b,n} := u_{b,n}^+ = z^n, \quad n \in \mathbb{N}_0. \quad (4.34)$$

The subscript b records the correlation window from which the phase coordinate is obtained. The monomial itself is dimensionless. By the classification (4.29) these are precisely the nonnegative-degree endomorphisms of the selected scalar carrier.

Proposition 4.4 (Analytic lift). *There are unique unitary maps*

$$\mathcal{A}_b^\pm : \mathcal{H}_b \longrightarrow H_\pm^2(\mathbb{T}) \quad (4.35)$$

such that

$$\mathcal{A}_b^\pm \phi_{b,n} = u_{b,n}^\pm \quad (n \geq 0), \quad (4.36)$$

and $\mathcal{J}\mathcal{A}_b^+ = \mathcal{A}_b^-$. Throughout the paper $\mathcal{A}_b := \mathcal{A}_b^+$ denotes the UV-to-IR channel. With $N_\pm z^{\pm n} := n z^{\pm n}$ on the polynomial core these maps also intertwine the normalized correlation generator and its entire response semigroup through

$$\mathcal{A}_b^\pm \mathcal{A}_b (\mathcal{A}_b^\pm)^{-1} = E_{b,1} N_\pm^2, \quad \mathcal{A}_b^\pm e^{-t\mathcal{A}_b/2} = e^{-tE_{b,1} N_\pm^2/2} \mathcal{A}_b^\pm. \quad (4.37)$$

Proof. Let $P_0 := \mathbf{1}_{\{0\}}(D_b)$ and retain the reflecting unitary $\mathcal{R}_b$ from (3.37). The lifts have the operator formula

$$\mathcal{A}_b^\pm = (P_0 + \sqrt{2} \mathbf{1}_{(0,\infty)}(\pm D_b)) \mathcal{R}_b. \quad (4.38)$$

Each nonconstant even spectral pair has equal norm in its two oriented components. The factor $\sqrt{2}$ restores its norm after one component has been selected while P_0 preserves the normalized constant. The families $\{\phi_{b,n} : n \geq 0\}$ and $\{z^{\pm n} : n \geq 0\}$ are orthonormal bases of their respective Hilbert spaces. Each basis correspondence extends uniquely to a unitary. The reflection identity follows from $\mathcal{J}z^n = z^{-n}$. The eigenvalue law $E_{b,n} = E_{b,1}n^2$ proves the generator identity in (4.37). Functional calculus gives the semigroup identity. $\square$

The lift chooses one member of each conjugate frequency pair and transports the entire normalized response semigroup to that oriented channel. The function operations therefore act on oriented representatives of the spectrum selected in Section 4.

The two lifts retain the opposite orientations of each eigencomponent. Their common energy and correlation weight are

$$\frac{E_{b,n}}{E_{b,1}} = n^2, \quad \Lambda_{b,n} = \exp\left(-\frac{b^2 E_{b,1}}{2} n^2\right). \quad (4.39)$$

4.5 Modulation and cascade

The generated natural-mode family is the image of the complete correlation spectrum under the chosen oriented lift. Thus

$$\mathcal{U}_b := \mathcal{A}_b(\mathcal{E}_b^0) = \{u_{b,n} : n \in \mathbb{N}_0\}. \quad (4.40)$$

The first equality is the intrinsic definition. The second is its spectral coordinate realization. Let 0_b be the constant circle endomorphism and let 1_b be the identity endomorphism. Under the same realization one has

$$0_b = u_{b,0}, \quad 1_b = u_{b,1}. \quad (4.41)$$

The first is the oriented lift of the unique zero-energy component and the second is the lift of its unique immediate successor in the response order. Thus 0_b is the constant function with value one on the circle while 1_b is the function $z \mapsto z$. The constant function adds no phase under pointwise modulation. The identity map passes its input unchanged through cascade. These roles give their arithmetic names. Linear combinations of modes use the ordinary zero vector in the ambient Hilbert space.

For $u, v \in \mathcal{U}_b$ define

$$u \boxplus_b v := m_{\mathbb{T}} \circ (u \times v) \circ \Delta_{\mathbb{T}} = uv, \quad u \boxtimes_b v := u \circ v. \quad (4.42)$$

Here $\Delta_{\mathbb{T}}(z) = (z, z)$ and $m_{\mathbb{T}}(z_1, z_2) = z_1 z_2$. Hence $\boxplus_b$ is the monoidal product induced by the circle-group law while $\boxtimes_b$ is categorical composition of endomorphisms. The first operation is pointwise multiplication of analytic signals. The second is composition of the associated circle maps. On basis modes one has

$$u_{b,m} \boxplus_b u_{b,n} = u_{b,m+n}, \quad u_{b,m} \boxtimes_b u_{b,n} = u_{b,mn}. \quad (4.43)$$

Both operations preserve $\mathcal{U}_b$. Their distributive laws follow from the group-endomorphism property.

For example the chosen circle coordinate gives

$$u_{b,2}(z)u_{b,3}(z) = z^2 z^3 = z^5, \quad u_{b,2}(u_{b,3}(z)) = (z^3)^2 = z^6.$$

Modulation accumulates the phases of two modes at the same input. Cascade feeds the output of the inner map into the outer map. The exponents record the distinct effects of these two function operations.

The character classification gives closure under both operations; the homomorphism law gives distributivity. These are identities on the modes themselves.

Theorem 4.5 (Intrinsic natural-mode semiring). *The quintuple*

$$(\mathcal{U}_b, \boxplus_b, \boxtimes_b, 0_b, 1_b) \quad (4.44)$$

is a commutative semiring. The additive identity is the constant mode 0_b and the multiplicative identity is the identity map 1_b .

Proof. Pointwise multiplication of circle-valued functions is associative and commutative with constant identity 0_b . Composition is associative with identity 1_b . The classification (4.29) shows that composition commutes on $\mathcal{U}_b$. Both operations preserve the positive oriented family. The constant homomorphism is absorbing under composition.

For $u, v, w \in \mathcal{U}_b$ the homomorphism property of u gives

$$u \boxtimes_b (v \boxplus_b w) = u \circ (vw) = (u \circ v)(u \circ w) = (u \boxtimes_b v) \boxplus_b (u \boxtimes_b w), \quad (4.45)$$

$$(u \boxplus_b v) \boxtimes_b w = (uv) \circ w = (u \circ v)(v \circ w) = (u \boxtimes_b w) \boxplus_b (v \boxtimes_b w). \quad (4.46)$$

These identities prove both distributive laws directly on the modes. $\square$

Proposition 4.6 (Minimal native closure). *In the normalized circle realization of the generated carrier T let $M(T)$ be the smallest subset of $\text{End}_{\text{cts}}(T)$ containing the constant homomorphism and id_T and closed under pointwise multiplication. Then*

$$M(T) = \mathcal{U}_b. \quad (4.47)$$

The set $M(T)$ is closed under composition and is the smallest subsemiring containing these two maps. Carrier group isomorphisms transport this characterization by conjugation.

Proof. The nonnegative power maps contain the constant and identity maps and are closed under pointwise multiplication. Conversely every such family contains each finite pointwise power of the identity. The analytic lift therefore gives $M(T) = \{z \mapsto z^n : n \geq 0\} = \mathcal{U}_b$. Composition of two power maps is again a power map. Conjugation by a carrier group isomorphism preserves both operations and the two distinguished maps. $\square$

The semiring and its minimal native closure are structures on functions. Degree identifies their operations with the familiar coordinate operations on nonnegative integers.

The orientation comparison retains source and target types. Let $\mathbb{T}_J$ and $\mathbb{T}_{-J}$ denote the same real compact carrier equipped with the complex structures J and $-J$. Choose normalized circle coordinates z_J and z_{-J} and let

$$C_J : \mathbb{T}_J \longrightarrow \mathbb{T}_{-J}, \quad z_{-J} \circ C_J = \overline{z_J} \quad (4.48)$$

be the orientation-reversing group isomorphism. For the power map $\chi_n(z) = z^n$ define

$$u_{b,n}^J := z_J^{-1} \circ \chi_n \circ z_J, \quad u_{b,n}^{-J} := z_{-J}^{-1} \circ \chi_n \circ z_{-J}, \quad n \in \mathbb{N}_0, \quad (4.49)$$

and write $\mathcal{U}_b^{\pm J} := \{u_{b,n}^{\pm J} : n \in \mathbb{N}_0\}$.

Proposition 4.7 (Typed orientation-reversal invariance). *Conjugation by C_J defines a semiring isomorphism*

$$\mathfrak{C}_J : \mathcal{U}_b^J \longrightarrow \mathcal{U}_b^{-J}, \quad \mathfrak{C}_J(u) := C_J \circ u \circ C_J^{-1}. \quad (4.50)$$

It preserves the monoidal product and composition together with zero and identity. It also preserves successor defined by $S_b u := u \boxplus_b 1_b$. Explicitly one has

$$\mathfrak{C}_J(u \boxplus_b v) = \mathfrak{C}_J(u) \boxplus_b \mathfrak{C}_J(v), \quad \mathfrak{C}_J(u \boxtimes_b v) = \mathfrak{C}_J(u) \boxtimes_b \mathfrak{C}_J(v), \quad (4.51)$$

$$\mathfrak{C}_J(0_b) = 0_b, \quad \mathfrak{C}_J(1_b) = 1_b, \quad \mathfrak{C}_J(S_b u) = S_b \mathfrak{C}_J(u). \quad (4.52)$$

Consequently the generated arithmetic is independent of the chosen orientation although its two coordinate realizations are complex conjugates.

Proof. Equation (4.48) gives $\mathfrak{C}_J(u_{b,n}^J) = u_{b,n}^{-J}$ for every n . Since C_J is a group isomorphism conjugation preserves the pointwise monoidal product. Conjugation preserves composition for arbitrary maps. This proves (4.51). It also carries the constant homomorphism and the

identity map to their counterparts on $\mathbb{T}_{-J}$. Since successor is modulation by the degree-one mode the final identity in (4.52) follows from the first operation law.

The pullback of the target character z_{-J}^n along C_J is z_J^{-n} . Thus the negative Hardy channel is the fixed-coordinate image of the positive character family on the oppositely oriented carrier. Composition in that family uses the reversed carrier as both source and target as in (4.50). $\square$

Proposition 4.8 (Winding-degree certification). *The map*

$$\deg_b(u) := \frac{1}{2\pi i} \int_{\mathbb{T}} u(z)^{-1} du(z) \quad (4.53)$$

is a semiring isomorphism

$$\deg_b : (\mathcal{U}_b, \boxplus_b, \boxtimes_b, 0_b, 1_b) \xrightarrow{\cong} (\mathbb{N}_0, +, \cdot, 0, 1). \quad (4.54)$$

The same map satisfies

$$\deg_b(u_{b,n}) = \sqrt{\frac{E_{b,n}}{E_{b,1}}} = n. \quad (4.55)$$

The map certifies the generated semiring by standard arithmetic coordinates. The elements 0_b and 1_b and the operations $\boxplus_b$ and $\boxtimes_b$ are already fixed on the endomorphism family at this stage.

Proof. The winding number of z^n is n . The classification theorem makes $\deg_b$ bijective on the positive oriented family. Pointwise product and composition satisfy

$$\deg_b(uv) = \deg_b(u) + \deg_b(v), \quad \deg_b(u \circ v) = \deg_b(u) \deg_b(v), \quad (4.56)$$

so degree intertwines the two intrinsic operations with the standard coordinate operations. The final identity follows from $E_{b,n} = n^2 E_{b,1}$. $\square$

4.6 Homotopy ring and additive group completion

Let $[\mathbb{T}, \mathbb{T}]_{\text{htpy}}$ denote the homotopy classes of continuous circle maps. Pointwise multiplication and composition descend to

$$[f] \oplus [g] := [fg], \quad [f] \odot [g] := [f \circ g]. \quad (4.57)$$

The standard winding classification [5, Section 1.1] also describes the additive completion of the natural modes. The degree map is a ring isomorphism

$$\deg : ([\mathbb{T}, \mathbb{T}]_{\text{htpy}}, \oplus, \odot) \xrightarrow{\cong} (\mathbb{Z}, +, \cdot). \quad (4.58)$$

The canonical representatives $u_{b,n}(z) = z^n$ identify the natural-mode semiring with its nonnegative cone by

$$\mathcal{U}_b \xrightarrow{\cong} [\mathbb{T}, \mathbb{T}]_{\text{htpy}}^{\geq 0}, \quad u_{b,n} \mapsto [z^n]. \quad (4.59)$$

Additive Grothendieck completion gives a canonical ring isomorphism

$$\Psi_b : \text{Gr}_{\boxplus_b}(\mathcal{U}_b) \xrightarrow{\cong} [\mathbb{T}, \mathbb{T}]_{\text{htpy}}, \quad [(u_{b,m}, u_{b,n})] \mapsto [z^{m-n}]. \quad (4.60)$$

Thus the additive group completion of the natural endomorphisms is canonically isomorphic to the ring of homotopy classes of circle maps. Winding number identifies each homotopy class with a unique monomial class $[z^k]$. Pointwise multiplication and composition preserve homotopies. On these representatives the two degree laws in (4.56) hold for all integer exponents and prove the ring isomorphism. Its nonnegative cone is the image of $\mathcal{U}_b$ by Proposition 4.8. Every integer class is a difference of two such classes. The universal property of additive group completion therefore gives Ψ_b . The degree identities prove its bijectivity and multiplicativity.

On the natural-mode family arithmetic value is topological degree. Addition is character modulation and multiplication is circle-map cascade. Exponentiation is iterated cascade according to

$$u_{b,n}^{\boxtimes_b r} := \underbrace{u_{b,n} \boxtimes_b \cdots \boxtimes_b u_{b,n}}_{r \text{ factors}} = u_{b,n^r}. \quad (4.61)$$

The power identity follows by induction under composition. These are the function laws established in Theorem 4.5.

The semiring elements are effective change endomorphisms and both operations are inherited from complete spectral functions. At the primitive level τ_a is reversible. Statistical composition followed by windowing and spectral resolution with analytic lift produces effective morphisms $u_{b,n}$ that are non-invertible for $n \geq 2$. The construction thereby passes from reversible automorphisms to an oriented endomorphism semiring.

4.7 The real cosine channel

The real cosine representatives exhibit both the sum and difference frequencies that the orientation separates. Write

$$c_{b,n}(d) := \cos(n\theta_b(d)). \quad (4.62)$$

Pointwise mixing gives

$$2c_{b,m}c_{b,n} = c_{b,m+n} + c_{b,|m-n|}. \quad (4.63)$$

Multiplying the positive oriented representatives gives $u_{b,m+n}$ and corresponds to the sum-frequency term in this identity. The real product also contains a difference-frequency term. The operation $\boxplus_b$ therefore multiplies the oriented representatives directly. Lifting their real product would retain both frequency terms.

Composition has an equally direct real form. If T_n denotes the n th Chebyshev polynomial then

$$c_{b,n} = T_n(c_{b,1}), \quad T_m \circ T_n = T_{mn}. \quad (4.64)$$

Hence $\boxtimes_b$ is the circle-map cascade law represented on the real coordinate by Chebyshev composition.

The operations in (4.42) are defined on the discrete mode set $\mathcal{U}_b$. Vector addition in $H^2(\mathbb{T})$ gives signal superposition. Arithmetic addition is pointwise modulation on $\mathcal{U}_b$ and arithmetic multiplication is cascade composition on the same family. These two operations have a different domain from vector superposition.

5 Peano arithmetic and canonical transport

The analytic semiring has a distinguished zero and unit. Modulation by the first non-constant mode advances one step in the response order and gives its Peano successor. Factorization concerns cascade composition.

5.1 Successor and recursion

Retain the intrinsically defined constant and identity endomorphisms 0_b and 1_b from (4.41). Define

$$S_b(u) := u \boxplus_b 1_b. \quad (5.1)$$

It follows that

$$S_b(u_{b,n}) = u_{b,n+1}. \quad (5.2)$$

The successor therefore advances to the unique next component in the change-response order given by (4.19). Recursion concerns maps out of the ordered family. An initial value and a rule for one further step determine a unique map on that family. This is the universal property of a natural-number object in sets [3].

Theorem 5.1 (Peano spectral system). *The triple $(\mathcal{U}_b, 0_b, S_b)$ satisfies the following Peano axioms.*

- (i) 0_b is not in the image of S_b .
- (ii) S_b is injective.
- (iii) If $B \subseteq \mathcal{U}_b$ contains 0_b and $S_b(B) \subseteq B$ then $B = \mathcal{U}_b$.

Together with $\boxplus_b$ and $\boxtimes_b$ it is a Peano semiring. Let X be any set with an element $x_0 \in X$ and a map $F : X \rightarrow X$. There is a unique map $h : \mathcal{U}_b \rightarrow X$ such that

$$h(0_b) = x_0, \quad h(S_b u) = F(h(u)). \quad (5.3)$$

Thus $(\mathcal{U}_b, 0_b, S_b)$ is a natural-number object in the category of sets.

Proof. If $S_b(u) = 0_b$ then $u(z)z = 1$ and hence $u(z) = z^{-1}$. This function belongs to the negative analytic channel and lies outside $\mathcal{U}_b$. Thus 0_b is not a successor. If $S_b(u) = S_b(v)$ then pointwise cancellation of the identity map gives $u = v$. The classification of positive analytic modes shows that every member of $\mathcal{U}_b$ is a finite pointwise product of 1_b . A subset containing 0_b and closed under S_b must therefore contain every natural mode. The semiring assertion is Theorem 4.5.

For recursion start with the value x_0 at 0_b and assign $F(x)$ to the successor of any mode assigned x . Every mode is reached by a finite successor chain. Repeated application of this rule assigns a value to every mode. The chain is unique. Two chains of different lengths ending at the same mode would make 0_b a successor by cancellation of successors. The assignments therefore define a map h satisfying (5.3). Any two such maps agree at 0_b and agreement is preserved by successor. The induction property therefore proves their equality. In the external spectral coordinate the constructed map is $h(u_{b,n}) = F^n(x_0)$. $\square$

The last formula is the external coordinate expression of the recursive map. Finite iteration and induction belong to the ambient metatheory. Within the generated family successor is modulation by the first positive mode and is already fixed by the response order.

The strict additive order of this semiring is

$$u <_b v \iff v = u \boxplus_b w \text{ for some } w \neq 0_b. \quad (5.4)$$

For natural modes one has

$$u_{b,m} <_b u_{b,n} \iff m < n \iff \mathcal{A}_b^{-1} u_{b,m} \prec_b \mathcal{A}_b^{-1} u_{b,n}. \quad (5.5)$$

Algebraic order and topological degree agree in direction with normalized energy and change response.

5.2 Composition irreducibility

For $u, v \in \mathcal{U}_b$ write $u \mid_b v$ if $v = u \boxtimes_b w$ for some $w \in \mathcal{U}_b$. A mode $\pi \in \mathcal{U}_b$ is called a spectral prime if

$$\pi \neq 0_b, 1_b, \quad \pi = u \boxtimes_b v \implies u = 1_b \text{ or } v = 1_b. \quad (5.6)$$

Let

$$\mathcal{P}_b := \text{Irr}(\mathcal{U}_b, \boxtimes_b) \quad (5.7)$$

denote the resulting family. Thus primality is irreducibility for the cascade operation constructed in the preceding section. Descent in the response order gives existence of a factorization. Euclidean division in the generated Peano semiring gives the divisor property for an irreducible mode and hence uniqueness. The energy criterion expresses primality in the spectral data attached to each mode.

For a positive natural mode define its normalized spectral coordinate by

$$\kappa_b(u_{b,n}) := \sqrt{\frac{E_{b,n}}{E_{b,1}}} = \sqrt{\frac{-\log \Lambda_{b,n}}{-\log \Lambda_{b,1}}} = n. \quad (5.8)$$

Both expressions use data of the correlation spectrum. The square root is positive and the second equality follows from (4.7).

Theorem 5.2 (Spectral irreducibility and factorization). *A natural mode $\pi \in \mathcal{U}_b$ is a spectral prime if and only if $\kappa_b(\pi) > 1$ and there are no modes $u, v \neq 0_b, 1_b$ such that*

$$\kappa_b(\pi) = \kappa_b(u) \kappa_b(v). \quad (5.9)$$

Equivalently there are no such u, v satisfying the energetic identity

$$\frac{E_b(\pi)}{E_{b,1}} = \frac{E_b(u)}{E_{b,1}} \frac{E_b(v)}{E_{b,1}}, \quad (5.10)$$

where $E_b(u_{b,n}) := E_{b,n}$.

Every $u \in \mathcal{U}_b \setminus \{0_b, 1_b\}$ has a factorization

$$u = \boxtimes_b \pi^{\boxtimes_b \nu_u(\pi)}, \quad \nu_u(\pi) \in \mathbb{N}_0, \quad (5.11)$$

in which only finitely many exponents are nonzero. The exponent profile ν_u is unique. Equivalently one has

$$(\mathcal{U}_b \setminus \{0_b\}, \boxtimes_b, 1_b) \cong \text{FinMult}(\mathcal{P}_b). \quad (5.12)$$

The prime family is

$$\mathcal{P}_b = \{u_{b,p} : p \text{ is prime}\}. \quad (5.13)$$

Proof. The cascade law gives $\kappa_b(u \boxtimes_b v) = \kappa_b(u)\kappa_b(v)$. A nonzero nonunit mode therefore admits a nontrivial cascade factorization exactly when (5.9) or equivalently (5.10) holds for two nonunits. Its negation is the primality condition (5.6).

Existence follows by well-founded descent in the response order. A nonunit mode is either irreducible or is a cascade of two strictly smaller positive modes. If $u = v \boxtimes_b w$ with v and w nonunits write $w = 1_b \boxplus_b a$ with a positive. Distributivity gives $u = v \boxplus_b (v \boxtimes_b a)$ and hence $v <_b u$. The same argument applies to w . Iteration therefore terminates after finitely many steps.

Uniqueness is derived entirely from the generated Peano order and the two semiring operations in Appendix B. Peano induction gives the intrinsic Euclidean division of Proposition B.1. Strict remainder descent then gives the intrinsic gcd and Bézout identity of Proposition B.3. Lemma B.4 deduces the divisor property for every composition-irreducible mode using the ordered Grothendieck ring. Proposition B.5 cancels common irreducible factors successively. These steps prove uniqueness of the exponent profile in (5.11).

The operation law (4.43) now certifies that $u_{b,n}$ is composition irreducible exactly when the positive integer n has no nontrivial factorization. $\square$

The proof separates construction from certification in the same way as the semiring argument. Divisibility and factorization are first expressed through cascade in the generated Peano semiring. The final coordinate statement identifies the resulting irreducibles with the usual prime indices.

The same internal divisibility gives infinitely many spectral primes. Suppose $\mathcal{P}_b = \{\pi_1, \dots, \pi_r\}$. Form

$$Q := \pi_1 \boxtimes_b \dots \boxtimes_b \pi_r, \quad N := Q \boxplus_b 1_b. \quad (5.14)$$

By Theorem 5.2 the mode N has a spectral-prime factor. None of the π_j can divide N . Intrinsic Euclidean division of $N = Q \boxplus_b 1_b$ by π_j has remainder 1_b because Q is divisible by π_j and $1_b <_b \pi_j$. This contradicts the existence of a prime factor in the assumed finite list.

