

A Spectral Criterion for Emergent Gravity

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ABSTRACT: We give a source-resolved spectral criterion for long-range tensor exchange between conserved matter sources. Exact channel elimination preserves the propagation kernel together with the source and readout maps and the direct response. Common-space estimates control the residue at the displaced critical root of a finite approximation. A positive Källén–Lehmann representation identifies this residue with a zero-invariant-mass atom even at a continuum endpoint. For a complete positive Lorentz-covariant conserved stress tensor we classify the null-shell measure and prove that its TT atomic weight vanishes. A reduced compact spectral realisation instead carries internal tensor vertices into an external two-helicity response. Conservation reduces the leading vertex of localised stationary matter to its total energy. The mixed residue then factorises into the two energies and a common propagation coefficient. Spatial resolvent estimates determine the potential and force with finite-size corrections. Normalisation through the matter vertex leaves one dimensional calibration for the overall Planck scale. The spectral residue fixes the dimensionless exchange strength. The resulting response admits an Einstein–matter effective description under the stated positivity and helicity assumptions together with universal soft coupling and local nonlinear consistency.

KEYWORDS: Emergent gravity; Massless spin-2 fields; Universal gravitational coupling; Newtonian limit; Spectral positivity; Källén–Lehmann representation

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1 Introduction

A spectral account of gravity begins with the response exchanged between matter configurations. Internal modes and their products determine which disturbances a source excites. The low-mass spectrum determines how those disturbances propagate over large distances. In the generated compact construction the gauge and matter channels arise from a common source algebra whose transport extends to a Lorentz event carrier [1]. We study how these ingredients enter a collective tensor response and how matter measures its residue. The tensor is emergent in the familiar sense of a collective mode such as a phonon. Its gravitational interpretation follows from its helicity content and universal energy coupling together with the consistency of its nonlinear interactions. An effective metric expresses this collective response at the macroscopic level.

Familiar spectral phenomena illustrate the physical question. A delta contribution to a photon correlator identifies a stable propagating channel. A multiparticle threshold instead gives continuous spectral weight with a different spatial tail. The Casimir effect likewise shows how a change in the spectrum of quantum fluctuations alters a macroscopic force [2]. For tensor exchange the complete energy–momentum vertex specifies how matter excites the response. A positive spectral measure records the states visible to that vertex. The causal correlator describes their propagation.

Internal eigenvalues and scale equations determine the modal content. A propagation pole concerns the response at external momentum. Their connection requires one sourced correlator. We use compact spectral data to determine the internal channel and its matter vertices. The external response has its own propagating state space. The complete covariant stress tensor gives a restrictive realisation whose null-shell sector can be classified directly. In each realisation the source changes the operator together with its state weights and observable. Differentiating this common rule determines the propagation kernel and the maps through which the source enters and the response is observed. It also fixes the direct response at fixed state. Exact block elimination preserves their combination.

The mathematical tools are classical. The Källén–Lehmann representation identifies invariant-mass atoms. Analytic Fredholm theory determines the singular part of an operator inverse. Schur–Feshbach reduction removes complementary channels [3–6]. Here these operations retain the physical source and observable. A critical direction contributes through the source that excites it and the readout that detects it. A finite approximation has its own critical root at which its residue must be evaluated. This comparison preserves the observed response through elimination and makes the internal calculation comparable with a positive external Wightman measure.

The connection to matter uses the complete energy–momentum vertex. Laue’s integrated stress identity [7] reduces the zero-momentum vertex of a localised stationary configuration to its total energy. Spatial stresses enter the finite-size remainder. The same reduced source map sends the leading source of each configuration to a common critical direction. The mixed residue then factorises into the two total energies and one propagation coefficient. The spatial inverse determines the leading exchange kernel. Separate estimates for that kernel and its gradient control the potential and force. Constraint contributions

and direct terms are retained before the static limit.

Compact spectral calculations supply endpoint derivatives and a mass-shift critical point together with explicit tensor vertices on scalar and one-form modes. A simple internal zero supplies a local coordinate on the susceptible channel. Matching the zero and first derivative of its inverse response to the external two-helicity inverse fixes the leading pole and its normalisation. Higher analytic terms give regular or derivative corrections. The compact TT channel thus determines the internal source data. The external helicity space determines the macroscopic propagation law. Their source and readout maps connect the two descriptions.

Induced gravity obtains local curvature terms by integrating out matter on a background geometry [8, 9]. The present criterion concerns the nonlocal response selected by the sourced spectrum. Its residue determines the strength entering matter exchange. Its tensor structure and interactions determine whether it admits an Einstein-type local completion. Local counterterms are separated before this spectral weight is extracted.

Composite tensor dynamics has been studied in matter-only constructions [10, 11]. Maitiniyazi and Yamada derived induced kinetic terms for auxiliary tensor channels in fermionic and scalar composite models [12]. Fehre et al. computed a positive Lorentzian graviton spectral function with a massless one-particle peak and a continuum [13]. The composite calculations establish tensor propagation in specified models. The spectral-function calculation starts from an identified graviton field. Our question is how a propagating object selected from a sourced response retains its physical identification under channel elimination and finite approximation. The source and readout maps provide this identification. Positive Wightman matching gives its spectral interpretation.

The null-shell classification is a structural result for the complete stress tensor. Positivity and exact Lorentz covariance together with conservation force its null-orbit measure to be longitudinal. The proof uses the tensor measure directly and therefore applies without an asymptotic one-particle decomposition. Its transverse-traceless quotient has zero atomic weight. The constructive response analysis treats a reduced two-helicity realisation with its own positive covariance and inherited source maps. It transports the complete response through channel elimination and controls its residue at the shifted root of a finite approximation. The compact realisation supplies explicit matter vertices and an internal simple zero for this construction. Massless representation theory gives the related restriction on states created by a covariant tensor [14]. The Weinberg–Witten theorem tests the particle stress-tensor matrix elements and translation charge of an infrared particle interpretation [15].

The gravitational completion uses established consistency results. Weinberg’s soft theorem fixes the universal coupling of a helicity-two pole under its scattering assumptions [16]. Deser’s self-coupling construction gives an Einstein completion [17]. Consistent local deformations with at most two derivatives are classified under the hypotheses of ref. [18]. The effective action adopts this completion with its normalisation fixed by the mixed source residue.

The general response theory and compact spectral constructions are developed in refs. [1, 19–21]. The finite evaluations use the Coarse-Graining Genesis Framework [22].

The technical appendices give the tensor-measure classification and operator estimates together with the source identities used in the spectral and exchange calculations.

2 The spectral system and its source variations

The compact modal datum specifies the spectral projections and modal products together with their weights and source derivatives. Once these variations are transported to the response space a closed reduced form supplies a positive resolvent. The covariant QFT realisation then fixes the Wightman measure and its physical TT quotient and Euclidean continuation produces the scalar form factor whose endpoint weight is tested. This sequence separates an internal channel calculation from the external propagation question while keeping both in one source-resolved construction.

The compact spectral system of refs. [1, 20] consists of an internal space with modal products and spectral operators whose heat semigroups define averaging. It serves as a finite-volume carrier for computing source derivatives. After the modal data have been transported to a common Hilbert space external propagation is formulated on the flat event carrier in the corresponding linear-response realisation. A gap gives a Yukawa tail. Spectral weight reaching the origin gives a continuum tail or a massless atomic contribution according to its measure. Source and readout maps decide which of these channels is visible to matter.

2.1 Compact modal data and their sourced variation

Write $M_L = \mathbb{RP}^3$ with round radius L and volume $V_3(L) = \pi^2 L^3$. Its scalar eigenvalues are $\lambda_l = l(l+2)/L^2$ with multiplicity $(l+1)^2$ for even l . The round carrier combines a discrete Laplace spectrum with a closed modal product algebra so source variations can be computed together with the modes and weights that carry them. The vector and spinor carriers are kept separately. A modal label j records the carrier and eigenmode together with the matter multiplicity. The datum includes its spectral projection and multiplication coefficients as well as its eigenvalue. For orthonormal scalar modes for example

$$\mathbf{m}_{ij}^k = \int_{M_L} \overline{u_k} u_i u_j \, dv_L. \quad (2.1)$$

The products and their spectral projections determine the corresponding modal couplings.

The positive spectral operator A_L of a specified carrier defines $T_v = e^{-vA_L}$ for $v \geq 0$. Its spectral resolution gives

$$T_{v_1} T_{v_2} = T_{v_1+v_2}, \quad T_v = \sum_j e^{-v\lambda_j} P_j. \quad (2.2)$$

These identities hold strongly including on any zero eigenspace. They give the internal Gaussian composition used in spectral observables. The source-space Gaussian is characterised independently by the convolution law together with orthogonal invariance and square-root scaling. Multiplication of an already formed external response by a Gaussian is a different operation.

To differentiate a geometric source first transport the modal spaces to one Hilbert space. For a homothety this transport includes the $L^{3/2}$ volume factor in the pullback of scalar wavefunctions. It gives $A_L = L^{-2}A_1$ on the transported domain. For a finite smoothly transported spectral subspace $\mathcal{I}$ denote its compressed operator by $A_{\mathcal{I}}(h)$ and its observable weight by $W_{\mathcal{I}}(h)$. A basic spectral functional is

$$\mathcal{Q}_{\mathcal{I}}(h) = \frac{1}{V(h)} \text{Tr}[W_{\mathcal{I}}(h)f(A_{\mathcal{I}}(h))]. \quad (2.3)$$

The weight includes the channel coefficients and Gaussian factor. The source rule includes variation of the weight and volume together with transport of the retained subspace. One may instead put the Gaussian in f provided its source dependence is then differentiated there.

For later source variations suppose $A_{\mathcal{I}}, W_{\mathcal{I}}, V$ are twice continuously differentiable with $V > 0$ and suppose f is holomorphic on a neighbourhood of the relevant spectra. Set $f = f(A_{\mathcal{I}})$ and $v_a = \partial_a \log V$ with

$$F_a = \text{Tr}[W_a f + W Df(A_{\mathcal{I}})[A_a]].$$

Twice differentiating $V\mathcal{Q}_{\mathcal{I}} = \text{Tr}[W_{\mathcal{I}}f(A_{\mathcal{I}})]$ gives

$$\begin{aligned} \mathcal{Q}_a &= V^{-1}F_a - v_a \mathcal{Q}, \\ \mathcal{Q}_{ab} &= V^{-1}F_{ab} - v_a \mathcal{Q}_b - v_b \mathcal{Q}_a - V^{-1}V_{ab} \mathcal{Q}, \\ F_{ab} &= \text{Tr} [W_{ab}f + W_a Df[A_b] + W_b Df[A_a] \\ &\quad + W D^2 f[A_a, A_b] + W Df[A_{ab}]]. \end{aligned} \quad (2.4)$$

where the derivatives of $A_{\mathcal{I}}$ include those of its comparison maps. For noncommuting variations one uses the standard Dunford formula

$$Df(A)[H] = \frac{1}{2\pi i} \oint_{\Gamma} f(\zeta)(\zeta - A)^{-1} H (\zeta - A)^{-1} d\zeta.$$

and the second derivative inserts the two variations in both orders. This formulation is independent of a choice of eigenvectors at a degenerate eigenvalue. The contour-calculus justification is standard.

For an infinite modal sum these identities pass to the limit if the summands and their first two source derivatives have locally uniform summable majorants. A Gaussian times a polynomial mode bound supplies such a majorant on compact positive-scale intervals. A moving hard window is treated piecewise between eigenvalue crossings with boundary contributions at a change of the retained subspace.

The compact TT spectral branch. The compact closure uses the following finite spectral sum. Let $c > 0$ be the dimensionless endpoint coordinate and let $\mu_* > 0$ be the fixed mass unit used to restore dimensions in this trace. Its comparison radius is $L_{\text{tr}} = c/\mu_*$ and $k_{\text{win}}^2 = \pi\mu_*^2/c^2$. The carrier above is evaluated at $L = L_{\text{tr}}$ in the following sum. At fixed content and spin-structure branch

$$\lambda_j = \frac{\mu_*^2 \hat{\lambda}_j}{c^2}, \quad m_j^2 = \frac{\mu_*^2 \hat{m}_j^2(\vartheta)}{c^2}, \quad K_{\text{TT}}(x, b) = \frac{x^2}{(x+b)^2}. \quad (2.5)$$

Here ϑ is the torsion parameter of the compact spectral model. It is distinct from the internal heat parameter v and the external averaging parameter τ used below. The TT channel weights are $w_j = 1$ for real scalar components and $w_j = 4$ for gauge and Weyl components with their modal and matter multiplicities in d_j . Ghost and tensor-carrier entries have zero weight in this particular matter sum. The dimensionless shifts on these retained carriers are

$$\widehat{m}_{\text{sc}}^2 = \frac{3}{4}, \quad \widehat{m}_{\text{g}}^2 = \frac{C_{2,\text{rep}}}{2} + \frac{\vartheta^2}{6}, \quad \widehat{m}_{\text{f}}^2 = \frac{3\vartheta^2}{8}. \quad (2.6)$$

Here $C_{2,\text{rep}}$ is the quadratic Casimir of the retained gauge representation. The symbol $C_2(Q^2)$ denotes the TT form factor. The spinor curvature is already contained in $\widehat{\lambda}_j$. The example uses the charged species trace with odd left-handed and even right-handed spatial branches. The one-generation charge moments give $\vartheta = 1/50$ in its source normalisation and the common weak and colour screening conditions select three identical copies within the replication family of ref. [20]. These choices specify the content of this trace. This matter sum defines the internal spectral closure. The dimensionless value $\kappa_{\text{TT}} = 4/27$ fixes its endpoint and internal readout. The matched sourced response separately determines the external pole and its zero-mass spectral weight.

For the retained index set $\mathcal{J}_{n_{\text{sp}}}$ and frequency bound R define

$$B_j = \sqrt{(R\pi - \widehat{\lambda}_j)_+}, \quad X_j(u) = u^2 + \widehat{\lambda}_j, \\ A_{n_{\text{sp}},R}(\vartheta) = \frac{1}{32\pi^3} \sum_{j \in \mathcal{J}_{n_{\text{sp}}}} d_j w_j \int_0^{B_j} \frac{X_j(u)^2 e^{-X_j(u)/\pi}}{[X_j(u) + \widehat{m}_j^2(\vartheta)]^2} du. \quad (2.7)$$

The source reads the spectral domain $\omega^2 + \lambda_j < Rk_{\text{win}}^2$ with projector $P_R = \mathbf{1}_{[0, Rk_{\text{win}}^2)}(\omega^2 + A_L)$ on the joint frequency and modal space. Thus R specifies its accepted bandwidth inside the Gaussian envelope. The integer response-level family of ref. [20] selects $R = 5$ on the three-copy branch by the shell-support and three-position conditions. We enumerate its modes with $n_{\text{sp}} = 60$. The eight nonzero rows are given in Appendix E.3. Since $k_{\text{win}}L_{\text{tr}} = \sqrt{\pi}$ variations of c and ϑ on this branch leave the retained set fixed. Quadrature approximates the integrals on this fixed domain. The Gaussian tail bound in Eq. (E.9) instead compares this response with the one accepting the full Gaussian spectrum.

At the emergence endpoint c equals the dimensionless coordinate $\varkappa$ of ref. [20]. That calculation uses the trace arguments $(\ell_{\text{end}}, \widehat{\Lambda}_{\text{tr}}) = (c, \sqrt{\pi}/c)$. Its physical Kaluza–Klein readout is

$$L_{\text{phys}} = \frac{c}{M_G}, \quad c_{\text{gen}} = c^2, \quad \Lambda_{\text{gen}} = \frac{c_{\text{gen}}}{L_{\text{phys}}} = cM_G.$$

Here c_{gen} selects the retained family positions and M_G is fixed by the dimensional calibration. The reference unit μ_* restores the trace spectrum. L_{tr} and L_{phys} coincide when this unit is chosen to be M_G . This correspondence distinguishes the trace bandwidth from the family-position cutoff and physical radius.

On this fixed compact branch the complete spectral sum obeys the exact radius and torsion identities

$$\mathcal{T}_{n_{\text{sp}},R}(c, \vartheta) := \frac{V_3 \Pi_{\text{TT}}^{n_{\text{sp}},R}}{32\pi^2 \mu_*} = \frac{A_{n_{\text{sp}},R}(\vartheta)}{c}, \quad c \partial_c \mathcal{T}_{n_{\text{sp}},R} = -\mathcal{T}_{n_{\text{sp}},R}. \quad (2.8)$$

The coefficient is positive if a positive-weight mode has a nonempty frequency interval. Where denominators are strictly positive

$$A'_{n_{\text{sp}},R}(\vartheta) = -\frac{1}{16\pi^3} \sum_j d_j w_j (\hat{m}_j^2)'(\vartheta) \int_0^{B_j} \frac{X_j(u)^2 e^{-X_j(u)/\pi}}{[X_j(u) + \hat{m}_j^2(\vartheta)]^3} du. \quad (2.9)$$

Thus $A'_{n_{\text{sp}},R} < 0$ for $\vartheta > 0$ whenever a gauge or fermion contribution is retained with positive integration length. For the first identity substitute $u = c\omega/\mu_*$. The frequency measure and V_3 leave the factor μ_*/c and the normalisation in Eq. (2.8) leaves c^{-1} . The kernel is homogeneous of degree zero. Differentiating the complete expression includes the volume and frequency-boundary terms together with the Gaussian variation. At fixed c the torsion variation acts only on the displayed mass shifts. This gives Eq. (2.9) and its strict sign.

The radius and torsion derivatives describe two specified variations of the spectral sum. To resolve a local tensor source we must also vary the modal operators and products using Eq. (2.4) for the mixed response.

A local tensor source on the scalar carrier. The scalar source follows by variation of its spectral operator. The metric in this construction serves as a comparison source on the compact carrier and defines the operator variation. External propagation is the response on the flat physical background. Let $g_\epsilon = g + \epsilon h$ be a smooth metric deformation on the same compact manifold and let $t = \frac{1}{2} \text{tr}_g h$. Transport $L^2(M, dv_{g_\epsilon})$ to $L^2(M, dv_g)$ by multiplication with $(dv_{g_\epsilon}/dv_g)^{1/2}$. Denote the transported positive scalar Laplacian by $\tilde{A}_\epsilon$.

Proposition 2.1 (Geometric modal source). For scalar eigenfunctions u_i, u_j of $A = -\Delta_g$ the first source vertex is

$$\begin{aligned} H_{ij}[h] &:= \langle u_i, \tilde{A}'_0 u_j \rangle \\ &= \int_M (tg^{mn} - h^{mn}) \nabla_m \bar{u}_i \nabla_n u_j dv_g - \frac{\lambda_i + \lambda_j}{2} \int_M t \bar{u}_i u_j dv_g. \end{aligned} \quad (2.10)$$

For trace-free h this reduces to $H_{ij}[h] = -\int h^{mn} \nabla_m \bar{u}_i \nabla_n u_j dv_g$. For a constant compression $h = -2\eta g$ it gives $H_{ij} = 2\eta \lambda_i \delta_{ij}$. On a finite complete-shell subspace the heat operator has derivative

$$[De^{-vA}[H]]_{ij} = -H_{ij} \int_0^v e^{-(v-r)\lambda_i - r\lambda_j} dr. \quad (2.11)$$

Thus its off-diagonal matter responses and diagonal spectral changes come from the same geometric source.

Proof. Differentiate the Dirichlet form $\int g_\epsilon^{mn} \nabla_m \bar{u} \nabla_n v dv_{g_\epsilon}$. The inverse metric contributes $-h^{mn}$ and the measure contributes tg^{mn} . The inverse comparison map has derivative $-t/2$ on each argument. Integration against the eigenfunctions gives the last term of Eq. (2.10). Smoothness and compactness justify these form derivatives on the common H^1 domain. Differentiating the finite matrix exponential gives the integral in Eq. (2.11) including equal eigenvalues. Compression yields the stated diagonal result by orthonormality and the eigenfunction equation. $\square$

This is the first local source realisation of the compact spectral data. The tensor variation changes both eigenvalue weights and off-diagonal modal mixing so the response retains channel conversion as well as endpoint shifts.

This calculation also distinguishes matter-resolved response from a homogeneous trace. On the round space the diagonal sum $\sum_{i \in l} \nabla_m \bar{u}_i \nabla_n u_i$ over a complete scalar shell is proportional to g_{mn} . The shell projector is isometry invariant and the isotropy group acts irreducibly on each tangent space. Consequently every complete-shell trace with a scalar shell weight has zero first variation under trace-free h while individual vertices in Eq. (2.10) need not vanish. Indeed for a nonconstant real mode u the smooth trace-free deformation $h = du \otimes du - |\nabla u|^2 g/3$ gives $H_{uu}[h] = -(2/3) \int_M |\nabla u|^4 dv_g < 0$. Such individual vertices can survive projection onto a transverse response space. The second variation and the matter-resolved mixed response therefore carry information absent from the homogeneous endpoint derivative.

For a simple positive scalar mass $M_i = (\lambda_i + b)^{1/2}$ its geometric derivative is $(H_{ii}[h] + b'[h])/(2M_i)$. Within a degenerate shell the corresponding first-order splitting is obtained by diagonalising the basis-independent compression $P_l(\tilde{A}'_0 + b')P_l$. A curvature or mass term must be varied according to its own spectral definition. These vertices live on the internal compact geometry. The tensor and constraint equations construct their propagating mixed response while the internal modal label and external momentum retain their respective roles.

2.2 Positive form-resolvent responses

The free-carrier propagation rule of ref. [1] is expressed through the internal spectrum. For internal projectors P_l and its scalar mass tower $M_l^2 = \lambda_l + m_\delta^2$ with $m_\delta > 0$ it is

$$\hat{G}_{\text{full}}(Q) = \sum_l \frac{P_l}{Q^2 + M_l^2} = (Q^2 + \Delta_M + m_\delta^2)^{-1}. \quad (2.12)$$

Here Q is the external Euclidean momentum and l labels an internal mode. The operator $\Delta_M = -\Delta_g$ is the positive scalar Laplacian. For internal source profiles $g_a \in L^2(M)$ its resolved matrix is $C_{ab}(Q) = \sum_l \langle g_a, P_l g_b \rangle / (Q^2 + M_l^2)$. Orthogonality and Cauchy-Schwarz bound the sum of the absolute numerators by $\|g_a\| \|g_b\|$ proving convergence for $Q^2 \geq 0$. At $Q = 0$ the internal trace of a resolvent power converges for powers greater than $3/2$. The same spectral construction applies to a collective tensor once its sourced response operator is specified.

Suppose elimination of the nonpropagating variables of the same source system gives the exact response

$$C(z) = C_{\text{dir}}(z) + B_{\text{src}}^*(H + zK_{\text{prop}})^{-1}B_{\text{src}}. \quad (2.13)$$

The equality concerns the full response including its source maps. The standard representation theorem for closed forms gives its positive spectral realisation [5, Ch. VI]. Let $\mathfrak{h}$ be the dense closed nonnegative form associated with H . Assume $0 < m_K I \leq K_{\text{prop}} \leq M_K I$ and take the source map $B_{\text{src}} : \mathcal{U} \rightarrow \mathcal{X}$ to be bounded. Define

$$\mathfrak{a}[u, v] = \mathfrak{h}[K_{\text{prop}}^{-1/2}u, K_{\text{prop}}^{-1/2}v], \quad D(\mathfrak{a}) = K_{\text{prop}}^{1/2}D(\mathfrak{h}), \quad V = K_{\text{prop}}^{-1/2}B_{\text{src}}. \quad (2.14)$$

The form is closed and dense and its associated operator A is nonnegative self-adjoint. If E_A is its spectral resolution then for $z \notin (-\infty, 0]$

$$C(z) - C_{\text{dir}}(z) = \int_{[0, \infty)} \frac{d\rho_{\text{nat}}(\mu)}{\mu + z}, \quad d\rho_{\text{nat}} = V^*dE_A V, \quad Z_{\text{nat}} = V^*P_{\ker A}V. \quad (2.15)$$

The measure is positive operator-valued and finite. Its total mass is V^*V . Moreover $x(C(x) - C_{\text{dir}}(x)) \rightarrow Z_{\text{nat}}$ strongly as $x \downarrow 0$. Indeed the bounded invertible change of variables makes the two form norms equivalent while the form equation gives

$$(H + zK_{\text{prop}})^{-1} = K_{\text{prop}}^{-1/2}(A + z)^{-1}K_{\text{prop}}^{-1/2}.$$

The spectral theorem proves Eq. (2.15). Spectral dominated convergence applied to $x/(\mu + x)$ gives the strong zero-mode limit.

The closed positive form associates an internal response rule with a positive spectral measure. Its zero-mode projection is tested through the source embedding.

The variable μ is the spectral parameter of the reduced form operator. Spacetime reconstruction supplies the invariant-mass interpretation used in the physical matching. The global affine form in Eq. (2.13) supplies the positive measure above. For an analytic direct term or a local expansion $H + zK_{\text{prop}} + O(z^2)$ the Laurent analysis of the reciprocal operator supplies the corresponding local singularity classification.

A concrete gapped response is supplied by the generated Yang–Mills construction of ref. [1] Section 8 on its condensed branch $L > L_c$. At the stationary mean intensity q_L the normalised radial response and the coexact connection response have mass operator

$$\mathcal{M}_{\text{YM}}^2 = m_{\text{YM,rad}}^2 \oplus (\Delta_1^\perp + \delta_{q_L} I), \quad \inf \text{Spec } \mathcal{M}_{\text{YM}}^2 = \min\{m_{\text{YM,rad}}^2, 4/L^2 + \delta_{q_L}\} > 0.$$

Here $\Delta_1^\perp$ is the coexact one-form Laplacian and $\delta_{q_L} > 0$ is the restoring shift at that stationary source state. For the curvature intensity $\mathcal{Q} = \|F_B\|^2$ of the generated connection sample B let $\sigma_L^2 = \text{Var}_{q_L} \|B\|^2$. Its source-mean derivative has radial coefficient $b_{\mathcal{Q}} = \sigma_L \partial_q \mathbb{E}_q \mathcal{Q}|_{q=q_L} > 0$. In its quadratic radial realisation Eq. (2.13) has $H = m_{\text{YM,rad}}^2$ and $K_{\text{prop}} = 1$ with $B_{\text{src}} = b_{\mathcal{Q}}$. Hence

$$C_{\text{rad}}(Q^2) = \frac{b_{\mathcal{Q}}^2}{Q^2 + m_{\text{YM,rad}}^2}.$$

The positive atom lies at a nonzero mass with weight fixed by the source overlap. It gives a finite susceptibility at $Q = 0$ and a Yukawa exchange in the spatial realisation. The original fixed-source curvature polynomial is represented in the same response completion with its Gaussian products and higher spectral components intact. The radial formula measures the change of its mean intensity. In the condensed quadratic field realisation of that reference Appendix C the Wick intensity itself also contains a linear radial term followed by a two-particle continuum. These examples show how the chosen source and observation determine which part of a generated positive spectrum is measured.

2.3 The covariant stress-tensor realisation and its TT quotient

The compact construction and the covariant QFT construction have complementary roles. The former supplies explicit internal spectral data and computable source derivatives. The latter fixes the positive measure and Lorentz orbit structure that give the stress response its physical interpretation. Their connection is formulated through the sourced response map and tested by covariant spectral matching.

Matching identifies the reduced operator and its source and readout maps with the TT quadratic form of the continued stress-tensor correlator. In the affine representation the form and source embedding together with the direct term come from the same sourced equations. Under this identification the reduced-form spectral parameter continues to the external correlator variable.

