

Certified Spectral Computation for Semilinear Evolution: Finite Records and Polynomial Bit Cost

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ABSTRACT: We study finite records whose accuracy for an infinite-dimensional nonlinear evolution can be checked without rerunning the producing algorithm. The model is a fixed Neumann reaction–diffusion equation with quadratic reaction on a fixed time interval. Inputs are rational cosine polynomials satisfying a common exponentially weighted coefficient bound. Exact rational tests of continuity and coefficient bounds together with the initial discrepancy and Galerkin residual certify the trajectory error for the original equation and the error of its terminal mean and point value. Acceptance depends on the submitted record independently of the algorithm that produced its coefficients. A rational Taylor construction yields an accepted record for every admissible input and requested accuracy. Construction cost and record length are polynomial in the complete input length and the number of requested accuracy bits. Checking is polynomial in the input and submitted record lengths. The proof controls quantized restarts and the bit lengths of intermediate numerators and denominators. A separate rational check certifies quadratic accounts with positive transport defects and signed injection and approximation increments. Finite response approximations also include operator-tail and rounding bounds in the acceptance test. These results give a computational realization of the generated spectral framework for the encoded Neumann family. A rational finite-dimensional system provides a second realization through quantized Picard iteration. Construction and checking programs accompany both models.

KEYWORDS: Polynomial bit complexity; Certified computation; Verifiable computation; Reaction–diffusion equations; Exact rational arithmetic; Proof certificates

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1 Introduction

A spectral calculation produces a finite approximation to an infinite-dimensional evolution. To use the result in a later calculation or observation one needs a bound on its error for the original equation. That bound must account for omitted modes as well as the accuracy of the computed trajectory. We study how the required evidence can be retained in a finite record whose construction and independent verification have controlled bit cost.

The encoded problem fixes the initial data and requested accuracy. A construction algorithm supplies coefficients for a trajectory together with the information needed to bound its residual and spectral tail. A separate checker tests the submitted evidence against the prescribed problem. Acceptance entails an error bound by the soundness theorem. Its proof uses the recorded coefficients and inequalities rather than the recurrence that produced them. The record can therefore be checked without repeating the construction of the trajectory.

Trustworthy computation has a precise operational meaning here. The input is finitely encoded and a fixed procedure produces a finite record. Acceptance implies the stated error bound by a proof independent of the producer recurrence. Construction and checking have controlled bit costs and the output has a controlled length. Effective approximation supplies a procedure converging to the solution. An independently checkable record and resource bounds are additional parts of the requirement.

A residual certificate measures how closely the recorded trajectory follows the equation. A quadratic ledger follows a specified positive observation through transport and injection as well as projection and finite approximation. The transported quantity has a retained part and a positive defect. Injection and approximation alter it through signed increments that include their interaction with the state. These accounts distinguish reductions caused by transport or omission from changes caused by a new contribution. The residual and ledger are checked separately because accuracy of a path and the distribution of a quadratic quantity along that path are distinct claims.

The retained variables must also determine the evolution being certified. In the history response of [1] an initial coefficient difference can be invisible to the value readout and affect it later. For a bounded linear response invariance of the readout kernel characterizes when the reduced values support a closed equation. Otherwise the eliminated contribution remains part of the state description or its memory. Even with the correct state variables approximating the operator or readout introduces errors distinct from the residual of the finite equation. The acceptance bound must include these contributions to concern the original equation and observation.

For a fixed Neumann reaction–diffusion equation we give rational trajectory records with an exact residual test. The checker receives the equation and the input data together with the accuracy request independently of the proposed record. It reconstructs the trajectory and checks a coefficient norm bound and continuity at the time nodes. It also checks the initial discrepancy and the expanded Galerkin residual. All these tests use exact rational arithmetic. Combined with the spectral tail estimate these checks bound the error over the whole interval and in the specified observations. The soundness proof uses these

recorded inequalities and the model estimates. It therefore applies to coefficients supplied by any algorithm within the prescribed record format and size bounds. Here independence means that acceptance and its proof do not use the Taylor recurrence of the producer. The producer and checker may share the canonical input and record formats.

For this record format we also prove uniform production and bit-cost bounds. The Neumann equation with quadratic reaction is fixed together with the time interval and exponentially weighted input bound. The rational cosine coefficient table and requested accuracy vary. A single rational Taylor algorithm produces an accepted record for every admissible input and accuracy request. Control of quantized restart values and intermediate integers gives polynomial bounds for the record length and for the bit costs of construction and checking in the full encoded data length and the number of requested accuracy bits. The checker terminates on arbitrary finite submissions with polynomial cost in the combined length of the external data and accuracy request together with the submitted record.

Writing n for the full data length and m for the requested accuracy bits we derive the spatial cutoff and number of time intervals along with the Taylor degree and working precision. Bounds on those parameters determine the record length. Bounding the arithmetic operations and the sizes of their intermediate integers then gives the running cost. These are separate steps in the proof of a polynomial bound in $n + m$.

There are two estimates behind the construction. The first compares the infinite-dimensional solution with its Galerkin solution in a weaker coefficient norm. The second compares the Galerkin solution with an arbitrary continuous piecewise polynomial satisfying the residual test. The acceptance conditions are the same for the conservative theoretical parameters and for practical parameter choices. Uniform success of the prescribed construction requires a rational Taylor recurrence and bounds on the error of rounding its coefficients onto a fixed dyadic grid. Each Taylor expansion starts from the endpoint actually stored in the record. The complex-time estimate must therefore remain valid at every computed node.

An arithmetic operation count alone would leave the size of the rational coefficients uncontrolled. We therefore give common denominators for the unrounded recurrence and bound its intermediate integers. Resetting the denominator calculation at each time node prevents accumulation across the full trajectory. The perturbation bound used here grows exponentially in the square of the cutoff but the required precision depends on its logarithm. Thus this bound can be accommodated with a polynomial number of working bits.

The second construction concerns a finite-dimensional coupled linear system with rational matrices and polynomial forcing on a fixed time interval and under fixed matrix and coefficient bounds. It implements Picard iteration on a space of trajectories. A common bound on the time coefficients controls the polynomial degree and truncation towards zero preserves the invariant set after quantization. These bounds give an invariant domain and control the error of each finite update. They also bound its evaluation cost. The record format remains fixed throughout the iteration.

Relation to validated integration. Nedialkov and collaborators survey enclosures for ordinary differential equations including Taylor methods and rounding control [2]. Jorba and Zou develop high-order Taylor integration using coefficient recurrences [3]. We use such a recurrence for a spectral Galerkin system and retain its coefficients as an independently checkable record.