The real-window representative of a spectral prime is $\mathcal{A}_b^{-1}u_{b,p} = \phi_{b,p}$. It remains an entire normalized single-frequency eigenfunction. Orthogonality distinguishes different modes while composition irreducibility selects the prime subfamily among them.

5.3 Cotangent directions of compositional change

Let

$$\mathfrak{A}_b := \text{span}_{\mathbb{C}}(\mathcal{U}_b \setminus \{0_b\}) \quad (5.15)$$

be the algebraic complex span of the positive natural modes with $\boxtimes_b$ extended bilinearly. Its unit is 1_b . Define the augmentation

$$\epsilon_b \left(\sum_u a_u u \right) := a_{1_b}, \quad I_b := \ker \epsilon_b. \quad (5.16)$$

Unique factorization records a positive mode by the multiplicity of each irreducible cascade direction in it. These are its prime occupations. The augmentation filtration isolates the individual directions from their products. Products of two nonunit modes span I_b^2 . Their classes vanish in I_b/I_b^2 while the prime classes remain. This quotient is the algebraic cotangent space of the composition algebra at the multiplicative vacuum 1_b in the standard sense for an augmented commutative algebra [17]. Here I_b is a space of finite linear combinations whereas $\mathcal{P}_b$ is a set of individual modes. Identifying the quotient with their linear span gives a basis that can also be completed in the inherited Hilbert norm.

Theorem 5.3 (Prime cotangent space and algebraic closure). *The augmentation filtration satisfies*

$$I_b = \text{span}_{\mathbb{C}}\{u \in \mathcal{U}_b : u \neq 0_b, 1_b\}, \quad (5.17)$$

$$I_b^2 = \text{span}_{\mathbb{C}}\{u \in \mathcal{U}_b : u = v \boxtimes_b w \text{ for some } v, w \neq 0_b, 1_b\}. \quad (5.18)$$

The quotient map consequently restricts to a vector-space isomorphism

$$\text{span}_{\mathbb{C}} \mathcal{P}_b \xrightarrow{\cong} I_b/I_b^2, \quad \pi \mapsto [\pi]. \quad (5.19)$$

After the quotient basis $\{[\pi] : \pi \in \mathcal{P}_b\}$ is declared orthonormal its Hilbert completion is canonically unitary to $\overline{\text{span}} \mathcal{P}_b$. The same prime-basis identification defines an algebra isomorphism

$$\text{Sym}_{\text{alg}}(I_b/I_b^2) \xrightarrow{\cong} \mathfrak{A}_b, \quad \prod_{\pi \in \mathcal{P}_b} [\pi]^{\nu(\pi)} \mapsto \boxtimes_b \pi^{\boxtimes_b \nu(\pi)}. \quad (5.20)$$

Thus the first-order prime directions generate the entire positive composition algebra by finite symmetric products.

Proof. The augmentation kills exactly the nonunit basis modes. Bilinearity of $\boxtimes_b$ makes I_b^2 the span of their pairwise products. A basis mode survives in I_b/I_b^2 precisely when it has no nontrivial cascade factorization. The surviving modes are the primes and their independence follows from orthonormality. This proves the quotient and Hilbert-space identifications.

Monomials in the symmetric algebra are indexed by finite occupation profiles. Spectral factorization identifies these profiles bijectively with positive natural modes and intertwines their products. This proves (5.20). $\square$

The cotangent description also determines every first-order response. Let M be a complex vector space regarded as an $\mathfrak{A}_b$ -module through ϵ_b . Then restriction to I_b induces a natural isomorphism

$$\text{Der}_{\epsilon_b}(\mathfrak{A}_b, M) \cong \text{Hom}_{\mathbb{C}}(I_b/I_b^2, M), \quad (5.21)$$

where an ϵ_b -derivation satisfies

$$D(ac) = \epsilon_b(a)D(c) + \epsilon_b(c)D(a). \quad (5.22)$$

An ϵ_b -derivation vanishes on 1_b and on I_b^2 by (5.22). Hence it defines a linear map on I_b/I_b^2 . Conversely suppose $L : I_b/I_b^2 \rightarrow M$ is linear. Write $a = \epsilon_b(a)1_b + i_a$ with $i_a \in I_b$ and set $D_L(a) := L([i_a])$. Expanding a product and discarding $i_a i_c \in I_b^2$ proves (5.22). The two assignments are inverse. Thus each first-order response factors uniquely through the prime cotangent space.

An algebra automorphism Φ satisfying $\epsilon_b \Phi = \epsilon_b$ preserves I_b and hence every I_b^r by multiplicativity. It therefore induces an automorphism of I_b/I_b^2 . If it also permutes the natural-mode basis then Theorem 5.3 shows that it preserves the prime modes. Their classes in this quotient are nonzero.

The ordered mode family carries a Peano successor and two arithmetic operations with unique prime factorization. Its modes retain their original correlation energies. Their arithmetic and spectral properties are compatible under changes of scale and orientation and supply the data for the ordered expression system and its analytic response.

5.4 The generation theorem

The preceding constructions give arithmetic together with its original correlation data. The generation theorem records both structures and their compatibility under scale and orientation transport.

Theorem 5.4 (Correlation-Spectral Arithmetic Theorem). *A minimal unbiased compositional change determines its Gaussian correlation law. Its unit local-information radius and total normalized correlation radius are the ultraviolet and infrared lengths of the finite correlation window. Within the stated boundary class preservation of the constant mode determines the Neumann closure and hence its zero-flux reflected spectrum. The resulting oriented mode family carries the following arithmetic and correlation data.*

- (i) *Its arithmetic reduct is an ordered Peano semiring of natural modes. Addition is pointwise modulation and multiplication is cascade. Winding degree is the isomorphism to the natural-number semiring. The prime elements are the cascade irreducibles and every positive mode has a unique finite prime expansion.*
- (ii) *The same modes retain their Hilbert inner product. They also retain their quadratic correlation energy and Gaussian response. The prime residue classes form the canonical basis of the augmentation cotangent space and their symmetric algebra recovers the positive composition algebra. The unique arithmetic isomorphism of Theorem 5.6 extends to the prime cotangent bases. It preserves normalized energies and intertwines the Gaussian flows after time rescaling.*

The arithmetic reduct consists of the ordered carrier with zero and unit together with successor and the two operations. The second assertion records the analytic and prime cotangent structures produced along with it. The response order and successor are native to the generated mode family as are modulation and cascade. Winding degree supplies their numerical certification.

The operations are defined on the functions, while spectral and winding coordinates certify their laws. This distinction also fixes the role of the ambient natural numbers in the construction.

Proposition 5.5 (Relative generation and degree certification). *Write $\mathbb{N}_0^{\text{meta}}$ for the natural-number object of the ambient analytic metatheory. It is used to state finite-dimensionality and to label a discrete spectrum after its unlabeled order has been obtained. It also supports finite induction in proofs. The generated carrier $\mathcal{U}_b$ is a different object. Its response order distinguishes the constant mode and the first positive mode. Its successor and modulation together with cascade and composition irreducibles are determined before $\deg_b$ is used as an arithmetic valuation. These structures are equivariant under every relabelling of $\mathcal{U}_b$ that preserves the response semigroup and pointwise multiplication together with map composition. Winding degree then gives the certification map*

$$\deg_b : \mathcal{U}_b \longrightarrow \mathbb{N}_0^{\text{meta}}, \quad (5.23)$$

which is an order-preserving semiring isomorphism. The response fixes zero and unit. Modulation defines successor and addition. Cascade defines multiplication and irreducibility on $\mathcal{U}_b$. The proofs use metatheoretic spectral coordinates and the classification of circle endomorphisms. Finite induction supplies the recursive steps. Elsewhere the superscript meta is suppressed once this distinction has been made.

Proof. Theorem 4.3 first selects the oriented analytic space by the rotation generator's positive spectral projection. Equations (4.18) and (4.19) obtain the order from the eigenvalue response. Theorem 4.5 defines the two operations on the mode family. Theorem 5.1 identifies modulation by the first positive mode with successor. Composition irreducibility and its factorization law are intrinsic to cascade by Theorem 5.2. Each definition is expressed in terms of the three structures required to be preserved. Thus each is carried unchanged by a structure-preserving relabelling. The identity (4.55) and Theorem 4.5 show that winding degree preserves the response order and both operations. $\square$

Thus construction and certification have different logical roles. The response determines the order while modulation and cascade supply the operations. Composition irreducibility selects the primes in $\mathcal{U}_b$. Spectral and winding coordinates are then available to prove its laws and identify its arithmetic reduct.

Theorem 5.6 (Canonical invariance of the generated arithmetic). *Let two minimal unbiased compositional change data have positive scales b and b' . After either orientation has been used as a complex coordinate there is a unique order-preserving semiring isomorphism*

$$\Phi_{b,b'} : \mathcal{U}_b \longrightarrow \mathcal{U}_{b'} \quad (5.24)$$

that preserves 0 and 1 and commutes with successor. It preserves spectral primes with their exponent profiles. It also preserves the cotangent quotient. Changing b alters the dimensional energy unit and rescales Gaussian time. The induced maps preserve normalized spectral energy. Reversing the carrier orientation transports the construction through the typed semiring isomorphism of Proposition 4.7.

Proof. Minimal carrier uniqueness identifies both real actions with the standard rotation plane. Gaussian uniqueness leaves only their positive covariance scales. The recursion property defines $\Phi_{b,b'}$ intrinsically by

$$\Phi_{b,b'}(0_b) = 0_{b'}, \quad \Phi_{b,b'}(S_b u) = S_{b'} \Phi_{b,b'}(u). \quad (5.25)$$

In spectral coordinates this is $\Phi_{b,b'}(u_{b,n}) = u_{b',n}$. Equations (4.43) and (5.5) show that this map preserves both operations and the response order. It also preserves 0_b and 1_b and commutes with successor. Every natural mode is a finite successor iterate of zero. Thus any map preserving zero and successor must equal $\Phi_{b,b'}$. This proves uniqueness.

Irreducibility and exponent profiles are expressed through cascade and are therefore preserved. The induced maps on the augmentation quotient and the finite occupation profiles are then canonical. To compare the analytic data extend $\phi_{b,n} \mapsto \phi_{b',n}$ linearly to the unitary $V_{b,b'} : \mathcal{H}_b \rightarrow \mathcal{H}_{b'}$. The energy formula gives

$$V_{b,b'} \frac{A_b}{E_{b,1}} V_{b,b'}^* = \frac{A_{b'}}{E_{b',1}}, \quad V_{b,b'} C_b(t) V_{b,b'}^* = C_{b'} \left(t \frac{E_{b,1}}{E_{b',1}} \right). \quad (5.26)$$

The generator identity holds on its spectral domain because the normalized eigenvalues and square-summability conditions agree. Functional calculus then gives the semigroup identity. Replacing the generated complex structure J by $-J$ invokes the typed conjugation map (4.50). Equations (4.51) and (4.52) prove the orientation statement with source and target transported together. $\square$

The comparison map is already determined by zero and successor. The normalized response gives a second characterization: its distinct values identify every mode without requiring the comparison map to preserve either arithmetic operation. Prime-occupation coordinates inherit the same arithmetic and correlation data.

Proposition 5.7 (Response rigidity and occupation transport). (i) For $\tau > 0$ define the normalized response on the mode family by

$$r_{b,\tau}(u) := \exp \left(-\frac{\tau E_b(u)}{2E_{b,1}} \right). \quad (5.27)$$

If a map $F : \mathcal{U}_b \rightarrow \mathcal{U}_{b'}$ satisfies $r_{b',\tau}(Fu) = r_{b,\tau}(u)$ for every u at one fixed $\tau > 0$ then $F = \Phi_{b,b'}$. A self-map preserving this response fixes every mode and every spectral prime.

(ii) Let $\mathcal{O}_b$ be the set of finite prime-occupation profiles. There is a unique monoid homomorphism

$$\text{ev}_b : (\mathcal{O}_b, +, 0) \longrightarrow (\mathcal{U}_b \setminus \{0_b\}, \boxtimes_b, 1_b)$$

sending the single occupation δ_π to π for every $\pi \in \mathcal{P}_b$. It is bijective. Extending it by $\text{ev}_b(\perp) = 0_b$ to a disjoint zero symbol uniquely transports the ordered semiring and successor together with the normalized Gaussian response to $\mathcal{O}_b \sqcup \{\perp\}$.

Proof. For (i) write $F(u_{b,n}) = u_{b',m}$. The square spectrum gives $e^{-\tau m^2/2} = e^{-\tau n^2/2}$ and hence $m = n$. Thus F is the map of Theorem 5.6.

Every finite profile is a sum of single-prime profiles. Its image is therefore forced to be the corresponding cascade product. Theorem 5.2 makes this evaluation bijective. The extended bijection uniquely transports the operations and order. For example its induced addition is

$$\nu \oplus \eta = \text{ev}_b^{-1}(\text{ev}_b(\nu) \boxplus_b \text{ev}_b(\eta)).$$

For a finite profile put $q_b(\nu) := \prod_{\pi \in \mathcal{P}_b} \deg_b(\pi)^{\nu(\pi)}$. The inherited energy and response are necessarily

$$\frac{E_b(\text{ev}_b \nu)}{E_{b,1}} = q_b(\nu)^2, \quad r_{b,\tau}(\text{ev}_b \nu) = e^{-\tau q_b(\nu)^2/2}. \quad (5.28)$$

The zero profile evaluates to 1_b whereas $\perp$ evaluates to 0_b . Thus the occupation coordinates retain the same diagonal response as their evaluated modes. $\square$

On a finite range the same response also gives a quantitative separation. For $N \geq 1$ put

$$\delta_N(\tau) := \min_{0 \leq n < N} \left(e^{-\tau n^2/2} - e^{-\tau(n+1)^2/2} \right) > 0.$$

A response measurement with error less than $\delta_N(\tau)/2$ determines its mode uniquely among $u_{b,0}, \dots, u_{b,N}$ by nearest response. Thus exact response rigidity has a finite resolution counterpart with an explicit separation depending on the retained range.

Proof of Theorem 5.4. Theorem 2.3 produces the real rotation plane with its normalized complex structure and the circle group $U(1)$. Theorem 2.5 gives the unique compositional tangent law of covariance b^2 . At this fixed covariance it also maximizes entropy by (2.27). The uniquely characterized local-information and global-capacity functionals determine its intrinsic local and global correlation radii denoted by the two endpoint lengths. Constant preservation selects the Neumann closure within the separated nondegenerate homogeneous local class. Its zero-flux realization then gives the ordered orthonormal basis $\mathcal{E}_b^0 = \{\phi_{b,n} : n \geq 0\}$. The reflecting double is compatible with the selected circle carrier. Its two analytic channels are exchanged by reflection and the chosen orientation gives $\mathcal{U}_b = \{u_{b,n} : n \geq 0\}$ with

$$\deg_b(u_{b,n}) = \sqrt{E_{b,n}/E_{b,1}} = n. \quad (5.29)$$

Theorem 4.5 defines modulation and cascade on this family. The constant and identity endomorphisms give zero and unit and modulation by the unit gives successor. Theorem 5.1 proves the recursion property while Proposition 4.8 gives the ordered semiring isomorphism

$$(\mathcal{U}_b, \boxplus_b, \boxtimes_b, 0_b, 1_b) \cong (\mathbb{N}_0, +, \cdot, 0, 1). \quad (5.30)$$

The internal Euclidean argument of Appendix B and Theorem 5.2 give unique finite expansions in

$$\mathcal{P}_b = \text{Irr}(\mathcal{U}_b, \boxtimes_b). \quad (5.31)$$

This family is infinite by the Euclidean argument following Theorem 5.2. Theorem 5.6 gives the unique arithmetic transport under changes of positive scale and orientation. This proves (i).

The spectral theorem and the analytic lift preserve the mode inner product and quadratic correlation energy. Equations (3.41) and (5.26) give the retained Gaussian flow and its scale transport. Theorem 5.3 gives

$$I_b/I_b^2 \cong \text{span}_{\mathbb{C}} \mathcal{P}_b, \quad \text{Sym}_{\text{alg}}(I_b/I_b^2) \cong \mathfrak{A}_b. \quad (5.32)$$

The arithmetic transport preserves cascade irreducibility and therefore the prime residue classes and their symmetric products. This proves (ii). $\square$

The arithmetic branch of Lean Formalization of Arithmetic and Basic Physical Interfaces [10] formalizes the source construction and arithmetic assertion in part (i), together with canonical scale and orientation transport. The analytic proof also gives the cotangent and Hilbert structures in part (ii).

5.5 Consequences of the main theorem

Proposition 4.6 identifies the minimal endomorphism closure with the spectral mode family. Proposition 5.7 gives its prime-occupation realization and shows that the exact response at one positive normalized parameter fixes the correspondence between generated families. The statement concerns the canonical mode representatives. Their normalization as group homomorphisms fixes the phase of each character.

Theorem 5.6 also separates the energy unit from the arithmetic. A change of scale rescales the dimensional energies and Gaussian time together while preserving normalized responses and factorizations. This compatibility identifies traces and operator realizations on the canonically linked spaces generated from the same mode family. Table 1 records the generated objects and the representations built from them.

The prime Hilbert space is $\mathcal{H}_{b,\mathcal{P}} := \overline{\text{span}} \mathcal{P}_b$ whereas the full positive-mode space has basis $\{\phi_{b,n} : n \geq 1\}$ by Proposition 4.4. Unique factorization connects these two bases through occupations.

5.6 Gaussian spectral zeta

Let

$$\mathcal{H}_b^+ := \overline{\text{span}}\{\phi_{b,n} : n \geq 1\} = \phi_{b,0}^\perp \subset \mathcal{H}_b. \quad (5.33)$$

The zero eigenvalue is omitted when a negative power of A_b is formed.

On $\phi_{b,n}$ the normalized operator $A_b/E_{b,1}$ has eigenvalue n^2 . Its $-s/2$ power therefore has eigenvalue n^{-s} . Absolute summability for $\Re s > 1$ therefore gives

$$\text{Tr}_{\mathcal{H}_b^+} \left(\frac{A_b}{E_{b,1}} \right)^{-s/2} = \sum_{n=1}^{\infty} n^{-s} = \zeta(s). \quad (5.34)$$

Table 1. Construction dictionary. Each row records a generated object, its determining relations and the result that establishes it.

Layer	Generated object	Determining relations	Establishing result
Minimal change	real rotation carrier with generated complex structure and tangent law	orthogonal action and convolution with exact self-similarity and covariance	Theorems 2.3 and 2.5
Correlation closure	permanent UV/IR window and reflecting double	unit local information, additive correlation mass and constant preservation	Proposition 3.1 and Theorem 3.3
Spectral carrier	ordered modes with $E_{b,n} = E_{b,1}n^2$	change response and oriented circle lift	Theorem 4.1
Arithmetic generation	Peano semiring and prime cotangent directions	modulation and cascade with composition irreducibility	Theorems 5.1 and 5.3 and 5.4
Expression resolution	finite ordered expressions over each natural mode	normalization and algebraic evaluation with uniform-difference splitting	Theorem 6.1 and Section 8.1
Resolved source	positive histories and finite phase fibres	equal root-channel weights, independent children and Gaussian pushforward	Proposition 7.1
Assembly response	parent-child incidence operator on complete histories	unit reverse splitting, detailed balance and parent-child differences	Theorem 11.4

The identity is independent of the scale b after normalization by the first positive energy.

The proof of this trace identity uses only the quadratic spectrum. Absolute convergence and unique factorization also identify this series with its Euler product over the generated prime modes.

5.7 Compatible appearances of π

Three normalizations occur in the construction.

- (i) the endpoint ratio $L_{\text{IR}}/\ell_{\text{UV}} = \sqrt{\pi}$ and the endpoint-time ratio $t_{\text{IR}}/t_{\text{UV}} = \pi$.
- (ii) the winding normalization $(2\pi i)^{-1} \int_{\mathbb{T}} u^{-1} du$.
- (iii) the Fourier-self-dual Gaussian $g(x) = e^{-\pi x^2}$ under the convention

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx. \quad (5.35)$$

This convention gives

$$\widehat{g} = g, \quad \sum_{n \in \mathbb{Z}} e^{-\pi t n^2} = t^{-1/2} \sum_{m \in \mathbb{Z}} e^{-\pi m^2/t}. \quad (5.36)$$

The endpoint factor comes from the Gaussian capacity integral. Self-duality under the fixed Fourier convention selects $e^{-\pi x^2}$ within the normalized Gaussian family. The winding integral uses the corresponding angular period 2π .

6 Ordered expressions and phase fibres

The arithmetic mode records the result of an operation. Its finite construction is retained by keeping the ordered inputs and brackets before evaluation. The resulting expression object has value fibres and a finite phase group at each positive root. A positive source on these objects assigns weights to the parent-child relations that define the assembly response.

6.1 Expressions before evaluation

At the expression level addition and cascade assemble two ordered subexpressions under a new root. Keeping the subexpressions preserves every intervening operation and bracket. We impose only the unit and absorbing-zero relations at this level. Evaluation identifies expressions with the same mode. The individual expressions remain available for the source and response constructions.

Let $\text{Expr}_b^{\text{raw}}$ be the set of finite ordered binary terms generated by the two constants $0_b, 1_b$ and by the syntactic operations $\boxplus_t$ and $\boxtimes_t$. These symbols denote syntactic nodes. The corresponding operations on $\mathcal{U}_b$ arise after evaluation. Let $\equiv_0$ be the least congruence containing

$$e \boxplus_t 0_b \equiv_0 e, \quad 0_b \boxplus_t e \equiv_0 e, \quad (6.1)$$

$$e \boxtimes_t 1_b \equiv_0 e, \quad 1_b \boxtimes_t e \equiv_0 e, \quad (6.2)$$

$$e \boxtimes_t 0_b \equiv_0 0_b, \quad 0_b \boxtimes_t e \equiv_0 0_b. \quad (6.3)$$

Associativity and commutativity are not imposed at this stage. Neither is distributivity. Input order and bracketing are therefore retained.

We use canonical representatives of the congruence classes. Define NF_b recursively. First normalize the children. At an additive node remove a child equal to 0_b . At a cascade node return 0_b if either child is 0_b . Otherwise remove a child equal to 1_b or retain the ordered node if neither child is a unit. The constants are fixed. Write

$$\text{Expr}_b := \text{NF}_b(\text{Expr}_b^{\text{raw}}) \quad (6.4)$$

and define the operations on reduced terms by normalized assembly through

$$e \boxplus_{\text{expr}} f := \text{NF}_b(e \boxplus_t f), \quad e \boxtimes_{\text{expr}} f := \text{NF}_b(e \boxtimes_t f). \quad (6.5)$$

The algebraic term evaluation is the recursive map

$$\text{val}_b : \text{Expr}_b \longrightarrow \mathcal{U}_b \quad (6.6)$$

that fixes $0_b, 1_b$ and replaces $\boxplus_{\text{expr}}$ and $\boxtimes_{\text{expr}}$ by $\boxplus_b$ and $\boxtimes_b$. It acts on reduced expressions themselves. The expression operations assemble histories; evaluation gives the corresponding modulation and composition of modes. Thus the evaluation quotient carries arithmetic, while its fibres retain the distinct constructions of each value.