The input in this realisation is a local unitary QFT in flat spacetime with a renormalised symmetric conserved stress tensor. Gauge theories are taken on their positive physical BRST cohomology with the relevant anomalies cancelled. We assume Wightman temperedness together with the spectral condition and usual Euclidean continuation. The source quotient removes representative ambiguities. Lorentz-orbit decomposition identifies the measure. Continuation defines its Euclidean coefficient. We work in four dimensions with metric signature $(+ - - -)$ and units $\hbar = 1$ with the velocity unit fixed. We use a Poincaré-invariant vacuum and require the complete symmetric tensor to transform exactly covariantly on the positive physical Hilbert space. These assumptions apply to the full tensor measure. The null-shell restriction and timelike spin decomposition use their respective orbit stabilisers.

Let $T_{\mu\nu}$ be a renormalised symmetric stress tensor satisfying

$$\partial^\mu T_{\mu\nu} = 0 \tag{2.16}$$

as an operator identity modulo the usual contact terms. In a gauge theory $T_{\mu\nu}$ denotes a ghost-number-zero BRST cohomology class. A convenient representative is obtained by coupling the microscopic theory to a nondynamical metric and differentiating the renormalised action

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{ren}}[g]}{\delta g^{\mu\nu}(x)} \Big|_{g=\eta}. \tag{2.17}$$

The metric in Eq. (2.17) is a nondynamical source used to define the insertion.

The connected spectral problem is expressed in terms of

$$\Theta_{\mu\nu}(x) := T_{\mu\nu}(x) - \langle \Omega | T_{\mu\nu}(x) | \Omega \rangle I. \tag{2.18}$$

Centring removes the constant vacuum expectation value from connected correlators. The same constant drops out of the commutator. The cosmological term in a gravitational effective action is a separate parameter.

The Wightman and retarded functions play different roles

$$W_{\mu\nu,\rho\sigma}(x) = \langle \Omega | \Theta_{\mu\nu}(x) \Theta_{\rho\sigma}(0) | \Omega \rangle, \quad (2.19)$$

$$G_{\mu\nu,\rho\sigma}^R(x) = i \theta(x^0) \langle \Omega | [\Theta_{\mu\nu}(x), \Theta_{\rho\sigma}(0)] | \Omega \rangle. \quad (2.20)$$

The first supplies a positive spectral measure. The second supplies causal linear response. The Fourier transform of the commutator is the difference of the positive- and negative-energy boundary values.

The physical TT source quotient. For nonzero Euclidean momentum define

$$\theta_{\mu\nu}(Q) = \delta_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2}, \quad (2.21)$$

$$\Pi_{\mu\nu,\rho\sigma}^{(2)}(Q) = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}. \quad (2.22)$$

It has trace $N_2 = \Pi_{\mu\nu,\mu\nu}^{(2)} = 5$. The scalar coefficient is formed at $Q^2 > 0$ before taking a radial or distributional limit. Its rank-five coefficient is the massive timelike spin-two coefficient. The null-shell quotient and source matching select the massless helicities $+2$ and -2 .

An equivalent definition avoids the singular projector. Let $\mathcal{J}_{\text{TT}}$ be the space of smooth compactly supported symmetric test sources satisfying

$$\partial_\mu J^{\mu\nu} = 0, \quad \eta_{\mu\nu} J^{\mu\nu} = 0. \quad (2.23)$$

The smeared insertion $\Theta(J) = \int d^4x J^{\mu\nu} \Theta_{\mu\nu}$ depends only on the physical TT class of the source. This formulation remains meaningful on the massless shell and is used to define positivity.

Stress tensors are not unique as local representatives. Consider the standard improvement and gauge-theory redundancies

$$T'_{\mu\nu} = T_{\mu\nu} + (\partial_\mu \partial_\nu - \eta_{\mu\nu} \square) X + s_B Y_{\mu\nu} + \mathcal{E}_{F,\mu\nu}^R, \quad (2.24)$$

where s_B is the BRST differential and $\mathcal{E}_F^R$ is the renormalised quantum EOM insertion generated by a local field redefinition. It includes the Jacobian and operator-mixing subtractions needed for the Schwinger–Dyson identity rather than an arbitrary bare product $F \delta S / \delta \phi$. Local counterterms quadratic in the background source have contact second derivatives.

Representative independence of the nonlocal TT datum. At separated points and between physical states the nonlocal quadratic form of Θ on $\mathcal{J}_{\text{TT}}$ is unchanged by Eq. (2.24). A finite local background-source counterterm changes only the subtraction polynomial. Consequently the existence and rank of a zero-mass TT spectral atom together

with its sign are independent of these choices. Indeed integration by parts and Eq. (2.23) give

$$\int d^4x J^{\mu\nu}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\square)X = 0. \quad (2.25)$$

BRST-exact insertions vanish between physical cohomology classes. The renormalised EOM insertion acts by the local field variation on the other insertions so its time-ordered products reduce to contact terms by the Schwinger–Dyson identity. Finally functional derivatives of a local source counterterm have support at coincident points and are polynomial in momentum. None of these operations changes the nonlocal Wightman measure on physical TT test sources. An invertible finite renormalisation of a physical operator basis acts by congruence on that measure and preserves its positivity and rank together with the presence of an atom. This statement concerns the explicit class Eq. (2.24). A more general conserved local redefinition must be resolved into physical improvements and BRST and EOM components together with contact terms before the conclusion is invoked.

2.4 Positive Wightman measure and the null-shell boundary

Insertion of the joint spectral resolution of the four-momentum operator in Eq. (2.19) gives a tensor-valued measure supported in the closed forward cone. The full measure is positive on every complex symmetric test tensor K for which the smearing is defined

$$\langle\Omega|\Theta(K)^\dagger\Theta(K)|\Omega\rangle = \int_{\overline{V}_+} \overline{\tilde{K}^{\mu\nu}(p)} dM_{\mu\nu,\rho\sigma}(p) \tilde{K}^{\rho\sigma}(p) \geq 0. \quad (2.26)$$

The operator conservation law gives the two distributional Ward identities

$$p^\mu dM_{\mu\nu,\rho\sigma}(p) = 0, \quad dM_{\mu\nu,\rho\sigma}(p)p^\rho = 0. \quad (2.27)$$

These are identities of the Wightman measure. Contact terms in time-ordered products enter the subtraction polynomial and do not modify them.

We first retain the matrix structure of the source response. Let $E(dp)$ be the joint projection-valued measure of translations. The tensor-valued tempered measure $dM_{\mu\nu,\rho\sigma}(p)$ is defined weakly by smearing the two local operator-valued distributions against test tensors. For a finite family $J_a \in \mathcal{J}_{\text{TT}}$ define

$$dM_{ab}(p) := \langle\Omega|\Theta(J_a)^\dagger E(dp)\Theta(J_b)|\Omega\rangle. \quad (2.28)$$

Theorem 2.2 (Covariant disintegration of the physical TT measure). Assume Wightman temperedness together with the spectral and Poincaré-covariance conditions stated above. Then

- (i) for every finite family of physical TT test sources $dM(p)$ is a positive-semidefinite matrix-valued Radon measure supported in the closed forward cone. More precisely for $c_a \in \mathbb{C}$ and every nonnegative Borel function f for which the expression is finite

$$\sum_{a,b} \bar{c}_a \int f(p) dM_{ab}(p) c_b = \left\| \left[\int f(p) E(dp) \right]^{1/2} \sum_b c_b \Theta(J_b) |\Omega\rangle \right\|^2 \geq 0. \quad (2.29)$$

- (ii) fix the invariant orbit measure $d\Omega_s(p) = (2\pi)^{-3}\theta(p^0)\delta(p^2 - s)d^4p$ on $H_s^+ = \{p : p^0 > 0, p^2 = s\}$. On the open timelike cone there are unique source-independent positive locally finite measures $d\rho_2(s)$ and $d\rho_0(s)$ such that for every symmetric Schwartz test tensor $K_{\mu\nu}$ supported away from $p = 0$

$$\langle K, MK \rangle_{p^2 > 0} = \sum_{A=2,0} \int_{(0,\infty)} d\rho_A(s) \int_{H_s^+} d\Omega_s(p) \overline{K^{\mu\nu}(p)} \Pi_{\mu\nu,\rho\sigma}^{(A)}(p) K^{\rho\sigma}(p), \quad (2.30)$$

where only for $s > 0$

$$\begin{aligned} \pi_{\mu\nu}(p) &= \eta_{\mu\nu} - \frac{p_\mu p_\nu}{s}, \\ \Pi_{\mu\nu,\rho\sigma}^{(2)}(p) &= \frac{1}{2}(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho}) - \frac{1}{3}\pi_{\mu\nu}\pi_{\rho\sigma}, \quad \Pi_{\mu\nu,\rho\sigma}^{(0)}(p) = \frac{1}{3}\pi_{\mu\nu}\pi_{\rho\sigma}. \end{aligned} \quad (2.31)$$

For $J_a, J_b \in \mathcal{J}_{\text{TT}}$ and Borel sets $B \subseteq (0, \infty)$ the pushforward of Eq. (2.28) is therefore

$$\rho_{ab}^J(B) = \int_B q_{ab}^{(2)}(s) d\rho_2(s), \quad q_{ab}^{(2)}(s) := \int_{H_s^+} d\Omega_s(p) \overline{\tilde{J}_a^{\mu\nu}(p)} \Pi_{\mu\nu,\rho\sigma}^{(2)}(p) \tilde{J}_b^{\rho\sigma}(p). \quad (2.32)$$

Thus all source dependence is in the positive matrix $q^{(2)}(s)$. The measure $d\rho_2$ is common to every source family.

- (iii) at $s = 0$ the pushforward itself rather than Eq. (2.31) defines the matrix atom

$$\mathbf{Z}_0[J] := [\rho_{ab}^J(\{0\})]_{a,b} \succeq 0. \quad (2.33)$$

It is defined without evaluating a massive rank-five projector at $p^2 = 0$.

Proof. Equation (2.29) is the spectral theorem applied to $\sum_b c_b \Theta(J_b)|\Omega\rangle$. On each timelike orbit conservation leaves the inequivalent $SO(3)$ representations $\text{Sym}_0^2(\mathbb{R}^3)$ and $\mathbb{R}$. Schur's lemma and positivity therefore give two unique nonnegative scalar measures. TT sources remove the scalar block and pushforward by $p \mapsto p^2$ preserves matrix positivity at the boundary. Appendix B gives the orbit disintegration and reconstruction in the conventions used here. This is the tensor Källén–Lehmann construction [3, 23, 24]. $\square$

The two measures in the theorem serve different purposes. The common tensor measure fixes $d\rho_2$ while a test-source family fixes $d\rho_{ab}^J$ through the factors in Eq. (2.32).

Theorem 2.3 (Null-shell exclusion for the covariant stress tensor). For a complete tensor satisfying positivity and exact Lorentz covariance together with conservation and the invariant-vacuum assumption the restriction of the Wightman tensor measure to $H_0^+ = \{p : p^0 > 0, p^2 = 0\}$ has the form

$$dM_{\mu\nu,\rho\sigma}(p)|_{H_0^+} = \alpha_{\text{null}} p_\mu p_\nu p_\rho p_\sigma d\Omega_0(p), \quad \alpha_{\text{null}} \geq 0. \quad (2.34)$$

It vanishes on every conserved test source. A possible measure at $p = 0$ has the form

$$dM_{\mu\nu,\rho\sigma}(p)|_{\{0\}} = \beta_0 \eta_{\mu\nu} \eta_{\rho\sigma} \delta^{(4)}(p), \quad \beta_0 \geq 0, \quad (2.35)$$

and vanishes on traceless sources. Consequently the TT matrix atom is

$$\mathbf{Z}_0[J] = 0 \quad (J_a \in \mathcal{J}_{\text{TT}}). \quad (2.36)$$

This is a classification of the Wightman measure itself prior to any asymptotic one-particle interpretation.

Proof. Proposition B.1 gives the complete orbit classification. Conservation annihilates the term in Eq. (2.34) while tracelessness annihilates Eq. (2.35). Pushing the restricted measure forward under $p \mapsto p^2$ therefore gives Eq. (2.36). The one-particle counterpart is the little-group result of ref. [14]. The proposition above works directly at the level of the positive Wightman measure. $\square$

The theorem concerns the complete covariant stress-tensor measure. Exact covariance confines its null-shell support to the conserved longitudinal sector. A physical two-helicity atom is represented instead on a reduced response space or a gauge-potential quotient carrying its own positive covariance law while nonatomic long-distance correlations remain part of the complete measure.

Let

$$G_{\mu\nu,\rho\sigma}^E(Q) = \langle \Theta_{\mu\nu}(Q) \Theta_{\rho\sigma}(-Q) \rangle_{E,c} \quad (2.37)$$

be the connected Euclidean correlator. At $Q^2 > 0$ its TT coefficient is

$$C_2(Q^2) := \frac{1}{N_2} \Pi_{\mu\nu,\rho\sigma}^{(2)}(Q) G_{\mu\nu,\rho\sigma}^E(Q). \quad (2.38)$$

All kinematic powers are included in this definition. C_2 has mass dimension four. Thus the scalar spectral density includes the kinematic powers associated with this normalisation.

Boundary construction of the scalar TT measure. Fix the orbit normalisation in Theorem 2.2 and the Euclidean normalisation in Eq. (2.38). The timelike measures obey for some $\epsilon > 0$

$$\int_{(0,\epsilon)} \frac{d\rho_2(s) + d\rho_0(s)}{s^2} < \infty. \quad (2.39)$$

Extend ρ_2 to $[0, \infty)$ by assigning zero mass to $\{0\}$. This positive locally finite extension supplies the subtracted Källén–Lehmann representation of C_2 . In particular

$$Z_0 := \rho_2(\{0\}) = 0. \quad (2.40)$$

The scalar and source-matrix atoms agree through their common vanishing

$$\mathbf{Z}_0[J] = 0, \quad Z_0 = 0. \quad (2.41)$$

The endpoint extension follows from positivity and local finiteness rather than from a formal limit of the massive projector. On a compact momentum patch near the nonzero null cone the positive components

$$\Pi_{00,00}^{(2)} = \frac{2|\mathbf{p}|^4}{3s^2}, \quad \Pi_{00,00}^{(0)} = \frac{|\mathbf{p}|^4}{3s^2}. \quad (2.42)$$

have a shell integral bounded uniformly below as $s \downarrow 0$. Local finiteness therefore gives Eq. (2.39). In the time-ordered reconstruction external TT contraction selects the timelike spin-two coefficient and removes the scalar and longitudinal null-shell pieces. Off-shell extensions differ only by local terms. Appendix B gives this reconstruction and the contact-term argument. Standard continuation then yields the Stieltjes transform [3, 4, 23, 25] whose measure is unique. The null-shell theorem fixes its agreement with the source-matrix atom.

Choose a subtraction point $Q_*^2 > 0$ and a nonnegative integer n_{sub} sufficient for UV convergence. Reflection positivity and analytic continuation give

$$C_2(Q^2) = P_2(Q^2) + (Q_*^2 - Q^2)^{n_{\text{sub}}} \int_0^\infty \frac{d\rho_2(s)}{(s + Q^2)(s + Q_*^2)^{n_{\text{sub}}}}. \quad (2.43)$$

The finite polynomial P_2 contains contact terms and subtraction constants. Changing n_{sub} or Q_*^2 or a finite local counterterm changes the analytic split but not $d\rho_2$. We write

$$C_2^{\text{nl}}(Q^2) := C_2(Q^2) - P_2(Q^2) \quad (2.44)$$

for the dispersive part in the chosen subtraction scheme.

The variable s is the independent invariant-mass variable in the spectral integral. The complexified external invariant is denoted by z . Its Euclidean value is $z = Q^2$ and its Lorentzian boundary value is $z = -p^2 - i0$ for momentum transfer p . Thus analytic continuation acts on z and never on the integration variable s . The compact modes carry discrete labels such as l or j and their window scale is k_{win} . The unadorned k is reserved for the FRG momentum scale. The Gaussian parameter τ has dimension inverse mass squared. The subtraction order is n_{sub} . The compact spectral cutoff is n_{sp} and the finite response dimension is N . With the stated stress-tensor normalisation C_2 has mass dimension four and Z_0 dimension six.

2.5 Extraction and classification of the zero-mass atom

The following extraction formula applies to any positive scalar measure with the stated integrability and source matching. For the covariant stress tensor the boundary construction above already gives zero atomic weight. Decompose the measure as

$$d\rho_2(s) = Z_0 \delta_0(ds) + d\rho_{2,c}(s), \quad \rho_{2,c}(\{0\}) = 0, \quad (2.45)$$

where $Z_0 \geq 0$. The subscript c denotes the complement of the zero atom. This remainder includes positive-mass atoms and singular continuous weight as well as any absolutely continuous part. The measure notation also fixes the full weight of an atom at the integration endpoint.

Proposition 2.4 (Euclidean extraction of the TT atom). Let $d\nu(s) = d\rho_{2,c}(s)/(s+Q_*^2)^{n_{\text{sub}}}$. Assume that ν is locally finite at $s = 0$ has no atom there and obeys $\int_1^\infty s^{-1} d\nu(s) < \infty$. Then

$$Z_0 = \lim_{Q^2 \downarrow 0} Q^2 C_2^{\text{nl}}(Q^2). \quad (2.46)$$

In particular $Z_0 > 0$ is equivalent to an atom of the positive scalar measure. Matching the measure to the physical source quadratic form identifies this weight with the TT atom. In the covariant stress-tensor realisation this matching is the boundary construction above and gives $Z_0 = 0$.

Proof. The atomic term in Eq. (2.43) multiplied by Q^2 tends to Z_0 . For the remaining measure split the integral at $s = 1$ in a fixed choice of mass units. On $(0, 1]$ the factor $Q^2/(s + Q^2)$ is bounded by one and tends pointwise to zero. Local finiteness and the absence of an atom permit dominated convergence. On $[1, \infty)$ it is bounded by Q^2/s so the assumed UV bound makes the tail vanish. The remaining subtraction prefactor is bounded and tends to $(Q_*^2)^{n_{\text{sub}}}$. This proves Eq. (2.46). $\square$

The limit in Eq. (2.46) is an infrared projection. A positive value isolates spectral weight concentrated exactly at zero invariant mass while every non-atomic contribution is suppressed by the factor $Q^2/(s + Q^2)$. It therefore distinguishes an atomic propagation channel from a continuum that merely reaches the origin. The source maps and spatial propagator determine the force carried by this channel.

The atomic weight and the support of the remainder distinguish the infrared possibilities. When $Z_0 > 0$ and $\rho_{2,c}((0, s_{\text{gap}})) = 0$ for some $s_{\text{gap}} > 0$ the atom is isolated and gives an analytic simple pole. If the remaining weight accumulates at zero the atom persists but there is no punctured neighbourhood free of other spectral singularities. This includes both continuous thresholds and accumulating positive-mass atoms.

When $Z_0 = 0$ a threshold reaching the origin may still produce a long-distance power law. A gapped remainder has no massless component. An absolutely continuous threshold can give logarithmic or fractional-power behaviour or a more general boundary singularity. The covariant stress tensor falls into these nonatomic cases by Theorem 2.3. Positivity and representative independence are assumptions of this spectral classification rather than additional spectral cases.

This distinction is already visible in free-field benchmarks. Free scalar and fermion stress tensors are bilinear as are those of free gauge fields. Its first non-vacuum matrix elements therefore lie in two-particle sectors. Massive constituents give a threshold at $s = 4m^2$ while massless constituents give a continuum beginning at the origin. In a four-dimensional scale-invariant theory dimensional analysis fixes the TT density to scale as s^2 up to normalisation. Logarithmic running can modify this scaling away from a fixed point [26]. The absence of a TT atom for the covariant tensor also holds nonperturbatively by Theorem 2.3. Cumulative spectral weights detect the finite-volume limit in which several vanishing levels concentrate into one atom. The precise limit and its Euclidean extraction are treated in Appendix E.5 and Proposition E.1.

3 Source response and changes of resolution

The observable depends on a source both explicitly and through the solution of the spectral equation. An eigenvalue of the unsourced operator identifies an available direction. Nonzero source and readout projections make that direction visible after the other response channels

have been eliminated. The implicit-function theorem separates these contributions and fixes the complete linear response.

The equation and measured observable are differentiated with respect to the same source. This fixes the direct and propagated contributions together with the readout before a pole is interpreted as a physical exchange channel.

This is the transport step between the modal and source data of Section 2 and the momentum-dependent response analysed in Section 4. Compact source derivatives enter the equation being solved while the observable is the quantity compared with the stress-tensor correlator. Differentiation of this common rule gives the complete response tuple.

3.1 Response from the generating equations

Proposition 3.1 (Response of the sourced equations). Let $\mathcal{E}(Y, h)$ and $\mathcal{O}(Y, h)$ be continuously Fréchet differentiable maps on Banach spaces near a solution $\mathcal{E}(Y_*, 0) = 0$. If $L = D_Y \mathcal{E}(Y_*, 0)$ is a bounded isomorphism there is a local solution branch $Y(h)$ and its measured linear response is

$$C = \left. \frac{d\mathcal{O}(Y(h), h)}{dh} \right|_0 = \mathcal{O}_h - \mathcal{O}_Y L^{-1} \mathcal{E}_h. \quad (3.1)$$

For any bounded isomorphism M from the response space to the equation space the operators

$$\mathcal{B} = I - M^{-1}L, \quad \mathcal{S} = -M^{-1}\mathcal{E}_h, \quad \mathcal{R} = \mathcal{O}_Y, \quad C_{\text{dir}} = \mathcal{O}_h \quad (3.2)$$

give the identical response $C = C_{\text{dir}} + \mathcal{R}(I - \mathcal{B})^{-1}\mathcal{S}$.

Proof. The implicit-function theorem gives $Y_h = -L^{-1}\mathcal{E}_h$. Substitution in the chain rule proves Eq. (3.1). Since $I - \mathcal{B} = M^{-1}L$ the inverse cancels the preconditioner in Eq. (3.2). $\square$

For the compact branch the closure equation selects the endpoint at which the accumulated channel strength reaches its prescribed TT value while the observable records a retained carrier at that endpoint. Set $\kappa_{\text{TT}} = 4/27$ and

$$\mathcal{E}(c, \vartheta) = \frac{A_{n_{\text{sp}}, R}(\vartheta)}{c} - \kappa_{\text{TT}}, \quad \mathcal{O}_a(c, \vartheta) = \frac{\mu_* \sqrt{\widehat{\lambda}_a + \widehat{m}_a^2(\vartheta)}}{c}. \quad (3.3)$$

The observable is the modal mass interpreted as the zero-momentum spectral energy. Its inertial interpretation requires the matter vertex and the reduced static response.

On the fixed branch described by Eq. (2.8) with $A = A_{n_{\text{sp}}, R} > 0$ the closure has exactly one positive root

$$c_*(\vartheta) = \frac{A(\vartheta)}{\kappa_{\text{TT}}}, \quad \mathcal{E}_c = -\frac{A}{c^2}, \quad \mathcal{E}_\vartheta = \frac{A'}{c}, \quad c'_* = \frac{A'}{\kappa_{\text{TT}}}. \quad (3.4)$$

Primes in this calculation denote ϑ derivatives. For a positive modal mass $M_a = \mathcal{O}_a(c_*(\vartheta), \vartheta)$

$$\frac{M'_a}{M_a} = \frac{(\widehat{m}_a^2)'}{2(\widehat{\lambda}_a + \widehat{m}_a^2)} - \frac{A'}{A}. \quad (3.5)$$

The second term is common to all such modes. The first is the direct response of the carrier mass shift. Since A/c decreases continuously from infinity to zero the crossing is unique. Implicit differentiation gives $c'_* = -\mathcal{E}_c^{-1}\mathcal{E}_\vartheta$ and the chain rule for $\mathcal{O}_a$ gives Eq. (3.5).

The torsion variation changes both the radius endpoint and the modal mass at fixed radius. At $\vartheta > 0$ the stated shifts and $A' < 0$ make $M'_a > 0$. The common endpoint contribution is related to the local energy source obtained by geometric compression.

The nonzero radius Jacobian makes the closure root regular in c . This is the internal endpoint selected by the compact closure convention. Propagation criticality instead concerns the external-invariant-dependent operator $L(z; c_*)$ evaluated at that endpoint. The two roots coincide through the specified source and inverse-response matching which maps the compact endpoint to the external response branch and keeps the two variables explicit.

Preconditioning preserves the response and the failure of invertibility of L . Algebraic eigenvalue multiplicities can change. For instance

$$L(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M^{-1}L(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (3.6)$$

The zero eigenvalue of $L(0)$ is simple whereas the preconditioned matrix has a double zero with a Jordan block. We therefore formulate the positive-residue theorem for the reciprocal response operator. At criticality the inverse-function argument is replaced by an analytic singular-response analysis.

3.2 Gaussian averaging and external resolution

Gaussian averaging is used as a resolution operation on a response that has already been defined. Its semigroup law makes finite-resolution comparisons composable and its multiplier is unity at zero momentum. The sourced dynamics rather than the averaging kernel determines whether an atom is present.

Let $D_r : x \mapsto rx$ denote dilation on $\mathbb{R}^d$. Consider a family of probability measures $\{\mu_\tau\}_{\tau \geq 0}$ that is weakly continuous at $\tau = 0$ obeys

$$\mu_{\tau_1} * \mu_{\tau_2} = \mu_{\tau_1 + \tau_2}, \quad \mu_0 = \delta_0, \quad (3.7)$$

is centred and orthogonally invariant and has a finite nonzero second moment at $\tau = 1$. Suppose it also satisfies exact square-root self-similarity

$$\mu_\tau = (D_{\sqrt{\tau}})_\# \mu_1. \quad (3.8)$$

The convolution law describes composition of independent averaging steps. Centering removes drift and square-root self-similarity fixes the relation between width and scale. Together these conditions specify the averaging class used here.

Independent averaging steps compose additively while their width scales as the square root of the resolution parameter. These requirements select the Gaussian semigroup within the stated class.

Proposition 3.2 (Gaussian averaging law). Under the preceding assumptions the characteristic function is

$$\widehat{\mu}_\tau(q) = \exp(-D\tau|q|^2) \quad (3.9)$$

for a constant $D > 0$. Thus the law is Gaussian up to the unit chosen for τ .

Proof. Weak continuity and reflection symmetry together with the convolution law give $\widehat{\mu}_\tau(q) = \exp[-\tau\psi(q)]$ with $\psi \geq 0$. Self-similarity makes ψ homogeneous of degree two. Orthogonal invariance then gives $\psi(q) = D|q|^2$. The nonzero covariance fixes $D > 0$. Appendix A supplies the positivity argument for the characteristic function and the Fourier inversion. $\square$

This is the Gaussian case of the classical classification of self-similar convolution semigroups [27, Chapter 3]. The generated-operation construction in ref. [21] supplies the broader mathematical setting for this averaging law. Proposition 3.2 and Appendix A give the self-contained form used here.

With a rescaling of τ we set $D = 1$. The corresponding heat semigroup is

$$(\mathcal{H}_\tau f)(x) = \frac{1}{(4\pi\tau)^{d/2}} \int d^d y e^{-|x-y|^2/(4\tau)} f(y), \quad \widehat{\mathcal{H}_\tau f}(Q) = e^{-\tau Q^2} \widehat{f}(Q). \quad (3.10)$$

If the coordinate-space width is denoted by σ our convention is $\tau = \sigma^2/2$ for a Fourier factor $e^{-\sigma^2 Q^2/2}$.

The quadratic exponent follows already from orthogonal invariance and square-root self-similarity. The nonzero second moment fixes its nondegenerate normalisation. If square-root self-similarity is replaced by α -self-similarity at the same time as the finite-second-moment condition is dropped the other stable semigroups have $\mu_\tau = (D_{\tau^{1/\alpha}})_\# \mu_1$ with $0 < \alpha < 2$. The averaging law fixes external resolution. The functional RG regulator R_k controls the integration of fluctuations. Gaussian resolution therefore preserves and resolves the spectral content of the defined response whereas the existence of a massless channel is decided by its unsmeared dynamics.