For semilinear parabolic equations van den Berg and collaborators combine spectral approximation and a posteriori validation to obtain rigorous trajectories [4]. Duchesne and collaborators use semigroup estimates to validate higher-dimensional parabolic trajectories over long time intervals and establish global existence in specific examples [5]. The integrator of van den Berg and Breden uses a piecewise-constant linearization to validate semilinear parabolic flows [6]. Our construction addresses the cost of producing independently checkable approximations uniformly over an encoded input family. The initial coefficient table is part of the input. A single algorithm must succeed throughout the family and both the record and its verification enter the bit bound. The Neumann model allows us to prove these statements with explicit parameter choices.

The use of a residual to estimate a solution or an observation is also standard in a posteriori analysis. Examples include the work of Giles and Süli [7] and the treatment of certified reduced models by Hesthaven and collaborators [8]. Here the residual is expanded exactly in rational coordinates. Spatial truncation is charged separately through the stronger coefficient norm so a small finite-dimensional residual is converted into an error bound for the original equation.

Regularity and bit complexity. Complexity is measured in the encoded data length and the number of requested accuracy bits as in the real-function framework of Ko [9]. Kawamura and collaborators study how regularity parameters supplied with a function representation control the uniform cost of operations on analytic functions and Gevrey classes [10]. Their bounds track regularity parameters as well as the requested precision. Here the exponential coefficient bound and the existence interval are fixed for the input family while the rational coefficient table and accuracy vary. We charge for reading that table as well as for constructing and checking the output record.

For linear evolutionary PDEs Koswara and collaborators derive bit-complexity bounds from convergent difference schemes and matrix-powering estimates [11]. For the quadratic Neumann equation we produce a piecewise polynomial trajectory together with an exact residual test. The cost proof includes the integers used to produce this record and those used to check it. The cost of an implemented Picard update likewise includes coefficient evaluation and integration together with access to supplied data. The analytic bounds and finite arithmetic estimates therefore enter the same cost calculation.

For a quadratic ledger rational matrices and vectors permit exact checks of the stage updates and positive-defect factorizations. The checks also include complete injection and approximation increments. Summing the checked identities gives the global account. Its verification cost is polynomial in the complete submitted length. A bound in the original problem size also requires control of the construction and size of that ledger.

Logical role	Quantities
Fixed model data	Equation with Neumann boundary condition. Fixed final time and common input bound
Encoded input	Rational cosine table with encoded length n and the unary accuracy request 1^m
Derived construction data	Spectral cutoff and time partition with Taylor degree and working precision
Certified output	Finite trajectory record and its accepted error bound
Resource output	Record length and bit costs of construction and checking

Table 1. Logical roles of the data in the Neumann computation.

Spectral constructions and finite records. The companion analysis paper [12] retains spectral estimates through local nonlinear evolution and finite observation. A common stronger bound controls tails and a weaker norm measures error. Here a finite record also specifies coefficients and arithmetic precision for which the acceptance inequalities can be proved. The construction and convergence rate give an effective presentation of the Neumann solution while each accepted record supplies evidence for its own accuracy claim. The source construction in [1] additionally supplies a normalized history law and a parent–child response. Their Gaussian tails give finite operator enclosures. Source normalization controls the analytic mass without charging for the number of histories. Explicit representation and finite-arithmetic costs are tracked separately.

Information and arithmetic readouts use continuity bounds in the appropriate spectral norms. Finite-witness readout also requires a successful terminal state and a quantitative decoding margin. These bounds translate certified state approximations into the corresponding output guarantees.

2 Encoded data and spectral bounds

We work on $[0, \pi]$ with the Neumann equation

$$\partial_t u = \partial_{xx} u + u^2, \quad \partial_x u(t, 0) = \partial_x u(t, \pi) = 0, \quad T = \frac{1}{16}. \quad (2.1)$$

The input is the complete rational coefficient table

$$u_0(x) = \sum_{j=0}^{N_0} a_j \cos(jx), \quad a_j \in \mathbb{Q}, \quad \sum_{j=0}^{N_0} 4^j |a_j| \leq 1. \quad (2.2)$$

Zero entries are retained. The encoded length n counts all numerators and denominators together with indices and fixed-format delimiters. A request for accuracy 2^{-m} is encoded by 1^m with $m \geq 1$ so the total input size is $n + m$. The initial coefficient table and the precision vary. The equation and dimension are fixed. So are the final time and initial norm bound. Table 1 separates these logical roles.

Retaining zero entries ensures that the encoded length controls the number of coefficients. Thus $N_0 + 1 = O(n)$. The bound in (2.2) can then be checked by exact rational arithmetic. The integer 4^{N_0} has $O(n)$ bits and comparison of the weighted sum with one has polynomial cost in the full input length.

2.1 Two coefficient algebras

For $f(x) = \sum_{j \geq 0} f_j \cos(jx)$ put

$$\|f\|_{\mathcal{A}_4} = \sum_{j \geq 0} 4^j |f_j|, \quad \|f\|_{\mathcal{A}_{\text{lin}}} = \sum_{j \geq 0} (1+j) |f_j|. \quad (2.3)$$

Each space consists of the series for which the indicated norm is finite. Absolute convergence identifies these series with continuous functions and makes their coefficients unique. Both spaces are complete weighted sequence spaces. We use the first norm to bound trajectories and the second to measure approximation error. A common bound in the stronger norm does not by itself make tails uniformly small in that same norm. The gap between the two weights gives a uniform tail bound in the weaker norm.

The elementary coefficient estimates used below are collected here. Both spaces in (2.3) are Banach algebras with product constant one. The heat semigroup $S(t)$ and the projection P_K onto $0 \leq j \leq K$ are contractions in both spaces. Furthermore

$$\|f\|_{\infty} \leq \|f\|_{\mathcal{A}_{\text{lin}}} \leq \|f\|_{\mathcal{A}_4}, \quad \|(I - P_K)f\|_{\mathcal{A}_{\text{lin}}} \leq 2^{-(K+1)} \|f\|_{\mathcal{A}_4}. \quad (2.4)$$

The product rule is

$$\cos(ix) \cos(jx) = \frac{1}{2} \cos((i+j)x) + \frac{1}{2} \cos(|i-j|x). \quad (2.5)$$

For either weight $w_j = 4^j$ or $w_j = 1+j$ both w_{i+j} and $w_{|i-j|}$ are at most $w_i w_j$. Summing absolute coefficients proves the product bound on finite series and passage to the weighted sequence completion proves it on the whole space. The diagonal factors of $S(t)$ are e^{-tj^2} so the contraction assertions follow coefficientwise. Since $1+j \leq 2^j \leq 4^j$ the norm comparison holds. For $j > K$ we have $(1+j)4^{-j} \leq 2^{-j} \leq 2^{-(K+1)}$. This gives the tail bound.