Theorem 6.1 (Finite-fibre expression resolution). *The reduction rules (6.1)–(6.3) generate the least operation congruence $\equiv_0$. They terminate. Every raw term has a unique reduced representative in its $\equiv_0$ -class. The reduced term object has the following properties.*

- (i) *It is initial in the category of sets with two distinguished elements and two ordered binary operations satisfying the unit and absorbing laws displayed in (6.1)–(6.3). Structure-preserving maps serve as morphisms. No further identities are required.*
- (ii) *The map val_b is a surjective morphism for both operations.*
- (iii) *The relation*

$$e \sim_{\text{val}} f \iff \text{val}_b(e) = \text{val}_b(f), \quad (6.7)$$

is a congruence for both expression operations. Hence the operations descend to the evaluation quotient by

$$[e] \boxplus_q [f] := [e \boxplus_{\text{expr}} f], \quad [e] \boxtimes_q [f] := [e \boxtimes_{\text{expr}} f]. \quad (6.8)$$

- (iv) *Evaluation with these descended operations induces a canonical semiring isomorphism*

$$\text{Expr}_b / \sim_{\text{val}} \xrightarrow{\cong} \mathcal{U}_b. \quad (6.9)$$

- (v) *Every fibre of val_b is finite.*

Thus Expr_b is a finite-fibre expression resolution of the natural-mode semiring. The arithmetic quotient is the one constructed in Sections 4.3–5. The theorem adds its pre-evaluation expression structure.

Proof. Every application of a reduction rule strictly lowers the number of nodes. The recursive definition of NF_b is therefore terminating. It is unchanged by each generating reduction in every term context and therefore by the congruence they generate. Its value has no reducible node. A term without a reducible node is fixed by the recursion. Hence two terms are $\equiv_0$ -equivalent exactly when they have the same normal form. For any operation congruence R containing the six displayed reductions structural induction gives $e R \text{NF}_b(e)$ for every raw term e . Equality of normal forms therefore implies $e R f$. Thus equality of normal forms is precisely the least generated congruence. Each congruence class has one reduced representative.

Recursive interpretation of raw terms in any two-operation algebra satisfying the displayed laws factors through $\equiv_0$. Uniqueness follows because every term is assembled from the two constants. This is the stated initial property. The mode operations satisfy the same unit and absorbing laws. Recursive evaluation is therefore well defined and preserves both operations. Every $u \in \mathcal{U}_b$ is a finite successor iterate of 0_b and is thus the evaluation of a reduced additive term. Evaluation is surjective. Whenever $e \sim_{\text{val}} e'$ and $f \sim_{\text{val}} f'$ the morphism laws give

$$e \boxplus_{\text{expr}} f \sim_{\text{val}} e' \boxplus_{\text{expr}} f', \quad e \boxtimes_{\text{expr}} f \sim_{\text{val}} e' \boxtimes_{\text{expr}} f'.$$

Thus (6.8) is well defined. The induced map to $\mathcal{U}_b$ preserves both descended operations. It is surjective by the preceding construction and injective by the definition of $\sim_{\text{val}}$. It is therefore the claimed semiring isomorphism.

To prove finiteness take the winding coordinate after the preceding construction. A positive reduced term of value n has no zero subterm. Its leaves are copies of 1_b and their number is at most n . The bound follows by recursion. Leaf counts add at an additive root. At a nontrivial cascade root with child values $m, \ell \geq 2$ the induction hypothesis and $m + \ell \leq m\ell$ give the same bound. There are finitely many ordered binary trees with at most n leaves and finitely many choices of the two node labels. The zero fibre consists only of 0_b . Hence every evaluation fibre is finite. Appendix D gives the recursion together with the growth estimate and depth filtration. $\square$

Two forgetful maps give useful value-level compressions of this resolution. Suppose the outer multiplicative skeleton of a reduced term has nonunit factors. Flattening its cascade nodes from left to right then gives a bar word with the same total mode. This map forgets multiplicative bracketing and the internal expression of each factor. Evaluating the two child terms at an additive root gives an ordered pair with a fixed sum. That map forgets the two child histories. The bar complex and the fixed-sum fibres are therefore two different value-level compressions of the expression object.

6.2 Positive histories and phase fibres

The expression resolution also defines a generated state space. For each $u \in \mathcal{U}_b$ write

$$\text{Expr}_b(u) := \text{val}_b^{-1}(u), \quad h_b(u) := |\text{Expr}_b(u)|, \quad h_n := h_b(u_{b,n}). \quad (6.10)$$

On the positive sector these are the complete ordered histories of a mode. Every leaf is 1_b . A nonleaf history has a unique root operation and two ordered positive child histories. At a cascade root both child values exceed 1_b . Thus a history can be recovered from its root channel and its children. This recursive description is equivalent to the positive normal forms already constructed and uses the same expression syntax. The zero fibre remains the singleton expression 0_b .

On histories the two operations are actual ordered assemblies. For positive u, v an additive root gives the injective map

$$\text{Expr}_b(u) \times \text{Expr}_b(v) \longrightarrow \text{Expr}_b(u \boxplus_b v), \quad (e, f) \longmapsto e \boxplus_{\text{expr}} f. \quad (6.11)$$

For $u, v >_b 1_b$ a cascade root similarly embeds this product into $\text{Expr}_b(u \boxtimes_b v)$. In these ranges normalization leaves the root and its ordered children intact. This proves injectivity. Evaluation intertwines the assemblies with the function operations through

$$\text{val}_b(e \boxplus_{\text{expr}} f) = \text{val}_b(e) \boxplus_b \text{val}_b(f), \quad (6.12)$$

$$\text{val}_b(e \boxtimes_{\text{expr}} f) = \text{val}_b(e) \boxtimes_b \text{val}_b(f). \quad (6.13)$$

For boundary inputs the normalized operations already enforce the unit and zero laws. Thus the same two operations have a resolved assembly level and a spectral-arithmetic evaluation. At the assembly level exchanging children or changing brackets generally changes the history. The corresponding semiring identities hold after evaluation.

There is also finite information in the circle map itself before its argument is evaluated. For a positive mode put

$$K_b^{\text{ph}}(u) := \ker(u : \mathbb{T} \longrightarrow \mathbb{T}), \quad X_b(u) := \text{Expr}_b(u) \times K_b^{\text{ph}}(u). \quad (6.14)$$

The kernel records the phase choices that have the same image under the generated endomorphism. For $u = u_{b,n}$ with $n \geq 1$ orientation gives

$$K_b^{\text{ph}}(u) = \{e^{2\pi i r/n} : r \in \mathbb{Z}/n\mathbb{Z}\}, \quad |X_b(u)| = nh_n. \quad (6.15)$$

The identity $u(z) = z^n$ shows that these are precisely its kernel elements. The group law is inherited from the circle. The phase group of 1_b is trivial. The constant zero endomorphism has the whole circle as its kernel. The zero mode is therefore retained separately from the finite positive phase fibres.

The spaces are related by the actual maps

$$\coprod_{u >_b 0_b} X_b(u) \longrightarrow \coprod_{u >_b 0_b} \text{Expr}_b(u) \xrightarrow{\text{val}_b} \mathcal{U}_b^+, \quad (6.16)$$

where the first map forgets phase. Assembly and normalization act on expressions; phase translation acts inside a fixed root fibre. Evaluation retains the mode and its correlation response, while the history and phase coordinates distinguish its preimages.

7 The positive history source

An evaluated mode fixes a value and its correlation energy, while its expression fibre records the operations that produced it. Equal root-channel weights and independent child assembly define a positive history law. Multiplication by the Gaussian root mass gives an intensity whose fibre sum recovers that mass exactly. Internal observations can then vary among histories without changing the root energy.

Gaussian time supplies the root weight. Expression depth measures a finite construction. Assembly time evolves its coefficient response at a fixed root weight. Table 2 distinguishes these parameters and the finite root cutoff. Increasing expression depth and changing the Gaussian observation scale are different operations.

Table 2. Correlation scale and the parameters of history construction. The Gaussian coordinates describe the root weight. Depth and assembly time describe different aspects of its histories.

Symbol	Range	Normalization	Role
b	$b > 0$	covariance scale of the tangent Gaussian	fixes correlation length and window endpoints together with the unnormalized spectrum
t	$t > 0$	semigroup scale with $t = \theta b^2$	supplies the root multiplier $C_b(t) = e^{-tA_b/2}$
θ	$\theta > 0$	$\theta = t/b^2$	compares Gaussian states independently of the representative scale b
β	$\beta > 0$	$\beta = tE_{b,1}/2$ for the root mass $e^{-\beta n^2}$	supplies Gaussian summability in the mode and history sources
r, N	$r, N \in \mathbb{N}_0$	expression depth and root-degree cutoff in that order	separate finite construction depth from finite observation of root values
s	$s \geq 0$	time for the assembly operator at fixed β	evolves history coefficients through actual parent-child relations

7.1 Root channels and the recursive history law

Choosing a root operation differs from choosing an entire expression. Value four has three additive root channels and one multiplicative root channel. These four channels produce six reduced expressions. Equal weights on root channels need not give equal probabilities to the completed trees. The local choice defines the law recursively before the histories are enumerated.

For $u >_b 1_b$ let $\mathcal{C}_b(u)$ be the disjoint union of the ordered channels

$$\begin{aligned} \mathcal{C}_b^+(u) &= \{(v, w) : v, w >_b 0_b, v \boxplus_b w = u\}, \\ \mathcal{C}_b^\times(u) &= \{(v, w) : v, w >_b 1_b, v \boxtimes_b w = u\}, \end{aligned} \quad c_b(u) := |\mathcal{C}_b(u)|. \quad (7.1)$$

The operation type is part of the channel. In degree coordinates the count is

$$c_b(u_{b,n}) = (n - 1) + (d(n) - 2) > 0 \quad (n \geq 2), \quad (7.2)$$

where $d(n)$ counts positive divisors. Both child values are strictly smaller than the parent. Write $\text{node}_c(e, f)$ for assembly in a specified channel with the ordered child histories e, f .

Give the available root channels equal conditional weight. Within each chosen channel draw the two children independently according to the same recursive law. Equal weighting concerns the local assembly choices at each node; the probability of a complete tree then follows recursively from its children. The unit leaf supplies the base case

$$p_b(1_b) = 1, \quad p_b(\text{node}_c(e, f)) = \frac{p_b(e)p_b(f)}{c_b(u)}, \quad u = \text{val}_b(\text{node}_c(e, f)). \quad (7.3)$$

Proposition 7.1 (The history source and its value marginal). *The recursion (7.3) determines a unique strictly positive probability law on every positive expression fibre. For $\beta > 0$ put*

$$G_{b,\beta}(u) := e^{-\beta \deg_b(u)^2}, \quad I_{b,\beta}(e) := G_{b,\beta}(\text{val}_b(e))p_b(e). \quad (7.4)$$

Then

$$\sum_{e \in \text{Expr}_b(u)} p_b(e) = 1, \quad \sum_{e \in \text{Expr}_b(u)} I_{b,\beta}(e) = G_{b,\beta}(u). \quad (7.5)$$

The total positive mass is

$$\mathcal{Z}_b^+(\beta) := \sum_{e: \text{val}_b(e) >_b 0_b} I_{b,\beta}(e) = \sum_{n \geq 1} e^{-\beta n^2} = Z_b(2\beta/E_{b,1}) - 1. \quad (7.6)$$

Here $Z_b(t) := \text{Tr}_{\mathcal{H}_b} C_b(t) = \sum_{n \geq 0} e^{-tE_{b,n}/2}$ is the correlation heat trace. Its zero-mode contribution is one. The normalized intensity is therefore a probability law on all positive histories.

Proof. At the unit leaf the assertion is immediate. At a higher root the unique root decomposition identifies its history set with the disjoint union over $c \in \mathcal{C}_b(u)$ of the two child-history products. The induction hypothesis gives total conditional mass one in each product. Thus the sum in (7.3) is $c_b(u)/c_b(u) = 1$. Positivity follows from the same recursion. Any law satisfying that recursion agrees at the leaf and then at every ordered node. This proves uniqueness under the recursive specification. Multiplying by the common root factor gives the second identity in (7.5). Summing these nonnegative identities gives (7.6). The Gaussian series converges. $\square$

At value four the channel count is $c_b(u_{b,4}) = 4$. The four unbalanced additive trees have probability 1/8 each. The balanced additive tree and the multiplicative tree each have probability 1/4. Representative probabilities are

$$\begin{aligned} p_b(1 + (1 + (1 + 1))) &= \frac{1}{8}, \\ p_b((1 + 1) + (1 + 1)) &= p_b((1 + 1) \times (1 + 1)) = \frac{1}{4}. \end{aligned} \quad (7.7)$$

Here the displayed arithmetic notation denotes the corresponding ordered syntactic nodes. Uniform counting instead assigns probability 1/6 to each tree. The quantity h_n counts histories above a mode without adding Neumann eigenvectors. Equation (7.6) retains the original simple-spectrum partition function.

The law p_b is independent of the observation parameter β . At a fixed parameter one can also weight root channels by their child Gaussian masses and renormalize by the local channel sum. This produces a different conditional law with the same root marginal. The recursive equal-channel law above is used throughout the history construction. Appendix D records the finite recursion and its relation to the expression normal forms.

7.2 Phase averaging and internal observations

On $X_b(u)$ require the history marginal to be p_b and the phase law to be invariant under translation by $K_b^{\text{ph}}(u)$. Translation acts transitively on each phase fibre. These requirements uniquely determine

$$p_b^X(e, \zeta) = \frac{p_b(e)}{n}, \quad I_{b,\beta}^X(e, \zeta) = G_{b,\beta}(u) \frac{p_b(e)}{n}, \quad n = \deg_b(u). \quad (7.8)$$

The sum of $I_{b,\beta}^X$ over $X_b(u)$ is again $G_{b,\beta}(u)$. Thus forgetting either phase alone or both phase and history has a specified probability pushforward.

The retained data have concrete observables. Let $\ell(e)$ count unit leaves and let $a(e), m(e)$ count additive and multiplicative internal nodes. They satisfy $\ell(e) = a(e) + m(e) + 1$. At a surviving node define the orientation score by

$$o(1_b) = 0, \quad o(\text{node}_c(e, f)) = \ell(e) - \ell(f) + o(e) + o(f). \quad (7.9)$$

Reflection $e \mapsto e^\leftarrow$ exchanges the two children at every node. Structural induction gives $p_b(e^\leftarrow) = p_b(e)$ and $o(e^\leftarrow) = -o(e)$. Reflection therefore makes the mean orientation score zero. Its even and odd observation spaces are orthogonal. On $X_b(u)$ this reflection leaves the phase coordinate fixed. It is distinct from reversal of circle orientation because the latter also conjugates phase.

At each internal node let ∂_v^E be its normalized square energy minus the two child energies. Summing these differences cancels every nonroot internal contribution. Since each unit leaf has normalized energy one, the whole-tree identity is

$$\ell(e) + J(e) = \deg_b(\text{val}_b(e))^2, \quad J(e) := \sum_{v \in V_{\text{int}}(e)} \partial_v^E. \quad (7.10)$$

Proposition D.2 gives the energy and occupation versions of this telescoping identity. Hence $e^{-\beta \ell(e)} e^{-\beta J(e)} = G_{b,\beta}(\text{val}_b(e))$. The factors retain the internal construction even when their product has descended to the root.

Proposition 7.2 (History observations beyond value evaluation). *The leaf observation does not factor through val_b . Its conditional variance on $\text{Expr}_b(u_{b,6})$ under p_b is strictly positive.*

Proof. Use the ordered expression operations to form

$$\begin{aligned} a_2 &= 1_b \boxplus_{\text{expr}} 1_b, & a_3 &= 1_b \boxplus_{\text{expr}} a_2, \\ e_+ &= a_3 \boxplus_{\text{expr}} a_3, & e_\times &= a_3 \boxtimes_{\text{expr}} a_2. \end{aligned} \quad (7.11)$$

Both evaluations are $u_{b,6}$. The first expression has six unit leaves and the second has five. The whole-tree identity (7.10) therefore gives $(\ell(e_+), J(e_+)) = (6, 30)$ and $(\ell(e_\times), J(e_\times)) = (5, 31)$. Their total quadratic energy is thirty-six in either case. The Gaussian root weights therefore agree. A value-dependent observation would agree on the two histories. Leaf count distinguishes them. Both have strictly positive probability under the recursive source law. Variance zero for a finite positive law would force leaf count to be constant on the fibre. The two displayed leaf counts exclude this possibility and establish strictly positive conditional variance. $\square$

Thus the complete history source contains observations that a single circle-mode value cannot recover. Value-dependent observables remain constant on each fibre. These include the Gaussian root energy and the arithmetic parity. The leaf and internal observations specify where the additional information is retained.

8 Evaluation and conditional information

Evaluation sends an expression to its mode. On amplitude and probability spaces it gives two different readouts: a weighted sum of amplitudes within each fibre and the sum of their probabilities. Conditional expectation identifies the observations recoverable from the root value; variance and entropy quantify the variation retained inside its history fibre.

8.1 Counting linearization of expressions

For the finite expression fibres (6.10) put

$$\mathcal{H}_{b,u}^{\text{expr}} := \ell^2(\text{Expr}_b(u)). \quad (8.1)$$

These spaces are finite-dimensional by Theorem 6.1. Their orthogonal sum is the counting linearization

$$\mathcal{H}_b^{\text{expr}} := \bigoplus_{u \in \mathcal{U}_b} \mathcal{H}_{b,u}^{\text{expr}} = \ell^2(\text{Expr}_b). \quad (8.2)$$

Let $\mathcal{L}_b^0 := \ell^2(\mathcal{U}_b)$ have basis vectors denoted $|u\rangle_b$. The normalized value compression is defined on an expression basis by

$$Q_b^{\text{expr}}|e\rangle := h_b(u)^{-1/2}|u\rangle_b, \quad u = \text{val}_b(e). \quad (8.3)$$

Weighting each fibre by a probability law also includes the recursive source. For a probability law π_u on each finite fibre set

$$Q_\pi|e\rangle = \sqrt{\pi_u(e)}|u\rangle_b, \quad \omega_{\pi,u} = \sum_{\text{val}_b(e)=u} \sqrt{\pi_u(e)}|e\rangle. \quad (8.4)$$

The vectors $\omega_{\pi,u}$ are orthonormal. Hence Q_π extends to a coisometry. Its adjoint sends $|u\rangle_b$ to $\omega_{\pi,u}$. The resulting orthogonal decomposition is

$$\mathcal{H}_b^{\text{expr}} = \overline{\text{span}}\{\omega_{\pi,u}\} \oplus \ker Q_\pi. \quad (8.5)$$

These assertions follow from $\sum_e \pi_u(e) = 1$ on each fibre and orthogonality between different fibres. Counting compression takes $\pi_u(e) = 1/h_b(u)$. Its adjoint sends a value basis vector to the uniform fibre vector

$$\omega_{b,u} := h_b(u)^{-1/2} \sum_{e \in \text{Expr}_b(u)} |e\rangle. \quad (8.6)$$

The counting decomposition is therefore

$$\mathcal{H}_b^{\text{expr}} = \mathcal{H}_{b,\text{av}}^{\text{expr}} \oplus \mathcal{H}_{b,\text{diff}}^{\text{expr}}, \quad \mathcal{H}_{b,\text{av}}^{\text{expr}} = \overline{\text{span}}\{\omega_{b,u} : u \in \mathcal{U}_b\}, \quad \mathcal{H}_{b,\text{diff}}^{\text{expr}} = \ker Q_b^{\text{expr}}. \quad (8.7)$$

The algebraic relation $\sim_{\text{val}}$ of (6.7) and the Hilbert subspace $\ker Q_b^{\text{expr}}$ have different types. The former identifies expressions in a quotient algebra. The latter is the linear space of zero-mean differences inside the finite evaluation fibres.

There is also a trace-preserving value readout. The isometry

$$J_b^{\text{expr}}|e\rangle := |\text{val}_b(e)\rangle_b \otimes |e\rangle \quad (8.8)$$

defines on trace-class operators the channel $\rho \mapsto \text{Tr}_{\mathcal{H}_b^{\text{expr}}} (J_b^{\text{expr}} \rho (J_b^{\text{expr}})^*)$. This channel differs from Q_b^{expr} by recording the value distribution without coherently averaging expressions in the same fibre.

8.2 Source-compatible readout and phase differences

Counting compression treats the members of a finite expression fibre uniformly. The recursive source of Proposition 7.1 supplies another unit vector on that fibre. Its readout retains the local channel weights when an expression is assembled. Let

$$\mathcal{H}_b^{\text{hist}} := \bigoplus_{u >_b 0_b} \mathcal{H}_{b,u}^{\text{expr}}, \quad \omega_{b,u}^{\text{hist}} := \sum_{e \in \text{Expr}_b(u)} \sqrt{p_b(e)} |e\rangle, \quad Q_b^{\text{hist}}|e\rangle := \sqrt{p_b(e)} |u\rangle_b. \quad (8.9)$$

The codomain is $\ell^2(\mathcal{U}_b^+)$. The zero expression can be adjoined as a separate basis vector.

For $u >_b 1_b$ collect the two child-history spaces over all root channels into

$$\mathcal{R}_{b,u}^{\text{hist}} := \bigoplus_{c=(o;v,w) \in \mathcal{C}_b(u)} \mathcal{H}_{b,v}^{\text{expr}} \otimes \mathcal{H}_{b,w}^{\text{expr}}. \quad (8.10)$$

Assembly $M_{b,u}^{\text{hist}}$ sends the basis pair in channel c to $|\text{node}_c(e, f)\rangle$. It is a unitary onto $\mathcal{H}_{b,u}^{\text{expr}}$ because every nonleaf history has one such root description. Define

$$\mathcal{B}_{b,u}(|e\rangle \otimes |f\rangle) := \sqrt{p_b(e)p_b(f)} |c\rangle, \quad \eta_{b,u} := c_b(u)^{-1/2} \sum_{c \in \mathcal{C}_b(u)} |c\rangle. \quad (8.11)$$

Proposition 8.1 (Source readout and root assembly). *The map Q_b^{hist} is a coisometry with initial space the closed span of the vectors $\omega_{b,u}^{\text{hist}}$. For $\beta > 0$ the vector*

$$\Psi_{b,\beta}^{\text{hist}} := \sum_{u >_b 0_b} \sum_{e \in \text{Expr}_b(u)} \sqrt{G_{b,\beta}(u)p_b(e)} |e\rangle \quad (8.12)$$

belongs to $\mathcal{H}_b^{\text{hist}}$ and satisfies

$$\|\Psi_{b,\beta}^{\text{hist}}\|^2 = \mathcal{Z}_b^+(\beta), \quad Q_b^{\text{hist}} \Psi_{b,\beta}^{\text{hist}} = \sum_{u >_b 0_b} \sqrt{G_{b,\beta}(u)} |u\rangle_b. \quad (8.13)$$

For every $u >_b 1_b$ the map $\mathcal{B}_{b,u}$ is a coisometry onto $\ell^2(\mathcal{C}_b(u))$. Writing $Q_{b,u}^{\text{hist}}$ for the restriction of Q_b^{hist} to the root fibre gives the assembly identity

$$Q_{b,u}^{\text{hist}} M_{b,u}^{\text{hist}} = |u\rangle_b \langle \eta_{b,u} | \mathcal{B}_{b,u}. \quad (8.14)$$

The lifted Gaussian multiplier commutes with the fibre decomposition and intertwines both the counting and source-compatible readouts with the corresponding value multiplier.