Preservation of the atom under external smearing. Smearing each stress-tensor insertion with $\mathcal{H}_{\tau/2}$ multiplies the Euclidean two-point form factor by

$$C_{2,\tau}^{\text{ext}}(Q^2) = e^{-\tau Q^2} C_2(Q^2). \quad (3.11)$$

The averaging acts on the external insertions of the fixed physical response.

The same multiplier gives the following atom-preservation identity at every finite $\tau \geq 0$

$$\lim_{Q^2 \downarrow 0} Q^2 e^{-\tau Q^2} C_2^{\text{nl}}(Q^2) = Z_0. \quad (3.12)$$

The multiplier is entire and nonzero with value one at the origin. Thus if C_2^{nl} is regular at $Q^2 = 0$ the product is regular there. If $C_2^{\text{nl}} = Z_0/Q^2 + C_{\text{rem}}$ with $Q^2 C_{\text{rem}} \rightarrow 0$ then

$$e^{-\tau Q^2} C_2^{\text{nl}}(Q^2) = \frac{Z_0}{Q^2} + C_{\text{rem}}(Q^2) - \tau Z_0 + o(1). \quad (3.13)$$

the residue is unchanged. A regular form factor remains regular after the same multiplication.

The filtered expression in Eq. (3.11) need not itself be a Stieltjes function and $e^{-\tau Q^2} P_2(Q^2)$ is not a contact polynomial. Positivity and contact subtraction are therefore imposed on the unfiltered correlator first. Equation (3.11) preserves the infrared limit. At finite τ the operator $\mathcal{H}_\tau$ is injective on tempered distributions. For $\tau > 0$ its inverse is unbounded on the natural Hilbert spaces. Its use therefore entails a loss of resolution at finite precision which is quantified separately from the functional truncation error.

3.3 The background-source flow and the TT response space

For the QFT realisation introduce in Euclidean signature a source $h_{\mu\nu}$ for the centred renormalised stress tensor and write

$$Z_k[j, h] = \int \mathcal{D}\phi \exp[-S[\phi] - \Delta S_k[\phi] + j \cdot \phi + h \cdot \Theta_R], \quad W_k = \log Z_k, \quad (3.14)$$

where $h \cdot \Theta_R = \int d^4x h^{\mu\nu} \Theta_{\mu\nu, R}$ and $\Delta S_k = \frac{1}{2} \phi \cdot R_k \cdot \phi$ is schematic superfield notation. Counterterms polynomial in h are included from the outset. They renormalise coincident composite insertions. The auxiliary source h probes the connected response of the stress operator and provides the QFT implementation of the same source-resolved logic used above. It remains a nondynamical background variable. A collective field supplies a candidate carrier in a separate response representation. Matching the operator and its source and observable maps relates that representation to the stress correlator.

Thus the FRG flow is a QFT realisation of the same source–equation–observable triple. It evolves the vertex data used to construct the response while the background source remains a nondynamical probe.

The modified Legendre transform is taken only with respect to j

$$\Gamma_k[\varphi, h] = j \cdot \varphi - W_k[j, h] - \Delta S_k[\varphi], \quad \varphi = \frac{\delta W_k}{\delta j}. \quad (3.15)$$

At fixed h it obeys the exact Wetterich equation [28]

$$\partial_t \Gamma_k[\varphi, h] = \frac{1}{2} \text{STr} [G_k[\varphi, h] \partial_t R_k], \quad G_k = (\Gamma_k^{(2,0)} + R_k)^{-1}, \quad t = \log(k/k_0). \quad (3.16)$$

Here the first superscript counts fundamental-field derivatives and the second counts h derivatives. Scale-dependent composite fields and their correlation functions can also be treated by the flow construction of ref. [29]. Here we retain the fixed composite source and differentiate its generating functional directly.

For a source-independent regulator differentiating Eq. (3.16) twice with respect to source components h_i, h_j gives

$$\begin{aligned} \partial_t \Gamma_{k, ij}^{(0,2)} = \frac{1}{2} \text{STr} [& G_k \Gamma_{k, j}^{(2,1)} G_k \Gamma_{k, i}^{(2,1)} G_k \partial_t R_k + G_k \Gamma_{k, i}^{(2,1)} G_k \Gamma_{k, j}^{(2,1)} G_k \partial_t R_k \\ & - G_k \Gamma_{k, ij}^{(2,2)} G_k \partial_t R_k]. \end{aligned} \quad (3.17)$$

The equation is part of the full vertex hierarchy. For a source-dependent regulator its derivatives include the corresponding derivatives of R_k .

The connected source response includes propagation through the fundamental-field sector. Differentiating Eq. (3.15) gives the Schur identity

$$W_{k,hh}^{(0,2)} = -\Gamma_{k,hh}^{(0,2)} + \Gamma_{k,h\varphi}^{(1,1)}(\Gamma_k^{(2,0)} + R_k)^{-1}\Gamma_{k,\varphi h}^{(1,1)}, \quad (3.18)$$

with signs fixed by Eq. (3.15). At $k = 0$ after renormalisation and continuation the TT projection of $W_{hh}^{(0,2)}$ is the quantity that must be matched to Eq. (2.43). Appendix E.5 relates this endpoint to the operator and source convergence required of finite vertex expansions.

The FRG realisation of the response tuple. At vanishing fundamental-field source j and for the source-independent regulator used in Eq. (3.17) define the exact stationarity map

$$\mathcal{E}_k(\varphi, h) := \Gamma_{k,\varphi}[\varphi, h] + R_k\varphi = 0, \quad \mathcal{A}_k := D_\varphi \mathcal{E}_k = \Gamma_{k,\varphi\varphi} + R_k = G_k^{-1}. \quad (3.19)$$

Choose a noncritical source-free reference branch φ_* at $h = 0$. Assume that $\mathcal{E}_k$ is continuously Fréchet differentiable near this branch and that its Jacobian $\mathcal{A}_k$ is a bounded isomorphism on a specified response space. The corresponding observable is $\mathcal{O}_k(\varphi, h) = -\Gamma_{k,h}[\varphi, h]$ evaluated on the stationary branch. All response maps are evaluated on this branch. External momentum labels Fourier blocks of the linearised response. It need not parametrise the translation-invariant vacuum field itself. Let $\mathcal{A}_{0,k}(z)$ be an analytic boundedly invertible preconditioner on the same space and set

$$\begin{aligned} \mathfrak{F}_k(\varphi; h) &:= \varphi - \mathcal{A}_{0,k}^{-1}\mathcal{E}_k(\varphi, h), & \mathcal{B}_k &:= D_\varphi \mathfrak{F}_k = I - \mathcal{A}_{0,k}^{-1}\mathcal{A}_k, \\ \mathcal{S}_k &:= -\mathcal{A}_{0,k}^{-1}\Gamma_{k,\varphi h}, & \mathcal{R}_k &:= -\Gamma_{k,h\varphi}, & \mathcal{C}_{k,\text{dir}} &:= -\Gamma_{k,hh}. \end{aligned} \quad (3.20)$$

Then an infinitesimal physical tensor-source variation obeys

$$\delta\varphi = (I - \mathcal{B}_k)^{-1}\mathcal{S}_k \delta h, \quad W_{k,hh} = \mathcal{C}_{k,\text{dir}} + \mathcal{R}_k(I - \mathcal{B}_k)^{-1}\mathcal{S}_k. \quad (3.21)$$

Thus the same source-dependent effective action fixes the kernel and both coupling maps as well as the direct term. Their response reproduces the Schur identity (3.18) exactly.

The preconditioner $\mathcal{A}_{0,k}$ is a bookkeeping isomorphism on the chosen response space. Its choice changes the factorisation while preserving the measured response. At a singular stationarity Jacobian continuation from the noncritical domain uses an analytic response family satisfying the analytic Fredholm hypotheses. This condition belongs to the response family and is independent of the preconditioner. For a finite vertex ansatz $\mathcal{E}_{k,N}$ and $\mathcal{O}_{k,N}$ give the response of the corresponding closure. Comparison with $W_{k,hh}$ uses common-space operator and source-map bounds. Direct differentiation of Eq. (3.19) at fixed vanishing fundamental source gives $\mathcal{A}_k \delta\varphi = -\Gamma_{k,\varphi h} \delta h$. Since $I - \mathcal{B}_k = \mathcal{A}_{0,k}^{-1}\mathcal{A}_k$ and the Jacobian is invertible on the stated noncritical domain the first identity in Eq. (3.21) follows. Substitution into the total h derivative of $W_{k,h} = -\Gamma_{k,h}$ gives its second identity which is Eq. (3.18). For the extended variables Proposition 3.1 supplies the general identity. Block elimination yields the reduced response with its source and direct terms given in Appendix C.

TT response space and complementary channels. A vertex expansion contains physical TT insertions together with trace and longitudinal components. BRST and EOM representatives and auxiliary variables complete the response space. Let P be a bounded projection of the response space $\mathcal{X}$ onto its TT block and $\bar{P} = I - P$. The two blocks reduce the full response operator $\mathcal{B}$ precisely when

$$\bar{P}\mathcal{B}P = P\mathcal{B}\bar{P} = 0. \quad (3.22)$$

Whenever $I - \bar{P}\mathcal{B}\bar{P}$ is invertible the exact projected resolvent uses the Schur–Feshbach kernel [6]

$$\mathcal{B}_{\text{TT}}^{\text{eff}} = P\mathcal{B}P + P\mathcal{B}\bar{P}(I - \bar{P}\mathcal{B}\bar{P})^{-1}\bar{P}\mathcal{B}P. \quad (3.23)$$

Retaining only $P\mathcal{B}P$ loses the second term whenever it is nonzero. One-sided invariance can make that term vanish without reducing both blocks. The source and observable maps together with the direct term reduce at the same time as given explicitly in Eqs. (C.9) and (C.10). Neither the source nor the observable map is assumed to vanish on the $\bar{P}$ block. In this reduction the effective maps $\mathcal{S}_{\text{eff}}$ and $\mathcal{R}_{\text{eff}}$ act on tensor sources before contraction with a physical TT polarisation.

The reduced response acts between the source spaces as

$$\mathcal{J}_{\text{TT}} \xrightarrow{\mathcal{S}_{\text{TT}}} \mathcal{X}_{\text{TT}} \xrightarrow{(I - \mathcal{B}_{\text{TT}}^{\text{eff}})^{-1}} \mathcal{X}_{\text{TT}} \xrightarrow{\mathcal{R}_{\text{TT}}} \mathcal{J}_{\text{TT}}^*. \quad (3.24)$$

The first map realises the stress-tensor source in the chosen response coordinates. The last map reconstructs the measured TT correlator. Both maps transform together under a change of operator normalisation so that the measured quadratic form is preserved. The singularities of this measured response determine the visible critical modes.

4 Visible critical modes and spectral residues

The compact spectral ladder has a simple internal crossing at which a collective direction becomes soft. This enhanced susceptibility produces spatial propagation when the same sourced equations supply the corresponding external-momentum dependence. The derivative of the inverse response with respect to the external invariant fixes the pole strength. The source and readout overlaps determine whether matter can excite and observe the critical direction. For the compact spectral model of Ref. [22] consider the mass-shift deformation $t_\chi = e^{-2\chi}$ for $\chi \geq 0$. We use the trace units $L_{\text{tr}} = c/\mu_*$ and $k_{\text{win}} = \mu_*\sqrt{\pi}/c$ of Section 2.1. The radius coordinate c and the closure coefficient $\kappa_{\text{TT}} = 4/27$ are dimensionless. The ladder below depends on their dimensionless spectral data and is independent of the reference mass unit μ_* . The deformation holds the matter eigenvalues and volume fixed together with the Gaussian window. It replaces each $\hat{m}_j^2$ by $t_\chi\hat{m}_j^2$ and scales the two entries $8\mu_*^2/c^2$ and $6\mu_*^2/c^2$ of the TT vertex by t_χ . Thus it is a specified spectral deformation distinct from a common radius rescaling of every operator. With the data of Eq. (2.7)

define

$$\mathcal{J}(\chi) = \frac{1}{\pi^3} \sum_j d_j w_j \int_0^{B_j} \frac{X_j(u)^2 e^{-X_j(u)/\pi}}{[X_j(u) + t_\chi \hat{m}_j^2]^2} du, \\ D(t) = 8t \left[\frac{8t}{1 - e^{-8t/\pi}} + 6t \right], \quad x(\chi) = \frac{\mathcal{J}(\chi)}{D(t_\chi)}. \quad (4.1)$$

The frequency substitution $u = c\omega/\mu_*$ gives $\Pi_\chi = \mu_*^4 c^{-4} \mathcal{J}(\chi)$ and $v_{\text{TT},\chi} = \mu_*^{-4} c^4 / D(t_\chi)$. Their product $x = v_{\text{TT},\chi} \Pi_\chi$ is therefore independent of c while $\mathcal{J}(0) = 32A_{n_{\text{sp}},R}$.

Its critical point is simple. Every summand of $\mathcal{J}$ is nondecreasing in χ and its limit is positive. For $t > 0$ the function $8t/(1 - e^{-8t/\pi})$ is strictly increasing since $e^y > 1 + y$. Consequently $D'(t) > 0$ and $x'(\chi) > 0$ and $D(t_\chi) \sim 8\pi t_\chi$ as $\chi \rightarrow \infty$. Equivalently $t_\chi \downarrow 0$ and $x(\chi) \rightarrow \infty$. For the eight rows of Appendix E.3 $\sum_j d_j w_j = 2080$. Bounding each rational kernel by one and $\int_0^\infty e^{-u^2/\pi} du = \sqrt{\pi} \int_0^\infty e^{-v^2} dv = \pi/2$ gives $x(0) < 65/(7\pi^2) < 1$ where $t_0 = 1$. There is exactly one $\chi_c > 0$ with $x(\chi_c) = 1$. At $\vartheta = 1/50$ independent quadrature of these same rows gives $x(0) \simeq 0.1007$ and $\chi_c \simeq 0.6572$ with $x'(\chi_c) \simeq 3.058$. The preceding inequalities establish existence and simplicity independently of the quadrature.

The scalar ladder response $\Pi_\chi/(1 - x(\chi))$ therefore obeys

$$\lim_{\delta \downarrow 0} \delta \frac{\Pi_{\chi_c - \delta}}{1 - x(\chi_c - \delta)} = \frac{\mu_*^4 \mathcal{J}(\chi_c)}{c^4 x'(\chi_c)} > 0. \quad (4.2)$$

This pole is in the internal detuning $\delta = \chi_c - \chi$. The radius endpoint fixes c . The external variable enters the sourced propagation response whose pole determines spatial exchange. Appendix E.3 records the corresponding scale derivatives at the radius endpoint.

The crossing $x(\chi_c) = 1$ marks unit feedback in the complete ladder and mixing kernel so a collective direction appears in the reduced operator. The sourced equations continue this direction to an external branch whose nonzero z -slope fixes the propagating residue. The Fredholm formulation states this mechanism independently of a diagrammatic grouping.

4.1 Visible tensor modes of the compact source

There is an explicit TT projection of the scalar source in Proposition 2.1. Identify the round quotient with $SO(3)$ and write its adjoint matrix as $R_{ib}(U)$. The nine real modes $u_{ib} = (3/V_3)^{1/2} R_{ib}$ are orthonormal and have scalar eigenvalue $8/L^2$. For a real symmetric trace-free matrix q form $h_q^+ = V_3^{-1/2} q_{ab} \theta^a \theta^b$ in the left-invariant orthonormal coframe and h_q^- in the right-invariant coframe. Both are transverse and traceless. They span orthogonal five-dimensional subspaces with rough tensor eigenvalue $6/L^2$.

The scalar operator source and its energy source on this shell are

$$H[h_q^+] = \frac{4}{L^2 \sqrt{V_3}} I_3 \otimes q, \quad H[h_q^-] = \frac{4}{L^2 \sqrt{V_3}} q \otimes I_3, \\ DH_{\text{sc}}[h_q^+] = \alpha_2 I_3 \otimes q, \quad DH_{\text{sc}}[h_q^-] = \alpha_2 q \otimes I_3, \quad \alpha_2 = \frac{2}{M_2 L^2 \sqrt{V_3}}, \quad (4.3)$$

where $H_{\text{sc}} = (-\Delta_g + \mathcal{R}_g/8)^{1/2}$ and $M_2 = \sqrt{35}/(2L)$. The curvature variation vanishes on these TT directions. All source factors therefore follow from the same scalar operator and

its common source normalisation. Appendix E.4 proves the normalisation transversality and the two matrix formulas.

For a real symmetric occupation matrix $\mathbf{N}_{\text{occ}}$ on this nine-dimensional shell let $\mathbf{N}_{\text{occ},L}$ and $\mathbf{N}_{\text{occ},R}$ be its partial traces. Its ten TT source components are the matrix pair

$$j_{\text{hom}}(\mathbf{N}_{\text{occ}}) = \alpha_2((\mathbf{N}_{\text{occ},L})_0, (\mathbf{N}_{\text{occ},R})_0), \quad B_0 = B - \frac{1}{3} \text{tr}(B)I_3. \quad (4.4)$$

They span both tensor subspaces so this source is visible on a ten-dimensional TT space. An isotropic occupation $\mathbf{N}_{\text{occ}} = I_9/9$ has zero TT projection despite positive energy M_2 . The constant scalar mode also has positive energy and zero TT projection. These examples show why a static matter response must retain the trace and constraint sectors.

The $(2,1)$ one-form band used in the compact normalisation also has an explicit source on this tensor space. In the right-invariant frame let $\mathcal{Q} = \text{Sym}_0(\mathbb{R}^3)$ and set $a_{q,v} = (3/V_3)^{1/2}(qRv)_a\theta_R^a$ for $q \in \mathcal{Q}$ and $v \in \mathbb{R}^3$. These forms span the $(2,1)$ band with Hodge eigenvalue $16/L^2$. On its identification with $\mathcal{Q} \otimes \mathbb{R}^3$ the compressed source for $h_t^- = V_3^{-1/2}t_{ab}\theta_R^a\theta_R^b$ is

$$H_{\text{lr}}[h_t^-] = \frac{16}{L^2\sqrt{V_3}}\mathcal{A}_t \otimes I_3, \quad \mathcal{A}_t q = (tq + qt)_0. \quad (4.5)$$

The source is obtained by differentiating the one-form spectral operator and its norm with respect to the tensor perturbation. Appendix E.4 derives it and its nonzero source on each real decomposable mode. The equal occupation of the full band again has zero TT source.

These pairings determine the source on compact tensor eigenspaces whose rough eigenvalue is positive at finite L . Projection onto the critical space of the sourced response determines their propagation residue.

4.2 A finite-dimensional critical space

The residue follows from the Schur complement of an isolated critical subspace as in analytic operator perturbation theory [5]. Keeping the source maps in this reduction identifies exactly which critical directions contribute to the measured response.

Theorem 4.1 (Positive residue on a finite critical space). Let $L(z)$ be bounded and analytic on a Hilbert space $\mathcal{X}$ near zero with $L(\bar{z})^* = L(z)$. Suppose that $L(0)$ has a finite-dimensional kernel with orthogonal projection P and that its restriction to $\bar{P}\mathcal{X}$ with $\bar{P} = I - P$ is boundedly invertible. Assume

$$G = PL'(0)P|_{P\mathcal{X}} > 0. \quad (4.6)$$

Then $L(z)$ is invertible for sufficiently small nonzero z and

$$L(z)^{-1} = \frac{PG^{-1}P}{z} + O(1). \quad (4.7)$$

For a Hilbert source space $\mathcal{U}$ if $S(z) : \mathcal{U} \rightarrow \mathcal{X}$ and $R(z) : \mathcal{X} \rightarrow \mathcal{U}$ are analytic and satisfy $R(x) = S(x)^*$ for real x and if C_{dir} is analytic then the residue of $C = C_{\text{dir}} + RL^{-1}S$ is

$$Z_{\text{nat}} = S(0)^*PG^{-1}PS(0) \succeq 0, \quad \text{rank } Z_{\text{nat}} = \text{rank}(PS(0)). \quad (4.8)$$

The finite rational case is the same Schur calculation in coordinates. Let H, K be real symmetric rational matrices with $H \succeq 0$. Let S be a rational source and let U be any basis of $\ker H$. Put $G_U = U^\top K U$. If $G_U \succ 0$ then

$$S^\top (H + zK)^{-1} S = \frac{Z_{\text{crit}}}{z} + O(1), \quad Z_{\text{crit}} = S^\top U G_U^{-1} U^\top S \succeq 0, \quad (4.9)$$

with $\text{rank } Z_{\text{crit}} = \text{rank}(U^\top S)$. If W_H contains a maximal independent set of columns of H the complementary gap obeys

$$\lambda_{\min}(H|_{(\ker H)^\perp}) \geq \left\{ \text{Tr} \left[(W_H^\top H W_H)^{-1} W_H^\top W_H \right] \right\}^{-1} > 0, \quad (4.10)$$

with the empty-kernel and $H = 0$ cases interpreted separately.

Proof. On $P\mathcal{X} \oplus \overline{P}\mathcal{X}$ the complementary block is analytically invertible and the Schur complement is $zG + O(z^2)$ [5]. Hence only the P block has a z^{-1} term. Inserting the source and readout gives the displayed residue whose factorisation through $G^{-1/2}PS(0)$ proves positivity and rank. The rational formulas are the same calculation in a nonorthonormal kernel basis. The last inequality is the trace bound for the inverse positive complementary matrix. $\square$

Equation (4.8) separates three physical ingredients. The operator G is the propagation slope on the critical space. The projected map $PS(0)$ carries the applied source into that space. Its adjoint gives the readout. A zero overlap makes an algebraic critical mode invisible whereas positivity of G makes the visible residue a positive quadratic form.

For rational matrices elimination and positive Schur pivots determine the kernel and residue together with the gap bound exactly [22]. The kernel identity fixes the critical space while positive eigenvalues contribute to the regular response. For $\epsilon > 0$ one has $\lim_{z \downarrow 0} z/(\epsilon + z) = 0$. If $K \succ 0$ on the full space the positive-form construction in Eq. (2.15) also gives a positive measure for the same finite affine response.

Reciprocity makes the slope Hermitian but does not make it positive. For example $L(z) = -zI$ has a negative residue despite reciprocity. Equation (4.6) must be calculated from the propagation rule. The projection P is taken after the physical constraints have been handled. A gauge or normalisation direction is not made observable by this theorem. For an unbounded form realisation the bounded result applies only after a justified form-space reduction. Eq. (2.15) gives the corresponding exact affine construction.

Appendix E.3 evaluates the residue for explicit finite matrices including a critical direction orthogonal to every source. Further finite calculations in Ref. [22] retain the direct term and the source-dependent complementary contribution. For approximations to an infinite-dimensional response the operator and source convergence estimates of Appendix D control the limiting critical space and residue.

For matter source columns $S_a(0)$ the critical mixed residue is the Gram matrix of $G^{-1/2}PS_a(0)$ whose rank is the dimension of the spanned critical source space. If an energy-source identity gives $PS_a(0) = q_a^{\text{src}} v_0$ with the same v_0 for every allowed matter state it instead gives

$$Z_{ab} = q_a^{\text{src}} q_b^{\text{src}} \langle v_0, G^{-1} v_0 \rangle \quad (4.11)$$

for real energy charges. For stationary conserved matter Laue's identity derives this common direction from the tensor vertex and the reduced source map.

4.3 Response factorisation

On a Euclidean ray $Q = \sqrt{z} \hat{Q}$ with $z > 0$ choose a real unit TT polarisation e transverse to the fixed direction $\hat{Q}$. In a covariant renormalisation scheme rotational covariance gives $e^* G^E(Q) e = C_2(z)$ independently of this choice of e . Thus contracting the effective source and observable maps gives $S_{\text{TT}} = S_{\text{eff}} e$ and $R_{\text{TT}} = e^* R_{\text{eff}}$. The direct term is contracted in the same way. In this section these are scalar-channel maps $\mathbb{C} \rightarrow \mathcal{X}_{\text{TT}}$ and $\mathcal{X}_{\text{TT}} \rightarrow \mathbb{C}$. Their analytic continuation near $z = 0$ is a hypothesis of the theorem below. If a truncation breaks rotational symmetry polarisation dependence must be retained until restoration is controlled. For several source amplitudes the response is instead a matrix and the norm estimates of Appendix D apply to it.

The polarisation multiplicity also matters for the kernel theorem. If exact rotational covariance identifies the spin-two response block as $V_2 \otimes \mathcal{X}_{\text{mult}}$ with $\mathcal{B} = I_{V_2} \otimes \mathcal{B}_{\text{red}}$ restriction to a fixed polarisation leaves $\mathcal{B}_{\text{red}}$ on the multiplicity space. Here $\dim V_2 = 5$ at nonzero Euclidean momentum. Simplicity refers to the eigenvalue of this reduced family on the multiplicity space. This reduction follows from the intertwining property and Schur's lemma. It must be verified in the chosen response representation. When it is unavailable the full critical multiplicity is retained and a finite-rank analysis replaces the simple-eigenvalue theorem.

Let $\mathcal{X}_{\text{TT}}$ be the response space selected either by the closed TT block or by the Schur–Feshbach reduction in Eq. (3.23) with any justified polarisation reduction made as above. Near a source-free solution X_* consider

$$X = \mathfrak{F}(X; z, \eta) + S_{\text{TT}}(z; \eta) J, \quad \mathcal{B}_{\text{TT}}(z; \eta) := D_X \mathfrak{F}(X_*; z, \eta). \quad (4.12)$$

Here J is the scalar amplitude of an infinitesimal background-source variation along e . The fundamental-field source is j in Eq. (3.14). Equation (4.12) is the linear-source form of the construction in Proposition 3.1. Its kernel and direct term $\mathcal{C}_{\text{reg}}$ come from the same source functional as S_{TT} and R_{TT} . The parameter z is the complexified external invariant with $z = Q^2$ on the Euclidean axis and $z = -p^2 - i0$ at the Lorentzian boundary. The symbol η denotes a control parameter which may include couplings or an RG scale. Differentiation with respect to z is not RG evolution. Only the first derivative at $J = 0$ is used. Terms beyond linear response are suppressed in Eq. (4.12).

If a measured response has regular direct term $\mathcal{C}_{\text{reg}}$ and observable map R_{TT} differentiation of Eq. (4.12) with respect to J gives

$$\mathcal{C}_{\text{resp}}(z; \eta) = \mathcal{C}_{\text{reg}}(z; \eta) + R_{\text{TT}}(z; \eta) [I - \mathcal{B}_{\text{TT}}(z; \eta)]^{-1} S_{\text{TT}}(z; \eta). \quad (4.13)$$

The symbol $\mathcal{C}_{\text{resp}}$ denotes the response produced by the specified physical source and read-out. It equals the stress coefficient C_2 only after those maps have been matched to the stress observable of Section 2.3.