The unnormalized cosines keep the product coefficients rational. To compare these bounds with Hilbert space estimates consider the normalized $L^2(0, \pi)$ basis

$$\phi_0 = \pi^{-1/2}, \quad \phi_j = (2/\pi)^{1/2} \cos(jx) \ (j \geq 1), \quad A_N \phi_j = j^2 \phi_j. \quad (2.6)$$

The coefficient norms control the corresponding spectral norms through

$$\begin{aligned} \left\| e^{(\log 2) A_N^{1/2}} (I + A_N)^{1/2} f \right\|_2 &\leq \sqrt{\pi} \|f\|_{\mathcal{A}_4}, \\ \left\| (I + A_N)^{1/2} f \right\|_2 &\leq \sqrt{\pi} \|f\|_{\mathcal{A}_{\text{lin}}}. \end{aligned} \quad (2.7)$$

Indeed the L^2 norm of a cosine is at most $\sqrt{\pi}$. The weight comparisons are $2^j \sqrt{1+j^2} \leq 4^j$ and $\sqrt{1+j^2} \leq 1+j$. Thus the first coefficient bound leaves a fixed positive analytic reserve for estimating tails while the second controls a first-order spectral norm. Formula (2.6)

Role	Coefficient estimate	Normalized Hilbert consequence
Common strong bound	$\ f\ _{\mathcal{A}_4} \leq M$	$\ f\ _{Y^+} \leq \sqrt{\pi}M$ with $\rho_+ = (1/2, \log 2, 1)$
Error and convergence	$\ f - g\ _{\mathcal{A}_{\text{lin}}} \leq e$	$\ f - g\ _{Y^-} \leq \sqrt{\pi}e$ with $\rho_- = (1/2, 0, 1)$
Spatial reserve	Common $\mathcal{A}_4$ bound	$\ (I - P_K)f\ _{\mathcal{A}_{\text{lin}}} \leq 2^{-K-1} \ f\ _{\mathcal{A}_4}$

Table 2. The coefficient estimates and the spectral resource pair. The Hilbert bounds follow from (2.7). The last row gives the coefficient tail used in the computation.

converts between the two coordinate conventions. The rational computations use the unnormalized cosines.

The companion analysis [12] uses resources $\rho = (\theta, \alpha, s)$. The associated norm is $\left\| e^{\alpha A_N^\theta} (I + A_N)^{s/2} f \right\|_2$. Here Y^+ and Y^- denote the Hilbert spaces equipped with the two norms in (2.7). The continuous embeddings transfer the coefficient bounds to these spectral norms. The algorithm uses the coefficient norms because their product estimates can be evaluated in rational arithmetic on the stored coefficients.

2.2 A common solution interval

Write $X_T = C([0, T]; \mathcal{A}_4)$ with its supremum norm. The mild equation is

$$u(t) = S(t)u_0 + \int_0^t S(t-s)u(s)^2 ds. \quad (2.8)$$

The Galerkin solution v_K has initial value $P_K u_0$ and satisfies

$$\partial_t v_K + A_K v_K = P_K(v_K^2), \quad A_K = \text{diag}(0^2, 1^2, \dots, K^2). \quad (2.9)$$

Proposition 2.1 (Common bound and Galerkin error). For every input (2.2) equation (2.8) has a unique solution in X_T . Every Galerkin solution exists on the same interval and

$$\sup_{t \leq T} \max\{\|u(t)\|_{\mathcal{A}_4}, \|v_K(t)\|_{\mathcal{A}_4}\} \leq 2. \quad (2.10)$$

Uniformly in the input and cutoff we have

$$\|u - v_K\|_{C([0, T]; \mathcal{A}_{\text{lin}})} \leq \frac{5}{3} 2^{-(K+1)}. \quad (2.11)$$

Proof. On the radius-two closed ball of X_T the right-hand side of (2.8) has norm at most $1 + 4T = 5/4$. Its Lipschitz constant is at most $4T = 1/4$ since $f^2 - g^2 = (f - g)(f + g)$. The contraction theorem gives a fixed point in that ball. The projected equation has precisely the same estimates. Uniqueness also holds on the full trajectory space. Any two solutions have bounded norms on the compact time interval. Local Lipschitz continuity of the square map therefore permits Gronwall's inequality to be applied to their difference. That difference must vanish.

Let $E_K = \|u - v_K\|_{C([0,T];\mathcal{A}_{\text{lin}})}$. The full-space residual of v_K is $-(I - P_K)v_K^2$. Subtracting the two mild equations gives

$$\begin{aligned} E_K &\leq \|(I - P_K)u_0\|_{\mathcal{A}_{\text{lin}}} + 4TE_K + T \sup_{t \leq T} \|(I - P_K)v_K(t)^2\|_{\mathcal{A}_{\text{lin}}} \\ &\leq (1 + 4T)2^{-(K+1)} + 4TE_K. \end{aligned} \quad (2.12)$$

Division by $1 - 4T = 3/4$ proves (2.11). $\square$

The two tail terms account for high modes in the input and those created by multiplication. In particular the estimate applies when the input has support above the computational cutoff. Since both trajectories are exact this estimate accounts only for spatial truncation. Replacing the Galerkin trajectory by a finite time polynomial introduces a separate approximation error.

The two observations used throughout the computations are

$$O_{\text{av}}(f) = \frac{1}{\pi} \int_0^\pi f(x) \, dx = f_0, \quad O_0(f) = f(0) = \sum_{j \geq 0} f_j. \quad (2.13)$$

Both have norm at most one on $\mathcal{A}_{\text{lin}}$ so the trajectory error bounds their errors at every time with the same constant.

3 Finite trajectories and independent certificates

A trajectory can be tested after its coefficients have been computed. We reconstruct it as a polynomial on each interval of a rational time partition. Agreement at the joins makes the path continuous and a coefficient norm bound keeps it in the domain of the nonlinear stability estimate. Its initial discrepancy and differential residual then control the error along the interval. These tests use the stored coefficients so their validity is independent of how the trajectory was obtained.

Let $0 = t_0 < t_1 < \dots < t_J = T$ and $h_i = t_{i+1} - t_i$. On the i th interval write

$$q_i(\tau, x) = \sum_{\ell=0}^{r_i} \sum_{j=0}^K q_{i,\ell,j} \tau^\ell \cos(jx), \quad w(t, x) = q_i((t - t_i)/h_i, x), \quad (3.1)$$

where all $q_{i,\ell,j}$ are rational. For a polynomial $p(\tau) = \sum_\ell p_\ell \tau^\ell$ with cosine-polynomial coefficients use the absolute coefficient bounds

$$\|p\|_{\text{t},4} = \sum_\ell \|p_\ell\|_{\mathcal{A}_4}, \quad \|p\|_{\text{t},\text{lin}} = \sum_\ell \|p_\ell\|_{\mathcal{A}_{\text{lin}}}. \quad (3.2)$$

These dominate the respective spatial norms on $0 \leq \tau \leq 1$. The product estimate of section 2 applies to these norms as well by taking the Cauchy product in time and summing the spatial bounds.