Proof. Apply (8.4) with $\pi_u = p_b$ on the positive fibres. Summing the squared coefficients of (8.12) gives $\mathcal{Z}_b^+(\beta)$. Applying Q_b^{hist} and summing p_b on each fibre proves the second assertion. On each channel $\mathcal{B}_{b,u}^*|c\rangle$ is the tensor product of the two normalized source vectors. These vectors are orthonormal over different channel summands. This proves coisometry. Evaluation on a basis pair reduces (8.14) to

$$\sqrt{p_b(\text{node}_c(e, f))} = c_b(u)^{-1/2} \sqrt{p_b(e)p_b(f)},$$

which is the source recursion. A multiplier depending only on the evaluated root is scalar on each fibre. The final intertwining therefore follows on the same bases and extends by continuity. $\square$

The resulting decomposition is $\mathcal{H}_b^{\text{hist}} = \overline{\text{span}}\{\omega_{b,u}^{\text{hist}} : u >_b 0_b\} \oplus \ker Q_b^{\text{hist}}$. This splitting uses the source weights. The earlier counting splitting uses $\omega_{b,u}$. The value-four example shows that the two initial lines can differ. Algebraic evaluation and these coherent Hilbert readouts retain different kinds of data. The trace-preserving channel of (8.8) retains the value distribution.

Phase averaging has an equally explicit description. On $L^2(X_b(u), p_b^X)$ define

$$(P_{b,u}^{\text{ph}}f)(e, \zeta) := \frac{1}{n} \sum_{\xi \in K_b^{\text{ph}}(u)} f(e, \xi), \quad n = \deg_b(u). \quad (8.15)$$

It is the orthogonal projection onto the phase-invariant observations. Its square equals itself and its residual has zero phase mean at each history. The product law $p_b(e)/n$ then makes the cross inner product vanish. It follows that

$$f = P_{b,u}^{\text{ph}}f + (I - P_{b,u}^{\text{ph}})f, \quad \|f\|^2 = \|P_{b,u}^{\text{ph}}f\|^2 + \|(I - P_{b,u}^{\text{ph}})f\|^2. \quad (8.16)$$

The invariant space has dimension h_n . The zero-phase-mean space has dimension $h_n(n-1)$. Quotienting by the invariant space identifies a class with its unique zero-mean representative. Quotienting by the zero-mean space instead retains the history observation. These are the two complementary readouts of the finite phase data.

8.3 Conditional readout and retained information

Normalize the positive source by

$$q_{b,\beta}(u) = \frac{G_{b,\beta}(u)}{\mathcal{Z}_b^+(\beta)}, \quad P_{b,\beta}(e) = q_{b,\beta}(\text{val}_b(e))p_b(e). \quad (8.17)$$

Conditioning on a root gives exactly the recursive history law. Thus the conditional expectation of a history observation is

$$(\mathbb{E}_b F)(u) = \sum_{\text{val}_b(e)=u} p_b(e)F(e). \quad (8.18)$$

Let $U_P F = \sum_e \sqrt{P_{b,\beta}(e)} F(e) |e\rangle$ and $U_q a = \sum_u \sqrt{q_{b,\beta}(u)} a(u) |u\rangle_b$. Positivity of all probabilities makes these maps unitaries from the respective weighted L^2 spaces to the counting Hilbert spaces. Direct substitution gives

$$Q_b^{\text{hist}} U_P = U_q \mathbb{E}_b. \quad (8.19)$$

In this identification the source readout is conditional expectation in amplitude coordinates. Probability pushforward of a diagonal state is instead the record channel defined by (8.8).

Proposition 8.2 (Value and conditional information). *Every real $F \in L^2(P_{b,\beta})$ satisfies*

$$\text{Var}_{P_{b,\beta}}(F) = \text{Var}_{q_{b,\beta}}(\mathbb{E}_b F) + \sum_u q_{b,\beta}(u) \text{Var}_{p_b(\cdot|u)}(F). \quad (8.20)$$

With H denoting Shannon entropy the source and its phase extension obey

$$H(P_{b,\beta}) = H(q_{b,\beta}) + \sum_u q_{b,\beta}(u) H(p_b(\cdot|u)), \quad (8.21)$$

$$H(P_{b,\beta}^X) = H(P_{b,\beta}) + \sum_u q_{b,\beta}(u) \log \deg_b(u). \quad (8.22)$$

All terms in these entropy identities are finite for $\beta > 0$.

Proof. Decompose F into its root conditional mean and $F - \mathbb{E}_b F$. The latter has mean zero on every fibre. Their cross inner product therefore vanishes. Applying this orthogonality after subtracting the overall mean proves (8.20). Substitute $\log P_{b,\beta}(e) = \log q_{b,\beta}(u) + \log p_b(e)$ in the entropy sum. The phase law adds $-\log \deg_b(u)$ to the logarithm of each probability and gives the second identity. The fibre bound $h_n < 8^n$ from Appendix D implies $H(p_b(\cdot|u_{b,n})) \leq \log h_n < n \log 8$. The root entropy equals $\log \mathcal{Z}_b^+(\beta) + \beta \sum_n q_{b,\beta}(u_{b,n}) n^2$. Gaussian summability controls the root and conditional entropies as well as the phase term. It also justifies the rearrangements. $\square$

For leaf count the second term of (8.20) is strictly positive because the conditional variance at root six is positive. Evaluation retains the conditional mean but does not determine these within-fibre fluctuations. The retained fluctuations supply internal observations and coefficient differences for actual assemblies.

Theorem 8.3 (History source and exact readout). *The reduced expressions over the generated mode semiring have finite value fibres and finite positive phase fibres. Equal root-channel probabilities and independent child assembly together with phase-translation invariance determine the positive laws p_b and p_b^X . Their Gaussian intensities push forward exactly to the original simple-spectrum mode mass. Algebraic evaluation recovers the mode semiring as its quotient. Counting and source-weighted Hilbert readouts have the initial-kernel decompositions (8.5) and (8.7). The source readout corresponds to conditional expectation by (8.19). The variance and entropy identities (8.20)–(8.22) measure the data retained before evaluation.*

Proof. Theorem 6.1 establishes the algebraic quotient and finite fibres. Equations (6.14) and (6.15) give the phase extension. Proposition 7.1 and translation invariance prove the source laws and their marginals. The weighted fibre construction proves the two Hilbert decompositions. Proposition 8.1 intertwines root assembly. Substitution proves the conditional readout. Orthogonality and the entropy calculation give the information formulas. $\square$

9 Connected response of the history source

At a fixed arithmetic value, perturbing the logarithm of the conditional source state gives a connected response: the Hessian of its log partition function. On diagonal observations this is the usual covariance; off-diagonal entries compare distinct histories and phases. Scalar perturbations change only the total intensity and form the kernel. Orthogonal reduction in this Hessian separates the response shared by two observations from the remaining variation.

9.1 Finite matrix response of a history source

A finite positive source supplies a reference state for observations on its fibre. The same calculation covers $F = \text{Expr}_b(u)$ with law p_b and $F = X_b(u)$ with law p_b^X . Put

$$D_F := \text{diag}(p_F), \quad \Phi_F(K) := \log \text{Tr} \exp(\log D_F + K), \quad K = K^*. \quad (9.1)$$

Matrices are complex Hermitian and the observation space is regarded as a real vector space. The reference has trace one and hence $\Phi_F(0) = 0$.

Proposition 9.1 (Connected history response and orthogonal reduction). (i) *At zero perturbation the Hessian of Φ_F is the symmetric form*

$$B_F(K, J) = \int_0^1 \text{Tr}(D_F^{1-v} K D_F^v J) dv - \text{Tr}(D_F K) \text{Tr}(D_F J). \quad (9.2)$$

It is nonnegative with kernel the real scalar line. It is strictly positive on every nonzero traceless Hermitian observation. Its rank is $|F|^2 - 1$. For diagonal observations it is the covariance under p_F .

(ii) *For Hermitian K, J with $B_F(J, J) > 0$ put*

$$a_F = \frac{B_F(K, J)}{B_F(J, J)}, \quad K_{\text{res}} = K - a_F J. \quad (9.3)$$

This is the unique coefficient for which $B_F(K_{\text{res}}, J) = 0$. It gives

$$B_F(K, K) = B_F(K_{\text{res}}, K_{\text{res}}) + \frac{B_F(K, J)^2}{B_F(J, J)}. \quad (9.4)$$

It minimizes $B_F(K - aJ, K - aJ)$ over real a . If $B_F(J, J) = 0$ then J is scalar and $B_F(K, J) = 0$ for every K .

Proof. Differentiating the matrix exponential by its integral formula and using cyclicity of trace gives (9.2). Write $p_i > 0$ for the entries of D_F and $\bar{K} = \sum_i p_i K_{ii}$. The diagonal and off-diagonal contributions are

$$B_F(K, K) = \sum_i p_i (K_{ii} - \bar{K})^2 + \sum_{i \neq j} \mathcal{L}(p_i, p_j) |K_{ij}|^2, \quad \mathcal{L}(x, y) := \int_0^1 x^{1-v} y^v dv > 0. \quad (9.5)$$

Thus the form vanishes precisely on scalar matrices. A scalar traceless matrix is zero. This proves strict positivity on that subspace and the rank assertion. For diagonal matrices only the first term remains.

For part (ii) expand the quadratic polynomial in a . Completing its square gives the minimum and its unique coefficient together with (9.4). In the zero case part (i) identifies J as a scalar. Substitution in (9.2) gives zero mixed response. $\square$

Using the unnormalized intensity instead replaces $\log D_F$ by $\log D_F + \log G_{b,\beta}(u)I$. It adds $\log G_{b,\beta}(u)$ to Φ_F and leaves the response derivatives unchanged. The finite response therefore measures the history and phase data inside a fixed root fibre. The root mass is recovered by the pushforward already proved.

9.2 Conditional response

A nonzero mixed response means that the two observations measure part of the same variation. Part (ii) removes this shared response. The source weights determine both the coefficient to subtract and the remaining quadratic response.

For several observations independent modulo scalars the same argument uses their positive Gram matrix and gives the corresponding Schur complement. Leaf count and ordered-root imbalance provide internal observations. Phase observations supply further choices on the phase extension. A root-dependent arithmetic character is constant within one such fibre and has zero connected response there. Its nonconstant response is evaluated on states containing more than one root. The same response-orthogonal reduction applies to any finite family of root-input observations once its connected form is specified.

10 Assembly dynamics of generated histories

At each assembly we compare the parent coefficient with the sum of its two ordered child coefficients. The children have smaller root values and lie in different fibres from the parent. The squared difference, weighted by the parent source intensity, retains all three entries and their cross terms. The source recursion controls the sum of these local comparisons over the complete history space.

10.1 Merge rates and detailed balance

Fix $\beta > 0$. For a root channel $c = (\circ; v, w) \in \mathcal{C}_b(u)$ take unit reverse split rate and put

$$r_{b,\beta}(c) := \frac{G_{b,\beta}(u)}{c_b(u)G_{b,\beta}(v)G_{b,\beta}(w)}. \quad (10.1)$$

This is a strictly positive rate for the specified channel. The recursive source gives the following identity for its actual child histories

$$I_{b,\beta}(e_L)I_{b,\beta}(e_R)r_{b,\beta}(c) = I_{b,\beta}(\text{node}_c(e_L, e_R)). \quad (10.2)$$

Thus the two-child source with the forward rate gives the same intensity as the parent source with the unit reverse rate. The equality follows by substituting (7.3). Cancelling the two positive child masses in this equality forces (10.1). This proves uniqueness for the given source and unit split rule. If a different positive reverse rate is assigned to an edge then detailed balance multiplies its forward rate by the same factor.

Unit reverse splitting treats each retained binary assembly as one reverse operation. It specifies the comparison of rates across channels. The Gaussian root mass and the equal-channel recursion determine the forward rates. Thus the source and the split rule determine the rates.

Multiplying the balance identity through an entire tree gives

$$I_{b,\beta}(e) = \prod_{\ell \in V_{\text{leaf}}(e)} G_{b,\beta}(1_b) \prod_{v \in V_{\text{int}}(e)} r_{b,\beta}(c_v). \quad (10.3)$$

Every internal child intensity cancels against its occurrence at the next assembly. The Gaussian exponents telescope to the root square energy. The channel factors give $p_b(e) = \prod_v c_b(u_v)^{-1}$. The source is therefore compatible with each actual assembly and with the complete finite construction.

10.2 The finite assembly form

Let $\mathcal{S}_{b,N}$ contain all positive histories whose root degree is at most N with $N \geq 1$. It is finite. Write $\mathcal{A}_{b,N}$ for its nonleaf histories and equip each with its unique ordered children. On real coefficient functions define

$$d_{\text{asm}}f(e) := f(e) - f(e_L) - f(e_R), \quad \mathcal{Q}_{b,\beta,N}(f) := \sum_{e \in \mathcal{A}_{b,N}} I_{b,\beta}(e) |d_{\text{asm}}f(e)|^2. \quad (10.4)$$

All children remain inside the cutoff because their root degrees are smaller. A repeated child occurs twice. The history of two unit leaves contributes $I_{b,\beta}(1+1)|f(1+1) - 2f(1_b)|^2$. The corresponding incidence vector is $|1+1\rangle - 2|1_b\rangle$.

For a general real triple x, y, z the summand expands as $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$. Equation (10.4) retains these terms with their common source weight. Its kernel has a simple interpretation.

Proposition 10.1 (The finite assembly kernel). *For $N \geq 1$ the kernel is given by*

$$\mathcal{Q}_{b,\beta,N}(f) = 0 \iff f(e) = \ell(e)f(1_b) \quad (e \in \mathcal{S}_{b,N}). \quad (10.5)$$

Prescribing the unit-leaf coefficient therefore gives a unique zero-response function. On the subspace $f(1_b) = 0$ every nonzero coefficient function has strictly positive response. Give the edge space $\mathbb{R}^{\mathcal{A}_{b,N}}$ the inner product $\langle \xi, \eta \rangle_I = \sum_e I_{b,\beta}(e) \xi(e) \eta(e)$ and give coefficient

functions counting inner product. With these conventions the assembly differential has adjoint d_{asm}^* . Then every coefficient function has the unique decomposition

$$f = a\ell_N + d_{\text{asm}}^*\xi, \quad a = \frac{\langle f, \ell_N \rangle}{\|\ell_N\|^2}, \quad \langle \ell_N, d_{\text{asm}}^*\xi \rangle = 0. \quad (10.6)$$

Its counting norm splits orthogonally. Its assembly response depends only on $d_{\text{asm}}^*\xi$.

Proof. Positivity of all edge weights makes zero response equivalent to $f(e) = f(e_L) + f(e_R)$ at every nonleaf history. Recursion from the unit leaf gives $f(e) = \ell(e)f(1_b)$. Conversely the leaf count adds at both root operations and satisfies each relation. The anchored assertion follows immediately. For any prescribed edge vector g set the unit coefficient to zero and recursively set $f(e) = g(e) + f(e_L) + f(e_R)$. Children have smaller roots. This gives $d_{\text{asm}}f = g$. Thus d_{asm} is surjective and its weighted adjoint is injective. Finite-dimensional orthogonality gives $\text{Ran } d_{\text{asm}}^* = (\ker d_{\text{asm}})^\perp = (\mathbb{R}\ell_N)^\perp$. This proves existence and uniqueness in (10.6). This includes the case $N = 1$ with zero edge space. Orthogonality gives the norm decomposition and $d_{\text{asm}}\ell_N = 0$ gives the assertion about response. $\square$

Zero assembly response means that the parent coefficient is the sum of the child coefficients at either root operation. Recursion therefore determines it from its value at the unit leaf. Leaf count gives the proportionality factor. These relations on history coefficients distinguish constructions with the same arithmetic value and different numbers of leaves. For a constant coefficient $d_{\text{asm}}f = -f$ at each nonleaf node.

10.3 The complete history operator

Let $\mathcal{S}_b$ be the set of all positive histories and use the real counting Hilbert space

$$\mathcal{K}_b^{\text{asm}} := \ell^2(\mathcal{S}_b; \mathbb{R}). \quad (10.7)$$

It is the real coefficient subspace of $\mathcal{H}_b^{\text{hist}}$. For a nonleaf history set

$$b_e := |e\rangle - |e_L\rangle - |e_R\rangle, \quad R_{b,\beta,n}^{\text{asm}} := \sum_{\substack{e \text{ nonleaf} \\ \deg_b(\text{val}_b(e))=n}} I_{b,\beta}(e) |b_e\rangle \langle b_e|. \quad (10.8)$$

The layer at root one is zero. For $n \geq 2$ let

$$\gamma_{b,n} := \sum_{\substack{\deg_b(\text{val}_b(e))=n \\ e_L=e_R}} p_b(e), \quad 0 \leq \gamma_{b,n} \leq 1. \quad (10.9)$$

The parent differs from each child. Consequently $\|b_e\|^2 = 3$ for distinct children and $\|b_e\|^2 = 5$ for repeated children. The finite cutoff is $R_{b,\beta}^{\text{asm},(N)} := \sum_{n \leq N} R_{b,\beta,n}^{\text{asm}}$ and acts on the complete space. Its quadratic form restricts to (10.4).

Proposition 10.2 (Trace-class assembly response and evolution). *For every $\beta > 0$ the series in*

$$\text{Tr } R_{b,\beta,n}^{\text{asm}} = (3 + 2\gamma_{b,n})e^{-\beta n^2}, \quad R_{b,\beta}^{\text{asm}} := \sum_{n \geq 1} R_{b,\beta,n}^{\text{asm}} \quad (10.10)$$

converges in trace norm to a nonnegative self-adjoint trace-class operator. The displayed trace formula is for $n \geq 2$. Its layer norm is bounded by $\|R_{b,\beta,n}^{\text{asm}}\|_1 \leq 5e^{-\beta n^2}$. The tail satisfies

$$\|R_{b,\beta}^{\text{asm}} - R_{b,\beta}^{\text{asm},(N)}\|_1 = \sum_{n>N} (3 + 2\gamma_{b,n})e^{-\beta n^2} \leq 5 \sum_{n>N} e^{-\beta n^2}, \quad N \geq 1. \quad (10.11)$$

Its complete form is

$$\mathcal{Q}_{b,\beta}(f) = \langle f, R_{b,\beta}^{\text{asm}} f \rangle = \sum_{e \text{ nonleaf}} I_{b,\beta}(e) |f(e) - f(e_L) - f(e_R)|^2. \quad (10.12)$$

At fixed β the evolution

$$f_s = e^{-sR_{b,\beta}^{\text{asm}}} f_0, \quad s \geq 0, \quad (10.13)$$

is a contraction and satisfies

$$\partial_s f_s = -R_{b,\beta}^{\text{asm}} f_s, \quad (10.14)$$

$$\frac{d}{ds} \frac{1}{2} \|f_s\|^2 = -\mathcal{Q}_{b,\beta}(f_s), \quad (10.15)$$

$$\frac{d}{ds} \frac{1}{2} \mathcal{Q}_{b,\beta}(f_s) = -\|R_{b,\beta}^{\text{asm}} f_s\|^2. \quad (10.16)$$

The finite-cutoff operators and their evolutions converge in operator norm. The evolution convergence is uniform on bounded assembly-time intervals. The complete operator has trivial kernel. Moreover $f_s \rightarrow 0$ in norm for each initial coefficient f_0 as $s \rightarrow \infty$.

Proof. For a positive rank-one operator $|v\rangle\langle v|$ both its trace and trace norm equal $\|v\|^2$. All histories at root $n \geq 2$ are nonleaf and their probabilities sum to one. Splitting them according to whether their children agree gives the trace in (10.10) and the bound by $5e^{-\beta n^2}$. Gaussian summability proves trace-norm convergence. Positivity of the tail makes its trace norm the sum of the layer traces. This proves (10.11). Positivity and self-adjointness pass to the limit. Taking inner products gives (10.12) with its full nonnegative series.

The exponential of this bounded operator is differentiable in norm. Differentiating the squared norm and the quadratic form gives the two dissipation identities. Nonnegativity makes the evolution contractive. For two nonnegative bounded generators the integral identity for their exponentials bounds the difference at time s by s times the generator difference. This proves the cutoff assertion. If $Rf = 0$ then every summand in the complete form vanishes. The same tree recursion as in the finite kernel proof gives $f(e) = \ell(e)f(1_b)$ for every history. Purely additive histories with arbitrarily many leaves show that ℓ is not square-summable. Hence $f(1_b) = 0$ and $f = 0$. If μ_{f_0} is the spectral measure of R at f_0 then $\|e^{-sR} f_0\|^2 = \int e^{-2s\lambda} d\mu_{f_0}(\lambda) \rightarrow 0$ by dominated convergence because $\mu_{f_0}(\{0\}) = 0$. Appendix E gives the explicit tail estimate and the high-root spectral edge of this injective operator. $\square$

Leaf count spans the finite kernel but is not square-summable on the complete history set. The complete operator is therefore injective even though every finite cutoff has a one-dimensional zero-response direction. The normalized source mass removes history counts

from the layer bound, while retaining every assembly and its cross terms. Gaussian root weights then control the trace-norm tail.

Assembly time s varies coefficients at a fixed Gaussian observation parameter. Changing the Gaussian parameter varies the root weights and therefore changes the assembly operator itself. At fixed parameter its evolution compares coefficients across the retained parent–child relations.

11 Value and history-difference dynamics

The weighted readout separates a history coefficient into its value component and a difference invisible to that readout. The assembly operator compares parents with children in different value fibres and can therefore couple the two components. The coupling has an explicit block form with a nonzero matrix element for the complete operator.

11.1 The value and difference blocks

Work in the real coefficient spaces of Section 10. Write $Q = Q_b^{\text{hist}}$ and $V = Q^*$. Set $P = VQ$. The map V embeds value coefficients in the source-weighted fibre lines. The map P projects onto their closed span. Thus every history coefficient has the unique representation

$$f = Va + z, \quad a = Qf, \quad z = (I - P)f \in \ker Q. \quad (11.1)$$

Fix $\beta > 0$ and abbreviate $R = R_{b,\beta}^{\text{asm}}$. Its blocks on this decomposition are

$$A = QRV, \quad B = QR|_{\ker Q}, \quad C = (I - P)R|_{\ker Q}. \quad (11.2)$$

The operators A, C are positive and self-adjoint. All three blocks are bounded. The off-diagonal adjoint is $B^* = (I - P)RV$.