This distinction is already visible in a reduced two-helicity branch. Let $\mathcal{H}_{\text{hel}} \simeq \mathbb{C}^2$ and suppose its field covariance has

$$\mathcal{D}_{\text{hel}}(z) = \frac{Z^{-1}I_2}{z} + \mathcal{D}_{\text{hel,reg}}(z), \quad Z > 0.$$

Dimensionful maps $S_T(z) : \mathcal{U}_T \rightarrow \mathcal{H}_{\text{hel}}$ and $R_T(z) : \mathcal{H}_{\text{hel}} \rightarrow \mathcal{U}_T$ match a physical TT stress source space $\mathcal{U}_T$ to this field branch. Let σ_T be the positive scalarisation fixed by the same TT polarisation convention as C_2 (an invariant normalised polarisation trace in the finite-dimensional case). The operator-valued response and its atomic coefficient are

$$\begin{aligned} \mathbf{C}_T^{\text{nl}}(z) &= R_T(z) \mathcal{D}_{\text{hel}}(z) S_T(z) + \mathbf{C}_{T,\text{reg}}^{\text{nl}}(z), \\ \mathbf{Z}_T &= R_T(0) Z^{-1} I_2 S_T(0), \\ C_{2,\text{red}}^{\text{nl}}(z) &= \sigma_T[\mathbf{C}_T^{\text{nl}}(z)], \quad Z_0^{(\text{red})} = \sigma_T[\mathbf{Z}_T]. \end{aligned} \quad (4.14)$$

In the dimensionless-field convention used for the effective realisation below Z has mass dimension two. The product of R_T and S_T carries the remaining dimensions required for $C_{2,\text{red}}$ and $Z_0^{(\text{red})}$ to have mass dimensions four and six respectively. Under reciprocity $R_T = S_T^*$ on the real axis so $\mathbf{Z}_T$ is positive semidefinite and the positive functional σ_T gives $Z_0^{(\text{red})} \geq 0$. Attaching two matter vertices of strength g_Φ gives the exchange weight g_Φ^2/Z . Equation (4.14) gives the stress-response atom whereas g_Φ^2/Z is the matter-exchange weight and is invariant under a renormalisation of the interpolating field. When the compact branch is matched to this stress response on the same scalar TT quadratic form the normalisations obey

$$\frac{M_P^2}{\widehat{Z}} = Z_{\text{prop}} = Z_0^{(\text{red})} = \sigma_T[R_T(0) Z^{-1} I_2 S_T(0)]. \quad (4.15)$$

Proposition 4.2 (Critical branch and propagation residue). Fix the radius at $c_* = A_{n_{\text{sp}},R}(\vartheta)/\kappa_{\text{TT}}$ and determine χ_c by $x(\chi_c) = 1$. Suppose the sourced propagation equations give analytic functions $\chi(z)$ and $\Pi_{\text{ext}}(z)$ together with $C_{\text{dir}}(z)$ near $z = 0$ with $z = Q^2$ and

$$\begin{aligned} \chi(0) &= \chi_c, \quad a = - \left. \frac{d\chi}{dz} \right|_{z=0} > 0, \\ \Pi_{\text{ext}}(0) &= \frac{\mu_*^4 \mathcal{J}(\chi_c)}{c_*^4}, \quad C_{\text{red}}(z) = C_{\text{dir}}(z) + \frac{\Pi_{\text{ext}}(z)}{1 - x(\chi(z))}. \end{aligned} \quad (4.16)$$

Then the external scalar response has the positive residue

$$C_{\text{red}}(Q^2) = \frac{Z_{\text{prop}}}{Q^2} + O(1), \quad Z_{\text{prop}} = \frac{\mu_*^4 \mathcal{J}(\chi_c)}{c_*^4 a x'(\chi_c)} > 0. \quad (4.17)$$

The same branch can be normalised directly against the massless tensor inverse. Let $M_P > 0$ be the reference mass used to make the external invariant dimensionless. The effective tensor realisation fixes this reference through its kinetic coefficient. With $\Pi(\chi) = \mu_*^4 \mathcal{J}(\chi)/c_*^4$ and $\widehat{z} = Q^2/M_P^2$ set

$$\ell(\chi) := \frac{1 - x(\chi)}{\Pi(\chi)}, \quad \ell(\chi(\widehat{z})) = \widehat{Z} \widehat{z}, \quad \widehat{Z} > 0. \quad (4.18)$$

The branch through χ_c is real analytic and unique and

$$\chi(0) = \chi_c, \quad -\frac{d\chi}{d\hat{z}}\Big|_{\hat{z}=0} = \frac{\hat{Z}\Pi(\chi_c)}{x'(\chi_c)} > 0, \quad \frac{\Pi(\chi(\hat{z}))}{1-x(\chi(\hat{z}))} = \frac{1}{\hat{Z}\hat{z}}. \quad (4.19)$$

Here $\hat{z}$ is dimensionless. In the stress-response normalisation $C_{\text{lad}} := \Pi/(1-x)$ has mass dimension four so $[\hat{Z}] = -4$ and its Q^2 -pole residue is

$$Z_{\text{prop}} = \frac{M_P^2}{\hat{Z}}. \quad (4.20)$$

Proof. The preceding crossing calculation fixes the unique χ_c and $x'(\chi_c) > 0$. The propagation branch has $\chi(z) = \chi_c - az + O(z^2)$ so $1 - x(\chi(z)) = ax'(\chi_c)z + O(z^2)$. Expanding the numerator and the analytic direct term proves the stated residue. Moreover $\ell'(\chi_c) = -x'(\chi_c)/\Pi(\chi_c) < 0$ so the analytic inverse-function theorem gives the matched branch and its derivative. $\square$

Equation (4.16) specifies the matching to be evaluated from the propagation equations. Its derivative a uses the external invariant at fixed radius and matter content. For a matrix-valued tensor response the corresponding calculation uses the physical critical projection and momentum derivative together with the source overlap on the critical subspace. Its spectral interpretation additionally requires a positive measure for the same sourced response.

Thus the relevant kernel is the dimensionless Jacobian of the complete self-consistent map. The Bethe–Salpeter bound-state equation [30] provides a two-particle representation. In the two-particle-irreducible effective-action formulation [31] differentiation of the stationarity equation yields the corresponding response kernel. Its product with the two-particle propagator represents this Jacobian when the sourced equations and observable map are retained consistently.

For an N -dimensional calculation with response space $\mathcal{X}_{\text{TT},N}$ we require a bounded comparison embedding $\iota_N : \mathcal{X}_{\text{TT},N} \rightarrow \mathcal{X}_{\text{TT}}$ and projection $P_N : \mathcal{X}_{\text{TT}} \rightarrow \mathcal{X}_{\text{TT},N}$ satisfying $P_N \iota_N = I_{\mathcal{X}_{\text{TT},N}}$. Its response is determined by

$$\mathfrak{T}_N = (\mathcal{B}_{\text{TT},N}, \mathcal{S}_{\text{TT},N}, \mathcal{R}_{\text{TT},N}, \mathcal{C}_{\text{reg},N}). \quad (4.21)$$

The renormalised operator basis $\{\mathcal{O}_a^{(2)}\}_{a=1}^N$ together with the effective action $\Gamma_{k,N}$ and regulator R_k define the projected stationarity equations $\mathcal{E}_{k,N}$ and observable $\mathcal{O}_{k,N}$. Proposition 3.1 then determines the kernel and direct term together with the source and observable maps with preconditioner $\mathcal{A}_{0,k,N}$ and subtraction polynomial $P_{2,N}$. The comparison maps place these finite operators in the common space of Appendix D where their approximation errors are measured.

4.4 Internal averaging and closure defect

External smearing changes the source and observable maps in Eq. (4.13) but not the spectrum of a fixed $\mathcal{B}_{\text{TT}}$. An internally averaged response can change the kernel only through a specified coarse-grained self-consistent map. Let $\mathcal{C}_\tau : \mathcal{X}_{\text{TT}} \rightarrow \mathcal{X}_{\text{TT},\tau}$ and define

$$E_\tau(X) := \mathcal{C}_\tau \mathfrak{F}(X) - \mathfrak{F}(\mathcal{C}_\tau X). \quad (4.22)$$

Write $\bar{Y}_*(z) = \mathcal{C}_\tau X_*(z)$ and $H_\tau(Y; z) = Y - \mathfrak{F}_\tau(Y; z)$. Since X_* is a source-free solution its transported point has residual

$$H_\tau(\bar{Y}_*; z) = E_\tau(X_*; z). \quad (4.23)$$

Thus the transported point is an actual coarse solution only when this residual vanishes. If $E_\tau = 0$ in a neighbourhood of X_* differentiation gives the exact intertwining relation

$$\mathcal{C}_\tau \mathcal{B}(X_*) = \mathcal{B}_\tau(\mathcal{C}_\tau X_*) \mathcal{C}_\tau. \quad (4.24)$$

For an approximate closure let $Y_{\tau,*}$ be an established coarse solution and let $\delta_{Y,\tau} := \|Y_{\tau,*} - \bar{Y}_*\|$ have a verified upper bound. Suppose that $D_Y \mathfrak{F}_\tau$ is Lipschitz with constant L_τ on a ball containing both points. On the specified invariant response comparison spaces where $\mathcal{C}_\tau$ is a bounded isomorphism the actual response Jacobian $\mathcal{B}_\tau(Y_{\tau,*})$ then obeys

$$\begin{aligned} & \|\mathcal{B}_\tau(Y_{\tau,*}) - \mathcal{C}_\tau \mathcal{B}(X_*) \mathcal{C}_\tau^{-1}\| \\ & \leq \|D_X E_\tau(X_*)\| \|\mathcal{C}_\tau^{-1}\| + L_\tau \delta_{Y,\tau}. \end{aligned} \quad (4.25)$$

Indeed differentiation of Eq. (4.22) compares Jacobians at X_* and $\bar{Y}_*$. Adding and subtracting $\mathcal{B}_\tau(\bar{Y}_*)$ gives the additional displacement term. Exact reference-branch matching $E_\tau(X_*; z) = 0$ permits the choice $Y_{\tau,*} = \bar{Y}_*$ and sets this term to zero. Proposition D.1 supplies one sufficient test for an approximate branch. At a singular stationarity point exact matching or a separate critical-branch construction supplies the branch. The inverse norm also makes explicit why strong smoothing can amplify a small closure error. Appendix D includes the resulting root and residue bounds.

4.5 An isolated Fredholm pole

Theorem 4.3 (Simple pole of the TT response kernel). At a fixed physical endpoint $\eta = \eta_c$ let U be a connected neighbourhood of $z = 0$ on which $\mathcal{B}_{\text{TT}}(z)$ is compact and analytic with $I - \mathcal{B}_{\text{TT}}(z)$ invertible somewhere in U . Suppose U is free of multiparticle thresholds and other nonmeromorphic singularities. Assume that the direct term and both source and readout maps in Eq. (4.13) are analytic on U . Then

- (i) if $1 \notin \text{spec } \mathcal{B}_{\text{TT}}(0)$ this kernel does not generate a pole at $z = 0$.
- (ii) if an algebraically simple eigenvalue satisfies

$$\lambda(0) = 1, \quad \lambda'(0) \neq 0, \quad (4.26)$$

then $(I - \mathcal{B}_{\text{TT}})^{-1}$ has a simple pole.

- (iii) with right and left eigenvectors normalised by $\langle \ell | r \rangle = 1$ that pole is visible in $\mathcal{C}_{\text{resp}}$ if and only if

$$\mathbf{R}_{\text{TT}}(0)|r\rangle \neq 0, \quad \langle \ell | \mathbf{S}_{\text{TT}}(0) \neq 0; \quad (4.27)$$

- (iv) its algebraic residue is

$$Z_{\text{alg}} = -\frac{\mathbf{R}_{\text{TT}}(0)|r\rangle \langle \ell | \mathbf{S}_{\text{TT}}(0)}{\lambda'(0)}. \quad (4.28)$$

The proof uses the rank-one Laurent expansion of the analytic Fredholm resolvent [5]. Appendix D gives the expansion with both source factors. A first-order transverse crossing in the control parameter η has $\partial_\eta \lambda(0; \eta_c) \neq 0$. Higher-order parameter crossings are covered by the same argument whenever the external-invariant slope remains nonzero. For example $\lambda(z; \eta) = 1 - z + \eta^3$ has $\partial_\eta \lambda(0; 0) = 0$ but $\partial_z \lambda = -1$. In an FRG realisation a finite- k crossing concerns the regulated response. The physical correlator is obtained in the renormalised $k \rightarrow 0$ limit. Its physical correlation scale remains fixed in this limit.

The remaining step is to identify the Laurent coefficient with the weight of the physical measure. This requires the sourced response rather than just the kernel spectrum to agree with the reconstructed correlator.

Theorem 4.4 (TT Fredholm–spectral matching). In addition to Theorem 4.3 assume that

- (i) the endpoint response $\mathcal{C}_{\text{resp}}$ in Eq. (4.13) is the unfiltered connected correlator of the same physical source observable that defines the response equation.
- (ii) this observable has an independently established positive scalar spectral measure $d\rho_{\text{resp}}$ on $[0, \infty)$ with atom $Z_0^{(\text{resp})} = \rho_{\text{resp}}(\{0\})$ matched to the same physical source quadratic form with common quotient and continuation and subtraction conventions. For the covariant stress-tensor realisation $\mathcal{C}_{\text{resp}} = C_2$ and these facts are supplied by the boundary construction in Section 2.5 and Eq. (2.43). For a directly reduced two-helicity realisation contraction with a fixed physical helicity source scalarises $\mathcal{C}_{\text{resp}} = \mathcal{D}_{\text{hel}}$ on its Fock space. The maps in Eq. (4.14) then produce the corresponding stress response.
- (iii) the direct term and complementary resolvent contributions are analytic at $z = 0$ after subtraction.
- (iv) the remaining spectral measure satisfies the hypotheses of Proposition 2.4.

Then

$$Z_{\text{alg}} = \lim_{Q^2 \downarrow 0} Q^2 \mathcal{C}_{\text{resp}}^{\text{nl}}(Q^2) = Z_0^{(\text{resp})} \geq 0. \quad (4.29)$$

In particular a matched scalar Laurent coefficient is real and nonnegative.

Proof. Theorem 4.3 gives $\mathcal{C}_{\text{resp}}^{\text{nl}}(Q^2) = Z_{\text{alg}}/Q^2 + A(Q^2)$ with A analytic under assumption (iii). Multiplication by Q^2 and passage to the origin gives Z_{alg} . Assumptions (i) and (ii) together with (iv) identify the same limit with the Wightman atom through Proposition 2.4. Positivity follows from the independently matched measure in assumption (ii). $\square$

The theorem is the point at which an operator pole acquires a physical spectral meaning. Analytic Fredholm theory fixes the Laurent coefficient. The independent positive-measure construction identifies the same coefficient as spectral weight. Together they connect the critical operator branch to the physical source measure.

For the complete covariant stress tensor the match includes Theorem 2.3 and gives $Z_{\text{alg}}^{(\Theta)} = Z_0 = 0$. Thus a matched exact limit has zero visible residue in this tensor class. For another response object the positive measure and null-shell match are supplied by that object’s spectral construction.

4.6 From compact spectral data to a propagation residue

Compact internal spectra provide bases and masses together with degeneracies and window weights for a finite response calculation. In the compact realisation the fields and their sources and observables are constructed in the same operator algebra and state. The spectral operations of Ref. [21] and the covariant transport of Ref. [1] fix the objects used in that construction. A limit of their responses uses the common domains and norm estimates in the proposition below. For specified semilinear generators ref. [32] controls the discarded nonlinear source on a common time interval using a stronger spectral bound and convergence in a weaker norm. Finite response calculations also approximate the operator and its readout. Appendix D compares these objects on common response spaces. Bounds on the same sourced equations and their variations supply the kernel and source-map errors. For example the prescription in Ref. [20] uses

$$K_{\text{TT}}^{\text{win}}(k_{\text{win}}^2, m^2) = \frac{k_{\text{win}}^4}{(k_{\text{win}}^2 + m^2)^2}, \quad y = \frac{m^2}{k_{\text{win}}^2 + m^2}, \quad \max_{0 \leq y \leq 1} y(1-y)^2 = \frac{4}{27}. \quad (4.30)$$

Here m is an internal spectral mass and k_{win} is the window scale. z remains the external invariant in the response equation. The sourced equations determine how these internal data enter the external-momentum operator in Eq. (4.13). The following proposition states sufficient conditions for a positive scalar physical residue from a finite approximation with a controlled error.

Proposition 4.5 (Physical residue from compact-spectrum approximations). Let a finite calculation supply internal eigenvalues m_n^2 and degeneracies d_n together with scale-window factors $K_n(k_{\text{win}}^2, m_n^2)$ and suppose that an extremum or fixed-point equation selects a model closure. Suppose the associated response has an independently constructed positive spectral measure whose null-shell sector carries the measured TT quadratic form. Let these data enter the response tuple Eq. (4.21) with analytic kernel and direct term together with analytic source and readout maps derived from common sourced equations. Suppose the external variable z satisfies the simple-crossing and visibility hypotheses of Theorem 4.3 and evaluate each finite residue $\mathcal{Z}_N$ at its displaced critical root z_N . Assume further that the common-space estimates of Appendix D control errors in the basis and vertices together with momentum discretisation and complementary-channel and closure errors and identify the limiting unfiltered correlator with its positive Wightman measure with any zero-threshold continuum separated from the atom. If for some finite N those estimates provide

$$|\mathcal{Z}_N - Z_0^{(\text{resp})}| \leq \Delta_N, \quad \text{Re } \mathcal{Z}_N - \Delta_N > 0,$$

then $Z_0^{(\text{resp})} > 0$. Here Δ_N is the controlled upper bound given by Eq. (D.18) assembled from the telescoping kernel bounds and the derivative and source-map errors in Eq. (D.7) together with the projection and root-displacement bounds and the slope estimates of Propositions D.2 and D.1. The simple-eigenvalue hypothesis is a sufficient route. Higher critical multiplicity is treated by the finite-rank theorem above.

Proof. The sourced response tuple and the external crossing give the finite pole and its residue at z_N . The common-space and cross-root estimates bound its difference from the

exact Laurent coefficient by Δ_N . Theorem 4.4 identifies that coefficient with the real spectral weight $Z_0^{(\text{resp})}$. Hence $Z_0^{(\text{resp})} \geq \text{Re } \mathcal{Z}_N - |\mathcal{Z}_N - Z_0^{(\text{resp})}| \geq \text{Re } \mathcal{Z}_N - \Delta_N > 0$. $\square$

A strict lower margin separates a positive limiting residue from a vanishing sequence. For example $C_N(z) = N^{-2}/z$ has positive finite residues tending to zero. The window extremum in Eq. (4.30) fixes an internal scale weight. The sourced propagation equation separately determines the external derivative and the two overlaps.

Higher critical multiplicity is possible. For example $\mathcal{B}(z) = (1 - z)I_2$ with $\mathbf{S} = (1, 1)^\top$ and $\mathbf{R} = (1, 1)$ gives $\mathbf{R}(I - \mathcal{B})^{-1}\mathbf{S} = 2/z$ although its critical eigenvalue has algebraic multiplicity two as allowed by Theorem 4.1.

4.7 Thresholds and spectral accumulation

If $d\rho_{2,c}$ has support arbitrarily close to $s = 0$ the analytically continued correlator has other spectral singularities arbitrarily close to the candidate pole. Continuous thresholds and accumulating positive-mass atoms both prevent an isolated meromorphic neighbourhood. For a continuous threshold the boundary behaviour may be logarithmic or of fractional-power type or more general. When spectral accumulation removes an isolated analytic neighbourhood the direct measure criterion still determines the atomic contribution. A limiting-absorption or weighted-resolvent analysis may also separate it from the remaining spectrum.

Positivity also constrains possible moment identities. A condition of the form

$$\int_0^\infty s d\rho_2(s) = 0 \quad (4.31)$$

forces a nonnegative measure to be supported at zero so it is incompatible with a nonzero positive continuum. Signed subtracted sum rules concern a different spectral object. Similarly Ward identities constrain tensor structures on the timelike cone. They do not provide spectral weight or force an eigenvalue crossing. On the null cone full positivity together with exact covariance and conservation gives Theorem 2.3.

The linear realisation fixes the field covariance and stress-response atom together with the kernel exchanged between matter sources.

4.8 A linear gauge-tensor realisation

The external representation uses the minimal gauge-invariant Fierz–Pauli realisation. Constraint reduction supplies two propagating helicities while the conserved-source numerator and spatial inverse provide the data against which the compact response is matched. Let $\Phi_{\mu\nu}$ be a dimensionless symmetric field on Minkowski space with signature $(+ - - -)$ and

$$\begin{aligned} S_{\text{FP}} &= Z \int d^4x \left[\frac{1}{2} \partial_\lambda \Phi_{\mu\nu} \partial^\lambda \Phi^{\mu\nu} - \partial_\mu \Phi^{\mu\nu} \partial^\rho \Phi_{\rho\nu} + \partial_\mu \Phi^{\mu\nu} \partial_\nu \Phi - \frac{1}{2} \partial_\lambda \Phi \partial^\lambda \Phi \right], \\ S_J &= g_\Phi \int d^4x \Phi_{\mu\nu} T^{\mu\nu}, \quad \Phi = \eta^{\mu\nu} \Phi_{\mu\nu}, \quad Z > 0. \end{aligned} \quad (4.32)$$

The effective action is local with two derivatives and the linear gauge symmetry $\delta\Phi_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. The background is the source-free flat vacuum. The coefficient Z and the

common matter vertex g_Φ specify the normalisation. This is the linear spin-two action underlying the self-coupling construction of Ref. [17].

For Eq. (4.32) the standard constraint reduction with a conserved source gives two physical helicities. Three residues must be distinguished. The reduced field covariance and its spectral measure together with the kernel exchanged between two matter sources are

$$\begin{aligned}\mathcal{L}_{\text{hel}}(Q^2) &= ZQ^2 I_2, & \mathcal{D}_{\text{hel}}(Q^2) &= \frac{1}{ZQ^2} I_2, \\ d\rho_\Phi(s) &= \frac{1}{Z} I_2 \delta_0(ds), & \mathcal{K}_{\text{matt}}(Q^2) &= g_\Phi^2 \mathcal{D}_{\text{hel}}(Q^2).\end{aligned}\tag{4.33}$$

The field atom is positive because $Z > 0$ and the matter exchange has weight g_Φ^2/Z . The dimensionful stress interpolating maps in Eq. (4.14) convert this field covariance into the spectral weight of a stress-tensor two-point function. The complete exchange between conserved sources in four dimensions is

$$\mathcal{A}[T, U](Q) = \frac{g_\Phi^2}{ZQ^2} (T_{\mu\nu}^* U^{\mu\nu} - \tfrac{1}{2} T^* U). \tag{4.34}$$

Its static energy-monopole potential is $V_{ab}(r) = -g_\Phi^2 E_a E_b / (8\pi Z r)$. These are the standard Fierz–Pauli reduction and conserved-source inverse [17]. The gauge kernel and four constraints leave two TT configuration degrees of freedom on which the quadratic form is $ZQ^2 I_2$ and the Hamiltonian is positive. The corresponding reflection form is a Gram form giving the positive helicity measure above. The trace-reversed de Donder inverse supplies the $-T^* U/2$ numerator. For static energy monopoles it is $E_a E_b/2$ and the transform of $|\mathbf{Q}|^{-2}$ yields the displayed potential.

We use the field-normalisation freedom to set $g_\Phi = 1$ and write the resulting kinetic coefficient as $Z = M_P^2$. Only the invariant combination g_Φ^2/Z enters the exchange. Comparing the static potential above with $-G_N E_a E_b / r$ therefore gives $G_N = (8\pi M_P^2)^{-1}$. The spectral calculation fixes the dimensionless residue while one dimensional calibration fixes M_P in physical units. The field-covariance residue is M_P^{-2} . The physical field measure in Eq. (4.33) is on the gauge-reduced helicity space. The off-shell conserved-source numerator and the two on-shell helicities are evaluated in their respective spaces.

Equation (4.18) specifies the identification of the ladder source normalisation with the reduced external field. The massless tensor inverse is proportional to Q^2 so the relation matches the internal inverse ladder to the leading infrared coordinate of the external inverse propagator including its zero and normalisation. The gauge action supplies the zero-momentum condition and propagation coefficient. The internal crossing supplies its unique coordinate and the conversion slope. Write $\bar{Z} = M_P^4 \hat{Z}$. In the convention used for the quoted spectral data the endpoint mass is identified with the Planck unit $\mu_* = M_P$. Then $\bar{Z} = 1$ gives the dimensionless slope $-d\chi/d\hat{z}|_0 \simeq 0.1082$. The matching is an input of the effective linear realisation whose quadratic external pole is given in Eq. (4.33). The propagated field is the macroscopic collective tensor represented by this reduced branch. The finite evaluations in Ref. [22] use these gauge identities and the reduced helicity covariance.

5 Matter coupling and long-range response

In the flat-background limit of the linear two-helicity realisation define the matter-normalised collective field $\tilde{\Phi}_{\mu\nu} := g_\Phi \Phi_{\mu\nu}$ from Eq. (4.32). Its vertex is

$$S_{\text{coupling}} = \int d^4x T_{\text{mat}}^{\mu\nu} \tilde{\Phi}_{\mu\nu}. \quad (5.1)$$

Here T_{mat} is the total matter energy–momentum tensor including interaction contributions. The coupling fixes the matter vertex and the spectral calculation determines the mediated response. The compact comparison geometry enters through the internal spectral operators.

In the compact construction the full one-family matter module has sixteen directions while its fifteen-dimensional charged quotient records the internal charge response [1]. The species trace used to fix the endpoint is taken on that charged sector. The energy–momentum source here is obtained by varying the complete matter action including the neutral direction and the interactions acting on the full module. Consequently the same tensor vertex carries component energies and their interaction energy into the exchange amplitude.

The interaction is read out in a definite order. Conservation first reduces the long-wavelength source to its total energy. The common response maps that source into the critical channel. The spatial inverse determines the potential and force. The helicity factorisation and soft limits then select the gravitational tensor structure. A positive spectral atom supplies the massless channel while the reduced static energy and the relative source directions determine the sign of the force.

5.1 Energy sources and the complete static response

Let $\mathbf{t}_a^{\mu\nu}(\mathbf{x})$ denote the vacuum-subtracted expectation of $T_{\text{mat}}^{\mu\nu}$ in a stationary localised matter configuration. The tensor source is kept complete including its stresses. At leading exchange order each configuration is evaluated in its unsourced stationary state. The following integrated stress identity is Laue’s theorem [7] with the long-wavelength estimate stated in the normalisation of the matter vertex.

Proposition 5.1 (Energy source, factorised residue, and static exchange). Suppose $\mathbf{t}_a$ is symmetric and time independent and is conserved in the fixed flat coordinates and $\int (1 + |\mathbf{x}|) \|\mathbf{t}_a(\mathbf{x})\| d^3x < \infty$. In its rest frame set $E_a = \int \mathbf{t}_a^{00} d^3x$ and $(e_{00})^{\mu\nu} = \delta_0^\mu \delta_0^\nu$. Then

$$\int \mathbf{t}_a^{ij} d^3x = 0, \quad \hat{\mathbf{t}}_a(\mathbf{p}) = E_a e_{00} + r_a(\mathbf{p}), \quad \|r_a(\mathbf{p})\| \leq |\mathbf{p}| \int |\mathbf{x}| \|\mathbf{t}_a(\mathbf{x})\| d^3x. \quad (5.2)$$

Thus the zero-momentum vertex in Eq. (5.1) has the same energy normalisation for every such configuration including bound systems when their complete conserved tensor is used.

Suppose in addition that on a fixed momentum ray the reduced source is $\hat{j}_a(\mathbf{p}) = \mathbf{V}(\mathbf{p})^* \hat{\mathbf{t}}_a(\mathbf{p})$ where $\|\mathbf{V}(\mathbf{p})^*\| \leq C_V$ and $\|\mathbf{V}(\mathbf{p})^* - \mathbf{V}_0^*\| \leq K_V |\mathbf{p}|$. For the critical projection P set $v_0 = P \mathbf{V}_0^* e_{00}$ and $I_a = \int |\mathbf{x}| \|\mathbf{t}_a(\mathbf{x})\| d^3x$. Then

$$\left\| P \hat{j}_a(\mathbf{p}) - E_a v_0 \right\| \leq |\mathbf{p}| (C_V I_a + K_V |E_a|). \quad (5.3)$$

If the critical inverse of Theorem 4.1 has positive propagation form G on $P\mathcal{X}$ its mixed residue is

$$Z_{ab}^{\text{mat}} = E_a E_b \mathfrak{b}_{\text{mat}}, \quad \mathfrak{b}_{\text{mat}} = \langle v_0, G^{-1} v_0 \rangle \geq 0, \quad (5.4)$$

and $\mathfrak{b}_{\text{mat}} > 0$ exactly when $v_0 \neq 0$.