The initial discrepancy and the expanded projected residual are

$$\begin{aligned} d_0 &= \|q_0(0) - P_K u_0\|_{\mathcal{A}_{\text{lin}}}, \\ \mathcal{R}_i(\tau) &= h_i^{-1} \partial_\tau q_i(\tau) + A_K q_i(\tau) - P_K(q_i(\tau)^2), \\ R_* &= \max_{0 \leq i < J} \|\mathcal{R}_i\|_{t, \text{lin}}. \end{aligned} \tag{3.3}$$

These quantities can be evaluated exactly in rational arithmetic. The initial discrepancy compares the record with the projected input and the residual tests the projected dynamics. The omitted spatial modes are accounted for separately by (2.11). We expand the product using (2.5) before taking absolute values. This retains cancellations within each residual coefficient and gives a bound on the whole interval. Appendix A gives the coefficient formulas for this expansion.

The accepted record supplies finite evidence for an approximation statement about the infinite-dimensional solution. Its coefficient inequalities are checked exactly. The soundness proof gives their analytic consequence for the trajectory and observations. This is the sense in which the record is a proof object for the asserted accuracy.

Theorem 3.1 (Soundness of the residual certificate). Fix an input (2.2) and a precision $m \geq 1$. Consider a record (3.1). Suppose its intervals cover $[0, T]$ and its coefficients have the declared finite format. Suppose also that

$$q_i(1) = q_{i+1}(0) \quad (0 \leq i < J-1), \quad \|q_i\|_{t,4} \leq 3 \quad (0 \leq i < J). \tag{3.4}$$

Define

$$E_{\text{cert}}(K, w; u_0) = \frac{5}{3} 2^{-(K+1)} + \frac{d_0 + T R_*}{1 - 5T}. \tag{3.5}$$

Then

$$\|u - w\|_{C([0,T]; \mathcal{A}_{\text{lin}})} \leq E_{\text{cert}}(K, w; u_0). \tag{3.6}$$

Consequently a checker accepting exactly when the structural tests and $E_{\text{cert}} < 2^{-m-2}$ hold certifies that accuracy for every accepted record. Both observations in (2.13) have the same certified error.

Proof. The endpoint tests make w continuous and piecewise differentiable. Integrating on each interval and joining the resulting identities shows that w satisfies the mild Galerkin equation with initial value $w(0)$ and forcing equal to its piecewise residual. The norm test bounds w by three in $\mathcal{A}_4$ and hence in $\mathcal{A}_{\text{lin}}$ while Proposition 2.1 bounds v_K by two. Subtracting their mild equations and using the algebra bound gives the following bound with $E_t = \|w - v_K\|_{C([0,t]; \mathcal{A}_{\text{lin}})}$.

$$E_T \leq d_0 + 5T E_T + T R_*. \tag{3.7}$$

The coefficient five is the sum of the two norm bounds in $w^2 - v_K^2 = (w - v_K)(w + v_K)$. Solving this inequality using $5T = 5/16 < 1$ and adding (2.11) gives (3.6). The observation errors satisfy the same bound because both observations have norm at most one. $\square$

For a given cutoff the spectral term in (3.5) is fixed by the input family. The quantities d_0 and R_* depend on the individual record. For records satisfying the structural tests time approximation and coefficient rounding are accounted for through the initial discrepancy and residual of the stored trajectory. For uniform production of accepted records the chosen degree and arithmetic precision must make these checked quantities small enough for every admissible input.

3.1 Input consistency and bounded verification

The residual estimate applies to every finite coefficient array satisfying the hypotheses of Theorem 3.1. Encoding and size limits also control the arithmetic required to check a submitted array. The equation and initial coefficient table as well as the requested precision are specified independently of the proposed record. Acceptance requires agreement with these data. Rational coefficients are compared in reduced form so the certificate refers to the prescribed initial condition.

Field lengths and array dimensions are checked before any dyadic denominator or rational polynomial is formed. The equation and input table together with the requested precision determine these limits. In particular the checker rejects an excessive precision field before constructing the integer it would specify.

Within these size bounds acceptance is decided by exact rational arithmetic. The coefficient sums in (3.2) bound the spatial norms throughout each time interval. These sums may exceed the actual suprema. The stability proof in Theorem 3.1 uses these computable upper bounds to establish the error guarantee for the exact trajectory.

3.2 Observations and rational output intervals

At T the stored coefficient vector is the exact rational sum $q_{J-1}(1)$. Thus the approximate mean and endpoint value are

$$z_{\text{av}} = \sum_{\ell=0}^{r_{J-1}} q_{J-1,\ell,0}, \quad z_0 = \sum_{j=0}^K \sum_{\ell=0}^{r_{J-1}} q_{J-1,\ell,j}. \quad (3.8)$$

The exact observations lie in the rational intervals centered at these values with radius E_{cert} . Rounding the lower endpoint down and the upper endpoint up preserves the enclosure in a decimal display.

As a direct check take constant input $u_0 = a$ with $|a| \leq 1$ and the constant trajectory $w(t) = a$. Its residual is $-a^2$ and $d_0 = 0$ and the time contribution to (3.5) is $a^2/11$. The exact solution is $u(t) = a/(1 - at)$. It permits a direct comparison with this estimate on $[0, 1/16]$. Increasing the formal cutoff changes the spectral part of the certificate but leaves this time residual unchanged. The two terms therefore describe different sources of error even for this constant record.

4 A uniform rational construction

The residual test can certify a proposed trajectory once its coefficients are known. To produce such a trajectory uniformly over (2.2) we use a rational Taylor construction. Its

local estimate must remain valid at the endpoints actually reached by the calculation. Each rounded polynomial therefore supplies its own exact endpoint as the next initial value and the bounds are propagated from that value rather than from an unrounded trajectory.

4.1 Parameters and recurrence

The cutoff first controls the omitted spatial modes. The time step is chosen small enough for a common complex-time estimate at the computed nodes. To control perturbations over the whole interval the local residual allowance also accounts for their possible growth. This allowance determines the Taylor degree and coefficient precision. For the request m set

$$\begin{aligned} \varepsilon &= 2^{-m}, & K &= m + 3, & D &= K + 1, & L_K &= 16D^2, & S &= 2^{2D^2}, \\ g &= \lceil \log_2(32L_K) \rceil, & h &= 2^{-g}, & J &= 2^{g-4}, & t_i &= ih, \\ R_K &= \sum_{j=0}^K 4^j, & \xi &= \frac{\varepsilon}{8S(1+T)}, \\ r &= \left\lceil \log_2 \frac{64(1+h^{-1})}{\xi} \right\rceil, & b &= \left\lceil \log_2 \frac{64(r+2)^2(1+h^{-1})R_K}{\xi} \right\rceil, & \delta_b &= R_K 2^{-b}. \end{aligned} \quad (4.1)$$

Here S bounds perturbation growth and ξ is the local residual allowance. The parameters r and b specify the degree and rounding precision. Integer comparisons with powers of two evaluate each logarithmic ceiling. The comparisons are given in Appendix A. The parameters satisfy

$$Jh = T, \quad 2L_K \leq J < 4L_K, \quad hL_K \leq \frac{1}{32}, \quad S \geq e^{L_K T}. \quad (4.2)$$

The last inequality follows from $L_K T = D^2$ and $e < 4$.