Proposition 11.1 (Exact coupled readout dynamics). *The coordinates (11.1) give*

$$\mathcal{Q}_{b,\beta}(f) = \langle a, Aa \rangle + 2\langle a, Bz \rangle + \langle z, Cz \rangle. \quad (11.3)$$

The complete assembly flow satisfies

$$\dot{a}_s = -Aa_s - Bz_s, \quad \dot{z}_s = -B^*a_s - Cz_s. \quad (11.4)$$

Its value component consequently obeys the exact equation

$$\dot{a}_s = -Aa_s - Be^{-sC}z_0 + \int_0^s Be^{-(s-t)C}B^*a_t dt. \quad (11.5)$$

The blocks obtained from $R_{b,\beta}^{\text{asm},(N)}$ converge in operator norm to these blocks. Their coupled flows converge uniformly on every bounded assembly-time interval.

Proof. The map $(a, z) \mapsto Va + z$ is an orthogonal isomorphism because $V^*V = I$ and $\text{Ran } V = (\ker Q)^\perp$. Expanding $\langle Va + z, R(Va + z) \rangle$ proves (11.3). Apply Q and $I - P$ to $\dot{f} = -Rf$ to obtain (11.4). Variation of constants in its second equation gives

$$z_s = e^{-sC}z_0 - \int_0^s e^{-(s-t)C}B^*a_t dt.$$

Boundedness of all generators makes the integral converge in norm and substitution proves (11.5). Compression by norm-one maps preserves operator-norm convergence. The coupled evolution is the orthogonal conjugate of the assembly evolution. Its cutoff convergence follows from Proposition 10.2. $\square$

The form in (11.3) remains nonnegative as a whole. The block B describes the value response to a difference that the initial readout cannot see. Even when $z_0 = 0$ the term $-B^*a_s$ can create a history-difference component. That component enters the value equation through $-Bz_s$. The last integral in (11.5) records this passage through the difference subspace. A nonzero matrix element of B therefore detects history dependence in the complete value response.

11.2 A difference with nonzero complete readout

Let $t_1, \dots, t_4$ be the four unbalanced additive histories of value four. Let $t_5 = (1+1)+(1+1)$ and $t_6 = (1+1) \times (1+1)$. All symbols in these trees denote ordered expression nodes. The recursive probabilities are $1/8$ for $t_1, \dots, t_4$ and $1/4$ for t_5, t_6 . Put

$$z = \frac{1}{2\sqrt{2}} \sum_{j=1}^4 |t_j\rangle - \frac{1}{2}(|t_5\rangle + |t_6\rangle). \quad (11.6)$$

Proposition 11.2 (History differences enter the value response). *The vector in (11.6) has $\|z\| = 1$ and $Qz = 0$. For every $\beta > 0$ the assembly response satisfies*

$$\langle u_{b,1}, QR_{b,\beta}^{\text{asm},(4)} z \rangle = -\frac{e^{-16\beta}}{4\sqrt{2}}, \quad (11.7)$$

$$\langle u_{b,1}, QR_{b,\beta}^{\text{asm}} z \rangle = -\frac{e^{-16\beta}}{4\sqrt{2}} - \frac{1 - 1/\sqrt{2}}{8} e^{-25\beta} < 0. \quad (11.8)$$

Here $u_{b,1}$ denotes the unit value basis vector. In particular the block B is nonzero.

Proof. The six histories are orthonormal. Their coefficients in z give $\|z\|^2 = 4/8 + 2/4 = 1$. Their source square-root weights give $Qz = (4/8 - 2/4)|u_{b,4}\rangle_b = 0$. The unit fibre is a singleton. Taking its coefficient after Q is the same as taking the unit history coefficient before Q . Each unbalanced tree has one unit child at its root. The balanced trees have no unit child there. The four root-four summands therefore contribute $-4e^{-16\beta}(1/8)(1/(2\sqrt{2}))$. This is (11.7).

A higher parent can contribute to this matrix element only if its incidence vector contains both a history of value four and the unit history. The parent must therefore have these histories as its two children. A retained multiplicative root has no unit child. Only the additive channels $(1, 4)$ and $(4, 1)$ at parent value five can therefore contribute. There are four channels at five. Each such parent over t_j has probability $p_b(t_j)/4$. Their combined contribution is

$$\frac{e^{-25\beta}}{2} \sum_{j=1}^6 p_b(t_j) z(t_j) = -\frac{1 - 1/\sqrt{2}}{8} e^{-25\beta}.$$

All other incidence vectors miss at least one of the two required supports. This proves the complete formula and its strict sign. $\square$

This example uses a difference invisible at the initial value readout. Its nonzero response comes from actual assemblies at roots four and five. The complete trace-class sum introduces no further contribution to this matrix element. Thus the retained history affects a value observation under the generated assembly response.

The recursive law fixes the exact coefficients in (11.8). The coupling itself persists under small changes of the conditional history weights. To compare such laws, write $\mathcal{S}_{b,n} := \text{Expr}_b(u_{b,n})$ and $p_0 := p_b$. For any strictly positive law p normalized on each $\mathcal{S}_{b,n}$, use the same readout and incidence formulas:

$$\begin{aligned} Q_p |e\rangle &:= \sqrt{p(e)} |u_{b,n}\rangle \quad (e \in \mathcal{S}_{b,n}), \\ R_{p,\beta} &:= \sum_{n \geq 2} e^{-\beta n^2} T_n(p), \quad T_n(p) := \sum_{e \in \mathcal{S}_{b,n}} p(e) |b_e\rangle \langle b_e|. \end{aligned} \quad (11.9)$$

Normalization makes Q_p a coisometry. Since $\|b_e\|^2 \leq 5$, the assembly sum converges in trace norm, with

$$\|R_{p,\beta}\|_1 \leq 5 \sum_{n \geq 2} e^{-\beta n^2} < \infty.$$

Proposition 11.3 (Stability of the history coupling). *Let z be the vector in (11.6). Set*

$$v_p := \sum_{e \in \mathcal{S}_{b,4}} \sqrt{p(e)} |e\rangle, \quad z_p := (I - |v_p\rangle \langle v_p|)z,$$

and put

$$\delta_n := \sum_{e \in \mathcal{S}_{b,n}} |p(e) - p_0(e)|, \quad \eta_4 := \|v_p - v_{p_0}\|.$$

Then $Q_p z_p = 0$, and $F_\beta(p) := \langle u_{b,1}, Q_p R_{p,\beta} z_p \rangle$ satisfies

$$|F_\beta(p) - F_\beta(p_0)| \leq 5 \left[e^{-16\beta} \delta_4 + e^{-25\beta} \delta_5 + (e^{-16\beta} + e^{-25\beta}) \eta_4 \right]. \quad (11.10)$$

In particular, if

$$\delta_4 + \delta_5 + 2\eta_4 < \frac{1}{20\sqrt{2}}, \quad (11.11)$$

then $F_\beta(p) < 0$ for every $\beta > 0$.

Proof. The vector v_p is a unit vector, and the restriction of Q_p to the root-four fibre is $f \mapsto \langle v_p, f \rangle |u_{b,4}\rangle$. Hence $Q_p z_p = 0$ and $\|z_p\| \leq 1$. Since $\langle v_{p_0}, z \rangle = 0$, one also has $\|z_p - z\| = |\langle v_p - v_{p_0}, z \rangle| \leq \eta_4$.

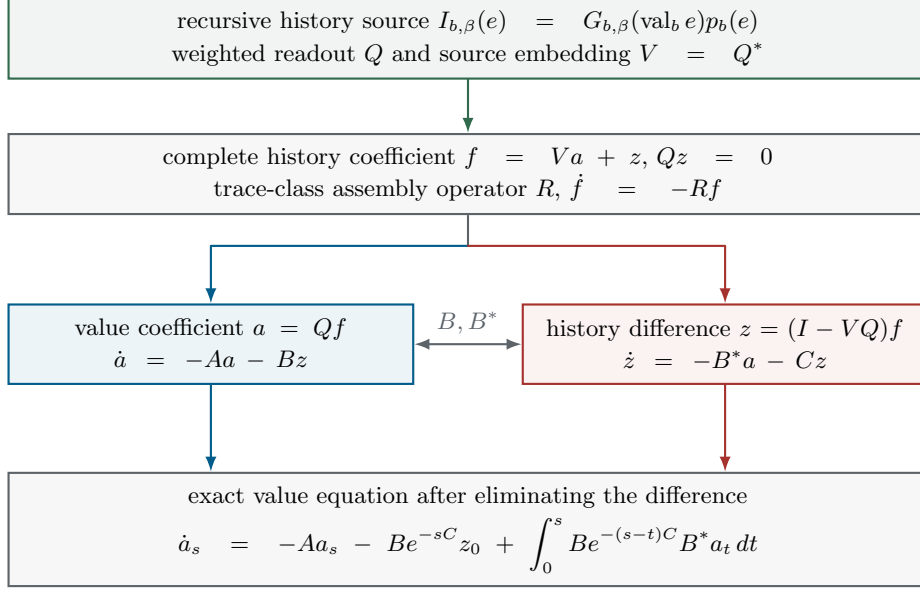
The unit fibre is a singleton. Its coefficient after Q_p is therefore independent of p . As in the preceding proof, an incidence vector meeting both the unit history and the support of z_p has root four or five. The layer bounds are

$$\|T_n(p) - T_n(p_0)\| \leq 5\delta_n, \quad \|T_n(p_0)\| \leq 5.$$

Apply these inequalities to $(T_n(p) - T_n(p_0))z_p + T_n(p_0)(z_p - z)$ for $n = 4, 5$ to obtain (11.10). Finally $e^{-25\beta} \leq e^{-16\beta}$, whereas (11.8) gives $F_\beta(p_0) \leq -e^{-16\beta}/(4\sqrt{2})$. Condition (11.11) preserves this strict sign. $\square$

Only the laws at roots four and five enter this estimate. The normalized weights at higher roots may be changed arbitrarily. Thus the value response to retained history is stable under local changes of the source weights, even though the readout and its difference subspace change with those weights.

Figure 2 displays the source readout and the coupled evolution.



Root observations and internal history observations use their specified readouts.

Figure 2. Source-weighted readout and the coupled assembly response. The orthogonal splitting separates the value coefficient from histories with zero readout. Compression of the same complete assembly operator gives all three blocks. The exact root-four example shows that the mixed blocks are nonzero. Eliminating the history difference leaves its initial contribution and the integral kernel in the value equation. The Gaussian observation parameter is held fixed during this assembly evolution.

11.3 The history-source and assembly theorem

Theorem 11.4 (History-source and assembly theorem). *The arithmetic system of Theorem 5.4 and its expression resolution have the following properties under the equal-root-channel recursion and unit reverse split rule.*

- (i) *Every positive expression fibre and its phase extension carry strictly positive conditional laws. Their Gaussian intensities push forward to the original mode mass with its original multiplicity and partition function.*
- (ii) *Algebraic evaluation and the counting and source-compatible coisometries together with phase averaging give the readouts described in Section 8. The source vector reads back exactly to the Gaussian mode vector. Complete tree transport depends*

only on the root. Leaf and internal observations distinguish histories with the same value.

- (iii) *The finite source states have connected Hermitian response with scalar kernel. Actual parent–child assemblies satisfy detailed balance. Their finite coefficient form has exactly the leaf-count kernel.*
- (iv) *For every positive Gaussian parameter the complete assembly layers define a nonnegative self-adjoint trace-class operator. Finite cutoffs converge to it in trace norm. Its contraction evolution satisfies (10.15) and (10.16).*
- (v) *Source readout splits that evolution into the value and history difference blocks of (11.4). Eliminating the difference gives (11.5). The nonzero matrix element (11.8) shows that these blocks are coupled on the complete space.*

At equal normalized Gaussian parameter canonical scale and orientation transport preserves the history and phase sources and intertwines their readouts and assembly operators.

Proof. Proposition 7.1 proves the history law and its marginal. Phase translation gives (7.8). Proposition 8.1 and (8.16) establish the readouts. Whole-tree telescoping proves the root transport. Proposition 7.2 gives the nonconstant internal observation. Proposition 9.1 and the detailed balance identity combine with Proposition 10.1 to give (iii). Proposition 10.2 proves (iv). Propositions 11.1 and 11.2 prove (v).

For transport apply the canonical arithmetic isomorphism recursively to each ordered expression while preserving its operation type. It bijects the root channels and hence preserves c_b and the recursive weights. The normalized carrier isomorphism maps $K_b^{\text{ph}}(u)$ onto the kernel of the transported mode. Its product with the expression map preserves phase probability. Equality of the normalized β values ensures that the induced basis isometry preserves $G_{b,\beta}$ and all edge intensities together with the incidence vectors. This isometry therefore intertwines the finite layers and their norm limit as well as the exponentials. The same basis calculation intertwines the two source readouts. These maps are written explicitly in Appendix D. $\square$

12 Completions and finite arithmetic

Formal differences and fractions of the generated arithmetic give an ordered field, whose metric completion supplies real scalars. Completing finite mode sums in the square-sum norm recovers the correlation Hilbert space. Finite rational coefficient lists remain construction certificates in its dense spectral core. The scalar metric and Hilbert norm determine the admitted limits; weighted coefficient bounds quantify their finite approximation.

12.1 Scale transport and finite Gaussian readout

The Hilbert completion uses the same scale map as the generated window. Its coordinate form extends directly to the completed spaces. For $b, b' > 0$ put

$$(U_{b,b'} f)(d') = \sqrt{\frac{b}{b'}} f\left(\frac{b}{b'} d'\right), \quad d' \in \mathcal{W}_{b'}. \quad (12.1)$$

Changing variables in the norm proves unitarity. The Neumann basis formulas give $U_{b,b'}\phi_{b,n} = \phi_{b',n}$. Differentiation on the Neumann domain gives

$$A_{b'}U_{b,b'} = (b/b')^2 U_{b,b'}A_b.$$

Thus this explicit map realizes the normalized spectral transport of Theorem 5.6.

For a finite readout write

$$\alpha_b(t) = \frac{tE_{b,1}}{2}, \quad Z_b(t) = \sum_{n \geq 0} e^{-\alpha_b(t)n^2}.$$

The normalized state and its mode probabilities are

$$\rho_b(t) = \frac{C_b(t)}{Z_b(t)}, \quad \varpi_{b,n}(t) = Z_b(t)^{-1} e^{-tE_{b,n}/2}.$$

Monotonicity of the Gaussian and $x/N \geq 1$ for $x \geq N$ give

$$\sum_{n > N} e^{-\alpha n^2} \leq \int_N^\infty e^{-\alpha x^2} dx \leq \frac{e^{-\alpha N^2}}{2\alpha N}, \quad \alpha > 0, \quad N \geq 1. \quad (12.2)$$

Every polynomial energy moment is finite at positive t . Substituting the logarithm of $\varpi_{b,n}$ into its Shannon sum therefore gives

$$S_{\text{sp}}(t) = - \sum_n \varpi_{b,n}(t) \log \varpi_{b,n}(t) = \log Z_b(t) + \frac{t}{2} \sum_n \varpi_{b,n}(t) E_{b,n} < \infty. \quad (12.3)$$

These estimates control probability and entropy in finite-resolution readouts.

12.2 From spectral natural modes to scalar fields

Additive cancellation permits a pair of natural modes to record a formal difference. Adding the same mode to both entries does not change that difference. The pair equivalence implements this identification using the generated addition. Distributivity then determines the product of pairs.

Let $\mathbb{Z}_b$ be the Grothendieck group of the cancellative commutative monoid $(\mathcal{U}_b, \boxplus_b, 0_b)$. Its elements are equivalence classes $[u, v]$ under the relation

$$(u, v) \sim (u', v') \iff u \boxplus_b v' = u' \boxplus_b v. \quad (12.4)$$

The semiring multiplication extends by

$$[u, v][u', v'] := [(u \boxtimes_b u') \boxplus_b (v \boxtimes_b v'), (u \boxtimes_b v') \boxplus_b (v \boxtimes_b u')]. \quad (12.5)$$

The degree map extends to

$$\deg_b^{\mathbb{Z}}([u, v]) := \deg_b(u) - \deg_b(v). \quad (12.6)$$

Division by nonzero elements produces the fraction field. Completion adds limits in the order metric. Complexification adjoins a chosen square root of minus one. Define

$$\mathbb{Q}_b := \text{Frac}(\mathbb{Z}_b), \quad \mathbb{R}_b := \text{Cauchy}(\mathbb{Q}_b)/\sim_0, \quad \mathbb{C}_b := \mathbb{R}_b[T]/(T^2 + 1). \quad (12.7)$$

Here two Cauchy sequences are related by $\sim_0$ when their difference converges to zero in the order metric. Write $i_b := [T] \in \mathbb{C}_b$ so that $i_b^2 = -1$.

Changing b rescales the window and carries each mode to the mode with the same response order. The scalar operations extend this transport. Changing the Gaussian width commutes with forming differences and fractions and with taking limits. Let **Scale** be the groupoid whose objects are the positive scales and whose unique arrow $b \rightarrow b'$ is the dilation

$$\delta_{b,b'}(d) := \frac{b'}{b} d. \quad (12.8)$$

Then $\delta_{b,b} = \text{id}$ and $\delta_{b',b''}\delta_{b,b'} = \delta_{b,b''}$.

The scale maps extend through differences, fractions, Cauchy completion and complexification. Uniqueness of these extensions gives their functorial compatibility.

Theorem 12.1 (Scalar completion and scale transport). (i) *The map $\deg_b^{\mathbb{Z}}$ is an ordered-ring isomorphism $\mathbb{Z}_b \cong \mathbb{Z}$ whose restriction along $u \mapsto [u, 0_b]$ is $\deg_b$. It extends uniquely to the ordered fields $\mathbb{Q}_b$ and $\mathbb{R}_b$ and to $\mathbb{C}_b$ after fixing $i_b \mapsto i$. Thus*

$$\mathbb{Q}_b \cong \mathbb{Q}, \quad \mathbb{R}_b \cong \mathbb{R}, \quad \mathbb{C}_b \cong \mathbb{C}. \quad (12.9)$$

The rational and real maps preserve order. The real map is an isometry for the order metric. The resulting scalar-completion chain is

$$\mathcal{U}_b \longrightarrow \mathbb{Z}_b \longrightarrow \mathbb{Q}_b \longrightarrow \mathbb{R}_b \longrightarrow \mathbb{C}_b. \quad (12.10)$$

(ii) *The assignments*

$$W(b) := (\mathcal{W}_b, d_{K_b}), \quad W(\delta_{b,b'}) := \delta_{b,b'}, \quad (12.11)$$

$$H(b) := \mathcal{H}_b, \quad H(\delta_{b,b'}) := U_{b,b'}, \quad (12.12)$$

$$U(b) := (\mathcal{U}_b, \boxplus_b, \boxtimes_b), \quad U(\delta_{b,b'})(u_{b,n}) := u_{b',n} \quad (12.13)$$

*define three functors from **Scale**. The first takes values in metric spaces with similarities. The second takes values in Hilbert spaces with unitaries. The third takes values in commutative semirings with isomorphisms. Their transport laws are*

$$d_{K_{b'}}(\delta_{b,b'}x, \delta_{b,b'}y) = \frac{b'}{b} d_{K_b}(x, y), \quad U_{b',b''}U_{b,b'} = U_{b,b''}. \quad (12.14)$$

The semiring functor extends uniquely through the scalar-completion chain.

$$U \xrightarrow{\text{Gr}} Z \xrightarrow{\text{Frac}} Q \xrightarrow{\text{Comp}} R \xrightarrow{[i]} C, \quad (12.15)$$

where the object at b is respectively $\mathcal{U}_b, \mathbb{Z}_b, \mathbb{Q}_b, \mathbb{R}_b, \mathbb{C}_b$. Under the degree isomorphisms all arrows in the last four functors become the identity on the corresponding standard scalar structure. Changing the Gaussian width therefore changes the metric realization by similarity and the state realization by a unitary. The induced arrows canonically identify the completed arithmetic objects.

Proof. Lemma B.2 establishes the ordered ring of difference pairs. Degree extends by subtraction and (12.5) maps to $(\deg_b(u) - \deg_b(v))(\deg_b(u') - \deg_b(v'))$. The pair relation proves injectivity. Every integer is a difference of natural degrees. Equation (B.5) proves order preservation. This gives the ordered-ring isomorphism in (i). It extends uniquely by fractions and then by limits of Cauchy sequences. The extensions preserve absolute value. If $\iota_b^{\mathbb{R}}$ is the completed real map then the complex extension sends $x + yi_b$ to $\iota_b^{\mathbb{R}}(x) + i\iota_b^{\mathbb{R}}(y)$. Fixing the image of i_b makes this extension unique.

The endpoint formulas show that $\delta_{b,b'}$ maps $\mathcal{W}_b$ onto $\mathcal{W}_{b'}$. The metric identity (3.33) proves the similarity law. Equation (12.1) gives the Hilbert arrow. Direct substitution gives its identity and composition laws. The basis formula $U_{b,b'}\phi_{b,n} = \phi_{b',n}$ and the analytic lift define the semiring arrow. Equations (4.43) show that it preserves both operations.

Applying the extensions in (i) to each semiring arrow gives (12.15). Their uniqueness makes them compatible with identities and composition. Degree preservation identifies every scalar arrow with the identity on the standard scalars. $\square$

These extensions have the usual universal properties. The first is universal for semiring homomorphisms to rings. The second inverts all nonzero elements of $\mathbb{Z}_b$. The third completes the order metric. The last adjoins the specified root of $T^2 + 1$. Equation (4.58) identifies the circle-homotopy ring with the spectral integer ring by

$$[\mathbb{T}, \mathbb{T}]_{\text{htpy}} \cong \text{Gr}_{\boxplus_b}(\mathcal{U}_b) \cong \mathbb{Z}_b. \quad (12.16)$$

The displayed identification follows from (4.60) and part (i). The group completion admits bilateral Fourier characters. Positive modes admit logarithmic multiplicative characters. Their construction uses the respective additive and multiplicative group laws just obtained.

12.3 Linear completion of spectral states

Let

$$\mathcal{V}_b^{\text{rat}} := c_{00}(\mathcal{E}_b^0; \mathbb{Q}_b) \quad (12.17)$$

be the finitely supported rational coefficient families on all complete correlation modes. Each coefficient has a finite certificate in the generated rational field through successor and difference operations followed by fractions. A state in this core is specified by finitely many such certificates and finitely many complete modes. Scalar extension allows arbitrary completed real coefficients. Hilbert completion then gives

$$\ell^2(\mathcal{E}_b^0; \mathbb{R}_b) = \left\{ a : \mathcal{E}_b^0 \rightarrow \mathbb{R}_b : \sum_{\phi \in \mathcal{E}_b^0} |a(\phi)|^2 < \infty \right\}. \quad (12.18)$$

The linear completion follows from Parseval's identity. Let $\iota_b : \mathbb{R}_b \rightarrow \mathbb{R}$ denote the ordered-field isomorphism in Theorem 12.1. The assignment $e_{\phi_{b,n}} \mapsto \phi_{b,n}$ extends to an isometric ι_b -linear isomorphism

$$J_b((a_n)_{n \geq 0}) := \sum_{n=0}^{\infty} \iota_b(a_n) \phi_{b,n}, \quad J_b : \ell^2(\mathcal{E}_b^0; \mathbb{R}_b) \xrightarrow{\sim} \mathcal{H}_{b,\mathbb{R}}, \quad (12.19)$$

where $\mathcal{H}_{b,\mathbb{R}}$ is the real form of $\mathcal{H}_b$. The space $\mathcal{V}_b^{\text{rat}}$ is countable and dense. Every $f \in \mathcal{H}_{b,\mathbb{R}}$ has a unique coefficient family $(a_n) \in \ell^2(\mathcal{E}_b^0; \mathbb{R}_b)$ satisfying

$$f = \sum_{n=0}^{\infty} \iota_b(a_n) \phi_{b,n}, \quad \sum_{n=0}^{\infty} |\iota_b(a_n)|^2 = \|f\|^2. \quad (12.20)$$

Parseval's identity identifies the completion of the finite coefficient families with $\ell^2(\mathcal{E}_b^0; \mathbb{R}_b)$ after scalars are transported through ι_b . The orthonormal expansion therefore defines an isometry onto $\mathcal{H}_{b,\mathbb{R}}$. Finite truncation followed by rational approximation of the retained coefficients proves density of $\mathcal{V}_b^{\text{rat}}$.