Moreover suppose the constraint reduction has a real Hilbert energy space $\mathcal{V}$ with a bounded symmetric coercive form $\mathfrak{l}$ and the source-dependent energy on that space is

$$E_{\text{stat}}[\gamma; J] = \frac{1}{2} \mathfrak{l}[\gamma, \gamma] - J[\gamma] + E_m + E_{\text{dir}}, \quad J \in \mathcal{V}^*. \quad (5.5)$$

Let $\mathcal{G} : \mathcal{V}^* \rightarrow \mathcal{V}$ be the inverse of its form operator. The unique stationary point is $\gamma = \mathcal{G}J$ and for $J = J_a + J_b$ the mediated cross term is

$$E_{ab}^{\text{med}} = -J_a[\mathcal{G}J_b]. \quad (5.6)$$

Direct matter cross terms if present are added separately.

Proof. Choose a smooth cutoff χ_R equal to one on the ball of radius R and zero outside the ball of radius $2R$. Conservation gives

$$\int \chi_R \mathfrak{t}_a^{ij} d^3x = - \int x^i (\partial_k \chi_R) \mathfrak{t}_a^{kj} d^3x.$$

The right-hand side tends to zero by integrability since $|x^i \partial_k \chi_R|$ is bounded on its receding annulus. The mixed components integrate to the total momentum which vanishes in the rest frame. The remaining component integrates to E_a . Subtracting the zero-momentum Fourier transform and using $|e^{-i\mathbf{p} \cdot \mathbf{x}} - 1| \leq |\mathbf{p}| |\mathbf{x}|$ proves the bound with the interaction and binding contributions included in the conserved tensor. Applying the two bounds on $\mathcal{V}^*$ to $\widehat{\mathfrak{t}}_a = E_a e_{00} + r_a$ gives Eq. (5.3). Multiplication of the critical inverse by $|\mathbf{p}|^2$ and passage to the source limit then gives Eq. (5.4). Positivity and strict positivity follow from the positive form G^{-1} on the visible critical space. The static statement follows from the Riesz representation theorem and completion of the square in the coercive form

$$\frac{1}{2} \mathfrak{l}[\gamma - \mathcal{G}J, \gamma - \mathcal{G}J] - \frac{1}{2} J[\mathcal{G}J].$$

Symmetry of the inverse gives Eq. (5.6). $\square$

For a slowly varying static field the corresponding position-space difference between $\int \mathfrak{t}_a^{\mu\nu}(\mathbf{x}) \widetilde{\Phi}_{\mu\nu}(\mathbf{x}) d^3x$ and $E_a \widetilde{\Phi}_{00}(0)$ is bounded by $\|\nabla \widetilde{\Phi}\|_\infty \int |\mathbf{x}| \|\mathfrak{t}_a\| d^3x$. The conserved system includes any external supports that carry momentum. The proposition applies to the resulting isolated stationary source. Stationary configurations on a compact domain instead retain the full modal vertex since the cutoff argument uses a noncompact matter domain. The complete conserved tensor incorporates stresses and interactions together with binding energy into the same monopole charge. The Ward identity therefore selects total energy as the long-wavelength source.

The overall minus sign has a direct variational origin. The positive quadratic form is the cost of a field configuration. Minimising it in the presence of a source lowers the energy

by one half of the source–response pairing. Attraction follows when two physical sources have positive cross pairing through $\mathcal{G}$ as in the common energy-factorised channel below. General source orientations retain the sign of $J_a[\mathcal{G}J_b]$.

After reducing the sourced constraints write the field seen by matter as $\tilde{\Phi} = \mathbf{V}\gamma$ in the retained coordinates. The static sign convention corresponding to the coupling above is $E_{\text{coupling}} = -J[\gamma]$ with $J_a = \mathbf{V}^*\mathbf{t}_a$. The mediated tensor response is consequently

$$\mathcal{D}_{\text{stat}} = \mathbf{V}\mathcal{G}\mathbf{V}^*, \quad E_{ab}^{\text{med}} = -\langle \mathbf{t}_a, \mathcal{D}_{\text{stat}}\mathbf{t}_b \rangle, \quad (5.7)$$

where $\mathcal{G}$ is the inverse of the reduced energy form in Proposition 5.1. Source-dependent eliminated components also produce the direct term of Eq. (C.12). The reduction retains both this term and the stresses. For a translation-invariant response Eq. (5.2) selects its 00,00 contraction at leading long wavelength. The full kernel including its constraint contributions determines the sign and spatial decay. A change $\tilde{\Phi} \mapsto a\tilde{\Phi}$ multiplies its covariance by a^2 and its vertex by a^{-1} leaving the exchange unchanged.

For the compact spectrum of Section 2.1 the modal energy is $M_a(c, \vartheta) = \mu_* \widehat{M}_a(\vartheta)/c$ where $\widehat{M}_a := \sqrt{\widehat{\lambda}_a + \widehat{m}_a^2}$. Apply the same geometric compression $c \mapsto ce^{-\eta}$ to all carriers with their dimensionless content fixed. Directly

$$\partial_\eta M_a(ce^{-\eta}, \vartheta)|_0 = M_a(c, \vartheta), \quad \partial_\eta \sum_a n_a M_a(ce^{-\eta}, \vartheta) \Big|_0 = \sum_a n_a M_a(c, \vartheta). \quad (5.8)$$

The occupations are held fixed and the sum is finite or absolutely convergent. This internal scale derivative measures the integrated modal energy at a fixed reference branch.

For the complete interacting spectral energy $\mathcal{E}_{\text{mat}}$ the same statement extends to its degree-one homogeneous part. If $\mathcal{E}_{\text{mat}}(c, \xi) = \mu_* c^{-1} \widehat{\mathcal{E}}_{\text{mat}}(\xi)$ with dimensionless $\widehat{\mathcal{E}}_{\text{mat}}$ then compression differentiates its full energy including binding. When dimensionless variables ξ change under the source variation their contribution $D_\xi \mathcal{E}_{\text{mat}} \partial_\eta \xi$ must also be retained. Equation (3.5) is an explicit example of such an additional contribution. These spectral derivatives refer to the specified internal comparison. For a nonlinear source map its second derivative also enters the chain rule. This internal susceptibility is distinct from the propagator of $\tilde{\Phi}$.

The scalar branch supplies a local representative of the same energy source. Appendix E.4 differentiates the conformally transported square-root operator on its common form domain. For a stationary occupation matrix commuting with that operator the result in a joint eigenbasis is

$$DE_{\text{occ}}(0)[\eta] = \int_M \eta \rho_{E, \text{occ}} dv_g, \quad \rho_{E, \text{occ}} = \sum_i n_i M_i |u_i|^2 \geq 0, \quad (5.9)$$

$$\int_M \rho_{E, \text{occ}} dv_g = E_{\text{occ}}(0).$$

The matrix version proved there also covers degeneracies and coherent entries. This density enters the reduced source map $\mathbf{V}^*$ while the common matter vertex remains Eq. (5.1).

Proposition 5.1 then shows how its long-wavelength projection acquires a common critical direction. The matter state enters through E_a while $\mathbf{b}_{\text{mat}}$ belongs to the shared propagating channel. Directional dependence of P and G together with that of $\mathbf{V}_0$ is retained until the isotropy of this contraction is established. The compact density has this matter interpretation after it is transported to the same source domain.

The quadratic equations can include constraint multipliers. For example a block with entries $K, C^*, C, 0$ is a saddle system rather than a positive form on all its variables. The construction in Eq. (2.15) applies to its positive reduced form on the physical quotient. Complementary sources and direct terms are retained through Eq. (C.12). Static matter enters through the complete conserved source. Its TT projection is one response channel while the energy monopole is selected by the zero-momentum Ward identity.

The sign of the mixed contraction depends on the relative source directions. For $\mathcal{G} = I$ opposite vectors have negative mutual contraction. A positive energy-factorised residue instead gives a negative mediated cross energy for positive energy charges.

For a centred Gaussian field of covariance $\mathcal{G}$ completing the square gives $W[J] - W[0] = J[\mathcal{G}J]/2$ and $\langle \gamma \rangle_J = \mathcal{G}J$. This is the same source pairing as in the reduced energy. This field measure is distinct from Gaussian scale averaging.

5.2 Spatial propagation and long-range exchange

Spatial propagation depends on the external momentum fibres of the response operator. Suppose separation of variables gives $L(\mathbf{p}) = A_{\text{int}} + \alpha_{\text{sp}}|\mathbf{p}|^2 I$ with $\alpha_{\text{sp}} > 0$ and $A_{\text{int}} \geq 0$. A visible internal kernel then gives the fibre term $P_{\ker A_{\text{int}}} / (\alpha_{\text{sp}}|\mathbf{p}|^2)$ whose distributional inverse supplies spatial propagation on infinite volume even though the full operator has no normalisable zero eigenvector.

The full $L^2(\mathbb{R}^3)$ Laplacian has no normalisable zero eigenvector but its distributional inverse is

$$G_{\text{sp}}(x) = \frac{1}{4\pi\alpha_{\text{sp}}|x|}, \quad -\alpha_{\text{sp}}\Delta G_{\text{sp}} = \delta_0. \quad (5.10)$$

with the coefficient fixed by the flux through a small sphere. It acts on compact sources in the homogeneous gradient energy space for which $\int |\hat{J}(\mathbf{p})|^2 / |\mathbf{p}|^2 d^3p < \infty$. The fibrewise $1/|\mathbf{p}|^2$ singularity therefore produces spatial propagation even though zero is not an L^2 eigenvalue of the full-space operator.

For the free-carrier spectral rule (2.12) the static kernel is obtained mode by mode. Put $a_{ab,l} = \langle g_a, P_l g_b \rangle$. The three-dimensional Fourier transform at zero external frequency is

$$G_{ab}^{\text{free}}(r) = \frac{1}{4\pi r} \sum_l a_{ab,l} e^{-M_l r}, \quad r > 0. \quad (5.11)$$

Absolute summability of $a_{ab,l}$ justifies the sum and its derivative away from zero. Since $M_l \geq m_\delta > 0$ it gives the explicit bounds

$$\begin{aligned} |G_{ab}^{\text{free}}(r)| &\leq \frac{\|g_a\| \|g_b\|}{4\pi r} e^{-m_\delta r}, \\ |\partial_r G_{ab}^{\text{free}}(r)| &\leq \frac{\|g_a\| \|g_b\|}{4\pi r^2} (1 + m_\delta r) e^{-m_\delta r}. \end{aligned} \quad (5.12)$$

The second bound uses that $(1 + Mr)e^{-Mr}$ decreases with $M \geq 0$. These are the spatial consequences of the scalar mass tower. They fix the elementary propagator used in composite spectral calculations. The collective tensor uses its resummed response and the matter vertex of Eq. (5.1).

On a connected compact space without boundary integration of $-\Delta\gamma = J$ gives $\int J = 0$. Its inverse is consequently defined only on a compatible source subspace up to a constant. Replacing a positive total energy source by its zero-mean part changes the physical problem unless a background or constraint equation supplies exactly that compensation. Finite-space distance windows and their boundary conditions are separate from the infinite-volume limit above.

Consider an observable whose matched physical TT measure $d\rho_{\text{resp}}$ contains an atom $Z_0^{(\text{resp})}\delta_0$. For $J^{\mu\nu} \in \mathcal{J}_{\text{TT}}$ its quadratic source response is

$$W[J] - W[0] = \frac{1}{2} \int \frac{d^4Q}{(2\pi)^4} J^{\mu\nu}(-Q) \Pi_{\mu\nu,\rho\sigma}^{(2)}(Q) \mathcal{C}_{\text{resp}}(Q^2) J^{\rho\sigma}(Q) + \dots, \quad (5.13)$$

with

$$\mathcal{C}_{\text{resp}}^{\text{nl}}(Q^2) = \frac{Z_0^{(\text{resp})}}{Q^2} + \mathcal{C}_{\text{resp,rem}}(Q^2), \quad Q^2 \mathcal{C}_{\text{resp,rem}}(Q^2) \longrightarrow 0. \quad (5.14)$$

Proposition 5.2 (Long-range TT response and static tail). Under the stated positive spectral representation a positive atom $Z_0^{(\text{resp})}$ contributes

$$\int \frac{d^4Q}{(2\pi)^4} \frac{e^{iQ \cdot x}}{Q^2} = \frac{1}{4\pi^2|x|^2} \quad (5.15)$$

to the connected response of physical TT sources. If the remaining static mixed measure ν on $(0, \infty)$ satisfies $\int e^{-r_0\sqrt{s}} d|\nu|(s) < \infty$ then

$$G_{\text{stat}}(r) = \frac{Z_0^{(\text{resp})}}{4\pi r} + g_{\text{rem}}(r), \quad g_{\text{rem}}(r) = \frac{1}{4\pi r} \int e^{-r\sqrt{s}} d\nu(s), \quad (5.16)$$

where $g_{\text{rem}} = o(r^{-1})$ and $g'_{\text{rem}} = o(r^{-2})$. External Gaussian averaging at finite τ leaves the atomic coefficient unchanged.

Proof. On conserved traceless sources the TT projector is the identity. Fourier transformation of the atomic term gives Eq. (5.15) and its zero-frequency restriction gives $1/(4\pi r)$. The Yukawa transform of the remaining measure is Eq. (5.16). Dominated convergence proves the potential limit while

$$4\pi r^2 g'_{\text{rem}}(r) = - \int (1 + r\sqrt{s}) e^{-r\sqrt{s}} d\nu(s) \quad (5.17)$$

and $(1 + u)e^{-u} \leq Ce^{-u/2}$ give the force limit. Finally

$$\int_{\tau}^{\infty} \frac{e^{-r^2/(4t)}}{(4\pi t)^2} dt = \frac{1 - e^{-r^2/(4\tau)}}{4\pi^2 r^2} \quad (5.18)$$

shows directly that finite Gaussian resolution changes only the short-distance profile. $\square$

The total-variation hypothesis includes mixed spectral measures which may be signed even when the matrix measure is positive. Source multipoles and direct terms together with other spatial corrections enter the total remainder with their own derivative bounds.

When source projection and constraint reduction together with spatial inversion give a common static energy channel their result takes the form

$$E_{ab}(r) = -\frac{\beta_{\text{eff}}^2}{4\pi\alpha_{\text{sp}}} \frac{q_a^{\text{src}} q_b^{\text{src}}}{r} + \mathcal{R}_{ab}(r), \quad \mathcal{R}'_{ab}(r) = o(r^{-2}). \quad (5.19)$$

For the tensor vertex and an isotropic critical contraction in Eq. (5.4) $q_a^{\text{src}} = E_a$ and $\beta_{\text{eff}}^2/\alpha_{\text{sp}} = \mathbf{b}_{\text{mat}}$. The coefficient is fixed by the same reduced propagation and source maps. If quasistatic work satisfies $\delta W = -\delta E_{ab}$ its radial force is $-\beta_{\text{eff}}^2 q_a^{\text{src}} q_b^{\text{src}} / (4\pi\alpha_{\text{sp}} r^2) + o(r^{-2})$. The leading force is attractive for positive product $q_a^{\text{src}} q_b^{\text{src}}$.

The acceleration depends also on the inertial mass. For a stable matter branch with

$$E_a(\mathbf{P}) = q_a^{\text{src}} + \frac{|\mathbf{P}|^2}{2m_{I,a}} + o(|\mathbf{P}|^2), \quad m_{I,a} > 0, \quad (5.20)$$

the ratio $q_a^{\text{src}}/m_{I,a}$ enters the acceleration. In the Lorentz-covariant realisation of Proposition 5.1 this ratio follows from the same total tensor. For the family obtained by boosting the isolated stationary state a boost Λ changes the equal-time volume by $1/\Lambda^0_0$. Stationarity and the integrated stress identity give

$$P_a'^{\mu} = \frac{\Lambda^0_{\alpha} \Lambda^{\mu}_{\beta}}{\Lambda^0_0} \int \mathbf{t}_a^{\alpha\beta} d^3x = E_a \Lambda^{\mu}_0, \quad E_a(\mathbf{P}) = \sqrt{E_a^2 + |\mathbf{P}|^2}. \quad (5.21)$$

For $E_a > 0$ this proves $m_{I,a} = E_a = q_a^{\text{src}}$ for the Lorentz-covariant boosted family used in this realisation. The equality follows from its dispersion relation together with the common conserved source map. For other dispersion relations Eq. (5.20) defines the relevant inertial mass. At finite coupling the total energy also includes energy transfer and the mediator contribution. A field redefinition $\gamma' = \xi\gamma$ sends α_{sp} to α_{sp}/ξ^2 and β_{eff} to β_{eff}/ξ . The measured combination $\beta_{\text{eff}}^2/\alpha_{\text{sp}}$ in Eq. (5.19) is unchanged. A zero-threshold continuum may add a further power-law tail while local subtraction terms remain at contact support. The atomic coefficient is isolated by the spectral limit and the derivative estimate above controls the force independently of the potential. The result is a source-visible long-range tensor response. Its helicity content and mixed-matter factorisation determine its gravitational interpretation.

5.3 Helicity, factorisation, and the physical coupling

The Euclidean projector in Eq. (2.22) has rank five and describes the continued transverse traceless spin-2 sector at nonzero invariant momentum. The physical helicity space of a massless spin-2 particle is two-dimensional. A helicity-two interpretation identifies the atom in the Lorentzian continuation with a positive-norm massless representation of helicities $+2$ and -2 . Longitudinal and negative-norm representatives then decouple from physical amplitudes. For the gravitational exchange considered here the same pole must couple to

matter in mixed correlators and the response to a generic conserved source must have no additional massless scalar residue.

The helicity step identifies the physical polarisation content of the covariant pole. Factorisation then tests whether the same polarisation space and source vertex occur for every matter state while the soft limit tests the resulting universal charge. These stages supply the polarisation and coupling data beyond the scalar residue.

The trace sector matters for coupling to ordinary matter. Between generic conserved sources a massless graviton in four dimensions has numerator

$$\mathcal{P}_{\mu\nu,\rho\sigma}^{\text{GR}} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma} + \text{terms proportional to } p_\mu, \quad (5.22)$$

where the last terms vanish on conserved sources. The nonzero-mass TT projector instead contains the coefficient $-1/3$ in four dimensions. Their contractions agree on traceless sources but not on general matter. The full gravitational tensor structure therefore involves the Ward identities and scalar sector as well as the pole in the matched response. Their combined massless amplitude has the Einstein form when its numerator is Eq. (5.22).

Factorisation across matter channels. Let a and b label matter states or gauge-invariant matter operators. A force interpretation requires the connected mixed amplitude to factorise at the same pole

$$\mathcal{A}_{ab}(p) = \frac{\mathcal{V}_a^{\mu\nu}(p)\mathcal{P}_{\mu\nu,\rho\sigma}^{\text{phys}}(p)\mathcal{V}_b^{\rho\sigma}(-p)}{-p^2 - i0} + \mathcal{A}_{ab}^{\text{reg}}(p), \quad (5.23)$$

with positive physical residue. This factorisation is a statement about mixed source derivatives of the same generating functional. The common denominator identifies a shared singular scale. The common physical projector and the two source vertices identify one exchanged channel.

If the helicity and soft-coupling conditions hold denote the canonically normalised field by $\gamma_{\mu\nu}^{\text{can}}$. The nonlinear effective metric and background source $h_{\mu\nu}$ of Section 3.3 have their respective roles. Choose the conventional normalisation

$$\mathcal{L}_{\text{int}} = -\frac{\kappa_{\text{phys}}}{2}\gamma_{\mu\nu}^{\text{can}}T_{\text{matter}}^{\mu\nu}. \quad (5.24)$$

This is the convention $\tilde{\Phi}_{\mu\nu} = -(\kappa_{\text{phys}}/2)\gamma_{\mu\nu}^{\text{can}}$ for the tensor vertex in Eq. (5.1). Then for conserved sources

$$\mathcal{A}_{ab}(p) \underset{p^2 \rightarrow 0}{\sim} \frac{\kappa_{\text{phys}}^2}{4} \frac{T_a^{\mu\nu}T_{b,\mu\nu} - \frac{1}{2}T_a T_b}{-p^2 - i0}, \quad \kappa_{\text{phys}}^2 = 32\pi G_N. \quad (5.25)$$

This equation defines G_N in the stated convention. A rescaling of the interpolating operator changes its Wightman weight $Z_0^{(\text{resp})}$ and compensates that change in the matter vertices. The mixed residue in Eq. (5.23) and hence G_N is invariant.

5.4 The Weinberg–Witten obstruction

The Weinberg–Witten theorem separates the spectral and particle questions. The positive vacuum measure concerns the states created by the operator whereas the theorem tests whether an ordinary massless particle can carry the translation charge measured by the same covariant stress tensor. In the form relevant here a positive TT atom interpreted as an asymptotic massless helicity-two particle in the same positive-metric Hilbert space is excluded whenever a Lorentz-covariant conserved stress tensor carries its nonzero four-momentum and the remaining hypotheses on states and matrix elements hold [15]. The conclusion is independent of whether the proposed particle is elementary or composite.

This particle obstruction and the vacuum spectral statement test different matrix elements. For the covariant stress-tensor input class of Section 2.3 the null-shell calculation gives zero TT atomic weight directly. For an independently constructed response the corresponding particle test uses its stress-tensor matrix elements and translation charge together with the remaining hypotheses of the standard theorem.

For any other response applicability is determined by its infrared states and the Lorentz covariance of their stress-tensor matrix elements together with the conserved translation charge acting on them including the forward matrix-element limit. Its infrared realisation therefore specifies the state space and charge together with the matrix elements that replace any standard-particle hypothesis. When ordinary asymptotic states are absent the spectral long-range response is still defined while Weinberg’s S -matrix soft theorem applies after an infrared scattering framework has been constructed. These restrictions are discussed for emergent gravity in ref. [33]. The corresponding higher-spin bounds and their relation to soft theorems are given in refs. [34, 35].

5.5 Soft universality and the Einstein form

Assume now that an infrared scattering framework exists and that the same positive-norm helicity-two pole factorises in amplitudes involving every matter species. Lorentz invariance and decoupling of spurious polarisations imply the leading soft identity. Weinberg’s argument then forces the coupling to be proportional to each external four-momentum and hence universal [16]. This supplies universality at leading soft order. Its extension to a local low-energy theory uses the consistency of the nonlinear gauge symmetry.

Once the exchanged channel has the helicity and universal soft structure of a massless spin-two field the remaining local question is the nonlinear completion of its gauge symmetry. For a local derivative expansion with ghost-free propagation and a single interacting massless spin-two field with at most two derivatives the consistent Poincaré-invariant completion is Einstein gravity up to field redefinitions and boundary terms [17, 18]. The low-energy action may therefore be written

$$S_{\text{IR}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-\gamma} (R[\gamma] - 2\Lambda) + S_{\text{matter}}[\gamma, \Psi] + \sum_i c_i \mathcal{I}_i^{(4+)}, \quad (5.26)$$

where $\mathcal{I}_i^{(4+)}$ are local higher-derivative operators after additional massive fields have been integrated out. Light matter fields are retained in S_{matter} . The vacuum density and

higher-derivative terms are fixed by additional coefficients of the effective action. An exact Minkowski background satisfies the vanishing renormalised tadpole condition including the cosmological term.

The chain from the spectral response to this effective action is now explicit. A positive TT atom supplies the long-range channel of Proposition 5.2. The helicity and conserved-source numerator select the massless spin-two exchange. Mixed-amplitude factorisation and the soft identity make its coupling universal. When the physical realisation lies outside an applicable Weinberg–Witten hypothesis and admits the local ghost-free single-field two-derivative completion used above the resulting infrared action is Eq. (5.26).

The pole and its helicity content determine the propagating field. The matter vertex and conserved-source numerator fix its exchange normalisation. A common metric–matter action carries these data into the nonlinear gauge identity and constraints with the same translation charge and cosmological normalisation.

5.6 A normalised Einstein–matter realisation

At leading two-derivative order the completion above can be normalised so that the same coefficients govern the spectral pole and matter vertex together with the macroscopic metric. The compact spectral system continues to fix the internal channel and its source normalisation. For $g_\Phi \neq 0$ introduce a metric $\gamma_{\mu\nu}$ and a positive coefficient M_*^2 by

$$\gamma_{\mu\nu} = \eta_{\mu\nu} - 2g_\Phi \Phi_{\mu\nu}, \quad M_*^2 = Z/g_\Phi^2. \quad (5.27)$$

We retain signature $(+---)$ and the curvature convention $R = -\gamma^{\mu\nu} \mathcal{R}_{\mu\nu}$. For real matter fields φ^A with field-space metric δ_{AB} the Einstein–scalar action is

$$\begin{aligned} S[\gamma, \varphi] &= -\frac{M_*^2}{2} \int d^4x \sqrt{-\gamma} \gamma^{\mu\nu} \mathcal{R}_{\mu\nu}(\gamma) + S_m[\gamma, \varphi], \\ S_m[\gamma, \varphi] &= \int d^4x \sqrt{-\gamma} \left[\frac{1}{2} \gamma^{\mu\nu} \delta_{AB} \partial_\mu \varphi^A \partial_\nu \varphi^B - V(\varphi) \right]. \end{aligned} \quad (5.28)$$

Assume $V(0) = \partial_A V(0) = 0$ and a nonnegative mass matrix $\partial_A \partial_B V(0)$. These conditions select the flat stationary matter background and a nonnegative quadratic matter spectrum. Define the Hilbert source by

$$T^{\mu\nu} = -\frac{2}{\sqrt{-\gamma}} \frac{\delta S_m}{\delta \gamma_{\mu\nu}}. \quad (5.29)$$

All species couple to the same metric. Other covariant matter functionals obey the corresponding Noether identity with their tensor or spinor Euler derivatives.

The family replication of Ref. [20] gives a concrete quadratic source. At zero Higgs background $K_{n_f}[h] = I_{n_f} \otimes K_1[h]$ on the common source-dependent domain so $W_{n_f}[h] = n_f W_1[h]$ up to a source-independent constant and

$$\partial_a \partial_b W_{n_f} = n_f (\langle S_{1,a} S_{1,b} \rangle_c - \langle S_{1,ab} \rangle), \quad (5.30)$$

where the derivatives include the domain and measure identification. The connected quadratic response and its direct term together with its TT compression thus carry the same family multiplicity. The compact representation and its three-family application are given in refs. [1, 20].

Classical completion and the cubic identity. The action (5.28) has quadratic tensor part (4.32) and leading matter vertex $g_\Phi \int \Phi_{\mu\nu} T_0^{\mu\nu} d^4x$ where T_0 is the flat Hilbert tensor. For compactly supported gauge parameters its cubic expansion obeys

$$\delta_0 S_3 + \delta_1 S_2 = 0, \quad (5.31)$$

with the matter and tensor terms included in each S_n . The full classical equations obey the corresponding off-shell Noether identity and propagate the gravitational constraints. Writing $h_{\mu\nu} = \gamma_{\mu\nu} - \eta_{\mu\nu}$ the quadratic curvature action is $M_*^2/4$ times the Fierz–Pauli bracket. Substitution of $h = -2g_\Phi \Phi$ gives $M_*^2 g_\Phi^2 = Z$ while $\delta S_m = -\frac{1}{2} \int T_0^{\mu\nu} \delta \gamma_{\mu\nu} d^4x$ gives the matter vertex with the same sign and normalisation. Deser’s self-coupling construction supplies Eq. (5.31) and the corresponding nonlinear gauge identity [17].