Write Round_b for coordinatewise truncation towards zero to a multiple of 2^{-b} . On D modes its $\mathcal{A}_4$ error is less than δ_b . Set $y_0 = \text{Round}_b(P_K u_0)$. On interval i compute exactly the rational vectors

$$\begin{aligned} c_{i,0} &= y_i, \\ c_{i,\ell+1} &= \frac{h}{\ell+1} \left(-A_K c_{i,\ell} + P_K \sum_{a=0}^{\ell} c_{i,a} c_{i,\ell-a} \right), \quad 0 \leq \ell < r. \end{aligned} \quad (4.3)$$

The coefficient products use (2.5). This is the Taylor recurrence for the Galerkin equation in the normalized time variable $\tau = (t - t_i)/h$. Coefficient recurrences of this kind are standard in Taylor integration [3].

After these coefficients have been formed set

$$q_{i,0} = y_i, \quad q_{i,\ell} = \text{Round}_b(c_{i,\ell}) \quad (1 \leq \ell \leq r), \quad q_i(\tau) = \sum_{\ell=0}^r q_{i,\ell} \tau^\ell, \quad y_{i+1} = q_i(1). \quad (4.4)$$

The exact endpoint sum remains on the same dyadic grid. It makes adjacent pieces agree without a further rounding error at the join.

Soundness covers every accepted submission. Uniform production requires an accepted record for every input in the declared family and every requested precision.

Theorem 4.1 (Uniform construction). For every input (2.2) and $m \geq 1$ the finite procedure defined by (4.1) (4.3) and (4.4) produces a record satisfying the acceptance conditions of Theorem 3.1. Its reconstruction satisfies

$$\|u - w_m\|_{C([0,T];\mathcal{A}_{\text{in}})} < 2^{-m-2}. \quad (4.5)$$

The record also satisfies the stronger rational tests

$$\|q_i\|_{t,4} < 3, \quad \|h^{-1}q'_i + A_K q_i - P_K(q_i^2)\|_{t,4} < \xi, \quad \|y_0 - P_K u_0\|_{\mathcal{A}_4} \leq \xi. \quad (4.6)$$

Controlling each Taylor polynomial requires a bound on its computed initial node. Its residual bound then proves that the next node satisfies the same assumption. This induction needs control in the stronger norm. The weaker residual test suffices for acceptance of the completed record.

4.2 Complex-time control of a local polynomial

A complex-time neighborhood gives geometric decay of the Taylor coefficients. This decay controls both the omitted terms and the residual of the truncated polynomial. Let $F_K(z) = -A_K z + P_K(z^2)$ on the complexified coefficient space. On the radius-four ball

$$\|F_K(z)\|_{\mathcal{A}_4} \leq 4K^2 + 16 \leq 2L_K, \quad \text{Lip}(F_K) \leq K^2 + 8 \leq L_K. \quad (4.7)$$

These inequalities also apply to absolutely summable time series with the norm in (3.2).

Lemma 4.2 (Bounds from a computed node). Suppose $\|y_i\|_{\mathcal{A}_4} \leq 9/4$. Then the local exact solution of $z' = F_K(z)$ based at y_i is holomorphic for complex time $|t - t_i| < (4L_K)^{-1}$. Its normalized Taylor coefficients satisfy

$$\|c_{i,0}\|_{\mathcal{A}_4} \leq \frac{9}{4}, \quad \|c_{i,\ell}\|_{\mathcal{A}_4} \leq \frac{11}{4} 8^{-\ell} \quad (\ell \geq 1). \quad (4.8)$$

The polynomial obtained from (4.4) has norm less than three in $\|\cdot\|_{t,4}$ and its expanded residual has that norm less than ξ .

Proof. Put $\rho = (4L_K)^{-1}$. On bounded holomorphic paths in the complex time disk with displacement at most one from y_i the integral map has displacement at most $\rho(2L_K) = 1/2$ and contraction constant at most $\rho L_K = 1/4$. Its values stay in the radius-four ball because $9/4 + 1 < 4$. Its fixed point has norm at most $11/4$. Applying the argument on smaller closed disks and using uniqueness on overlaps gives this bound throughout the open disk of radius ρ .

As $\rho/h \geq 8$ Cauchy's coefficient estimate gives (4.8). Comparison of coefficients in the equation gives (4.3) so these are precisely the rational coefficients used by the construction. The full Taylor series and hence its truncation has time coefficient norm at most

$$\frac{9}{4} + \frac{11}{4} \sum_{\ell \geq 1} 8^{-\ell} = \frac{37}{14} < 3. \quad (4.9)$$

Quantization adds at most $r\delta_b < \xi/64$. The bound remains below three.

Let p be the full Taylor series and $p^{[r]}$ its degree- r truncation. The geometric tails give

$$\|p - p^{[r]}\|_{t,4} \leq \frac{11}{28}8^{-r}, \quad \|p' - (p^{[r]})'\|_{t,4} \leq \frac{11(7r+8)}{196}8^{-r}. \quad (4.10)$$

Since $h^{-1}p' = F_K(p)$ the two tail estimates and the Lipschitz bound (4.7) control the residual introduced by truncation. Rounding changes the function norm by at most $r\delta_b$ and the derivative norm by at most $r(r+1)\delta_b/2$. Combining these estimates and using $hL_K \leq 1/32$ yields

$$\|h^{-1}q'_i - F_K(q_i)\|_{t,4} \leq h^{-1}((r+2)8^{-r} + (r+1)^2\delta_b) < \xi. \quad (4.11)$$

Indeed $r+2 \leq 4^r$ for $r \geq 1$ so the first term is bounded by $h^{-1}2^{-r} \leq \xi/64$. The definition of b bounds the second by $\xi h^{-1}(r+1)^2/[64(r+2)^2(1+h^{-1})] < \xi/64$. The bound applies to the absolute sum of the expanded residual coefficients and hence holds throughout the interval. $\square$

4.3 Induction and the common acceptance test

Proof of Theorem 4.1. Truncation towards zero gives $\|y_0\|_{\mathcal{A}_4} \leq 1$ and $\|y_0 - P_K u_0\|_{\mathcal{A}_4} \leq \delta_b \leq \xi$. Suppose the nodes through y_i have norm at most $9/4$. Lemma 4.2 gives the norm and residual tests on the next interval. The computed pieces through this interval form a continuous path. Compare that path with v_K in $\mathcal{A}_4$ using the finite vector field F_K . The local norm bound and the common Galerkin bound put both paths in its radius-four Lipschitz region. Gronwall's inequality therefore gives the following bound up to time t_{i+1} .

$$\|w_m(t) - v_K(t)\|_{\mathcal{A}_4} \leq e^{L_K t}(\delta_b + t\xi) \leq S\xi(1+T) = \frac{\varepsilon}{8}. \quad (4.12)$$

At the new endpoint $\|y_{i+1}\|_{\mathcal{A}_4} \leq 2 + \varepsilon/8 < 9/4$. This completes the induction and proves all the strong tests in (4.6).