The real scalar field $\mathbb{R}_b$ and the real Hilbert space $\mathcal{H}_{b,\mathbb{R}}$ therefore arise at different completion levels. The former consists of Cauchy classes of spectral rationals. The latter consists of square-summable coefficient families over the complete correlation spectrum. After complexification and analytic lift the closed span of the prime modes is the prime Hilbert subspace $\mathcal{H}_{b,\mathcal{P}}$.

The two completion routes from the correlation spectrum are

$$\text{correlation spectrum} \longrightarrow \begin{cases} (\mathcal{U}_b, \boxplus_b, \boxtimes_b) \longrightarrow \mathbb{Z}_b \longrightarrow \mathbb{Q}_b \longrightarrow \mathbb{R}_b, & \text{scalar completion,} \\ c_{00}(\mathcal{E}_b^0; \mathbb{Q}_b) \longrightarrow \ell^2(\mathcal{E}_b^0; \mathbb{R}_b) \cong \mathcal{H}_{b,\mathbb{R}}, & \text{linear completion.} \end{cases} \quad (12.21)$$

12.4 Spectral entropy and finite bands

For a normalized state $f = \sum_{n \geq 0} a_n \phi_{b,n} \in \mathcal{H}_{b,\mathbb{R}}$ put

$$p_n(f) := |a_n|^2, \quad S_{\mathcal{E}}(f) := - \sum_{n \geq 0} p_n(f) \log p_n(f) \in [0, \infty]. \quad (12.22)$$

The Shannon bounds for $(|a_n|^2)$ [12] give

$$\begin{aligned} S_{\mathcal{E}}(f) = 0 &\iff f = \pm \phi_{b,n} \text{ for some } n, \\ 0 \leq S_{\mathcal{E}}(f) &\leq \log N \quad \text{if } \# \text{supp}(a) \leq N. \end{aligned} \quad (12.23)$$

The upper bound is attained exactly for N equal nonzero weights. For complex states the sign in the zero-entropy characterization is replaced by an arbitrary unit phase. The entropy may be infinite for a general ℓ^2 state.

Finite spectral support makes this entropy finite. Spectral bands give finite-dimensional approximations to the full Hilbert space. For $E > 0$ define

$$\mathcal{H}_{b,\leq E} := \mathbf{1}_{[0,E]}(A_b) \mathcal{H}_{b,\mathbb{R}}. \quad (12.24)$$

The eigenvalue formula $E_{b,n} = (n\pi/\Delta_b)^2$ gives by counting

$$\dim \mathcal{H}_{b,\leq E} = \left\lfloor \frac{\Delta_b \sqrt{E}}{\pi} \right\rfloor + 1, \quad (12.25)$$

and Parseval gives $\mathbf{1}_{[0,E]}(A_b)f \rightarrow f$ in $\mathcal{H}_b$ as $E \rightarrow \infty$.

For a finite prime set $P \subset \mathcal{P}_b$ and $R \in \mathbb{N}_0$ let

$$\mathcal{F}_{b,P,R} := \bigoplus_{r=0}^R \text{Sym}^r(\text{span}_{\mathbb{C}} P). \quad (12.26)$$

If $|P| = K$ then

$$\dim \mathcal{F}_{b,P,R} = \binom{K+R}{R}. \quad (12.27)$$

These finite symmetric sectors bound both the available prime directions and the total occupation number. Give their occupation basis the labels $(\nu_\pi)_{\pi \in P}$ with $\sum_{\pi \in P} \nu_\pi \leq R$. Evaluation sends such a basis vector to the natural mode $\boxtimes_{b\pi \in P} \pi^{\boxtimes_b \nu_\pi}$. The empty profile is sent to 1_b . Unique factorization makes this map injective on the basis. The space $\mathcal{H}_{b,\leq E}$ imposes an energy bound on the evaluated mode and includes its zero mode. A restriction on prime directions and occupation number therefore specifies different data from a quadratic energy cutoff.

12.5 Continuous reconstruction and regularity

By the Parseval isometry (12.19) the rational finite-state space $\mathcal{V}_b^{\text{rat}}$ is countable and dense in $\mathcal{H}_{b,\mathbb{R}}$. One coordinate embeds $\mathbb{R}$ into this Hilbert space. The coefficient map embeds the Hilbert space into $\mathbb{R}^{\mathbb{N}}$. Thus

$$\text{card}(\mathcal{H}_{b,\mathbb{R}}) = \text{card}(\mathbb{R}). \quad (12.28)$$

A countable spectral basis therefore supports a continuum of completed states. A state is computable relative to the rational spectral core when an algorithm produces for each $j \geq 1$ a finite rational spectral sum within 2^{-j} in Hilbert norm. This requires effective control of the tail as well as of the retained coefficients.

For $s \geq 0$ define

$$\mathcal{H}_b^s := \left\{ f = \sum_{n \geq 0} a_n \phi_{b,n} : \sum_{n \geq 0} (1 + E_{b,n})^s |a_n|^2 < \infty \right\}. \quad (12.29)$$

For $N \geq 0$ set

$$P_{b,N} := \sum_{n=0}^N |\phi_{b,n}\rangle \langle \phi_{b,n}| = \mathbf{1}_{[0, E_{b,N}]}(A_b), \quad \alpha_b(t) := \frac{\pi^2 t}{2\Delta_b^2}. \quad (12.30)$$

Let

$$\mathcal{V}_b^{\text{fin}} := \text{span}_{\mathbb{R}} \mathcal{E}_b^0 \subset C(\mathcal{W}_b; \mathbb{R}). \quad (12.31)$$

We identify $\mathcal{V}_b^{\text{rat}}$ with its image under J_b . Finite construction certificates remain distinct from the two analytic completions they generate. Spectral and Gaussian tail estimates quantify the rates of reconstruction in those completions.

Theorem 12.2 (Reconstruction and regularity with effective finite resolution). *The rational spectral core and its finite real-coefficient extension have the following properties in their analytic completions.*

- (i) $\mathcal{V}_b^{\text{rat}}$ has finite mode and rational-coefficient certificates. The space $\mathcal{V}_b^{\text{fin}}$ consists exactly of finite spectral-support functions with arbitrary real coefficients. Moreover $\mathcal{V}_b^{\text{rat}} \subsetneq \mathcal{V}_b^{\text{fin}}$.

- (ii) Both finite spaces have uniform completion

$$\overline{\mathcal{V}_b^{\text{fin}}}^{\|\cdot\|_\infty} = \overline{\mathcal{V}_b^{\text{rat}}}^{\|\cdot\|_\infty} = C(\mathcal{W}_b; \mathbb{R}). \quad (12.32)$$

- (iii) If $P_{b, \leq E} := \mathbf{1}_{[0, E]}(A_b)$ and $f \in \mathcal{H}_b^s$ then

$$\|f - P_{b, \leq E} f\|_{\mathcal{H}_b} \leq (1 + E)^{-s/2} \|f\|_{\mathcal{H}_b^s}. \quad (12.33)$$

For $s > 1/2$ the spectral series converges absolutely and uniformly. For $s > m + 1/2$ it may be differentiated term by term through order m .

- (iv) For every $t > 0$ and $N \geq 0$ the operator tail satisfies

$$\|(I - P_{b, N})C_b(t)\|_{\mathcal{B}(\mathcal{H}_b)} = e^{-\alpha_b(t)(N+1)^2}, \quad (12.34)$$

and hence every $f \in \mathcal{H}_b$ satisfies

$$\|(I - P_{b, N})C_b(t)f\|_{\mathcal{H}_b} \leq e^{-\alpha_b(t)(N+1)^2} \|f\|_{\mathcal{H}_b}. \quad (12.35)$$

For the normalized correlation state $\rho_b(t)$ and $N \geq 1$ the probability tail satisfies

$$\text{Tr}((I - P_{b, N})\rho_b(t)) \leq \frac{e^{-\alpha_b(t)N^2}}{2\alpha_b(t)NZ_b(t)}. \quad (12.36)$$

Thus $C_b(t)$ has infinite rank and is injective. Every prescribed error tolerance nevertheless determines a finite effective band.

Thus a finite construction certificate records the retained coefficients as well as the modes. Uniform completion and quantitative spectral-tail control determine approximation in the two analytic completions.

Proof. The certificate assertion follows from the generated rational field and the definition of c_{00} . An irrational multiple of $\phi_{b,0}$ has finite spectral support and lies outside the rational core. This proves the strict inclusion in part (i).

For part (ii) the coordinate $y = \cos(\pi(d - \ell_{UV})/\Delta_b)$ is a homeomorphism from $\mathcal{W}_b$ to $[-1, 1]$ and satisfies $\cos(n\theta_b) = T_n(y)$. Finite cosine sums are therefore polynomials in y . The Weierstrass approximation theorem gives the uniform density of $\mathcal{V}_b^{\text{fin}}$. If $f = \sum_{n \in F} a_n \phi_{b,n}$ then rational approximants q_n satisfy

$$\left\| f - \sum_{n \in F} q_n \phi_{b,n} \right\|_\infty \leq \sum_{n \in F} |a_n - q_n| \|\phi_{b,n}\|_\infty. \quad (12.37)$$

The right-hand side can be made arbitrarily small. Hence the rational core has the same uniform closure.

For part (iii) Parseval's identity and $(1 + E_{b,n})^s \geq (1 + E)^s$ on the omitted modes give (12.33). The growth $E_{b,n} \asymp n^2$ and uniformly bounded eigenfunction amplitudes give absolute uniform convergence by the Cauchy–Schwarz inequality when $s > 1/2$. Multiplying the coefficients by $E_{b,n}^{j/2}$ gives by the same argument termwise differentiation through order m when $s > m + 1/2$.

Finally $C_b(t)$ is diagonal in the Neumann basis with eigenvalues $e^{-\alpha_b(t)n^2}$. Its norm on the omitted subspace is the first omitted eigenvalue. This proves (12.34) and (12.35). Dividing the Gaussian tail estimate (12.2) by $Z_b(t)$ gives (12.36). Every eigenvalue is positive. Thus $C_b(t)$ is injective and has infinite rank. Moreover its coefficients decay faster than every polynomial weight. Thus $C_b(t)\mathcal{H}_b \subset \bigcap_{s \geq 0} \mathcal{H}_b^s$. This completes the connection between reconstruction and regularity with effective resolution. $\square$

The countable pure-point spectrum supports both the Hilbert completion and the uniform completion $C(\mathcal{W}_b; \mathbb{R})$. Weighted summability or membership in a Gaussian image strengthens density to a quantitative approximation rate.

12.6 Admissibility conditions for continuous and discrete use

The continuous and discrete conditions meet in the coefficient sequence. Square summability defines a Hilbert state. Stronger weighted summability gives the continuous representatives above. Finite-resolution use specifies either a band or a tolerance in the corresponding tail estimate. The endpoints control spatial extent. These coefficient conditions control spectral approximation. Appendix C gives the general formulation and its compact-resolvent extension.

Probability requires normalization. Finite entropy requires a convergent Shannon sum. Uniform counting weight on a countably infinite set has divergent mass. Finite-band states satisfy the entropy bound of (12.23). The Gaussian state has finite entropy by (12.3) because its partition function and first energy moment are finite. General normalized Hilbert states may have infinite spectral entropy. These entropies refer to specified probability laws. The dimensionless endpoint response action measures retained scale.

On these completed spaces multiplication adds prime-occupation profiles. The source readouts extend by their established norm bounds to the corresponding Hilbert completions. Ordered child values remain available before the root evaluation described in Section 8.

12.7 Finite residues and phase coordinates

The phase group in (6.14) is defined directly from the generated circle map. The difference-ring completion equips its residue coordinates with multiplication and gives the finite harmonic interface.

Proposition 12.3 (Canonical finite-residue interface). *Let $a \in \mathcal{U}_b^+$ and put $A = \deg_b(a)$. Write*

$$R_{b,a} := \mathbb{Z}_b / j_b(a) \mathbb{Z}_b.$$

Euclidean division gives $R_{b,a}$ the unique representatives $\bar{r}^a$ with $0_b \leq_b r <_b a$ and hence $|R_{b,a}| = A$. When $A = 1$ this is the trivial ring with its trivial character. For $A > 1$ the chosen orientation of the circle carrier determines the primitive additive character

$$\psi_{b,a}(\bar{X}^a) = \exp\left(2\pi i \deg_b^{\mathbb{Z}}(X)/A\right), \quad X \in \mathbb{Z}_b,$$

and the perfect pairing

$$R_{b,a} \longrightarrow \widehat{R}_{b,a}, \quad t \longmapsto [x \longmapsto \psi_{b,a}(tx)].$$

Equivalently every $t \in R_{b,a}$ satisfies

$$\sum_{x \in R_{b,a}} \psi_{b,a}(tx) = A \mathbf{1}_{\{0\}}(t).$$

If $A > 1$ is odd then 2 has a unique inverse $h_{b,a} \in R_{b,a}$. Hence each $c \in R_{b,a}$ determines the quadratic phase

$$q_{b,a,c}(x) = \psi_{b,a}(h_{b,a}cx^2).$$

For intrinsically coprime a, d put $D = \deg_b(d)$ and choose $\alpha, \gamma \in \mathbb{Z}$ with $\alpha D + \gamma A = 1$. Simultaneous reduction gives $R_{b,a \boxtimes_b d} \cong R_{b,a} \times R_{b,d}$. Its character factors with the Bézout twist

$$\psi_{b,a \boxtimes_b d}(x) = \psi_{b,a}(\alpha x_a) \psi_{b,d}(\gamma x_d).$$

If $d \mid_b f$ write $f = d \boxtimes_b e$ and put $E = \deg_b(e)$. The quotient $R_{b,f} \rightarrow R_{b,d}$ then has dual frequency embedding $h \mapsto Eh$. The scale arrows induced by $Z(\delta_{b,b'})$ preserve all these objects. Under $\deg_b^{\mathbb{Z}} : \mathbb{Z}_b \cong \mathbb{Z}$ they are the standard residue rings $\mathbb{Z}/A\mathbb{Z}$ with their normalized additive characters and perfect pairings together with the odd-modulus quadratic phases.

Proof. Theorem 12.1 and Propositions B.1 and B.3 give the integer coordinate and unique representatives together with the Chinese remainder and divisibility maps. The character is well defined because changing a representative changes its degree by a multiple of A . Its orthogonality is the finite geometric sum. That identity makes the displayed pairing non-degenerate. Equality of the two finite group cardinalities makes it perfect. Substituting the Bézout identity proves the CRT factorization. The identity $e^{2\pi i h x / \deg_b(d)} = e^{2\pi i E h x / \deg_b(f)}$ proves the dual frequency formula. The functorial scale transport of Theorem 12.1 descends to the quotients and preserves the degree coordinate and circle orientation. The same character identifies the additive group of $R_{b,a}$ with $K_b^{\text{ph}}(a)$. Its image consists of the A th roots of unity and its kernel is zero by primitivity. Thus it is a group isomorphism. Transporting multiplication through this isomorphism supplies the residue-ring structure on the phase coordinates. The phase group used in the history construction and this residue interface are thereby canonically related. $\square$

13 Conclusion

The correlation construction gives an ordered Peano semiring whose elements are modes, whose operations are modulation and cascade, and whose arithmetic is certified by winding

degree. The source geometry fixes its order, Gaussian response and quadratic energy. Canonical scale and orientation transport preserves these data together with the arithmetic operations and prime factorization.

Resolving a value into ordered expressions retains information that arithmetic evaluation identifies. The six unit and zero reductions give finite expression fibres, and evaluation is a congruence for their assembly operations. The resulting quotient has the original arithmetic operations. Whole-tree defect identities preserve the root Gaussian response, while leaf counts and local defects distinguish histories in the same fibre. The recursive source assigns them positive probabilities and has exactly the original Gaussian marginal. Conditional expectation and coherent Hilbert compression express the corresponding readouts; variance and entropy identities quantify the internal information.

The same source weights on parent–child relations define a nonnegative self-adjoint trace-class assembly operator. Gaussian root mass controls its trace tails and finite-cutoff error. Its coefficient flow dissipates norm and action, but value readout alone does not close the evolution: the exact root-four matrix element couples an initially invisible history difference to a value observation. This coupling persists under small changes of the conditional weights at roots four and five, independently of the weights at higher roots. Eliminating the difference component retains its initial contribution and an integral kernel in the value equation.

Table 3 distinguishes the information used by these arithmetic and analytic constructions.

Table 3. Information retained by the arithmetic and history construction.

Level	Retained information	Analytic use
Arithmetic	response order and successor with two operations and unique factorization	recursion and divisibility with degree certification
Correlation spectrum	quadratic energy and inner product with scale and orientation transport	Gaussian response with spectral approximation and trace analysis
Expression fibres	ordered and bracketed children with operation types and local defects	whole-tree identities and pre-evaluation comparison
History source	recursive probabilities and finite phase information	exact Gaussian pushforward and conditional information
Assembly relations	parent–child incidence and common source weights	trace-class response and coupled value–difference evolution
Completed arithmetic	finite rational certificates with Hilbert limits and residue characters	finite-resolution estimates and finite harmonic interfaces

Finite rational coefficient families retain explicit construction certificates within the completed spaces; norm and regularity control general Hilbert states. Canonical transport

preserves the history laws and intertwines the readouts and assembly operators. Gaussian resolution, expression depth and assembly time remain distinct parameters: the first weights arithmetic values, the second describes their construction, and the third governs the coupled coefficient evolution.

A Gaussian and Neumann-kernel calculations

Gaussian integration gives the endpoint constants in Section 3. The reflecting heat kernel realizes the same change law on the finite interval. The Fourier convention fixes the normalization of the Gaussian–Poisson identity.

A.1 Endpoint integrals

For

$$K_b(d) = e^{-d^2/(2b^2)}, \quad (\text{A.1})$$

the substitution $u = d/b$ gives

$$\int_{\mathbb{R}} K_b(d) dd = b \int_{\mathbb{R}} e^{-u^2/2} du = \sqrt{2\pi} b. \quad (\text{A.2})$$

The identity $-\log K_b(d) = d^2/(2b^2)$ gives the unit local-information equation. The Gaussian integral gives the total normalized correlation-radius identity. These are respectively

$$\frac{d^2}{2b^2} = 1, \quad L = \sqrt{2\pi} b. \quad (\text{A.3})$$

These relations determine the local and global correlation radii denoted by the two endpoint lengths in Section 3.

A.2 Fourier self-duality and the π normalization

Direct Gaussian integration under the Fourier convention (5.35) gives the following identity for $a > 0$.

$$\mathcal{F}(e^{-ax^2})(\xi) = \sqrt{\frac{\pi}{a}} e^{-\pi^2 \xi^2/a}. \quad (\text{A.4})$$

Within the positive Gaussian family Ce^{-ax^2} the conditions

$$\mathcal{F}g = g, \quad \int_{\mathbb{R}} g(x) dx = 1 \quad (\text{A.5})$$

have the unique solution

$$g(x) = e^{-\pi x^2}. \quad (\text{A.6})$$

Equality of the exponents in (A.4) requires $a = \pi^2/a$ and hence $a = \pi$. The normalization then gives $C = 1$.

Applying Poisson summation to $x \mapsto e^{-\pi t x^2}$ gives

$$\sum_{n \in \mathbb{Z}} e^{-\pi t n^2} = t^{-1/2} \sum_{m \in \mathbb{Z}} e^{-\pi m^2/t}. \quad (\text{A.7})$$

The coefficient π thus selects the unit-mass Fourier fixed point within the positive Gaussian family. Formula (A.7) uses this same normalization.

A.3 Reflecting heat kernel

Set $x = d - \ell_{\text{UV}}$ and $y = d' - \ell_{\text{UV}}$ so that $x, y \in [0, \Delta_b]$. The integral kernel of $e^{-tA_b/2}$ is

$$H_{b,t}^N(x, y) = \frac{1}{\Delta_b} + \frac{2}{\Delta_b} \sum_{n=1}^{\infty} e^{-t(n\pi/\Delta_b)^2/2} \cos\left(\frac{n\pi x}{\Delta_b}\right) \cos\left(\frac{n\pi y}{\Delta_b}\right). \quad (\text{A.8})$$

The method of images uses the line kernel

$$h_t(z) := \frac{1}{\sqrt{2\pi t}} e^{-z^2/(2t)}, \quad (\text{A.9})$$

and gives the equivalent formula

$$H_{b,t}^N(x, y) = \sum_{m \in \mathbb{Z}} [h_t(x - y + 2m\Delta_b) + h_t(x + y + 2m\Delta_b)]. \quad (\text{A.10})$$

At the intrinsic ultraviolet time $t = b^2$ the line kernel h_t is the normalized Gaussian density G_b . Formula (A.10) is therefore the reflecting closure of the Gaussian tangent-change law on the active interval.

For $t \geq t_0 > 0$ each differentiated summand is bounded by a polynomial in n times $e^{-t_0(n\pi/\Delta_b)^2/2}$. The resulting summable majorant justifies termwise differentiation and gives

$$\partial_x H_{b,t}^N(0, y) = \partial_x H_{b,t}^N(\Delta_b, y) = 0. \quad (\text{A.11})$$

Integration in x leaves only the constant term and gives

$$\int_0^{\Delta_b} H_{b,t}^N(x, y) dx = 1. \quad (\text{A.12})$$

This verifies zero boundary flux and conservation of total mass.

A.4 Trace and long-time limit

For $t > 0$ the trace satisfies

$$\text{Tr}(e^{-tA_b/2}) = 1 + \sum_{n=1}^{\infty} e^{-t(n\pi/\Delta_b)^2/2} < \infty. \quad (\text{A.13})$$

The operator is therefore trace class. Its long-time limit is

$$\lim_{t \rightarrow \infty} e^{-tA_b/2} f = \langle f, \phi_{b,0} \rangle \phi_{b,0} \quad (\text{A.14})$$

in L^2 . For probability densities the limiting state is the uniform density $1/\Delta_b$. This density maximizes entropy on the finite carrier.