More generally define $\mathcal{E}^{\mu\nu} = (\sqrt{-\gamma})^{-1} \delta S / \delta \gamma_{\mu\nu}$ and $\mathcal{E}_A = (\sqrt{-\gamma})^{-1} \delta S / \delta \varphi^A$. For the displayed matter action these are

$$\begin{aligned} T_{\mu\nu} &= \delta_{AB} \partial_\mu \varphi^A \partial_\nu \varphi^B - \gamma_{\mu\nu} \left(\frac{1}{2} \gamma^{\rho\sigma} \delta_{AB} \partial_\rho \varphi^A \partial_\sigma \varphi^B - V \right), \\ \mathcal{E}_A &= -\square_\gamma \varphi_A - \partial_A V. \end{aligned}$$

Integrating the metric Lie derivative by parts gives

$$2\nabla_\mu \mathcal{E}^\mu{}_\nu = \mathcal{E}_A \partial_\nu \varphi^A, \quad 2\mathcal{E}^{\mu\nu} = M_*^2 \mathcal{G}^{\mu\nu} - T^{\mu\nu}. \quad (5.32)$$

Here $\mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \gamma_{\mu\nu} \gamma^{\rho\sigma} \mathcal{R}_{\rho\sigma} / 2$. On the matter equations the metric Euler derivative is covariantly conserved. Its lapse and shift equations are the four gravitational constraints. The identity supplies their homogeneous propagation equations. Their preservation with the four diffeomorphism freedoms leaves two gravitational configuration degrees of freedom near the flat vacuum. In particular the quadratic pole and matter normalisation are those of Eq. (4.33). The action fixes the source and propagation operator of the classical effective realisation. Eq. (4.18) fixes its compact critical coordinate. The metric and matter fields are varied independently in this effective realisation. When a generated construction prescribes relations between these fields variation is taken after pulling the action back to the admissible generated variables. Finite rational evaluations of the density identities are given in Ref. [22].

On an asymptotically flat Cauchy surface the differentiable translation generator is

$$H[N, N^i] = \int_\Sigma (N \mathcal{H} + N^i \mathcal{H}_i) d^3x + B_\infty[N, N^i]. \quad (5.33)$$

Compact lapse and shift generate gauge transformations. For an asymptotic time translation the energy on the constraint surface is the boundary charge. With $\gamma_{ij}^{(3)} = \delta_{ij} + s_{ij}$

$$P_{\text{ADM}}^0 = \frac{M_*^2}{2} \lim_{r \rightarrow \infty} \int_{S_r} (\partial_j s_{ij} - \partial_i s_{jj}) n^i dS. \quad (5.34)$$

The Hilbert source (5.29) supplies the matter contribution while Eq. (5.34) is the translation charge on the physical constraint space [36]. This is the charge realisation entering the Weinberg–Witten comparison.

Cosmological and stationary limits. The same action determines the background and its constraints together with the response about that background. Allow a constant vacuum density by adding $-\int \sqrt{-\gamma} \rho_\Lambda d^4x$ to Eq. (5.28). Let ρ and p_{mat} be the energy density and isotropic pressure of the remaining covariant matter action. On a spatially flat FLRW solution

$$3M_*^2 H^2 = \rho + \rho_\Lambda, \quad -2M_*^2 \dot{H} = \rho + p_{\text{mat}}, \quad \dot{\rho} + 3H(\rho + p_{\text{mat}}) = 0. \quad (5.35)$$

The last equation is the matter Noether identity. Together with the second equation it propagates the first including through $H = 0$.

For tensor propagation let the background solve Eq. (5.35) with constant $M_*^2 > 0$ and write

$$ds^2 = dt^2 - a(t)^2 (e^h)_{ij} dx^i dx^j, \quad \partial_i h_{ij} = h_{ii} = 0.$$

For unit-norm spatial polarizations set $h_{ij} = \sum_{\lambda=1}^2 h_\lambda e_{ij}^\lambda$ and $e_{ij}^\lambda e_{ij}^{\lambda'} = \delta_{\lambda\lambda'}$. The quadratic tensor action and its linear anisotropic-stress vertex are

$$\begin{aligned} S_T^{(2)} &= \frac{M_*^2}{8} \sum_\lambda \int a^3 \left[\dot{h}_\lambda^2 - a^{-2} |\nabla h_\lambda|^2 \right] dt d^3x \\ &\quad + \frac{1}{2} \sum_\lambda \int a^3 h_\lambda \Pi_\lambda dt d^3x, \end{aligned} \quad (5.36)$$

where Π_{ij} is the physical transverse traceless anisotropic stress and $\Pi_\lambda = e_{ij}^\lambda \Pi_{ij}$. Consequently

$$\ddot{h}_\lambda + 3H\dot{h}_\lambda - a^{-2} \Delta h_\lambda = \frac{2}{M_*^2} \Pi_\lambda. \quad (5.37)$$

In the flat background with $a = 1$ the substitution $h_\lambda = 2g_\Phi \phi_\lambda$ recovers the reduced action and residue in Eq. (4.33). For stationary weak perturbations of the flat background with nonrelativistic sources and negligible anisotropic stress the same metric equations give

$$\Delta \Psi = \frac{\rho}{2M_*^2} = 4\pi G_N \rho, \quad G_N = \frac{1}{8\pi M_*^2} = \frac{g_\Phi^2}{8\pi Z}. \quad (5.38)$$

The ADM constraint reduction gives the positive tensor kinetic term and for a decaying stationary solution with negligible anisotropic stress $\Psi = \Upsilon$. The Green function is therefore $-G_N \int \rho(\mathbf{y}) |\mathbf{x} - \mathbf{y}|^{-1} d^3y$ with leading interaction $-G_N E_a E_b / r$. Thus the tensor action and the static limit use the same M_*^2 while the retarded tensor equation continues the flat-background spectral pole.

Constant condensate and cold source. A radial scalar and an irrotational pressureless current test the same effective action in its scalar and cold-matter sectors. With the conventions of Eq. (5.28) take

$$S_{r\Sigma} = \int d^4x \sqrt{-\gamma} \left[\frac{1}{2} \gamma^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - U(\varphi) + \frac{\varrho}{2} (\gamma^{\mu\nu} \partial_\mu T \partial_\nu T - 1) \right]. \quad (5.39)$$

The multiplier ϱ imposes a unit timelike velocity $u_\mu = \partial_\mu T$. Its current equation is $\nabla_\mu (\varrho u^\mu) = 0$ and its stress is $T_{\mu\nu}^\Sigma = \varrho u_\mu u_\nu$. Its normalisation and gradient form imply $u^\mu \nabla_\mu u_\nu = 0$. This is the irrotational sector of the canonical dust description [37].

Thus T is the velocity potential of the matter flow and $T = t$ selects proper-time coordinates adapted to that flow. For a generated matter phase such a unit-gradient clock exists locally when the normalised phase one-form is closed [20, Proposition “Congruence of a generated matter phase”]. The phase-clock construction of ref. [1] Section 10 calibrates timelike translations through their energy-dependent phase. The velocity potential here is the flow-adapted clock of this particular matter realisation.

Let $U(\varphi_*) = U'(\varphi_*) = 0$ and $U''(\varphi_*) = m_\delta^2 > 0$. The constant vacuum energy is included in ρ_Λ of Eq. (5.35). For the compact scalar of Ref. [1] these conditions follow from $F_y(\sigma_*, y_*) = 0$ and $Z_* > 0$ with $m_\delta^2 = 4y_* F_{yy}(\sigma_*, y_*)/Z_*$. Coupling Eq. (5.39) to the Einstein action and the constant density ρ_Λ gives on a spatially flat FLRW solution with $T = t$ and $\varphi = \varphi_*$ and $\varrho = \rho(t) > 0$

$$3M_*^2 H^2 = \rho + \rho_\Lambda, \quad -2M_*^2 \dot{H} = \rho, \quad \dot{\rho} + 3H\rho = 0. \quad (5.40)$$

Consider a real nonzero spatial Fourier mode of physical wavenumber $\kappa = |\mathbf{k}|/a$ on an interval with $H \neq 0$. In scalar spatially flat gauge write $N = 1 + n$ and $T = t + \pi_T$ together with $\varrho = \rho + d$ and $b = \partial_i N^i$. Before the auxiliary variables are eliminated the quadratic Lagrangian per unit a^3 is

$$\begin{aligned} \mathcal{L}_2 = & -3M_*^2 H^2 n^2 - 2M_*^2 H n b + \frac{1}{2} \dot{\delta\varphi}^2 - \frac{1}{2} (\kappa^2 + m_\delta^2) \delta\varphi^2 \\ & + d(\dot{\pi}_T - n) + \frac{\rho}{2} (\dot{\pi}_T - n)^2 - \frac{\rho}{2} \kappa^2 \pi_T^2 + \rho b \pi_T. \end{aligned} \quad (5.41)$$

The shift equation gives $n = B\pi_T$ and the lapse equation fixes b . The cold canonical momentum is $\pi_\rho = d + \rho(\dot{\pi}_T - n)$. Preservation of the primary multiplier constraint sets $d = \pi_\rho$. After substituting the lapse and shift solutions and integrating $a^3 \pi_\rho \dot{\pi}_T$ by parts the remaining π_T dependence is $-\left[\dot{\pi}_\rho + (3H + B)\pi_\rho\right]\pi_T - D\pi_T^2/2$. Its algebraic equation yields the reduced quadratic action

$$\begin{aligned} S_{r\Sigma}^{(2)} = & \int dt a^3 \left\{ \frac{1}{2} \dot{\delta\varphi}^2 - \frac{1}{2} (\kappa^2 + m_\delta^2) \delta\varphi^2 + \frac{[\dot{\pi}_\rho + (3H + B)\pi_\rho]^2}{2D} \right\}, \\ B = & \frac{\rho}{2M_*^2 H}, \quad D = \rho\kappa^2 + \frac{3\rho^2}{2M_*^2} > 0. \end{aligned} \quad (5.42)$$

The kinetic matrix $\text{diag}(1, D^{-1})$ is positive for $M_*^2, \rho > 0$. The radial mode has unit propagation speed and mass m_δ whereas the cold mode has vanishing intrinsic short-wavelength sound speed. In the regime $\kappa^2 \gg \rho/M_*^2$ writing $\pi_\rho = \rho\delta$ gives the standard growing-density equation $\ddot{\delta} + 2H\dot{\delta} - \rho\delta/(2M_*^2) = 0$ at leading order.

The dynamical-amplitude realisation of Ref. [20, Appendix G.6, Proposition “Congruence of a generated matter phase” and Proposition “Occupied common background and physical propagation”] contains a matter pair carrying a phase and an auxiliary collective radius. Its regular algebraic radius equation is solved in the common action and the lapse and shift constraints are removed from its quadratic variation. The resulting physical scalar kinetic and gradient matrices are positive. Their characteristic speeds and the phase-clock rate have explicit positive bounds on a finite interval. This gives a coupled realisation in which matter evolution and the collective amplitude together with the auxiliary radius satisfy the same source balance.

6 Conclusions

A tensor pole contributes to matter exchange through the source that creates it and the observable that reads it. On an isolated critical subspace the propagation slope and projected source maps determine the residue. Exact channel elimination preserves the complete response. Common-space estimates control the residue at the shifted critical root of a finite approximation. Positive spectral matching identifies this coefficient with the weight of an invariant-mass atom.

The null-shell theorem classifies the complete microscopic stress tensor. Its positive Lorentz-covariant conserved measure is longitudinal on the massless orbit and has zero TT atomic weight. The reduced two-helicity response has its own physical state space and inherits its source maps from the effective spectral realisation. The compact construction gives an internal spectral zero and explicit matter vertices for this response. Its charged trace fixes a response domain and endpoint. The complete matter action supplies the conserved energy vertex. The gapped radial example gives a different spectral outcome. Its visible positive-mass mode has finite static susceptibility. In both cases source overlap fixes the measured weight.

For localised stationary matter conservation and Laue’s identity reduce the leading long-wavelength vertex to the total energy including binding contributions. A common critical direction then gives an energy-factorised mixed residue. Spatial resolvent estimates determine the leading potential and force with controlled finite-size corrections. A collective response with the required helicity content and universal soft coupling admits the Einstein–matter completion under the stated local nonlinear consistency assumptions. The effective geometry represents the macroscopic collective tensor.

A Gaussian averaging and spectral atoms

Gaussian resolution smooths an already defined response while preserving the weight of its infrared atom. The characteristic exponent fixes the averaging law and its Fourier multiplier gives the inverse on the range and the identity (3.12).

Semigroup characterisation. Let $\varphi_\tau(q) = \int e^{iq \cdot x} d\mu_\tau(x)$. The convolution law gives

$$\varphi_{\tau_1+\tau_2}(q) = \varphi_{\tau_1}(q)\varphi_{\tau_2}(q), \quad \varphi_0(q) = 1. \quad (\text{A.1})$$

Orthogonal invariance includes reflection and makes φ_τ real. For every integer n the semigroup identity gives $\varphi_\tau(q) = \varphi_{\tau/(2n)}(q)^{2n} \geq 0$. A zero at finite τ would give zeros arbitrarily close to $\tau = 0$ by the semigroup law. Weak continuity excludes this possibility. Hence

$$\varphi_\tau(q) = e^{-\tau\psi(q)} \quad (\text{A.2})$$

with a continuous nonnegative ψ .

Square-root self-similarity also gives $\varphi_\tau(q) = \varphi_1(\sqrt{\tau}q)$ so $\psi(rq) = r^2\psi(q)$ for $r > 0$. Orthogonal invariance implies that ψ depends only on $|q|$ and quadratic homogeneity then

yields $\psi(q) = D|q|^2$. The covariance is

$$\int x_i x_j d\mu_\tau(x) = 2D\tau \delta_{ij}, \quad (\text{A.3})$$

so the nonzero second moment gives $D > 0$ and fixes D once the scale convention is chosen. For $\tau > 0$ Fourier inversion gives

$$d\mu_\tau(x) = \frac{d^d x}{(4\pi D\tau)^{d/2}} \exp\left[-\frac{|x|^2}{4D\tau}\right]. \quad (\text{A.4})$$

Finite resolution and the infrared germ. The operator $\mathcal{H}_\tau$ on $L^2(\mathbb{R}^d)$ acts by Fourier multiplication by $e^{-\tau Q^2}$. For finite τ the multiplier is strictly positive so

$$\text{Ker } \mathcal{H}_\tau = \{0\}. \quad (\text{A.5})$$

Its inverse has multiplier $e^{\tau Q^2}$ and is unbounded for every $\tau > 0$. Fourier data supported on $n < |Q| < n+1$ are amplified by at least $e^{\tau n^2}$. The same local Fourier argument gives injectivity on tempered distributions. Thus finite Gaussian resolution loses stability at high momentum while retaining all Fourier directions.

For the infrared statement let F be meromorphic near zero and let w be analytic with $w(0) = 1$. If

$$F(z) = \sum_{j=-m}^{\infty} a_j z^j, \quad a_{-m} \neq 0, \quad (\text{A.6})$$

then multiplication by w preserves the pole order and leading coefficient. Taking $w(z) = e^{-\tau z}$ proves the meromorphic case of Eq. (3.12). Its direct limit also covers an atom at a continuous threshold. Since the same multiplication sends a contact polynomial to the generally nonpolynomial entire function $e^{-\tau Q^2} P_2(Q^2)$ contact subtraction is performed first. Positivity and the Stieltjes representation therefore refer to the unfiltered response.

B The TT quotient, contact terms, and spectral measure

The physical source quotient is formed before the massless limit. Local counterterms and changes of stress-tensor representative then separate from the nonlocal measure whose timelike disintegration and null-shell restriction give Eq. (2.43) without assigning a massive projector to the null orbit.

B.1 Renormalisation and the physical source quotient

Multiple stress-tensor insertions are renormalised by allowing local counterterms in the background source. In four dimensions their purely geometric part has the schematic form

$$S_{\text{ct}}[g] = \int d^4x \sqrt{g} (c_0 + c_1 R + c_2 R^2 + c_3 R_{\mu\nu} R^{\mu\nu} + c_4 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \dots). \quad (\text{B.1})$$

Two metric derivatives at $g_{\mu\nu} = \delta_{\mu\nu}$ are tensor polynomials in Q and contribute only to $P_2(Q^2)$. They have neither a positive-mass discontinuity nor a $1/Q^2$ term. Matter-dependent local counterterms have the same status after operator mixing is closed. An

anomaly-free source Ward identity makes the nonlocal correlator transverse whereas a trace anomaly changes the trace and local terms.

For operators with the same exact quantum numbers write $\mathcal{O}_{R,i} = Z_{ij}\mathcal{O}_{B,j}$ and $h_{B,i} = Z_{ji}h_{R,j}$ and form the physical quotient only after this mixing problem is closed. BRST-exact and renormalised EOM insertions then give their standard separated-point contact identities [38–40]. The quotient is formed at nonexceptional momentum before continuation to the infrared limit. Likewise the scalar improvement $I_{\mu\nu} = (\partial_\mu\partial_\nu - \eta_{\mu\nu}\square)X$ has $J^{\mu\nu}I_{\mu\nu} = 0$ for every conserved traceless source. It may alter the trace and contact terms but leaves the TT response unchanged.

For a finite physical operator family bold symbols denote its matrix-valued Wightman measure and atom. These are distinct from the scalar orbit measure $d\rho_2$ and from a source-family pushforward. Under a finite nonsingular change of physical operator basis $\mathcal{O}' = M\mathcal{O}$ the matrix-valued measure and its atom transform by congruence

$$d\boldsymbol{\rho}'_2 = M d\boldsymbol{\rho}_2 M^\dagger, \quad \mathbf{Z}'_0 = M \mathbf{Z}_0 M^\dagger. \quad (\text{B.2})$$

Thus positivity and rank are invariant under nonsingular basis changes.

B.2 Spectral resolution on physical sources

Let $E(dp)$ be the joint projection-valued measure of translations. For a finite family of TT sources J_i define

$$dM_{ij}(p) := \langle \Omega | \Theta(J_i)^\dagger E(dp) \Theta(J_j) | \Omega \rangle. \quad (\text{B.3})$$

For every $c_i \in \mathbb{C}$ and nonnegative Borel function f

$$\sum_{i,j} \bar{c}_i \int f(p) dM_{ij}(p) c_j \geq 0. \quad (\text{B.4})$$

The spectral condition restricts the support to the closed forward cone. This matrix is the source-smeared image of one common tensor-valued tempered measure. The source-independent scalar measure is obtained by restricting the common tensor measure to the open timelike cone. The map $p \mapsto s = p^2$ decomposes that cone into the Lorentz orbits H_s^+ . After the orbit measure $d\Omega_s$ is fixed uniqueness of disintegration leaves a matrix coefficient transforming under the stabiliser $SO(3)$. Conservation removes time components in the rest frame and

$$\text{Sym}^2(\mathbb{R}^3) = \text{Sym}_0^2(\mathbb{R}^3) \oplus \mathbb{R}. \quad (\text{B.5})$$

The two summands are inequivalent irreducible representations. Schur's lemma therefore makes the measure scalar on each block and forbids a mixed block. These two positive coefficients are precisely the common measures $d\rho_2$ and $d\rho_0$ in Eq. (2.30). The chosen normalisation of $d\Omega_s$ fixes their uniqueness independently of the test source.

For a family of TT sources the spin-zero block vanishes and the remaining matrix pushforward is $d\rho_{ab}^J(s) = q_{ab}^{(2)}(s)d\rho_2(s)$ on $(0, \infty)$ as in Eq. (2.32). Changing the test family therefore changes $q^{(2)}$ while preserving the common scalar measure. Polarisation of diagonal quadratic forms recovers all off-diagonal entries.

The null orbit has a separate stabiliser and requires its own measure classification. For the complete covariant tensor it is longitudinal and vanishes on conserved sources. Positivity and local finiteness of its $00, 00$ component near any nonzero null momentum yield Eq. (2.39). Assigning zero weight to $\{0\}$ therefore extends ρ_2 to a positive locally finite measure.

To reconstruct the time-ordered distribution continue each timelike shell using the polynomial numerator built from $\eta_{\mu\nu} - p_\mu p_\nu / s$ at fixed $s > 0$. Two polynomial extensions that agree on the shell differ by a factor $p^2 - s$ which cancels the propagator and leaves local tensor terms. The infrared inverse moment above controls the s^{-2} coefficients. UV divergences are removed by the fixed local subtractions. External TT contraction of this reconstruction kills the spin-zero and null-shell pieces and selects the timelike spin-two coefficient. A possible origin measure is removed on traceless test sources before any momentum projection. Wick rotation then gives Eq. (2.43) whose nonlocal Lorentzian discontinuity recovers $d\rho_2$. The commutator is the positive-energy measure minus its negative-energy reflection and determines the retarded response. Positivity in the atom criterion belongs to the positive-energy Wightman measure.

B.3 The little group on the null shell

Let $G = SO^+(1, 3)$ and $E = \text{Sym}^2(\mathbb{C}^4)$ and denote by D the induced finite-dimensional tensor representation of G on E . The following proposition supplies the orbit statement used in Theorem 2.3.

Proposition B.1 (Positive covariant measures on the null orbit). Let M be a positive $\text{End}(E)$ -valued Radon measure on H_0^+ such that for every Borel set $A \subset H_0^+$

$$M(gA) = D(g)M(A)D(g)^\dagger, \quad g \in G, \quad (\text{B.6})$$

and suppose that the two Ward identities Eq. (2.27) hold. Then M has a smooth positive density relative to $d\Omega_0$ and

$$dM_{\mu\nu, \rho\sigma}(p) = \alpha_{\text{null}} p_\mu p_\nu p_\rho p_\sigma d\Omega_0(p), \quad \alpha_{\text{null}} \geq 0. \quad (\text{B.7})$$

If a positive covariant measure supported at the origin is added it has the form

$$dM_{\mu\nu, \rho\sigma}^{(0)}(p) = \beta_0 \eta_{\mu\nu} \eta_{\rho\sigma} \delta^{(4)}(p), \quad \beta_0 \geq 0. \quad (\text{B.8})$$

Proof. Orbit regularity. Fix a positive Hermitian inner product on E and introduce the scalar control measure $\tau(A) = \text{tr } M(A)$. If $\tau(A) = 0$ positivity gives $M(A) = 0$. The scalar entries of M are therefore absolutely continuous with respect to τ . The Radon–Nikodym theorem gives $M = K d\tau$ with $K(p) \succeq 0$ for τ -almost every p . For fixed g the smallest and largest eigenvalues of $D(g)^\dagger D(g)$ give positive constants m_g, M_g such that

$$m_g \tau(A) \leq \tau(gA) \leq M_g \tau(A). \quad (\text{B.9})$$

Thus τ is quasi-invariant. The uniqueness of the quasi-invariant measure class on the homogeneous space $H_0^+ \simeq G/E(2)$ makes every nonzero τ equivalent to the standard

invariant orbit measure $d\Omega_0$ [41, ch. 2]. If $\tau = 0$ the result already holds with $\alpha_{\text{null}} = 0$. Otherwise $M = H d\Omega_0$ for a locally integrable positive matrix density H .

Define the compensated action on tensor-valued measures by

$$(\Pi(g)M)(A) := D(g)M(g^{-1}A)D(g)^\dagger. \quad (\text{B.10})$$

Equation (B.6) says that $\Pi(g)M = M$. If $\varphi \in C_c^\infty(G)$ has Haar integral one then $\Pi(\varphi)M := \int_G \varphi(g)\Pi(g)M dg = M$. The action map $G \times H_0^+ \rightarrow H_0^+$ is a submersion so in a local section this is the usual convolution of a distribution with a compactly supported smooth function. The convolution-smoothing theorem [41, 42] therefore gives a smooth density. Thus H has a representative satisfying

$$H(gp) = D(g)H(p)D(g)^\dagger \quad (\text{B.11})$$

pointwise. This establishes regularity along the null orbit. The transverse invariant-mass dependence remains encoded in the scalar spectral measure.

The little group and the Ward identity. Choose a null tetrad $\{\ell, n, e_1, e_2\}$ with $\ell \cdot n = 1$ and $e_i \cdot e_j = -\delta_{ij}$. All other inner products vanish. At $p = \ell$ factor the positive matrix as $H(\ell) = VV^\dagger$ with V of full column rank. For an element h of the little group Eq. (B.11) implies $D(h)V = VU(h)$. Multiplication by a left inverse of V shows that $U(h)U(h)^\dagger = I$ and uniqueness makes U a representation of the little group. A null rotation is unipotent in the finite tensor representation D . Its restriction to $\text{Ran } V$ is therefore both unipotent and unitary and hence is the identity.

To determine the common fixed space expand a symmetric contravariant tensor as

$$\begin{aligned} u = & A\ell\ell + Bnn + C(\ell n + n\ell) + \sum_i D_i(\ell e_i + e_i\ell) \\ & + \sum_i E_i(ne_i + e_i n) + \sum_{i,j} F_{ij}e_i e_j, \quad F_{ij} = F_{ji}. \end{aligned} \quad (\text{B.12})$$

Here juxtaposition denotes the ordinary tensor product with both orders written explicitly for mixed symmetric terms. Apply the relations $N_k\ell = 0$ and $N_k n = e_k$ together with $N_k e_i = \delta_{ki}\ell$ to both indices. The coefficients of ne_k and $e_k e_i$ together with those of $\ell\ell$ and $\ell e_i + e_i\ell$ in $N_k u = 0$ give

$$B = 0, \quad E_i = 0, \quad D_i = 0, \quad F_{ij} = -C\delta_{ij}. \quad (\text{B.13})$$

The null-rotation fixed tensors are therefore $u = A\ell\ell + C\eta$.

Let $C_\ell : E \rightarrow \mathbb{C}^4$ be contraction with ℓ in the first index. The measure Ward identity and $M = H d\Omega_0$ first give $C_p H(p) = 0$ almost everywhere. Smoothness promotes this identity to every point of the orbit. In particular

$$0 = C_\ell H(\ell)C_\ell^\dagger = (C_\ell V)(C_\ell V)^\dagger, \quad (\text{B.14})$$

so $C_\ell V = 0$. Conservation therefore holds for every column of V and removes the metric term from its null-rotation invariant form. Hence $H(\ell) = \alpha_{\text{null}}|\ell\ell\rangle\langle\ell\ell|$ with $\alpha_{\text{null}} \geq 0$. Transitivity of G and Eq. (B.11) give Eq. (B.7) on the whole orbit.

The origin. A positive Radon measure supported at one point is $K\delta^{(4)}(p)$ for a positive matrix K . Derivatives of the delta distribution do not occur in a measure. Covariance gives $K = D(g)KD(g)^\dagger$. Factor $K = V_0V_0^\dagger$ on its range. The same argument as above produces a continuous finite-dimensional unitary representation U_0 of G on $\text{Ran } V_0$. Its differential is a homomorphism from the noncompact simple Lie algebra $\mathfrak{so}(1,3)$ to the compact Lie algebra $\mathfrak{u}(r)$. An injection would pull back a positive Ad-invariant inner product and make $\mathfrak{so}(1,3)$ compact. Noncompactness therefore excludes an injection. Simplicity then makes the differential zero and connectedness of G makes U_0 trivial. Rotational invariance sets the mixed components of each column of V_0 to zero and makes its spatial block scalar. Invariance under one boost fixes the remaining coefficient leaving a multiple of η . This proves Eq. (B.8). $\square$

This orbit classification is independent of the limiting massive source factors $q^{(2)}(s)$. The one-particle theorem of ref. [14] is its representation-theoretic analogue. Proposition B.1 is formulated directly for the Wightman measure.