To check acceptance use $\|\cdot\|_{\mathcal{A}_{\text{lin}}} \leq \|\cdot\|_{\mathcal{A}_4}$. The strong tests imply $d_0 \leq \xi$ and $R_* \leq \xi$. Consequently

$$E_{\text{cert}} \leq \varepsilon \left(\frac{5}{48} + \frac{2}{11S} \right) \leq \varepsilon \left(\frac{5}{48} + \frac{1}{11} \right) < \frac{\varepsilon}{4}, \quad (4.13)$$

Here we used $K = m+3$ and $T = 1/16$ with $S \geq 2$. Thus the record satisfies the acceptance test and Theorem 3.1 gives (4.5). $\square$

The prescribed parameters succeed throughout the input family. Other parameter choices must pass the same tests on their completed records.

5 Record length and bit cost

As in section 2 n is the full encoded data length and the unary request 1^m specifies tolerance 2^{-m} . The equation and dimension are fixed. So are the final time and initial norm bound. Thus n and m measure the variable input while the cutoff and partition together with the Taylor degree and working precision are derived by the construction. We estimate the parameter sizes separately from the output length and the running costs.

Step	Production	Independent checking
1	Choose (4.1) and quantize $P_K u_0$.	Check the field bounds and compare the equation and input table as well as the precision with the external request.
2	Compute the rational Taylor coefficients by (4.3) at the current node.	Check the time partition and array shapes together with coefficient norms and exact joins in (3.4).
3	Quantize the coefficients and take their exact sum as the next node as in (4.4).	Expand (3.3) and compute d_0 and R_* in rational arithmetic.
4	Repeat over the time partition and store the coefficient arrays with their reconstruction parameters.	Test $E_{\text{cert}} < 2^{-m-2}$ using (3.5) and form the observation intervals from (3.8).

Table 3. Construction and checking of a Neumann record. The check uses the submitted coefficients and residual inequalities without requiring the coefficients to satisfy the Taylor recurrence.

Finiteness alone permits records and computations of uncontrolled size. The bit bounds below are therefore an additional quantitative conclusion rather than a consequence of using finite data.

A spectral tail bound must also be distinguished from the size of the retained representation. A normalized history source can assign a fixed Gaussian mass to each root while distributing it over many expressions. This removes a multiplicity factor from the source norm and operator-tail estimates but an explicit coefficient table still contains those expressions. In such a representation the dimension is the sum of the fibre cardinalities and enumerating it has a cost. Storing and updating its coefficients also enter the cost. The Neumann calculation uses one cosine coefficient per retained mode. The dimension bound proved here refers to that representation.

The proof estimates the construction parameters first. It then bounds the stored coefficient array and the integers formed during arithmetic. The seventh-power bound below concerns the array’s length. Running time also charges for operations on the larger intermediate integers and for processing the original rational input. Resetting the denominator at each time node is what keeps those integer lengths under control.

Theorem 5.1 (Cost of construction and verification). Fixed polynomials in $n + m$ bound the bit cost of the algorithm in Theorem 4.1. They also bound the length of its complete output record and the cost of its independent verification. The trajectory payload with an optional expanded residual table has length $O((m + 1)^7)$. Including a canonical copy of the input gives length $O(n + (m + 1)^7)$. The verifier is total on arbitrary submitted finite records. Its parsing and rejection cost is polynomial in the complete input and record length and accepted records satisfy Theorem 3.1.

5.1 Denominators on one interval

The arbitrary rational denominators in u_0 enter only the initial dyadic quantization. Every subsequent node lies on the 2^{-b} grid. To absorb the divisions in the Taylor recurrence on

a single interval set

$$q_{\text{den}} = \max\{b, g + 1\}, \quad Q = 2^{q_{\text{den}}}, \quad \mathfrak{D}_\ell = \ell! Q^{2\ell+1}, \quad C_\ell = \mathfrak{D}_\ell c_{i,\ell}. \quad (5.1)$$

The bilinear map $M(v, w) = 2P_K(vw)$ has integer structure coefficients in $\{0, 1, 2\}$. The factor two clears the halves in the cosine product including terms where the two output frequencies coincide.

With these denominators all C_ℓ are integer vectors and satisfy

$$C_0 = Qy_i, \quad C_{\ell+1} = -hQ^2 A_K C_\ell + \frac{hQ}{2} \sum_{a=0}^{\ell} \binom{\ell}{a} M(C_a, C_{\ell-a}). \quad (5.2)$$

The coordinates and products as well as the partial sums needed to compute the recurrence have

$$O(r(q_{\text{den}} + \log(r + 1)) + \log(D + 1)) \quad (5.3)$$

bits uniformly over the time intervals. The identities

$$\frac{\mathfrak{D}_{\ell+1}}{(\ell+1)\mathfrak{D}_\ell} = Q^2, \quad \frac{\mathfrak{D}_{\ell+1}}{(\ell+1)\mathfrak{D}_a \mathfrak{D}_{\ell-a}} = Q \binom{\ell}{a} \quad (5.4)$$

transform (4.3) into (5.2). The quantities Qy_i and hQ^2 as well as $hQ/2$ are integral because $q_{\text{den}} \geq \max\{b, g + 1\}$. This proves integrality by induction.

It remains to bound the integers arising before cancellation. By Lemma 4.2 every scalar Taylor coefficient has absolute value at most three. Hence $|C_{\ell,j}| \leq 3\ell! Q^{2\ell+1}$. This bound has $O(\ell \log(\ell+1) + (2\ell+1)q_{\text{den}})$ bits. A product in the bilinear map has at most the sum of the operand bit lengths while a binomial coefficient has at most $\ell+1$ bits. Summing over at most $D^2(\ell+1)$ terms increases the largest summand's bit length by at most the logarithm of this number plus a fixed constant. The linear term adds $O(g + q_{\text{den}} + \log(K+1))$ bits. These estimates give (5.3) for all intermediate values.

Rounding $c_{i,\ell}$ uses integer division after multiplication by 2^b . Summing the rounded coefficients at $\tau = 1$ is exact. The node induction bounds the integer part of the endpoint. Its fractional part still has b bits. Each interval therefore starts with the same denominator Q so denominators accumulated within one interval do not propagate into the next.