B Internal Euclidean and factorization lemmas

The divisibility argument for Theorem 5.2 starts from the ordered Peano semiring of Section 5. Cancellation and division with remainder yield a greatest common divisor and a Bézout identity in the difference ring. Their application to a composition-irreducible mode gives Euclid's divisor property and uniqueness of factorization. Integer divisibility then provides a coordinate description of the intrinsic relation.

Write $u \leq_b v$ when $u <_b v$ or $u = v$.

Addition is cancellative. Multiplication is cancellative for every nonzero common factor. Multiplication by a positive mode preserves the strict Peano order. Induction on the common summand proves additive cancellation. The base case is the additive identity. Injectivity of successor gives the induction step. For positive c and w write $w = 1_b \boxplus a$. Distributivity gives $c \boxtimes_b w = c \boxplus_b (c \boxtimes_b a) >_b 0_b$. Now $x <_b y$ means $y = x \boxplus_b w$ for a positive w . Multiplication by c gives $c \boxtimes_b y = (c \boxtimes_b x) \boxplus_b (c \boxtimes_b w) >_b c \boxtimes_b x$. Totality of the order then gives multiplicative cancellation.

Proposition B.1 (Intrinsic Euclidean division). *For $x \in \mathcal{U}_b$ and $d \in \mathcal{U}_b \setminus \{0_b\}$ there are unique $q, r \in \mathcal{U}_b$ such that*

$$x = (d \boxtimes_b q) \boxplus_b r, \quad 0_b \leq_b r <_b d. \quad (\text{B.1})$$

Proof. Use Peano induction on x . For $x = 0_b$ take $q = r = 0_b$. Suppose (B.1) holds for x . If $S_b(r) = d$ then

$$S_b(x) = d \boxtimes_b (q \boxplus_b 1_b) \quad (\text{B.2})$$

with remainder 0_b . Otherwise discreteness of the order and $r <_b d$ give $S_b(r) <_b d$ together with

$$S_b(x) = (d \boxtimes_b q) \boxplus_b S_b(r). \quad (\text{B.3})$$

This proves existence.

To prove uniqueness suppose that two quotient–remainder pairs occur. Consider first $q <_b q'$ and write $q' = q \boxplus_b w$ with $w \neq 0_b$. Every positive w has the form $1_b \boxplus_b a$. Distributivity then gives $d \leq_b d \boxtimes_b w$. Equality of the two decompositions and additive cancellation would then give $r = (d \boxtimes_b w) \boxplus_b r'$. This implies $d \leq_b r$ in contradiction with $r <_b d$. The case $q' <_b q$ is symmetric. Totality of the spectral order therefore gives $q = q'$. Additive cancellation gives $r = r'$. $\square$

Let

$$\mathbb{Z}_b := \text{Gr}_{\boxplus_b}(\mathcal{U}_b), \quad j_b : \mathcal{U}_b \longrightarrow \mathbb{Z}_b, \quad j_b(u) := [u, 0_b]. \quad (\text{B.4})$$

Converting a Bézout identity back into a natural-mode divisor requires control of the sign of a difference. The Peano order supplies the required sign through the order on the difference ring.

Lemma B.2 (Internal order on the difference ring). *The Grothendieck completion is an ordered integral domain with nonnegative cone $\mathbb{Z}_{b,+} := j_b(\mathcal{U}_b)$. The map j_b is an injective order-preserving semiring map satisfying*

$$[u, v] \leq [u', v'] \iff u \boxplus_b v' \leq_b u' \boxplus_b v. \quad (\text{B.5})$$

Every strictly positive element has a unique expression $j_b(w)$ with $w \neq 0_b$. The product of two strictly positive elements is strictly positive.

Proof. Additive cancellation identifies the completion with pairs subject to $(u, v) \sim (u', v')$ when $u \boxplus_b v' = u' \boxplus_b v$. The operations are

$$\begin{aligned} [u, v] + [u', v'] &= [u \boxplus_b u', v \boxplus_b v'], \\ [u, v][u', v'] &= [(u \boxtimes_b u') \boxplus_b (v \boxtimes_b v'), (u \boxtimes_b v') \boxplus_b (v \boxtimes_b u')]. \end{aligned} \quad (\text{B.6})$$

Substitution of the equivalence relation and distributivity show that both operations are well defined. The semiring identities then give the ring identities. Additive cancellation makes j_b injective.

Totality of the internal Peano order gives either $u = v \boxplus_b w$ or $v = u \boxplus_b w$ for some $w \in \mathcal{U}_b$. Hence every difference is either $j_b(w)$ or $-j_b(w)$. If $j_b(a) = -j_b(c)$ then injectivity gives $a \boxplus_b c = 0_b$. The Peano order forces $a = c = 0_b$. Thus $\mathbb{Z}_{b,+} \cap (-\mathbb{Z}_{b,+}) = \{0\}$ and $\mathbb{Z}_{b,+} \cup (-\mathbb{Z}_{b,+}) = \mathbb{Z}_b$. Also $\mathbb{Z}_{b,+}$ is closed under addition and multiplication by the two semiring laws. Defining $X \leq Y$ by $Y - X \in \mathbb{Z}_{b,+}$ therefore gives a translation-invariant total order and yields (B.5).

For $a, c >_b 0_b$ the internally proved strict monotonicity of positive multiplication gives $a \boxtimes_b c >_b 0_b$. Therefore $j_b(a)j_b(c) > 0$. Every nonzero ring element has one of these two strict signs. A product of nonzero elements is therefore nonzero. Injectivity of j_b gives uniqueness of the positive representative and shows that its induced order is exactly $\leq_b$. $\square$

The homotopy-degree realization in (4.58) agrees with this internally constructed ordered ring.

Proposition B.3 (Intrinsic gcd and Bézout identity). *For positive modes x, y repeated intrinsic Euclidean division terminates at a greatest common divisor g . There are also $A, B \in \mathbb{Z}_b$ such that*

$$j_b(g) = Aj_b(x) + Bj_b(y). \quad (\text{B.7})$$

Proof. Starting from $r_0 = x$ and $r_1 = y$ apply Proposition B.1 to obtain

$$r_{j-1} = (r_j \boxtimes_b q_j) \boxplus_b r_{j+1}, \quad r_{j+1} <_b r_j. \quad (\text{B.8})$$

Every strict initial segment of the spectral order is finite. A zero remainder must therefore occur. The last nonzero remainder g divides the preceding remainders by backward substitution. Every common divisor of x and y divides every remainder and hence divides g . Thus g is a greatest common divisor. In the difference ring each equation in (B.8) expresses the next remainder as a linear combination of the previous two. Backward substitution gives (B.7). $\square$

Lemma B.4 (Spectral Euclid lemma). *If $\pi \in \mathcal{P}_b$ and*

$$\pi \mid_b (u \boxtimes_b v), \quad (\text{B.9})$$

then $\pi \mid_b u$ or $\pi \mid_b v$.

Proof. If either u or v is 0_b then the divisibility conclusion follows from the absorbing law. Suppose both are positive and $\pi \nmid_b u$. Any common divisor of π and u is a unit because a nonunit divisor of the irreducible mode π must equal π and would contradict $\pi \nmid_b u$. Proposition B.3 therefore gives $A, B \in \mathbb{Z}_b$ with

$$1 = Aj_b(\pi) + Bj_b(u). \quad (\text{B.10})$$

Write $u \boxtimes_b v = \pi \boxtimes_b w$. Multiplying (B.10) by $j_b(v)$ gives

$$j_b(v) = j_b(\pi)(Aj_b(v) + Bj_b(w)). \quad (\text{B.11})$$

Put $c := Aj_b(v) + Bj_b(w)$. Both $j_b(v)$ and $j_b(\pi)$ are positive. Equation (B.11) excludes $c = 0$. Lemma B.2 shows that $c < 0$ would imply $-c > 0$ and $-j_b(v) = j_b(\pi)(-c) > 0$. This contradicts the positivity of $j_b(v)$ in the ordered ring. Hence c belongs to the positive cone and equals $j_b(v')$ for some $v' \in \mathcal{U}_b$. Injectivity of j_b now gives $v = \pi \boxtimes_b v'$ and proves $\pi \mid_b v$. This establishes the assertion together with the alternative $\pi \mid_b u$. $\square$

Proposition B.5 (Uniqueness of spectral factorization). *If*

$$\pi_1 \boxtimes_b \cdots \boxtimes_b \pi_r = \eta_1 \boxtimes_b \cdots \boxtimes_b \eta_s \quad (\text{B.12})$$

with all factors in $\mathcal{P}_b$ then $r = s$ and the two lists agree after a permutation.

Proof. The factor π_1 divides the product on the right. Repeated application of Lemma B.4 shows that it divides some η_j . Irreducibility of η_j and the fact that π_1 is a nonunit give $\eta_j = \pi_1$. Cancellation removes this common factor. Repeat until one list is exhausted. The other must then be exhausted as well because a nonempty product of nonunit positive factors exceeds 1_b . Thus the two lists have the same length and the same factors with multiplicity. $\square$

The proof uses cancellation and the Peano order together with the two semiring operations. Their coordinate description identifies the resulting divisibility relation with integer divisibility.

For positive natural modes $u, v \in \mathcal{U}_b \setminus \{0_b\}$ the relation is

$$u \mid_b v \iff \deg_b(u) \mid \deg_b(v). \quad (\text{B.13})$$

The identity $v = u \boxtimes_b w$ implies $\deg_b(v) = \deg_b(u) \deg_b(w)$. Conversely the identity $\deg_b(v) = \deg_b(u)r$ implies $v = u \boxtimes_b u_{b,r}$ because $\deg_b$ is injective.

For natural modes $u_{b,m}$ and $u_{b,n}$ the following inequalities are equivalent.

$$m < n, \quad \deg_b(u_{b,m}) < \deg_b(u_{b,n}), \quad E_{b,m} < E_{b,n}, \quad R_{b,\tau}(\phi_{b,m}) > R_{b,\tau}(\phi_{b,n}). \quad (\text{B.14})$$

For every $\tau > 0$ the equivalence follows from $\deg_b(u_{b,n}) = n$ together with $E_{b,n} = n^2 E_{b,1}$ and $R_{b,\tau}(\phi_{b,n}) = e^{-\tau E_{b,n}/2}$.

C Continuous and discrete admissibility

Finite expressions and finite rational coefficient lists certify the construction of the approximants. The bounded source readout maps their coefficients to the mode space and extends continuously to the Hilbert limit. Spectral tail bounds refer to this value space. The history response has its own trace-norm tail given by (10.11).

The topology of completion determines convergence; the error functional determines finite resolution. Compact support and finite discrete support are additional properties of the represented objects.

C.1 The reconstruction cycle

For the present correlation system the analytic channel is

$$\begin{aligned}
 \text{bounded correlation carrier} &\longrightarrow \text{compact-resolvent generator} \\
 &\longrightarrow \text{countable orthonormal spectrum} \\
 &\longrightarrow \text{finite spectral projections} \\
 &\longrightarrow \text{Hilbert completion} \\
 &\longrightarrow \text{finite-resolution projections.}
 \end{aligned} \tag{C.1}$$

The first three arrows resolve a continuous carrier into discrete spectral coordinates. Finite linear combinations form a dense subspace whose Hilbert completion reconstructs the full separable state space. Projection then returns a completed state to a finite description whose accuracy depends on the state and the chosen norm.

Section 12 distinguishes algebraic completion from norm completion. A countable basis and a countable dense core give different information. Finite approximation and computability are also distinct properties. A computable state requires an effective sequence of finite rational approximants together with a specified norm-error rate.

C.2 Topology and summability

Generated continuous–discrete admissibility. Continuous realizations carry a topology of convergence and a regularity class. A finite approximation also carries an error functional and a tolerance, or has an explicitly finite-dimensional range. Infinite discrete expansions carry a convergence or summability law. Probabilistic weights are nonnegative and normalized; their Shannon entropy is finite precisely when $\sum_j p_j |\log p_j| < \infty$. Thus

$$\begin{aligned}
 \text{continuous use} &\implies \text{topology, regularity, and an error scale when approximated,} \\
 \text{discrete use} &\implies \text{convergence or summability, and normalization when probabilistic.}
 \end{aligned} \tag{C.2}$$

These data determine how a realization enters the generated category. Pointwise, uniform, norm, weak and distributional convergence support different limit operations. Pointwise values, derivatives and nonlinear products require the corresponding domain regularity. The same distinctions apply on compact and noncompact carriers: compactness, finite spectral support and finite entropy are separate properties.

C.3 Finite precision and hard spectral cutoffs

Let P_N be increasing finite-rank orthogonal projections on a Hilbert space $\mathcal{H}$ such that $P_N \rightarrow I$ strongly. Then

$$\forall f \in \mathcal{H} \ \forall \varepsilon > 0 \ \exists N = N(f, \varepsilon) : \|(I - P_N)f\| < \varepsilon. \quad (\text{C.3})$$

The corresponding class-wide hard-cutoff statement is

$$\exists N \ \forall f \in \mathcal{H} : P_N f = f. \quad (\text{C.4})$$

Equation (C.4) holds if and only if $\mathcal{H} = \text{Ran } P_N$ and hence if and only if the represented space is finite dimensional. An exact countably infinite spectrum therefore supports finite precision at every tolerance through the object-dependent indices in (C.3). A single projector acts as the identity on the whole class precisely in the finite-dimensional case.

On a full norm ball the unresolved directions have the full available size. A positive heat time restricts the admissible tails and permits a uniform approximation. This gives an operator distinction between an exact finite band and a finite approximation.

Proposition C.1 (Finite resolution on norm balls and heat-smoothed states). (i) *Let $\mathcal{H}$ be infinite dimensional. Let P be a finite-rank orthogonal projection and let $B_R = \{f \in \mathcal{H} : \|f\| \leq R\}$ with $R > 0$. Then*

$$\sup_{f \in B_R} \|(I - P)f\| = R. \quad (\text{C.5})$$

Consequently no $0 \leq \varepsilon < 1$ allows a finite-rank projector to satisfy

$$\|(I - P)f\| \leq \varepsilon R \quad \text{for every } f \in B_R. \quad (\text{C.6})$$

A class containing a full infinite-dimensional norm ball cannot possess a class-wide fixed hard spectral resolution. Finite resolution must depend on the object or on additional compactness or regularity. It may also use a weaker integrated topology.

(ii) *Let $A \geq 0$ be self-adjoint with compact resolvent on a separable Hilbert space. Its eigenvalues are $0 \leq \lambda_0 \leq \lambda_1 \leq \dots \uparrow \infty$ and are counted with multiplicity. Let P_N project onto the first $N + 1$ eigenvectors. If e^{-tA} is trace class for every $t > 0$ then*

$$\|(I - P_N)e^{-tA}\| = e^{-t\lambda_{N+1}} \longrightarrow 0. \quad (\text{C.7})$$

For the normalized Gibbs state $\rho_A(t) = e^{-tA} / \text{Tr}(e^{-tA})$ one has

$$\text{Tr}((I - P_N)\rho_A(t)) \longrightarrow 0, \quad (\text{C.8})$$

and

$$S(\rho_A(t)) = \log \text{Tr}(e^{-tA}) + t \text{Tr}(A\rho_A(t)) < \infty. \quad (\text{C.9})$$

Proof. For part (i) the upper bound in (C.5) follows from $\|I - P\| = 1$. The range $\text{Ran } P$ is finite dimensional whereas $\mathcal{H}$ is infinite dimensional. Choose a unit vector $e \in (\text{Ran } P)^\perp$. Then $Re \in B_R$ and $\|(I - P)Re\| = R$.

For part (ii) spectral calculus gives (C.7). The largest omitted eigenvalue of e^{-tA} is $e^{-t\lambda_{N+1}}$. Trace-class summability and monotone convergence give (C.8). The entropy identity follows from $-\log \rho_A(t) = tA + \log \text{Tr}(e^{-tA})$ on the spectral basis. The inequality $xe^{-tx} \leq C_t e^{-tx/2}$ and the trace-class assumption at time $t/2$ imply $\text{Tr}(Ae^{-tA}) < \infty$ and prove (C.9). $\square$

Part (ii) yields a canonical hierarchy of finite-resolution layers indexed by (t, ε) . At each layer choose the least N for which the right-hand side of (C.7) is at most ε and use the hard projector P_N on that layer. The directed union of the finite-dimensional ranges is dense. Its Hilbert completion recovers the full exact space. A hard cutoff applies at each specified resolution. The complete construction retains the infinite successor chain.

C.4 Gaussian realization

For the Gaussian generator (12.34) gives the exact operator tail with $\alpha_b(t) = \pi^2 t / (2\Delta_b^2)$. For $t > 0$ and $0 < \varepsilon < 1$ solving that bound for the least nonnegative integer gives

$$N_{\text{eff}}(t, \varepsilon) := \max \left\{ 0, \left\lceil \sqrt{\frac{\log(1/\varepsilon)}{\alpha_b(t)}} - 1 \right\rceil \right\}. \quad (\text{C.10})$$

Every (t, ε) therefore determines a finite hard spectral layer. The exact Gaussian operator has strictly positive weight on every mode. The compact active interval and the largest retained spectral index describe different structures. The latter is determined after the resolution pair (t, ε) is specified. This relation preserves the finite operational content of each layer and the unbounded Peano successor structure of the complete natural-mode system.

D The finite-fibre expression resolution

The evaluation fibres of Expr_b retain the ordered expression data identified in $\mathcal{U}_b$ by Section 6. On positive fibres their normal forms are exactly the complete histories of Section 7. This correspondence connects expression normalization to the source-weighted readouts of Section 8.

D.1 Recursive histories and normal forms

A positive complete history is defined recursively as a unit leaf or a root channel with its two child histories. The root channels are those of (7.1). Replacing an additive channel by an additive syntactic node and a multiplicative channel by a cascade node defines a map to $\text{Expr}_b(u)$. Normalization retains these nodes because additive children are positive and multiplicative children exceed the unit. Conversely the normal-form cases in the proof of Theorem 6.1 recover exactly a unit leaf or one such root channel and its ordered children. Induction proves that these two recursive maps are inverse. They preserve evaluation and node types as well as leaf count and depth. The root values strictly decrease along every child edge. The recursion therefore terminates. There is no positive history of value zero. The original zero expression is retained separately.

This correspondence also identifies the recursively defined probability with the explicit node product

$$p_b(e) = \prod_{v \in V_{\text{int}}(e)} c_b(u_v)^{-1}, \quad (\text{D.1})$$

with empty product one at the unit leaf. Partitioning the histories by their root channels proves its normalization as in Proposition 7.1. Any probability family with the same unit value and node recursion agrees by structural induction. Reflection exchanges the ordered children and preserves all root values and channel counts in this product. It consequently preserves p_b .

The phase kernel of a positive root is defined in (6.14). Choosing the oriented angle identifies its elements with $\exp(2\pi i r/n)$ where r is taken modulo n . This is an isomorphism of finite groups. It supplies exactly the phase coordinate used by the history–phase product. The integer completion supplies the full residue ring structure.

D.2 Fibre recursion and growth

Write

$$h_n := h_b(u_{b,n}) = |\text{val}_b^{-1}(u_{b,n})|. \quad (\text{D.2})$$

The value is independent of b because scale transport acts recursively on constants and operation nodes.

Proposition D.1 (Expression-fibre recursion). *The first two fibres are singletons with $h_0 = h_1 = 1$. For $n \geq 2$*

$$h_n = \sum_{\substack{a+c=n \\ a,c \geq 1}} h_a h_c + \sum_{\substack{ac=n \\ a,c \geq 2}} h_a h_c. \quad (\text{D.3})$$

The initial values are

$$(h_0, h_1, \dots, h_{12}) = (1, 1, 1, 2, 6, 16, 52, 160, 536, 1796, 6216, 21752, 77504). \quad (\text{D.4})$$

For every $n \geq 1$ one has

$$h_n < 8^n. \quad (\text{D.5})$$

Proof. A reduced positive term is either 1_b or a term with an additive or cascade root. The two child values are positive at an additive root and at least two at a cascade root. These cases are disjoint because the root operation is part of the syntax. Ordered choices of the two children give (D.3). It produces the displayed initial values.

Every leaf of a positive reduced term is 1_b . A term of value n has at most n leaves. Leaf counts add at both roots. At an additive root the assertion follows from the sum of the two child values. At a nontrivial cascade root the child values $a, c \geq 2$ satisfy $a+c \leq ac$.

An ordered full binary tree with r leaves has Catalan number $C_{r-1} \leq 4^{r-1}$ possible shapes. Its $r-1$ internal nodes have at most 2^{r-1} choices of operation label. Hence the number of reduced terms with at most n leaves is bounded by

$$\sum_{r=1}^n C_{r-1} 2^{r-1} \leq \sum_{r=1}^n 8^{r-1} < 8^n.$$

The n -th evaluation fibre is a subset of this finite collection. □

Relation with arithmetic-formula enumeration. For $n \geq 1$ the fibre cardinality h_n coincides with the arithmetic-formula count recorded as OEIS A214833 [18]. A positive reduced expression is an ordered full binary tree with leaves 1_b and internal vertices labelled by addition or cascade. Multiplication by 1_b has been removed. The syntax retains order and bracketing. No quotient by associativity or commutativity is taken. Distributivity also remains a law of values rather than an identification of expressions. The additional value $h_0 = 1$ records the isolated zero expression in the present resolution.

Let

$$H(z) := \sum_{n \geq 1} h_n z^n.$$

For $|z| < 1/8$ summation of (D.3) gives

$$H(z) = z + H(z)^2 + \sum_{d \geq 2} h_d (H(z^d) - z^d). \quad (\text{D.6})$$

The arithmetic-formula analysis of Gnan, Radziwiłł and Sanna [9] yields constants $c > 0$ and $\rho > 4$ such that

$$h_n \sim c \rho^n n^{-3/2}, \quad c = 0.145691854699979 \dots, \quad \rho = 4.076561785276046 \dots \quad (\text{D.7})$$

Thus (D.5) gives an elementary internal finiteness estimate. The asymptotic formula (D.7) describes the actual growth of the fibres.

These numbers are the fibre sizes of the native evaluation map onto the generated spectral carrier. The counting linearization has the uniform and difference sectors of Section 8. The Gaussian response is constant on each evaluation fibre. The integers h_n measure history multiplicity. The Neumann multiplicities and the trace and partition functions remain those of the original mode space.

D.3 Parallel depth

Define the parallel depth $g(e)$ of a reduced expression by

$$g(0_b) = g(1_b) = 0, \quad g(e \circ_{\text{expr}} f) = 1 + \max\{g(e), g(f)\}, \quad (\text{D.8})$$

where $\circ_{\text{expr}}$ is either expression operation and the displayed node survives normalization. If $\text{val}_b(e) = u_{b,n}$ then structural induction gives

$$g(e) \leq n. \quad (\text{D.9})$$

At an additive root the child values add. At a nontrivial cascade root they are at least two and their sum is at most their product. These two estimates give the induction step for (D.9). Put

$$\text{Expr}_{b, \leq r} := \{e \in \text{Expr}_b : g(e) \leq r\}, \quad \mathcal{A}_{b,r} := \text{val}_b(\text{Expr}_{b, \leq r}). \quad (\text{D.10})$$

Each $\text{Expr}_{b, \leq r}$ is finite. The sets $\mathcal{A}_{b,r}$ increase with r and satisfy

$$\bigcup_{r \geq 0} \mathcal{A}_{b,r} = \mathcal{U}_b. \quad (\text{D.11})$$

A bounded-depth binary tree has a bounded number of nodes. Equation (D.9) places each fixed-value fibre in one such finite set. Every natural mode is represented by a finite right-successor expression. The index r records parallel expression depth. This differs from the arithmetic value n and from the number of operation nodes. Gaussian time and the spectral observation scale are also distinct parameters.