B.4 Infrared limit as a quadratic-form statement

For several physical operators write $d\boldsymbol{\rho}_2$ and $\mathbf{C}_2$ for the matrix-valued invariant measure and response. These operator indices differ from the test-source indices of Eq. (B.3). The atom extraction is first established on each quadratic form $v^\dagger \mathbf{C}_2 v$. With

$$d\nu_v(s) = \frac{v^\dagger d\boldsymbol{\rho}_{2,c}(s)v}{(s + Q_*^2)^{n_{\text{sub}}}}, \quad (\text{B.15})$$

the dominated-convergence argument of Proposition 2.4 applies to each quadratic form. Polarisation of the resulting quadratic-form limit gives the matrix limit. Positivity belongs to the physical matrix quadratic form. Individual off-diagonal coefficients need not be positive measures.

Local integrability at the origin is essential to the dominated-convergence argument. Without this integrability the infrared limit requires an estimate of Tauberian or limiting-absorption type. For the covariant tensor considered here Eq. (2.39) supplies the required integrability.

C Composite-source flow and connected response

The Schur identity differentiates the source together with the propagating variables and observable in one generating functional. The Legendre transform and source derivatives fix the signs of the reduced source and observable maps in Section 3.3.

C.1 Source derivatives of the Wetterich equation

Condense the symmetric tensor source and its spacetime argument into an index i and define

$$A_i := \frac{\delta \Gamma_k^{(2,0)}}{\delta h_i} = \Gamma_{k,i}^{(2,1)}, \quad A_{ij} := \frac{\delta^2 \Gamma_k^{(2,0)}}{\delta h_i \delta h_j} = \Gamma_{k,ij}^{(2,2)}. \quad (\text{C.1})$$

For a source-independent regulator $\delta_i G_k = -G_k A_i G_k$. Applying this identity twice to the Wetterich equation gives

$$\partial_t \Gamma_{k,ij}^{(0,2)} = \frac{1}{2} \text{STr} [G_k A_j G_k A_i G_k \partial_t R_k + G_k A_i G_k A_j G_k \partial_t R_k - G_k A_{ij} G_k \partial_t R_k], \quad (\text{C.2})$$

with the superfield signs contained in the supertrace and vertex ordering. The first two terms contain separate insertions while A_{ij} is the renormalised coincident double insertion required by the contact and Ward identities. Their flows involve higher field vertices. Finite closures of this hierarchy are compared with the full response in Appendix D.

C.2 Legendre transform and resolvent response

Write

$$\Gamma_k[\varphi, h] = j \cdot \varphi - W_k[j, h] - \Delta S_k[\varphi], \quad \varphi = W_{k,j}. \quad (\text{C.3})$$

At fixed φ the first derivative is $\Gamma_{k,h} = -W_{k,h}$ and $\delta_h j = -W_{k,jj}^{-1} W_{k,jh}$ while $W_{k,jj} = G_k = (\Gamma_{k,\varphi\varphi} + R_k)^{-1}$. Differentiation therefore gives $\Gamma_{k,hh} = -W_{k,hh} + W_{k,hj} G_k^{-1} W_{k,jh}$. Using $\Gamma_{k,h\varphi} = -W_{k,hj} G_k^{-1}$ and its transposed mixed derivative yields

$$W_{k,hh} = -\Gamma_{k,hh} + \Gamma_{k,h\varphi} G_k \Gamma_{k,\varphi h}, \quad (\text{C.4})$$

The second term propagates through the fundamental-field sector. It vanishes when the mixed source-field Hessians vanish. The converse holds for a strictly positive propagator on the coupled sector.

At vanishing fundamental source the effective average field satisfies

$$\mathcal{E}_k(\varphi, h) = \Gamma_{k,\varphi} + R_k \varphi = 0. \quad (\text{C.5})$$

Choose an analytic invertible preconditioner $\mathcal{A}_{0,k}$ and rewrite this equation as the fixed-point problem

$$\varphi = \mathfrak{F}_k(\varphi; h), \quad \mathfrak{F}_k = \varphi - \mathcal{A}_{0,k}^{-1} \mathcal{E}_k. \quad (\text{C.6})$$

Linearising in a tensor-source variation gives the response operators in Eq. (3.20). In particular

$$\begin{aligned} (I - \mathcal{B}_k)^{-1} \mathbf{S}_k &= -\mathcal{A}_k^{-1} \Gamma_{k,\varphi h}, \\ R_k (I - \mathcal{B}_k)^{-1} \mathbf{S}_k &= \Gamma_{k,h\varphi} G_k \Gamma_{k,\varphi h}. \end{aligned} \quad (\text{C.7})$$

The preconditioner cancels from the response though the compactness and approximation estimates refer to the chosen kernel representation. Proposition 3.1 applies the same construction to enlarged response coordinates. For a finite projected hierarchy it gives the response of the closure with errors controlled as in Appendix D.

C.3 Complete channel reduction

For a renormalised block $\mathcal{O}_{R,i} = Z_{ij}\mathcal{O}_{B,j}$ the sources transform contragrediently so the flow produces a matrix response. Internal mixing survives the external TT projection. The renormalised block includes the BRST and EOM representatives required by its counterterms. The physical quotient is formed after renormalisation.

Let the complete response variables be $X = (X_P, X_{\bar{P}})$ where P is the physical TT block and $\bar{P} = I - P$ its complement. Linearisation gives

$$\begin{pmatrix} \delta X_P \\ \delta X_{\bar{P}} \end{pmatrix} = \begin{pmatrix} \mathcal{B}_{PP} & \mathcal{B}_{P\bar{P}} \\ \mathcal{B}_{\bar{P}P} & \mathcal{B}_{\bar{P}\bar{P}} \end{pmatrix} \begin{pmatrix} \delta X_P \\ \delta X_{\bar{P}} \end{pmatrix} + \begin{pmatrix} \mathcal{S}_P \\ \mathcal{S}_{\bar{P}} \end{pmatrix} \delta J. \quad (\text{C.8})$$

Write the full measured response as $\delta\mathcal{O} = C_{\text{dir}}\delta J + R_P\delta X_P + R_{\bar{P}}\delta X_{\bar{P}}$. Assume $D := I_{\bar{P}} - \mathcal{B}_{\bar{P}\bar{P}}$ is invertible. The reduced kernel and source map together with the observable map and direct term are

$$\begin{aligned} \mathcal{B}_{\text{eff}} &:= \mathcal{B}_{PP} + \mathcal{B}_{P\bar{P}}D^{-1}\mathcal{B}_{\bar{P}P}, \\ \mathcal{S}_{\text{eff}} &:= \mathcal{S}_P + \mathcal{B}_{P\bar{P}}D^{-1}\mathcal{S}_{\bar{P}}, \\ R_{\text{eff}} &:= R_P + R_{\bar{P}}D^{-1}\mathcal{B}_{\bar{P}P}, \\ C_{\text{dir,eff}} &:= C_{\text{dir}} + R_{\bar{P}}D^{-1}\mathcal{S}_{\bar{P}}. \end{aligned} \quad (\text{C.9})$$

Wherever the remaining inverse exists the full correlator is

$$\begin{aligned} C &= C_{\text{dir}} + R(I - \mathcal{B})^{-1}\mathcal{S} \\ &= C_{\text{dir,eff}} + R_{\text{eff}}(I_P - \mathcal{B}_{\text{eff}})^{-1}\mathcal{S}_{\text{eff}}. \end{aligned} \quad (\text{C.10})$$

Solving the second block equation and substituting it into the first and into $\delta\mathcal{O}$ gives Eqs. (C.9)–(C.10). Analyticity is preserved while D remains invertible and the remaining inverse has the Laurent expansion of Section 4 under its Fredholm hypotheses.

Equivalently before preconditioning write the linearised equations as

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad L \begin{pmatrix} x_P \\ x_{\bar{P}} \end{pmatrix} = \begin{pmatrix} S_P \\ S_{\bar{P}} \end{pmatrix} h, \quad \mathcal{O} = C_{\text{dir}}h + R_P x_P + R_{\bar{P}} x_{\bar{P}}. \quad (\text{C.11})$$

If D is invertible direct substitution gives

$$\begin{aligned} L_{\text{eff}} &= A - BD^{-1}C, & S_{\text{eff}} &= S_P - BD^{-1}S_{\bar{P}}, \\ R_{\text{eff}} &= R_P - R_{\bar{P}}D^{-1}C, & C_{\text{dir,eff}} &= C_{\text{dir}} + R_{\bar{P}}D^{-1}S_{\bar{P}}. \end{aligned} \quad (\text{C.12})$$

The signs differ from the iteration-kernel entries in Eq. (C.9) because the off-diagonal blocks of $I - \mathcal{B}$ are negative. The complete response is the same. On an analytic invertibility domain

$$L'_{\text{eff}} = A' - B'D^{-1}C + BD^{-1}D'D^{-1}C - BD^{-1}C'. \quad (\text{C.13})$$

Here $(D^{-1})' = -D^{-1}D'D^{-1}$. Differentiating the source and observable maps in Eq. (C.12) retains the same dependence on the eliminated block.

In the FRG representation the renormalised physical response is obtained as $k \rightarrow 0$. This regulator limit is distinct from the Gaussian spectral weight and the finite correlation scale of the compact construction. On the Euclidean axis TT projection is performed at $Q^2 > 0$ and the continuation uses

$$z = Q^2 \quad \longrightarrow \quad z = -p^2 - i0. \quad (\text{C.14})$$

The discontinuity of the nonlocal boundary value determines the Wightman measure. For finite Euclidean data the uncertainty of analytic continuation adds to the regulator and truncation errors.

D Fredholm pole and truncation control

Solution branches and pole residues are compared on a common space. The spectral input is analytic perturbation theory [5, 43, Ch. VII]. The residue bounds retain the motion of the critical root and both source maps.

D.1 Laurent expansion at a simple eigenvalue

Let $\mathcal{B}(z)$ be analytic and compact on a neighbourhood of the origin and suppose $\lambda(z)$ is an isolated algebraically simple eigenvalue with $\lambda(0) = 1$ and $\lambda'(0) \neq 0$. Its Riesz projector is

$$\Pi(z) = |r(z)\rangle\langle\ell(z)| = \frac{1}{2\pi i} \oint_{\gamma} (\zeta - \mathcal{B}(z))^{-1} d\zeta, \quad (\text{D.1})$$

where γ encloses no other spectrum and the analytic left and right eigenvectors are normalised by $\langle\ell|r\rangle = 1$. Analytic Fredholm theory gives

$$[I - \mathcal{B}(z)]^{-1} = \frac{\Pi(z)}{1 - \lambda(z)} + \mathcal{R}(z) = -\frac{\Pi(0)}{\lambda'(0)z} + \mathcal{O}(1), \quad (\text{D.2})$$

with $\mathcal{R}$ analytic near zero. Insertion between $\mathbf{R}_{\text{TT}}$ and $\mathbf{S}_{\text{TT}}$ proves Eqs. (4.27) and (4.28). If either overlap vanishes the operator resolvent is singular but the measured TT correlator is regular in that direction.

If $1 \notin \text{spec } \mathcal{B}(0)$ the response is regular near the origin. The complete Schur reduction in Eqs. (C.9)–(C.10) preserves this response while including shifts caused by complementary channels.

D.2 Solution branches, critical roots and residues

Let $\mathcal{X}$ be the exact response Banach space. Let $\mathcal{Y}$ be a fixed Hilbert space of source amplitudes with its dual identified by its inner product. The scalar criterion uses $\mathcal{Y} = \mathbb{C}$. The Riesz projector $\Pi(z)$ acts on $\mathcal{X}$ whereas the tensor projector $\Pi^{(2)}(Q)$ acts on external Lorentz indices. For each finite space $\mathcal{X}_N$ use the maps in Eq. (4.21) and lift all finite objects to the same comparison spaces

$$\tilde{\mathcal{B}}_N(z) := \iota_N \mathcal{B}_N(z) P_N, \quad \tilde{\mathcal{S}}_N(z) := \iota_N \mathcal{S}_N(z), \quad \tilde{\mathcal{R}}_N(z) := \mathcal{R}_N(z) P_N. \quad (\text{D.3})$$

Thus $\tilde{\mathcal{B}}_N \in \mathcal{L}(\mathcal{X})$ and $\tilde{\mathcal{S}}_N \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ with $\tilde{\mathcal{R}}_N \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. On a compact set K in the analytic z -domain define the common-space kernel error

$$\epsilon_{\mathcal{B},N} := \sup_{z \in K} \|\tilde{\mathcal{B}}_N(z) - \mathcal{B}(z)\|_{\mathcal{L}(\mathcal{X})}. \quad (\text{D.4})$$

The convergence hypothesis $\epsilon_{\mathcal{B},N} \rightarrow 0$ therefore refers to operators on the same space through the fixed comparison maps. Let γ enclose the simple exact eigenvalue uniformly on K and exclude zero and all other exact spectrum and let Π_N be the contour projector of $\tilde{\mathcal{B}}_N$ on the same contour. Set

$$M_\gamma := \sup_{z \in K, \zeta \in \gamma} \|(\zeta - \mathcal{B}(z))^{-1}\|, \quad L_\gamma := \text{length}(\gamma). \quad (\text{D.5})$$

If $M_\gamma \epsilon_{\mathcal{B},N} < 1$ the resolvent identity and a Neumann series give

$$\|\Pi_N(z) - \Pi(z)\| \leq \frac{L_\gamma}{2\pi} \frac{M_\gamma^2 \epsilon_{\mathcal{B},N}}{1 - M_\gamma \epsilon_{\mathcal{B},N}}. \quad (\text{D.6})$$

The same Neumann bound holds along $\mathcal{B} + t(\tilde{\mathcal{B}}_N - \mathcal{B})$ $0 \leq t \leq 1$ so the contour projectors have constant rank one. Consequently an isolated simple eigenspace is stable under operator-norm convergent truncations.

Residue convergence also involves the following errors on the same spaces

$$\begin{aligned} \epsilon_{\partial \mathcal{B},N} &:= \sup_{z \in K} \|\partial_z \tilde{\mathcal{B}}_N(z) - \partial_z \mathcal{B}(z)\|_{\mathcal{L}(\mathcal{X})}, \\ \epsilon_{\mathcal{S},N} &:= \sup_{z \in K} \|\tilde{\mathcal{S}}_N(z) - \mathcal{S}(z)\|_{\mathcal{L}(\mathcal{Y}, \mathcal{X})}, \\ \epsilon_{\mathcal{R},N} &:= \sup_{z \in K} \|\tilde{\mathcal{R}}_N(z) - \mathcal{R}(z)\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})}, \\ \epsilon_{\mathcal{C}_{\text{reg}},N} &:= \sup_{z \in K} \|\mathcal{C}_{\text{reg},N}(z) - \mathcal{C}_{\text{reg}}(z)\|_{\mathcal{L}(\mathcal{Y})}. \end{aligned} \quad (\text{D.7})$$

The derivative and source-map errors control the residue. Convergence of the regular part also uses $\epsilon_{\mathcal{C}_{\text{reg}},N} \rightarrow 0$. All are evaluated on the actual solution branches. For nonnormal kernels the resolvent constant M_γ can be large even for separated eigenvalues [43, 44].

Proposition D.1 (Controlled critical branch). Let $H_\tau(Y; z) = Y - \mathfrak{F}_\tau(Y; z)$ be the coarse stationarity residual and $\bar{Y}_*(z)$ a transported reference branch. One sufficient condition for a nearby solution at each z is that H_τ is continuously differentiable near the closed Banach-space ball $\bar{B}(\bar{Y}_*, r_Y)$ and for a bounded isomorphism A

$$\begin{aligned} \|A^{-1} H_\tau(\bar{Y}_*)\| &\leq b, \quad \sup_{Y \in \bar{B}(\bar{Y}_*, r_Y)} \|I - A^{-1} D_Y H_\tau(Y)\| \leq q < 1, \\ b + q r_Y &\leq r_Y. \end{aligned} \quad (\text{D.8})$$

Then the solution in the ball is unique and

$$\|Y_{\tau,*} - \bar{Y}_*\| \leq \frac{b}{1 - q}. \quad (\text{D.9})$$

Uniform bounds and analytic data give an analytic solution branch. Alternatively exact matching $H_\tau(\bar{Y}_*(z); z) = 0$ gives the branch directly including at singular stationarity points.

On a solution branch established by either route or by a separate existence argument suppose the preceding response families and rank-one contour projectors are defined on a neighbourhood of the closed disk $D_r = \{z : |z| \leq r\} \subset K$. Let $\lambda(0) = 1$ and set $\alpha = |\lambda'(0)| > 0$. Assume

$$m_r := \inf_{z \in D_r} \left| \frac{1 - \lambda(z)}{z} \right| > 0, \quad (\text{D.10})$$

where the quotient at zero is its analytic value $-\lambda'(0)$. Assume the uniform bound $\sup_{D_r} |\lambda_N - \lambda| \leq e_{\lambda,N} < m_r r$. Then $1 - \lambda_N$ has exactly one zero z_N in the disk counted with multiplicity and

$$|z_N| \leq d_{z,N} := e_{\lambda,N}/m_r < r. \quad (\text{D.11})$$

For example admissible choices from the common-space operator estimates are

$$e_{\lambda,N} := \epsilon_{\mathcal{B},N}(\Pi_* + d_{\Pi,N}) + 2B_* d_{\Pi,N}, \quad (\text{D.12})$$

$$e_{\lambda',N} := \epsilon_{\partial\mathcal{B},N}(\Pi_* + d_{\Pi,N}) + 2B'_* d_{\Pi,N}, \quad (\text{D.13})$$

where $d_{\Pi,N}$ is the right side of Eq. (D.6) and the constants are $\Pi_* := \sup_{D_r} \|\Pi\|$ and $B_* := \sup_{D_r} \|\mathcal{B}\|$ together with $B'_* := \sup_{D_r} \|\partial_z \mathcal{B}\|$. The second line bounds $\sup_{D_r} |\lambda'_N - \lambda'|$.

Define the numerators $\mathcal{N} = \mathbf{R}\Pi\mathbf{S}$ and $\mathcal{N}_N = \tilde{\mathbf{R}}_N \Pi_N \tilde{\mathbf{S}}_N$ and set $R_* := \sup_{D_r} \|\mathbf{R}\|$ and $S_{N,*} := \sup_{D_r} \|\tilde{\mathbf{S}}_N\|$ with $\Pi_{N,*} := \sup_{D_r} \|\Pi_N\|$. A same-point numerator error and its cross-root extension are

$$e_{\text{num},N} := \epsilon_{\mathbf{R},N} \Pi_{N,*} S_{N,*} + R_* d_{\Pi,N} S_{N,*} + R_* \Pi_* \epsilon_{\mathbf{S},N}, \quad (\text{D.14})$$

$$D_{\text{num},N} := e_{\text{num},N} + L_{\text{num},N} d_{z,N}, \quad (\text{D.15})$$

$$D_{\text{slope},N} := e_{\lambda',N} + L_{\lambda',N} d_{z,N}, \quad (\text{D.16})$$

with constants $L_{\text{num},N} \geq \sup_{D_r} \|\mathcal{N}'_N\|$ and $L_{\lambda',N} \geq \sup_{D_r} |\lambda''_N|$. Assume $D_{\text{slope},N} < \alpha$. Then

$$|\lambda'_N(z_N)| \geq \alpha - D_{\text{slope},N} > 0. \quad (\text{D.17})$$

If the complementary resolvents and direct terms are analytic on D_r the residues at the respective roots satisfy

$$\begin{aligned} \mathcal{Z} &= -\frac{\mathcal{N}(0)}{\lambda'(0)}, & \mathcal{Z}_N &= -\frac{\mathcal{N}_N(z_N)}{\lambda'_N(z_N)}, \\ \|\mathcal{Z}_N - \mathcal{Z}\| &\leq \frac{D_{\text{num},N}}{\alpha - D_{\text{slope},N}} + \frac{\|\mathcal{N}(0)\| D_{\text{slope},N}}{\alpha(\alpha - D_{\text{slope},N})}. \end{aligned} \quad (\text{D.18})$$

For a scalar response $|\mathcal{N}(0)| > D_{\text{num},N}$ also gives $\mathcal{N}_N(z_N) \neq 0$.

Proof. For the solution estimate $T(Y) = Y - A^{-1}H_\tau(Y)$ is a contraction mapping the ball into itself since $\|T(Y) - \bar{Y}_*\| \leq b + q\|Y - \bar{Y}_*\| \leq r_Y$. Its fixed point satisfies $\delta_Y \leq b + q\delta_Y$ and

uniform analytic iterations give the parameter dependence. Exact matching is substitution into $H_\tau = 0$.

The sole zero of $1 - \lambda$ in D_r is simple and lies at the origin. On its boundary $|1 - \lambda(z)| \geq m_r r > e_{\lambda,N}$ so Rouché's theorem gives exactly one zero of $1 - \lambda_N$. At that zero $m_r |z_N| \leq |1 - \lambda(z_N)| \leq e_{\lambda,N}$. The identities $\lambda = \text{tr}(\mathcal{B}\Pi)$ and $\lambda' = \text{tr}(\mathcal{B}'\Pi)$ together with their truncated counterparts give Eq. (D.13). Here one uses $|\text{tr } T| \leq \text{rank}(T)\|T\|$ for the finite-rank differences. Telescoping the three factors in $\mathcal{N}_N - \mathcal{N}$ and integrating along the segment from zero to z_N then gives

$$\begin{aligned} \|\mathcal{N}_N(z_N) - \mathcal{N}(0)\| &\leq D_{\text{num},N}, \\ |\lambda'_N(z_N) - \lambda'(0)| &\leq D_{\text{slope},N}. \end{aligned} \quad (\text{D.19})$$

The lower slope bound is the reverse triangle inequality. Subtracting the two quotient residues gives Eq. (D.18). $\square$

At a singular stationarity point $I - A^{-1}D_Y H_\tau$ has eigenvalue one so the contraction alternative is unavailable. Exact branch matching or a separate existence argument supplies the branch there. Once it is known a state-variable Lipschitz bound on $D_Y \mathfrak{F}_\tau$ and Eq. (D.9) give the displacement term in Eq. (4.25).

Cauchy's estimates provide derivative bounds from uniform convergence on a strictly larger analytic disk. Under a real-analytic symmetry the unique root in the symmetric disk is real. Otherwise z_N may be complex. The proposition compares the finite and exact residues. Their physical spectral interpretation is supplied by Theorem 4.4.

Proposition D.2 (Telescoping operator-error bound). Let $\mathcal{B}_N^{[j]}(z) \in \mathcal{L}(\mathcal{X})$ for $j = 0, \dots, 4$ with $\mathcal{B}_N^{[0]} = \mathcal{B}$ and $\mathcal{B}_N^{[4]} = \tilde{\mathcal{B}}_N$. The successive operators implement vertex truncation and momentum discretisation followed by response-complement approximation and closure. Define

$$\delta_{j,N} := \sup_{z \in K} \|\mathcal{B}_N^{[j]}(z) - \mathcal{B}_N^{[j-1]}(z)\|_{\mathcal{L}(\mathcal{X})}, \quad j = 1, \dots, 4. \quad (\text{D.20})$$

Then

$$\epsilon_{\mathcal{B},N} \leq \delta_{1,N} + \delta_{2,N} + \delta_{3,N} + \delta_{4,N}. \quad (\text{D.21})$$

For a last link implemented on the finite response space $\mathcal{X}_N$ let $\mathcal{C}_{\tau,N}$ be boundedly invertible on that space. Define the finite response operators and their common-space lifts by

$$\begin{aligned} B_{3,N} &:= \mathcal{C}_{\tau,N} \mathcal{B}_N^{\text{base}} \mathcal{C}_{\tau,N}^{-1}, & B_{4,N} &:= \mathcal{B}_{\tau,N}(Y_{\tau,N,*}), \\ \mathcal{B}_N^{[3]} &:= \iota_N B_{3,N} P_N, & \mathcal{B}_N^{[4]} &:= \iota_N B_{4,N} P_N. \end{aligned} \quad (\text{D.22})$$

At the transported reference point set $DE_{\tau,N} = \mathcal{C}_{\tau,N} \mathcal{B}_N^{\text{base}} - \mathcal{B}_{\tau,N}(\bar{Y}_{N,*}) \mathcal{C}_{\tau,N}$. Let $Y_{\tau,N,*}$ be a solution branch with displacement bound $\delta_{Y,N} \geq \|Y_{\tau,N,*} - \bar{Y}_{N,*}\|_{\mathcal{X}_N}$. Let $L_{\tau,N}$ bound the state-variable Lipschitz constant of $\mathcal{B}_{\tau,N}$ in $\mathcal{L}(\mathcal{X}_N)$ with respect to this finite-space state norm. Then

$$\delta_{4,N} \leq \|\iota_N\| \|P_N\| \sup_{z \in K} (\|DE_{\tau,N}(z)\|_{\mathcal{L}(\mathcal{X}_N)} \|\mathcal{C}_{\tau,N}^{-1}(z)\|_{\mathcal{L}(\mathcal{X}_N)} + L_{\tau,N}(z) \delta_{Y,N}(z)). \quad (\text{D.23})$$

The embedding and projection norms account for the passage from the finite-space error to $\mathcal{L}(\mathcal{X})$. Consequently let $M_\gamma \leq \overline{M}_\gamma$ and $\delta_{j,N} \leq \overline{\delta}_{j,N}$ be upper bounds and put $\overline{\epsilon}_N = \sum_j \overline{\delta}_{j,N}$. If $\overline{M}_\gamma \overline{\epsilon}_N < 1$ then the computable estimate is

$$\|\Pi_N(z) - \Pi(z)\| \leq \frac{L_\gamma}{2\pi} \frac{\overline{M}_\gamma^2 \overline{\epsilon}_N}{1 - \overline{M}_\gamma \overline{\epsilon}_N}. \quad (\text{D.24})$$

Proof. The lifted telescoping identity gives Eq. (D.21) by the triangle inequality. Differentiation of the closure defect followed by composition with $\mathcal{C}_{\tau,N}^{-1}$ and inclusion of the actual-branch displacement gives Eq. (D.23). The lift norms account for the change of space. Exact branch matching removes the displacement term and the final Riesz estimate is Eq. (D.6). $\square$

Applying the same decomposition to $\partial_z \mathcal{B}$ and the source maps $\mathbf{S}$ and $\mathbf{R}$ with their solution-branch dependence supplies the uniform estimates needed in Proposition D.1. The resulting bounds depend on the intermediate operators and comparison norms together with the contour resolvent constant of the chosen approximation. Appendix E illustrates the root and residue shifts in an exactly solvable finite response.

Errors supplied by a finite evolution. When the response is obtained from an evolution its approximation is compared with the full sourced equation on a common interval. For semilinear spectral generators the stronger spectral bound controls both the omitted initial modes and the discarded nonlinear source in a weaker forcing norm. The Duhamel stability estimate then propagates these errors to the state [32].

Let $\mathbf{L} \geq 0$ be bounded and self-adjoint and put $f(\sigma) = e^{-\sigma \mathbf{L}} f_0$. Let w be an absolutely continuous finite approximation on $[0, T_{\text{obs}}]$. If $\|\mathbf{L} - \widehat{\mathbf{L}}_N\| \leq \delta_{L,N}$ put $r_N = \partial_\sigma w + \widehat{\mathbf{L}}_N w$. Contractivity and variation of constants give

$$\|f(\sigma) - w(\sigma)\| \leq \|f_0 - w(0)\| + \int_0^\sigma \|r_N(s)\| ds + \delta_{L,N} \int_0^\sigma \|w(s)\| ds =: E_N(\sigma).$$

For a bounded observation C_{obs} with $\|(C_{\text{obs}} - \widehat{C}_{\text{obs}})w\| \leq \eta_C \|w\|$ it follows that

$$\|C_{\text{obs}} f - \widehat{C}_{\text{obs}} w\| \leq \|C_{\text{obs}}\| E_N + \eta_C \|w\|.$$

The operator discrepancy acts on the same computed path as the finite residual and the observation has its own approximation error. Finite matrix and path enclosures bound these terms separately. Source derivatives are controlled through the differentiated equations including the variation of the generator and state domain together with the observation. Uniform operator estimates for the resulting response maps supply Eqs. (D.4) and (D.7). Estimates on a larger analytic parameter domain also control their z derivatives by Cauchy's formula. The critical-root comparison above then transfers these bounds to the residue at the actual finite root.