5.2 Parameter sizes and total operations

Write $M = m + 1$. The explicit formulas imply

$$D = O(M), \quad J = O(M^2), \quad g = O(\log(M+1)), \quad r, b, q_{\text{den}} = O(M^2). \quad (5.5)$$

The stability factor S enters the precision requirements through its logarithm. In fact $\log_2 S = 2D^2$ and $\log_2 R_K = O(D)$. Also $\log_2 \xi^{-1} = m + 2D^2 + O(1)$. Substitution into the formulas for r and b proves the asserted orders. The integer comparisons that compute these parameters operate on polynomially many bits.

There are $O(D^2)$ elementary coefficient operations in one cosine product and $O(r^2)$ pairs in the time convolution for a whole interval. Thus the construction has $O(JD^2r^2)$

Quantity	Uniform upper bound
Retained modes D	$O(m + 1)$
Time intervals J	$O((m + 1)^2)$
Taylor degree and dyadic precision r, b	$O((m + 1)^2)$
Intermediate integer length	$O((m + 1)^4)$
Arithmetic operations for construction	$O(JD^2r^2)$
Trajectory and residual payload	$O((m + 1)^7)$
Complete record with input copy	$O(n + (m + 1)^7)$
Construction and independent checking	$\text{poly}(n + m)$

Table 4. Cost bounds for the prescribed rational construction.

arithmetic operations on integers with the length in (5.3). Both factors are polynomial in $m + 1$. Schoolbook multiplication and division have quadratic cost in the operand length so including their bit cost preserves this polynomial bound. Reading the complete input and checking its weighted norm bound have polynomial cost. Projecting and rounding the retained coefficients also have polynomial cost in $n + m$. This proves the construction part of Theorem 5.1.

The stored coefficient integers have $O(b + 1)$ bits because the norm bound limits each coefficient to three and the grid is 2^{-b} . The number of entries is $JD(r + 1)$. Consequently the payload length is

$$O(JD(r + 1)(b + 1)) = O(M^7). \quad (5.6)$$

This seventh-power bound concerns stored bits. The running-time estimate also includes the operation count and the bit lengths of the intermediate integers. The optional expanded residual table has degree at most $2r$ in time and its dyadic denominators have exponent at most $2b + g$. Coefficient products and sums add $O(\log(Dr + 1))$ bits so its length obeys the same order. When present the canonical copy of the input adds $O(n)$ bits. Binary metadata and delimiters fit within these bounds.

5.3 A total verifier on finite records

Before forming large integers the verifier bounds the record length and the number of coefficients. It also bounds the lengths of the integer strings and the dyadic exponents. The estimates in Table 4 provide fixed polynomial limits containing every output of the construction. The external precision request and input length determine these limits. A submitted header cannot enlarge them. Scans and bounded integer comparisons suffice for these preliminary checks. Appendix B specifies the corresponding record bounds and their validation order.

Once these preliminary checks succeed expanding all residual polynomials uses $O(JD^2r^2)$ rational coefficient operations. The norm sums and continuity tests use fewer operations. Comparisons with the externally supplied initial table involve rationals of polynomial bit length in $n + m$. An unreduced common denominator can be taken as the product of the denominators appearing in that table. The bit length of this product

is at most their total input length. The checker then evaluates (3.5) exactly. Records outside the permitted bounds are rejected and records inside them are checked within a fixed polynomial budget. For arbitrary submitted records the cost includes their complete length. A streaming length check makes parsing at most linear in a rejected oversized record. This proves the verification claims of Theorem 5.1.

6 Uniform computation and composition

The rational Taylor construction gives complete error and cost bounds for the Neumann family. Another approach is to approximate an evolution by iteration on a space of trajectories. Contraction of the exact trajectory map then gives convergence. A polynomial implementation also requires a quantitative contraction gap and bounds on each finite update. It also needs a common domain preserved after approximation.

When local evolutions are composed both the state error and the quantities observed along the path have to be carried between stages. Stability controls the accumulated state error. A quadratic ledger records how transport and injection together with approximation change a chosen observation including the cross terms between old and new contributions.

6.1 An implemented contraction

Let x be a finite description of an equation together with its data and terminal time T_x . Its full length is $n = |x|$. On the declared admissible input set $\mathcal{D}$ let u_x be the exact trajectory with values in a Banach space X_x . The path norm is

$$\mathfrak{X}_x = C([0, T_x]; X_x), \quad \|w\|_{\mathfrak{X}_x} = \sup_{t \leq T_x} \|w(t)\|_{X_x}. \quad (6.1)$$

For evolution equations u_x is typically the fixed point of a Duhamel map $\mathcal{P}_x$. A finite path record z has a specified reconstruction $\text{Rec}_x z \in \mathfrak{X}_x$. A finite state record a has a specified reconstruction $\langle a \rangle_x \in X_x$. All coefficient and grid fields as well as precision fields count towards the record length. Working precision requests b are supplied to the procedures in unary as are the accuracy requests m .

Definition 6.1 (Uniform computation bounds). The encoded family has uniform computation bounds if fixed programs and fixed polynomials satisfy the following requirements for every $x \in \mathcal{D}$.

- (i) A polynomial-time compiler specifies a nonempty closed set $\mathcal{C}_x \subset \mathfrak{X}_x$ carrying a common stronger norm bound. The map $\mathcal{P}_x$ preserves this set and has fixed point u_x and satisfies

$$\text{diam } \mathcal{C}_x \leq 2^{B_x}, \quad \|\mathcal{P}_x v - \mathcal{P}_x w\|_{\mathfrak{X}_x} \leq c_x \|v - w\|_{\mathfrak{X}_x}, \quad 0 \leq c_x \leq 1 - a_x^{-1}. \quad (6.2)$$

The integers $a_x \geq 1$ and $B_x \geq 0$ as well as $b_{0,x} \geq 0$ are computable in polynomial time and have polynomial value bounds in n .

- (ii) For every $b \geq b_{0,x}$ a fixed record format defines $\mathcal{Z}_{x,b}$ with $\text{Rec}_x \mathcal{Z}_{x,b} \subseteq \mathcal{C}_x$. Every record has length at most a fixed polynomial $S(n, b)$. The initialization and update procedures in (iii) must satisfy these reconstruction and membership bounds.
- (iii) Uniform initialization and update procedures satisfy

$$\begin{aligned}
& \text{Init}(x, b) \in \mathcal{Z}_{x,b}, \\
& \text{Step}(x, z, b) \in \mathcal{Z}_{x,b}, \\
& \|\text{Rec}_x \text{Step}(x, z, b) - \mathcal{P}_x \text{Rec}_x z\|_{\mathfrak{X}_x} \leq 2^{-b} \quad (z \in \mathcal{Z}_{x,b}).
\end{aligned} \tag{6.3}$$

Their costs are polynomial in their complete input lengths. The update cost includes truncation and time approximation as well as rounding and coefficient evaluation and computation of the retained bounds.

- (iv) A uniform polynomial-time terminal procedure returns a state record $\text{End}(x, z, b)$ satisfying

$$\|\langle \text{End}(x, z, b) \rangle_x - (\text{Rec}_x z)(T_x)\|_{X_x} \leq 2^{-b}. \tag{6.4}$$

Its output length is polynomial in $n + b + |z|$.