D.4 Whole-tree defect identities

For a reduced term e let $V_{\text{int}}(e)$ and $V_{\text{leaf}}(e)$ denote its internal vertices and leaves. Each internal vertex v has ordered child values u_v^L, u_v^R and parent value u_v . Let $\nu_b(u)$ be the finite prime-occupation vector supplied by Theorem 5.2. With e_π denoting the coordinate vector of a prime π write

$$\nu_b(u) = \sum_{\pi \in \mathcal{P}_b} \nu_u(\pi) e_\pi \in \mathbb{N}_0^{(\mathcal{P}_b)} \subset \mathbb{Z}^{(\mathcal{P}_b)}, \quad \nu_b(0_b) = 0. \quad (\text{D.12})$$

For positive input modes define their additive occupation defect by

$$\delta_{+,b}(u, v) = \nu_b(u) + \nu_b(v) - \nu_b(u \boxplus_b v). \quad (\text{D.13})$$

This signed vector records the change in occupation at an additive root. Put

$$\mathbf{e}_b(u) := \frac{E_b(u)}{E_{b,1}} = \deg_b(u)^2. \quad (\text{D.14})$$

Define the local occupation and energy defects by

$$\partial_v^\nu := \nu_b(u_v) - \nu_b(u_v^L) - \nu_b(u_v^R), \quad (\text{D.15})$$

$$\partial_v^E := \mathbf{e}_b(u_v) - \mathbf{e}_b(u_v^L) - \mathbf{e}_b(u_v^R). \quad (\text{D.16})$$

At a cascade node $\partial_v^\nu = 0$. At an additive node it is the negative of the additive charge defect in (D.13).

Proposition D.2 (Whole-tree telescoping). *If $u = \text{val}_b(e)$ then*

$$\nu_b(u) - \sum_{\ell \in V_{\text{leaf}}(e)} \nu_b(u_\ell) = \sum_{v \in V_{\text{int}}(e)} \partial_v^\nu, \quad (\text{D.17})$$

$$\mathbf{e}_b(u) - \sum_{\ell \in V_{\text{leaf}}(e)} \mathbf{e}_b(u_\ell) = \sum_{v \in V_{\text{int}}(e)} \partial_v^E. \quad (\text{D.18})$$

For a positive reduced term all leaves have value 1_b . The first leaf sum therefore vanishes and the second equals the number of leaves.

Proof. Sum either local defect over the internal vertices. Every value attached to an internal nonroot vertex occurs once as a parent and once as a child with opposite signs. Only the root value and the leaf values remain. $\square$

The second identity gives the multiplicative factorization

$$e^{-\beta \mathbf{e}_b(u)} = \prod_{\ell \in V_{\text{leaf}}(e)} e^{-\beta \mathbf{e}_b(u_\ell)} \prod_{v \in V_{\text{int}}(e)} e^{-\beta \partial_v^E}. \quad (\text{D.19})$$

The factors depend on the expression. Their product depends only on its evaluated value.

D.5 Gaussian compatibility and value observations

On the counting linearization define

$$\tilde{C}_{b,\beta}|e\rangle := e^{-\beta \mathbf{e}_b(\text{val}_b(e))}|e\rangle, \quad \beta \geq 0. \quad (\text{D.20})$$

Here $\beta = tE_{b,1}/2$ as in (7.4). Let $\hat{C}_{b,\beta}$ denote the corresponding diagonal response on $\mathcal{L}_b^0$.

Theorem D.3 (Fibrewise response and observable rigidity). *The lifted response is scalar on each evaluation fibre and*

$$Q_b^{\text{expr}} \tilde{C}_{b,\beta} = \hat{C}_{b,\beta} Q_b^{\text{expr}}. \quad (\text{D.21})$$

Equivalently the expression-difference space $\mathcal{H}_{b,\text{diff}}^{\text{expr}}$ is invariant under $\tilde{C}_{b,\beta}$. The induced response on the quotient is the original Gaussian response of the evaluated mode. Equation (D.19) gives the same value for every expression in a fixed fibre.

Let $f : \mathcal{U}_b \rightarrow \mathbb{C}$ and define on the algebraic expression span

$$\widetilde{M}_f|e\rangle := f(\text{val}_b(e))|e\rangle. \quad (\text{D.22})$$

For every $u \in \mathcal{U}_b$ one has

$$\widetilde{M}_f|_{\mathcal{H}_{b,u}^{\text{expr}}} = f(u)I_{\mathcal{H}_{b,u}^{\text{expr}}}. \quad (\text{D.23})$$

For a positive mode u its unique prime profile defines the Liouville sign $\lambda_b(u) := (-1)^{\sum_{\pi} \nu_u(\pi)}$ and the square-free selector $\mathbf{q}_b(u) := \mathbf{1}_{\{\nu_u(\pi) \leq 1 \ \forall \pi\}}$. On the positive expression subspace write $\tilde{\Gamma}_b^{\text{Lio}} := \widetilde{M}_{\lambda_b}$ and $\tilde{Q}_{\text{sf}} := \widetilde{M}_{\mathbf{q}_b}$. For $n \geq 1$ the lifted Liouville sign and square-free selector obey

$$\tilde{\Gamma}_b^{\text{Lio}}|_{\mathcal{H}_{b,u_{b,n}}^{\text{expr}}} = \lambda_b(u_{b,n})I, \quad \tilde{Q}_{\text{sf}}|_{\mathcal{H}_{b,u_{b,n}}^{\text{expr}}} = \mathbf{q}_b(u_{b,n})I. \quad (\text{D.24})$$

Thus every observable that factors through arithmetic evaluation is constant on each value fibre. Expression multiplicity records construction history. The arithmetic sign is fixed by the evaluated value.

Proof. All basis vectors in $\mathcal{H}_{b,u}^{\text{expr}}$ have the same evaluated mode u and hence the same Gaussian multiplier $e^{-\beta \mathbf{e}_b(u)}$. Applying the two sides of (D.21) to an expression basis vector proves the intertwining identity. The invariant splitting and the quotient statement follow from (8.7) and (D.19).

The same basis calculation applies to an arbitrary value observable. Every expression basis vector in $\mathcal{H}_{b,u}^{\text{expr}}$ evaluates to u . Thus (D.23) follows directly from (D.22). Taking $f = \lambda_b$ and $f = \mathbf{q}_b$ gives the two arithmetic specializations. $\square$

Consequently the Gaussian response and every value observable descend through expression evaluation without selecting a representative inside a fibre.

The source-compatible readout obeys the corresponding identity

$$Q_b^{\text{hist}} \tilde{C}_{b,\beta} = \hat{C}_{b,\beta} Q_b^{\text{hist}}. \quad (\text{D.25})$$

The proof uses the same basis calculation with coefficient $\sqrt{p_b(e)}$ in place of $h_b(u)^{-1/2}$. The two readouts have different initial lines but the same root-dependent Gaussian multiplier.

D.6 Transport of the complete history source

Let $\Phi_{b,b'}$ be the canonical arithmetic comparison. Its expression lift $\Phi_{b,b'}^{\text{expr}}$ fixes the two syntactic constants in their respective carriers and sends an ordered node to the same node with transported children. Since $\Phi_{b,b'}$ preserves the native operations and their unit and zero laws the lift preserves normal forms and satisfies

$$\text{val}_{b'} \Phi_{b,b'}^{\text{expr}} = \Phi_{b,b'} \text{val}_b, \quad p_{b'}(\Phi_{b,b'}^{\text{expr}} e) = p_b(e). \quad (\text{D.26})$$

The second equality follows from the node product because the map is bijective on each set of root channels. If $C : \mathbb{T}_b \rightarrow \mathbb{T}_{b'}$ is the typed carrier isomorphism and the transported endomorphism is CuC^{-1} then $\zeta \mapsto C\zeta$ identifies their phase kernels. Its product with $\Phi_{b,b'}^{\text{expr}}$ preserves the phase law as well.

The corresponding maps on counting bases are unitary. At equal β let U^{hist} be the history unitary and U^{val} the value unitary. The source coefficients and incidence vectors give

$$Q_{b'}^{\text{hist}} U^{\text{hist}} = U^{\text{val}} Q_b^{\text{hist}}, \quad R_{b',\beta}^{\text{asm}} U^{\text{hist}} = U^{\text{hist}} R_{b,\beta}^{\text{asm}}. \quad (\text{D.27})$$

For the operator identity one first transports every finite root layer and then takes its norm limit. Functional calculus transports the assembly evolution. Typed orientation reversal has this same property because it transports the circle and its endomorphisms together. The separate history mirror exchanges the ordered children and need not change the circle coordinate.

D.7 Successor assembly

Let $s_R(e) := \text{NF}_b(e \boxplus_t 1_b)$. For $n \geq 1$ right insertion gives an injective map $\text{Expr}_b(u_{b,n}) \rightarrow \text{Expr}_b(u_{b,n+1})$ and therefore an isometry

$$V_n : \mathcal{H}_{b,u_{b,n}}^{\text{expr}} \longrightarrow \mathcal{H}_{b,u_{b,n+1}}^{\text{expr}}, \quad V_n |e\rangle = |s_R(e)\rangle. \quad (\text{D.28})$$

Set $V_0 |0_b\rangle := |1_b\rangle$ separately because the right-inserted zero expression normalizes directly to 1_b . With

$$\mathcal{W}_{b,n+1} := \mathcal{H}_{b,u_{b,n+1}}^{\text{expr}} \ominus V_n \mathcal{H}_{b,u_{b,n}}^{\text{expr}}, \quad (\text{D.29})$$

one has the exact orthogonal assembly

$$\mathcal{H}_{b,u_{b,n+1}}^{\text{expr}} = V_n \mathcal{H}_{b,u_{b,n}}^{\text{expr}} \oplus \mathcal{W}_{b,n+1}. \quad (\text{D.30})$$

Thus

$$\mathcal{A}_n(\psi, \xi) := V_n \psi + \xi \quad (\text{D.31})$$

is a unitary identification of $\mathcal{H}_{b,u_{b,n}}^{\text{expr}} \oplus \mathcal{W}_{b,n+1}$ with the next fibre. It specifies the deterministic assembly of old and new expression directions.

The uniform fibre vectors satisfy

$$\langle V_n \omega_{b,u_{b,n}}, \omega_{b,u_{b,n+1}} \rangle = \sqrt{\frac{h_n}{h_{n+1}}}. \quad (\text{D.32})$$

When $h_{n+1} > h_n$ this gives the normalized decomposition

$$\omega_{b,u_b,n+1} = \sqrt{\frac{h_n}{h_{n+1}}} V_n \omega_{b,u_b,n} + \sqrt{1 - \frac{h_n}{h_{n+1}}} \eta_{b,n+1}, \quad \eta_{b,n+1} \in \mathcal{W}_{b,n+1}. \quad (\text{D.33})$$

If the two fibre dimensions agree then the second term is absent.

E The history assembly operator

The assembly operator of Section 10 acts on positive histories with counting measure. Source normalization bounds the total conductance at each root and hence controls the trace tail and high-root response. Fix $\beta > 0$ throughout.

E.1 Incidence and finite kernels under root cutoffs

For a nonleaf history e let $b_e = |e\rangle - |e_L\rangle - |e_R\rangle$. The parent is different from either child because root values strictly increase under a retained assembly. Distinct children give $\|b_e\|^2 = 3$. Repeated children give $\|b_e\|^2 = 5$. Thus $\|b_e\|^2 \leq 5$ in both cases. For real coefficient functions one has

$$\langle f, R_{b,\beta}^{\text{asm},(N)} g \rangle = \sum_{e \in \mathcal{A}_{b,N}} I_{b,\beta}(e) d_{\text{asm}} f(e) d_{\text{asm}} g(e). \quad (\text{E.1})$$

This follows directly from the rank-one definition with the child multiplicities retained.

The weighted edge inner product in Proposition 10.1 makes the finite coefficient operator $d_{\text{asm}}^* d_{\text{asm}}$. Its kernel and orthogonal decomposition are therefore exactly those of (10.6). Surjectivity of the differential fixes the coexact representative uniquely.

The full leaf-count function is not square-summable in counting measure. Each positive root admits an additive history whose leaf count equals that root. The complete form consequently has trivial kernel as proved in Proposition 10.2. Its flow converges strongly to zero for each initial coefficient, while high-root histories make its spectral infimum zero.

E.2 Trace tails and the complete quadratic form

At root n normalization of the conditional law gives

$$\sum_{\substack{e \text{ nonleaf} \\ \deg_b(\text{val}_b(e))=n}} I_{b,\beta}(e) \leq e^{-\beta n^2}. \quad (\text{E.2})$$

For $n \geq 2$ every history in the fibre is nonleaf and equality holds. The layer at $n = 1$ is zero. The rank-one norm formula now yields

$$\sum_{n \geq 1} \|R_{b,\beta}^{\text{asm}}\| \leq \sum_{n \geq 2} \|R_{b,\beta}^{\text{asm}}\|_1 \leq 5 \sum_{n \geq 2} e^{-\beta n^2} < \infty. \quad (\text{E.3})$$

For $N \geq 1$ this gives

$$\|R_{b,\beta}^{\text{asm}} - R_{b,\beta}^{\text{asm},(N)}\| \leq \|R_{b,\beta}^{\text{asm}} - R_{b,\beta}^{\text{asm},(N)}\|_1 \leq \frac{5e^{-\beta N^2}}{2\beta N}. \quad (\text{E.4})$$

The last inequality follows by comparison with the Gaussian integral. The exact trace tail is (10.11). The first two repeated-child probabilities are $\gamma_{b,2} = 1$ and $\gamma_{b,3} = 0$. The next is $\gamma_{b,4} = 1/2$. These give layer traces $5e^{-4\beta}$ at root two and $3e^{-9\beta}$ at root three. The trace at root four is $4e^{-16\beta}$. Trace-norm convergence in Proposition 10.2 gives compactness. Here the trace counts the weighted assembly incidence vectors. The Gaussian multiplier on a counting history fibre instead has trace $h_n e^{-\beta n^2}$. The source intensity on that fibre has trace $e^{-\beta n^2}$.

Every $f \in \mathcal{K}_b^{\text{asm}}$ satisfies $|\langle f, b_e \rangle|^2 \leq 5\|f\|^2$. The conductance bound therefore justifies summing the complete nonnegative form and gives

$$0 \leq \mathcal{Q}_{b,\beta}(f) \leq 5 \sum_{n \geq 2} e^{-\beta n^2} \|f\|^2. \quad (\text{E.5})$$

Polarization gives its bilinear version and the equality with the norm-limit operator. The estimate is controlled by the normalized history mass. It therefore applies despite the exponential growth of h_n .

E.3 Evolution and convergence of finite responses

Write $R = R_{b,\beta}^{\text{asm}}$ and $R_N = R_{b,\beta}^{\text{asm},(N)}$. Their contraction evolutions satisfy the dissipation identities of Proposition 10.2.

The identity

$$e^{-sR_N} - e^{-sR} = \int_0^s e^{-(s-v)R_N} (R - R_N) e^{-vR} dv \quad (\text{E.6})$$

implies

$$\sup_{0 \leq s \leq S} \|e^{-sR_N} - e^{-sR}\| \leq S\|R - R_N\| \longrightarrow 0. \quad (\text{E.7})$$

Thus finite-root responses approximate the complete evolution on every bounded time interval with the explicit error supplied by (E.4).

These formulas have a complex Hilbert realization as well. Complexify the real counting space and extend R complex linearly. Its real symmetric matrix coefficients make the extension self-adjoint with $\langle f, Rf \rangle = \sum_e I_{b,\beta}(e) |f(e) - f(e_L) - f(e_R)|^2 \geq 0$. The extension preserves the norm bounds and cutoff convergence as well as the differentiated norm and action identities. It therefore agrees with the real coefficient formulation in the text.

E.4 High-root response

The complete operator has small responses along high-root histories. Let e_n be any history with root degree greater than N and set $v_n = |e_n\rangle$. No incidence vector with parent degree at most N contains this history. Thus $R_N v_n = 0$. With $T_N := R - R_N \geq 0$ and $\delta_N := \|T_N\|$ one obtains

$$\|Rv_n\| \leq \delta_N, \quad \mathcal{Q}_{b,\beta}(v_n) \leq \delta_N, \quad \|Rv_n\|^2 \leq \delta_N \mathcal{Q}_{b,\beta}(v_n). \quad (\text{E.8})$$

The last estimate follows from $T_N^2 \leq \|T_N\|T_N$. Choose the root greater than $\max\{N, 1\}$. Then v_n vanishes at the unit leaf and has norm one. Its quadratic response is positive because its own parent assembly contributes a strictly positive term.

Since $\delta_N \rightarrow 0$ the spectrum has infimum zero even on coefficients anchored at the unit leaf. Thus the best global coercivity constant on that subspace is zero. The final ratio in (E.8) likewise determines the optimal uniform exponential decay rate of the quadratic action. Differentiation at time zero shows that a rate $c > 0$ would require $\|Rf\|^2 \geq c\mathcal{Q}_{b,\beta}(f)/2$ for every f . The ratio tends to zero along the displayed histories. The complete flow therefore has strong decay for each coefficient. The trace tail controls cutoff and finite-time errors. The high-root response has no positive uniform decay rate.

F Structural comparison and analytic context

The construction retains correlation data and generation histories alongside arithmetic values. These are the additional structures in the comparison with standard realizations below.

F.1 Comparison with standard realizations

Classical realizations of the natural numbers include recursive constructions and realizations in set theory and category theory [1–3]. Table 4 compares their primitive data. It also records how successor and order enter each construction and how multiplication is defined.

The closest elementary comparison starts with an oriented circle and its nonnegative-degree endomorphisms $z \mapsto z^n$. Pointwise multiplication and composition already realize the natural-number semiring. Here the Gaussian law and closure rules also determine the inner product and energy together with the response order. Proposition 5.5 separates these constructions from their degree certification. Theorem 5.6 proves compatibility under scale and orientation transport. Proposition 4.6 characterizes the mode family as the smallest family containing the constant homomorphism and the identity and closed under pointwise multiplication. Proposition 5.7 shows that a map between the generated families preserving each mode’s normalized response at one positive parameter is necessarily the canonical correspondence. The prime cotangent construction identifies the irreducible directions on the resulting arithmetic carrier. Rasekh’s construction from the loop space of the circle [8] addresses a natural-number object in a higher-topos setting. The present construction uses correlation data to select the circle and its positive modes together with their response order. The resulting analytic and decomposition structures are carried through the arithmetic identification.

F.2 Assembly before arithmetic evaluation

The direct circle construction identifies each nonnegative integer with one endomorphism. In the present construction this is the evaluated level. The source first determines the spectral carrier and its correlation response. Ordered expression assembly then retains the inputs and root operation before their evaluation into that carrier. Equations (6.12) and (6.13) state the exact relation between the two levels. Modulation and cascade are their spectral outputs. The expression operations retain order and bracketing. Commuta-

Realization	Primitive datum	Successor/order	Multiplication
Peano system	zero and successor	primitive successor	recursive definition
von Neumann ordinals	membership hierarchy	membership order	arithmetic recursion
Natural-number object	initial pointed endomorphism	universal recursion	induced internally
Spectral system	minimal real change with correlation and boundary closure	next response mode with modulation by z	cascade of circle maps

Table 4. Structural roles in standard and correlation-spectral realizations of the natural numbers.

tivity and associativity identify outputs. Distributivity gives further identifications. The corresponding assembly histories can remain distinct.

Arithmetic formula trees and their enumeration have an established theory [9]. The additional structures studied here comprise the source law on those fibres and its exact Gaussian marginal together with the parent–child response using the same source weights.

This distinction is detectable within a single fibre. In the notation of Proposition 7.2 two histories above mode six have leaf counts six and five. Their accumulated local quadratic defects are thirty and thirty-one. The sum of leaf contribution and local defects is thirty-six in both cases. Hence the Gaussian response agrees although the separate history observables differ. Both histories have positive probability. The leaf observable therefore has strictly positive variance conditional on that root. It cannot be a function of the evaluated mode. The history resolution carries this additional information while evaluation preserves the mode’s own correlation data.

The recursive channel law also distinguishes this construction from counting expressions uniformly. At value four the six histories have probabilities $1/8, 1/8, 1/8, 1/8, 1/4, 1/4$. Their total mass is one. Their Gaussian-weighted total is the original root mass. The law is determined by equal root-channel weights and independent ordered subhistory assembly. These specifications define the resolved source. Its weighted readout, connected response and parent–child operator use the same probabilities. The assembly form couples coefficients of different root values and retains their cross terms. The diagonal Gaussian multiplier alone has no such parent–child incidence data. The complete operator and its flow are established in Section 10. Proposition 11.2 makes the coupling explicit on the complete space. A history difference with zero initial value readout has a strictly nonzero value response after the assembly operator acts. The retained history therefore enters the value evolution through the mixed block and the integral term of (11.5).

F.3 The Gaussian π -system

The length ratio $L_{\text{IR}}/\ell_{\text{UV}} = \sqrt{\pi}$ from Section 3 gives the squared ratio $t_{\text{IR}}/t_{\text{UV}} = \pi$ used to compare endpoint responses. With the Fourier convention (5.35) the normalized Gaussian $e^{-\pi x^2}$ is self-dual and the Jacobi–Poisson relation takes the symmetric form (5.36). These

are mutually compatible normalizations of the same Gaussian system.

The selection statements have different hypotheses. Theorem 2.5 uses convolution composition with exact same-shape self-similarity and unit covariance. Variance additivity first forces square-root scaling. Independence of the three stated conditions is proved in Proposition 2.6. Reflection symmetry and maximum entropy follow in Section 2 together with the central-limit description. The Fourier fixed-point calculation in (A.5)–(A.6) works within the positive quadratic family with unit mass and Fourier self-duality. The local information and global correlation-capacity requirements uniquely characterize the local and global radii of Section 3. Their UV and IR notation records their endpoint roles. Within the stated separated homogeneous local boundary class constant preservation selects the reflecting Neumann closure. This closure is self-adjoint and has zero endpoint flux.

Finite-resolution use of the discrete–continuous reconstruction cycle is governed by the conditions in Appendix C.

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Declarations

Data availability. The arithmetic branch of Lean Formalization of Arithmetic and Basic Physical Interfaces [10] formalizes the source Gaussian law, correlation endpoints, reflecting spectrum and native arithmetic, including recursion and unique factorization. It also covers expression normalization and evaluation, finite fibres, canonical spectral transport, history and phase laws, and whole-tree identities. The response formalizations include finite Hermitian response, assembly detailed balance, and the complete assembly operator and flow. The archive contains the proof sources, dependencies, theorem–manuscript correspondence and axiom audit, and is available at <https://doi.org/10.5281/zenodo.23254521>.

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