D.3 Spectral accumulation at zero

If a continuum begins at $s = 0$ the continued response generally has a nonmeromorphic threshold singularity or boundary behaviour. Such behaviour can prevent a common analytic neighbourhood for $\mathcal{B}(z)$ and $\mathcal{C}_{\text{reg}}(z)$ and the complementary resolvent. Atom extraction at that boundary uses spectral integrability rather than the isolated Laurent decomposition of Eq. (D.2).

An atom can coexist with the threshold. For example a measure may have the form

$$d\rho_{\text{resp}}(s) = Z_0^{(\text{resp})} \delta_0(ds) + f(s)\theta(s)ds, \quad (\text{D.25})$$

with both terms nonzero. Proposition 2.4 can still identify $Z_0^{(\text{resp})}$ when the continuum satisfies its integrability hypotheses although the isolated Fredholm theorem does not apply. A kernel analysis at the threshold uses weighted resolvent estimates or a limiting-absorption principle.

E Spectral examples, compact sources, and finite approximations

The examples distinguish a gapped continuum from an atomic contribution. They also exhibit a hidden-channel shift under Schur reduction and collective concentration of finite-volume levels. The compact calculations supply the spectral sum and matter vertices used in Sections 2.1 and 4.1.

E.1 Free scalar stress tensor

For a free real scalar of mass m the stress tensor is bilinear and its first non-vacuum matrix element creates two particles. TT contraction gives $2\varepsilon^{\mu\nu}p_{1\mu}p_{2\nu}$ independent of scalar improvements. The two-body phase-space integral yields

$$\rho_2^{\text{scalar}}(s) = \frac{s^2}{960\pi^2} \left(1 - \frac{4m^2}{s}\right)^{5/2} \theta(s - 4m^2), \quad (\text{E.1})$$

with one-particle normalisation $\langle p|p' \rangle = 2E_p(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')$ and the orbit measure of Section 2.4. The angular average for a unit TT polarisation is $2/15$ and the identical-particle factor is $1/2!$. Thus $Z_0 = 0$ both for $m > 0$ and for $m = 0$ where the continuum has density proportional to s^2 down to zero.

Massless free fermion and gauge-field stress tensors likewise have positive TT densities proportional to s^2 with normalisations fixed by their two-point functions [26]. By contrast for $Z_{\text{ref}}, A, M^2 > 0$ the reference measure $d\rho_{\text{ref}} = Z_{\text{ref}}\delta_0 + As^2e^{-s/M^2}ds$ defines $C_{\text{ref}}(Q^2) := \int_0^\infty d\rho_{\text{ref}}(s)/(s + Q^2)$ and has atomic weight $\lim_{Q^2 \downarrow 0} Q^2 C_{\text{ref}}(Q^2) = Z_{\text{ref}}$ unchanged by finite Gaussian averaging even though its continuum reaches zero.

E.2 Finite response and volume limits

Let $P = \text{diag}(1, 0)$ retain the first channel of the nonnormal family with $|\delta| < 1/4$

$$\mathcal{B}_\delta(z) = \begin{pmatrix} 5/4 - z & 1/2 \\ -1/4 + \delta & 1/2 \end{pmatrix}, \quad \|\mathcal{B}_\delta - \mathcal{B}_0\|_2 = |\delta|. \quad (\text{E.2})$$

Direct projection gives the false pole $z = 1/4$ whereas Schur reduction gives $\mathcal{B}_{\text{eff}} = 1 - z + \delta$. For $\mathbf{S} = (s_1, s_2)^\top$ and $\mathbf{R} = (r_1, r_2)$ complete elimination yields

$$C(z) = C_{\text{dir}} + 2r_2s_2 + \frac{[r_1 + (-1/2 + 2\delta)r_2](s_1 + s_2)}{z - \delta}. \quad (\text{E.3})$$

The critical root moves by δ and the residue contains both overlaps. At $\delta = 0$ the normalised right and left eigenvectors $|r\rangle = (2, -1)^\top$ and $\langle\ell| = (1, 1)$ have $\lambda'(0) = -2$ reproducing the numerator in Eq. (E.3). For real maps its sign is the sign of the overlap product. Either vanishing factor removes the pole. The choice $\mathbf{S} = (1, 2)^\top$ with $\mathbf{R} = (3, 4)$ and $C_{\text{dir}} = 5$ gives $21 + 3/z$ including the complementary direct term.

Cumulative spectral weight. For a finite-volume regularisation with discrete spectrum write

$$d\rho_{2, \ell_{\text{box}}}(s) = \sum_n Z_n(\ell_{\text{box}}) \delta_{s_n(\ell_{\text{box}})}(ds). \quad (\text{E.4})$$

A sequence of low-lying levels can mimic Z_0/Q^2 over a restricted momentum range. The limiting atom is determined by cumulative spectral weight which also includes concentration of individually vanishing levels.

Proposition E.1 (Cumulative finite-volume atom criterion). Let $\rho_{2, \ell_{\text{box}}}$ be positive Radon measures on $[0, \infty)$ converging locally vaguely to a locally finite measure ρ_2 . Thus their integrals converge against every continuous compactly supported test function. For each $\epsilon > 0$ with $\rho_2(\{\epsilon\}) = 0$

$$\lim_{\ell_{\text{box}} \rightarrow \infty} \rho_{2, \ell_{\text{box}}}([0, \epsilon]) = \rho_2([0, \epsilon]). \quad (\text{E.5})$$

Consequently along any decreasing sequence of such continuity radii

$$Z_0 = \rho_2(\{0\}) = \lim_{\epsilon \downarrow 0} \lim_{\ell_{\text{box}} \rightarrow \infty} \rho_{2, \ell_{\text{box}}}([0, \epsilon]) = \lim_{\epsilon \downarrow 0} \liminf_{\ell_{\text{box}} \rightarrow \infty} \rho_{2, \ell_{\text{box}}}([0, \epsilon]). \quad (\text{E.6})$$

Local finiteness suffices for these statements even with infinite total UV weight. Multiplication by a fixed continuous positive function $w(s)$ with $w(0) = 1$ preserves the criterion and the atomic weight.

Proof. The compact interval $[0, \epsilon]$ has boundary $\{\epsilon\}$ in the relative topology of $[0, \infty)$. The Portmanteau argument for local vague convergence therefore gives Eq. (E.5) at continuity radii. Continuity from above then yields the atom formula. Such radii can be chosen arbitrarily close to zero because a locally finite measure has at most countably many atoms. Multiplication by a continuous w with $w(0) = 1$ preserves local vague convergence and the atomic weight. $\square$

Under these convergence assumptions a line with $s_0(\ell_{\text{box}}) \rightarrow 0$ and $Z_0(\ell_{\text{box}}) \rightarrow Z_{\text{line}} > 0$ is sufficient for $\rho_2(\{0\}) \geq Z_{\text{line}}$ although an atom may also be formed collectively. The elementary example

$$d\rho_m^{(D)}(s) = \frac{1}{m} \sum_{n=1}^m \delta_{n/m^2}(ds) \longrightarrow \delta_0(ds), \quad \max_n Z_n^{(D)}(m) = m^{-1} \longrightarrow 0 \quad (\text{E.7})$$

has unit limiting weight because its entire support lies in $[0, 1/m]$. For $R_m(Q^2) := Q^2 \int d\rho_m^{(D)}(s)/(s + Q^2)$ the two ordered limits are

$$\lim_{m \rightarrow \infty} \lim_{Q^2 \downarrow 0} R_m(Q^2) = 0, \quad \lim_{Q^2 \downarrow 0} \lim_{m \rightarrow \infty} R_m(Q^2) = 1. \quad (\text{E.8})$$

Thus the infinite-volume limit precedes the infrared limit when many finite-volume levels collectively form an atom. By contrast a nonatomic continuum has cumulative weight $\rho_{2,c}([0, \epsilon]) \downarrow 0$ even though its largest individual weights also vanish. The cumulative limit distinguishes these cases.

E.3 The finite compact spectral branch

The response bandwidth $R = 5$ in Eq. (2.7) has nonempty support only when $\hat{\lambda} < 5\pi$. For the three-generation assignment the scalar eigenvalues are $l(l+2)$ with even l . The one-form eigenvalues are $(l+1)^2$ with odd l . The spinor eigenvalues are $(n+3/2)^2$ with even right-handed and odd left-handed n . Their respective multiplicities are $(l+1)^2$ and $2l(l+2)$ and $(n+1)(n+2)$. Thus the finite sum reduces exactly to the eight terms in Table 1.

Carrier and degree	$\hat{\lambda}$	$d_j w_j$	$\hat{m}_j^2$
scalar $l = 0$	0	4	3/4
scalar $l = 2$	8	36	3/4
colour vectors $l = 1$	4	192	$3/2 + \vartheta^2/6$
weak vectors $l = 1$	4	72	$1 + \vartheta^2/6$
abelian vector $l = 1$	4	24	$\vartheta^2/6$
right-handed spinors $n = 0$	9/4	168	$3\vartheta^2/8$
right-handed spinors $n = 2$	49/4	1008	$3\vartheta^2/8$
left-handed spinors $n = 1$	25/4	576	$3\vartheta^2/8$

Table 1. Nonzero contributions to the compact spectral response with bandwidth $R = 5$. Eigenvalues and mass shifts are in the dimensionless endpoint units. $d_j w_j$ combines modal multiplicity with matter content and channel weight.

The weights include four scalar components. The vector content consists of eight colour vectors and three weak vectors together with one abelian vector. The Weyl content consists of twenty-one right-handed and twenty-four left-handed components. The channel weights are those of Section 2.1. Higher modes have empty frequency intervals at this bandwidth so the eight rows exhaust the retained sum. Tensor-carrier and ghost terms have zero weight in the prescription of Section 2.1.

For these entries all denominators are positive near $\vartheta = 1/50$ including the scalar constant mode because of its 3/4 shift. Consequently the endpoint formula and its source derivatives are smooth on an open interval around that value.

Bandwidth comparison and numerical error. Let A_R denote the sum over all modal indices accepted by the fixed response bandwidth R . At $R = 5$ the table exhausts this sum

so $A_{60,5} = A_5$ exactly. An interval quadrature enclosure $|\hat{A}_5 - A_5| \leq \epsilon_A$ gives the endpoint error

$$|\hat{c}_5 - c_5| \leq \epsilon_A / \kappa_{\text{TT}}, \quad c_5 = A_5 / \kappa_{\text{TT}}.$$

This is the numerical error at the declared response level.

The full Gaussian response provides a useful comparison of source domains. For the sum without modal or frequency restrictions suppose the modal weights have polynomial growth and $\hat{m}_j^2 \geq 0$. The bound $0 \leq K_{\text{TT}} \leq 1$ proves absolute convergence. For a retained set $\mathcal{J}_{n_{\text{sp}}}$ write A_∞ for the complete sum over the same untruncated spectral towers and $A_{n_{\text{sp}},R}$ for the finite sum. Then

$$0 \leq A_\infty - A_{n_{\text{sp}},R} \leq \frac{1}{64\pi^2} \left[\sum_{j \in \mathcal{J}_{n_{\text{sp}}}} d_j w_j e^{-\hat{\lambda}_j/\pi} \operatorname{erfc}\left(\frac{B_j}{\sqrt{\pi}}\right) + \sum_{j \notin \mathcal{J}_{n_{\text{sp}}}} d_j w_j e^{-\hat{\lambda}_j/\pi} \right]. \quad (\text{E.9})$$

Indeed the omitted frequency integral is bounded by $e^{-\hat{\lambda}_j/\pi} \int_{B_j}^\infty e^{-u^2/\pi} du$ and the integral is $\pi \operatorname{erfc}(B_j/\sqrt{\pi})/2$. For omitted modes use $B_j = 0$. Positivity permits termwise summation. By the radius equation of Section 2.1 the corresponding difference of endpoint coordinates is exactly $(A_\infty - A_{n_{\text{sp}},R})/\kappa_{\text{TT}}$. When the modal enumeration exhausts the domain P_R this measures the change in the endpoint on admitting the Gaussian spectrum outside that domain. The finite-response-level construction in ref. [20] keeps R fixed while controlling the evaluation of its integrals.

Critical multiplicity and the complete slope. The finite-dimensional critical space in Theorem 4.1 can have several source-visible directions. Consider

$$L(z) = \begin{pmatrix} 2z & 0 & z \\ 0 & 3z & 2z \\ z & 2z & 1+z \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R = S^*. \quad (\text{E.10})$$

Here $P = \operatorname{diag}(1, 1, 0)$ with $G = \operatorname{diag}(2, 3)$ on its range and $\det L = z^2(6 - 5z)$. Eliminating the last component gives

$$L_{\text{eff}}(z) = z \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} - \frac{z^2}{1+z} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad \operatorname{Res}_{z=0} S^* L(z)^{-1} S = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 5/6 \end{pmatrix}. \quad (\text{E.11})$$

The residue has determinant $1/6$ so it has rank two despite a common pole. Replacing the source columns by $(1, 0, 1)^\top$ and $(2, 0, -1)^\top$ gives the rank-one residue $\begin{pmatrix} 1/2 & 1 \\ 1 & 2 \end{pmatrix}$. The alignment of the projected sources determines the rank.

Internal scale dependence. Along $L(k_{\text{win}}) = \zeta/k_{\text{win}}$ keep the dimensionless product $\zeta := k_{\text{win}}L$ fixed. The selected compact branch has $\zeta = \sqrt{\pi}$. The TT response uses $p_{\text{int}}^2 = 8/L^2$ and $m_{\text{curv}}^2 = 6/L^2$ with exponential regulator $R_{k_{\text{win}}} = p_{\text{int}}^2/(e^{p_{\text{int}}^2/k_{\text{win}}^2} - 1)$. Put

$$u = \frac{8}{\zeta^2}, \quad v = \frac{6}{\zeta^2}, \quad d(\zeta) = u + v + \frac{u}{e^u - 1}. \quad (\text{E.12})$$

Direct substitution gives

$$G_{\text{TT}}(k_{\text{win}}) = \frac{1}{k_{\text{win}}^2 d(\zeta)}, \quad p_{\text{int}}^2 G_{\text{TT}}(k_{\text{win}}) = \frac{u}{d(\zeta)}. \quad (\text{E.13})$$

Along this family the radius and curvature change together with the internal momentum. Comparison with a response at varying external momentum uses the corresponding state map together with source and observable transport between scales.

The mass-shift deformation including its fixed window and rescaled vertex is evaluated in Eq. (4.1).

E.4 Compact matter sources and tensor vertices

Local scalar energy under geometric compression. Use $\Delta_g = \text{div}_g \nabla$ in this subsection. In dimension three the standard conformal transformation formulas give

$$\begin{aligned} \Delta_{g_\omega} f &= e^{-2\omega} (\Delta_g f + \nabla \omega \cdot \nabla f), \\ \mathcal{R}_{g_\omega} &= e^{-2\omega} (\mathcal{R}_g - 4\Delta_g \omega - 2|\nabla \omega|^2). \end{aligned} \quad (\text{E.14})$$

Applying the product rule to $\mathcal{A}_g(e^{\omega/2} f)$ yields

$$\mathcal{A}_{g_\omega} = e^{-5\omega/2} \mathcal{A}_g e^{\omega/2}.$$

For compression $\omega = -\eta$ and the unitary volume transport is $U_\eta = e^{-3\eta/2}$. Hence

$$U_\eta \mathcal{A}_{g_\eta} U_\eta^{-1} = e^\eta \mathcal{A}_g e^\eta, \quad (\text{E.15})$$

including all curvature and derivative terms of the local source.

At the round reference $\mathcal{A}_g \geq 3/(4L^2)$ on its form domain $H^1(M)$. Multiplication by a smooth e^η is an isomorphism of this domain so the transported forms remain closed and satisfy

$$\langle u, e^\eta \mathcal{A}_g e^\eta u \rangle \geq \frac{3}{4L^2} e^{2\min \eta} \|u\|^2.$$

The positive square root is therefore defined for every smooth real η . Differentiation at zero gives $\dot{\mathcal{A}} = \eta \mathcal{A} + \mathcal{A} \eta$. For smooth reference eigenvectors differentiating the square root may be written as the convergent resolvent pairing

$$\langle u_i, \dot{H} u_j \rangle = \frac{1}{\pi} \int_0^\infty \frac{t^{1/2} \langle u_i, \dot{\mathcal{A}} u_j \rangle}{(M_i^2 + t)(M_j^2 + t)} dt = \frac{\langle u_i, \dot{\mathcal{A}} u_j \rangle}{M_i + M_j}. \quad (\text{E.16})$$

The common elliptic form domain and its positive lower bound justify the derivative. Equivalently the matrix pairings solve $H\dot{H} + \dot{H}H = \dot{\mathcal{A}}$. Substituting $\dot{\mathcal{A}}_{ij} = (M_i^2 + M_j^2)\eta_{ij}$ gives

$$(DH_g[\eta])_{ij} = \frac{M_i^2 + M_j^2}{M_i + M_j} \langle u_i, \eta u_j \rangle. \quad (\text{E.17})$$

In particular it remains regular for equal eigenvalues. Finite occupation support makes the energy trace a finite sum of these pairings.

For distinct energies the off-diagonal coefficient is

$$\frac{M_i^2 + M_j^2}{M_i + M_j} = \frac{M_i + M_j}{2} + \frac{(M_i - M_j)^2}{2(M_i + M_j)}. \quad (\text{E.18})$$

Thus the local positive density in Eq. (5.9) uses stationarity of the occupation matrix. Coherent transitions are still determined by the same source but need not have that diagonal density representation. For the constant mode the density is $\sqrt{3}/(2LV_3) > 0$. It has a positive local compression charge and zero TT vertex.

For a scalar section of a Hermitian vector bundle with unitary connection B write $D_B = d + B$ and

$$\mathcal{A}_{g,B} = D_B^* D_B + \mathcal{R}_g/8. \quad (\text{E.19})$$

The connection is held fixed as a one-form under the metric source. Since $D_B(fu) = dfu + fD_Bu$ the calculation in Eq. (E.14) applies with covariant derivatives and gives $U_\eta \mathcal{A}_{g_\eta,B} U_\eta^{-1} = e^\eta \mathcal{A}_{g,B} e^\eta$. At the round reference $D_B^* D_B \geq 0$ gives the same strict lower bound. All linear and quadratic connection terms are retained in this identity. The local energy density is consequently $\sum_i n_i M_i \|u_i(x)\|^2$ for any such bundle with this scalar operator. The eigenvalues M_i already include the connection dependence.

Compact TT tensors and scalar matrix elements. Let E_a be the left-invariant orthonormal frame with $[E_a, E_b] = (2/L)\epsilon_{abc}E_c$ and coframe θ^a . The round connection is $\nabla_{E_a} E_b = L^{-1}\epsilon_{abc}E_c$. For constant $q = q^\top$ with $\text{tr } q = 0$ set $h_q^+ = V_3^{-1/2} q_{ab} \theta^a \theta^b$. Its trace is zero and

$$\nabla^a (h_q^+)_{ab} = -\frac{1}{L\sqrt{V_3}} \sum_{a,c} \epsilon_{abc} q_{ac} = 0.$$

Write $(J_a)_{bc} = \epsilon_{bac}$ for the three infinitesimal rotation matrices. On these constant tensors the rough Laplacian is $-L^{-2} \sum_a [J_a, [J_a, q]]$. Direct multiplication gives

$$-\sum_a [J_a, [J_a, q]] = 6q - 2\text{tr}(q)I_3 = 6q, \quad \Delta_{TT} h_q^+ = \frac{6}{L^2} h_q^+. \quad (\text{E.20})$$

The tensors descend through the central quotient. The right-invariant frame gives h_q^- . Their inner products within each family are $\text{tr}(qr)$. Between families the Haar average is proportional to $\text{tr}(q)\text{tr}(r)$ and vanishes. Choosing a Frobenius-orthonormal basis of trace-free symmetric matrices therefore gives the ten-dimensional internal TT space used in Section 4.1.

For $R(U) = \text{Ad}(U)$ normalized Haar integration gives

$$\int_M R_{ib} R_{je} dv_g = \frac{V_3}{3} \delta_{ij} \delta_{be}, \quad E_a R = \frac{2}{L} R J_a. \quad (\text{E.21})$$

The Haar identity follows by invariance and the trace. The derivative follows from right multiplication by $\exp(2tJ_a/L)$. Thus $u_{ib} = (3/V_3)^{1/2} R_{ib}$ is the nine-dimensional scalar shell with eigenvalue $8/L^2$. Substitution in Eq. (2.10) uses

$$\sum_{a,d,c} q_{ad} (J_a)_{cb} (J_d)_{ce} = \text{tr}(q) \delta_{be} - q_{be} = -q_{be}$$

and proves $H[h_q^+] = 4I_3 \otimes q/(L^2\sqrt{V_3})$. Left multiplication proves the second vertex with the factors interchanged. On a round Einstein metric the scalar-curvature variation is $D\mathcal{R}[h] = -\langle \text{Ric}, h \rangle + \nabla^a \nabla^b h_{ab} - \Delta \text{tr } h$ which vanishes on both TT families. Hence the same vertices apply to $\mathcal{A}_g$ including its curvature shift. The square-root derivative on this equal-energy shell divides the vertex by $2M_2$ proving Eq. (4.3).

For a real occupation matrix $\mathbf{N}_{\text{occ}}$ the energy is $M_2 \text{Tr } \mathbf{N}_{\text{occ}}$. Its source derivative is the trace of $\mathbf{N}_{\text{occ}}$ with the energy vertex so partial traces give Eq. (4.4). Arbitrary small perturbations of $I_9/9$ by $q \otimes I_3$ and $I_3 \otimes r$ remain positive and vary the two trace-free partial traces independently. The map from occupations to compact tensor sources therefore has rank ten.

For real unit vectors a, b a pure product occupation has source $\alpha_2((aa^\top)_0, (bb^\top)_0)$. Two such occupations obey

$$\langle j_{\text{hom}}(a, b), j_{\text{hom}}(a', b') \rangle = \alpha_2^2[(a \cdot a')^2 + (b \cdot b')^2 - 2/3]. \quad (\text{E.22})$$

The contraction can have either sign for states of the same positive energy while an isotropic occupation has zero tensor source. In Section 5.1 these vertices enter the complete source together with the compression density and constraint terms. The reduced response determines the spatial dependence.

The tensor source of the $(2, 1)$ one-form band. Choose the right coframe conventions $d\theta_R^a = L^{-1}\epsilon_{abc}\theta_R^b \wedge \theta_R^c$ and $E_a R_{bi} = -(2/L)\epsilon_{abc}R_{ci}$. For $a = (qRv)_b\theta_R^b$ symmetry and tracelessness of q give

$$\delta a = 0, \quad *da = \frac{4}{L}a, \quad \Delta_1 a = \frac{16}{L^2}a. \quad (\text{E.23})$$

Indeed the derivative part of the curl is $2qRv/L$ and the coframe part contributes the same amount. Left translation acts on q by orthogonal conjugation and right translation acts on v . The span is therefore $(2, 1)$. Haar orthogonality gives $\int Rv(Rw)^T dv_g = (V_3/3)(v \cdot w)I_3$ so the factor $(3/V_3)^{1/2}$ in the text makes this identification isometric.

To differentiate the Hodge operator retain a fixed differential one-form while varying the internal comparison metric by h . At the round point put $B = *da = 4a/L$ and write $N_g(a) = \|a\|_g^2$ and $Q_g(a) = \|da\|_g^2 + \|\delta_g a\|_g^2$. For trace-free h inverse-metric differentiation gives

$$\dot{N}_g(a) = - \int h^{ab} a_a a_b dv_g, \quad \dot{Q}_g(a) = \int h^{ab} B_a B_b dv_g. \quad (\text{E.24})$$

The codifferential term has zero first derivative because $\delta a = 0$. In dimension three the two-form norm gives the plus sign in the second identity. The source compressed to the degenerate eigenband is the polarization of $\dot{Q}_g - (16/L^2)\dot{N}_g$. Consequently for $\|q\|_{\text{HS}} = |v| = 1$

$$\langle a_{q,v}, H_{\text{lr}}[h_t^-] a_{q,v} \rangle = \frac{32}{L^2\sqrt{V_3}} \text{tr}(tq^2). \quad (\text{E.25})$$

Polarization and the same Haar identity prove Eq. (4.5) on the full fifteen-dimensional band. This is the derivative after unitary comparison of the varying one-form spaces. Within

a degenerate band the transport commutator vanishes. The source is right-translation invariant and hence does not mix this band with the opposite $(1, 2)$ band at the same eigenvalue. Its eigenvalues give the first-order splittings in the $(2, 1)$ band.

For the unshifted energy operator $\sqrt{\Delta_1}$ division by $2\sqrt{16/L^2}$ gives the tensor source $j_{\text{lr}}(q, v) = 4(q^2)_0/(L\sqrt{V_3})$ on these unit decomposable states. A real trace-free 3×3 matrix satisfies $\text{tr } q^4 = (\text{tr } q^2)^2/2$ whence

$$\|(q^2)_0\|_{\text{HS}}^2 = \frac{1}{6}, \quad \|j_{\text{lr}}(q, v)\|_{\text{HS}} = \frac{4}{\sqrt{6}L\sqrt{V_3}} > 0. \quad (\text{E.26})$$

The trace $\text{Tr}_{\mathcal{Q}} \mathcal{A}_t$ is an invariant linear functional of a trace-free matrix and therefore vanishes. Equal occupation of the complete band therefore has zero TT source. This gives the bilinear coupling of the one-form spectrum to the internal tensor source. A prescribed mass shift contributes its own derivative. The one-form eigenvalue $16/L^2$ and the rough tensor eigenvalue $6/L^2$ belong to different operators. The scale-dependent response is given by Eq. (E.13).

The Coarse-Graining Genesis Framework [22] supplies the finite matrix and Haar-integral evaluations together with spectral quadrature and source variations.

E.5 Continuum and truncation limits

For a reflection-positive lattice channel let $C_2(Q^2; a_{\text{lat}}, \ell_{\text{box}})$ be the Ward-normalised TT coefficient and P_2 its local subtraction. The continuum and volume limits are taken at fixed nonzero momentum in the window

$$a_{\text{lat}}|Q| \ll 1 \ll \ell_{\text{box}}|Q|, \quad (\text{E.27})$$

before sending $Q^2 \downarrow 0$ in

$$R_2(Q^2; a_{\text{lat}}, \ell_{\text{box}}) := Q^2[C_2(Q^2; a_{\text{lat}}, \ell_{\text{box}}) - P_2(Q^2; a_{\text{lat}}, \ell_{\text{box}})]. \quad (\text{E.28})$$

Proposition 2.4 identifies this ordered limit with the atom while Proposition E.1 gives its cumulative-weight form.

For functional approximations the common-space operator and source bounds of Appendix D control residues at their displaced roots. Endpoint convergence and Theorem 4.4 then identify the physical atomic weight. The strict lower bound in Proposition 4.5 gives its positivity. Scalar two- and four-point spectral functions have been computed using source flows [45]. For the complete covariant stress tensor of Theorem 2.3 the corresponding atomic weight is zero.

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Declarations

Conflict of interest. The author declares no competing interests.

Data and code availability. The synthetic spectral sequences and finite response models are defined in Appendix E. The finite-dimensional source responses and constrained systems together with the critical residues use the Coarse-Graining Genesis Framework [22]. No external numerical dataset is used.

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