All bounds and programs are fixed for the encoded family. The cost bounds include evaluation of nonrational basis functions together with integration and access to external data whenever these operations are used.

The integer a_x bounds the inverse contraction gap and therefore controls the number of updates. The diameter bound enters through its exponent B_x while $b_{0,x}$ gives the minimum working precision. These quantities need polynomial value bounds as stipulated in the definition. Bounds on the lengths of their encodings alone would not control the number of updates or working bits.

The implemented update must preserve the domain despite its approximation error. For example a strict margin between the exact image $\mathcal{P}_x \mathcal{C}_x$ and the boundary of a norm ball can accommodate the error in (6.3). A corresponding estimate must preserve the stronger norm bound at the specified precision. Record lengths must also remain within the stated bound after every update.

Theorem 6.2 (Uniform approximate iteration). Under Definition 6.1 one deterministic algorithm takes $x \in \mathcal{D}$ and 1^m and returns a path record z and a state record a such that

$$\|\text{Rec}_x z - u_x\|_{\mathfrak{X}_x} < 2^{-m}, \quad \|\langle a \rangle_x - u_x(T_x)\|_{X_x} < 2^{-m}. \tag{6.5}$$

Its cost and output length are polynomial in $n + m$. The records retain the common bounds declared by the computation interface.

Proof. Choose the precision to control accumulated update and terminal errors.

$$b = \max\{b_{0,x}, m + \lceil \log_2 a_x \rceil + 3\}, \quad N_{\text{it}} = a_x(B_x + m + 3). \tag{6.6}$$

Starting from $z_0 = \text{Init}(x, b)$ perform exactly N_{it} updates. Every record belongs to $\mathcal{Z}_{x,b}$ by induction. For $e_j = \|\text{Rec}_x z_j - u_x\|_{\mathfrak{X}_x}$ the diameter and update bounds give

$$e_0 \leq 2^{B_x}, \quad e_{j+1} \leq c_x e_j + 2^{-b}, \quad e_j \leq c_x^j 2^{B_x} + a_x 2^{-b}. \quad (6.7)$$

Since $c_x^{N_{\text{it}}} \leq e^{-N_{\text{it}}/a_x} \leq 2^{-N_{\text{it}}/a_x}$ the final path error is at most $2^{-m-3} + 2^{-m-3} = 2^{-m-2}$. The same assertion holds for $a_x = 1$ where $c_x = 0$ and $N_{\text{it}} \geq 1$. Terminal evaluation adds at most $2^{-b} \leq 2^{-m-3}$. This proves (6.5).

Let p_{comp} bound compilation and let p_{op} be a nondecreasing polynomial bounding initialization and a single update as well as terminal evaluation and integer bookkeeping. The total cost is at most

$$p_{\text{comp}}(n) + (N_{\text{it}} + 2)p_{\text{op}}(n + b + S(n, b)). \quad (6.8)$$

Every argument and multiplier is polynomially bounded in $n + m$. Membership in $\mathcal{Z}_{x,b}$ bounds each intermediate record so this estimate covers the entire computation and its output. $\square$

The same construction can be retained as finite execution evidence. The algorithm in Theorem 6.2 admits a polynomial-length execution record with a polynomial-time checker. On $\mathcal{D}$ every accepted execution record certifies (6.5). Record the compiled data and parameters together with the iterates and terminal output. The checker enforces their length bounds and recomputes the parameters. It replays the fixed programs and compares the result of each call with the recorded word. The number of calls and their costs obey (6.8). A polynomial clock makes the checker total on proposed records with parsing charged to their complete length. Acceptance identifies the record with the stated algorithm's output so the theorem supplies its accuracy.

The Neumann checker in section 3 verifies a trajectory through its residual inequalities regardless of the construction used to obtain its coefficients. The execution checker above verifies the prescribed sequence of calls. Its accuracy guarantee then uses the uniform update bounds of Definition 6.1.

6.2 A rational coupled system

The abstract requirements can be verified explicitly for a finite-dimensional linear system with polynomial forcing. The input consists of a rational d -by- d matrix B and a rational vector polynomial $p(t) = \sum_{\ell=0}^N p_\ell t^\ell$ with

$$\|B\|_{\infty \rightarrow \infty} \leq \frac{1}{8}, \quad \sum_{\ell=0}^N 2^\ell \|p_\ell\|_\infty \leq \frac{1}{2}. \quad (6.9)$$

Both the matrix and the coefficient vectors are stored densely including zero entries. Their complete encoding length n therefore bounds d and $N + 1$ as well as the total lengths of the rational numerators and denominators. The associated equation is

$$y'(t) = By(t) + p'(t), \quad y(0) = p(0), \quad 0 \leq t \leq 1, \quad (6.10)$$

with trajectory map

$$(\mathcal{P}v)(t) = p(t) + \int_0^t Bv(s) \, ds. \quad (6.11)$$

The state norm is the maximum norm on $\mathbb{R}^d$. We control the degree of the time approximation through the stronger coefficient bound

$$\|v\|_{\mathcal{T}_2} = \sum_{\ell \geq 0} 2^\ell \|v_\ell\|_\infty, \quad \mathcal{C} = \left\{ v(t) = \sum_{\ell \geq 0} v_\ell t^\ell : \|v\|_{\mathcal{T}_2} \leq 1 \right\}. \quad (6.12)$$

Each series converges absolutely on $[0, 1]$ with a geometric bound on its time tail. This stronger coefficient bound determines how many time coefficients suffice at each precision. Contraction in the path norm alone would not provide that bound.

Proposition 6.3 (A complete Picard realization). The family (6.9) satisfies Definition 6.1 with $X_x = (\mathbb{R}^d, \|\cdot\|_\infty)$ and $T_x = 1$. The remaining parameters are $a_x = 2$ and $B_x = 1$ with $b_{0,x} = 1$. At working precision $b \geq 1$ put

$$R = \max\{N, b + 3\}, \quad \beta = b + R + 4. \quad (6.13)$$

The records are vector polynomials of degree at most R with coefficients in $2^{-\beta}\mathbb{Z}^d$ and $\mathcal{T}_2$ norm at most one. Initialization is the zero polynomial. An update computes $\mathcal{P}v$ exactly and discards terms of degree greater than R and truncates each retained coefficient towards zero to this dyadic grid. Terminal evaluation sums the stored coefficients exactly. These procedures have polynomial bit cost in $n+b$ and their records have length $O(d(R+1)(\beta+1))$.

Proof. Invariant domain. The stronger ball (6.12) is closed in $C([0, 1]; \mathbb{R}^d)$ with the maximum norm. Suppose $v^{(j)} \in \mathcal{C}$ converges uniformly there. Each coefficient sequence is bounded by $2^{-\ell}$ so a diagonal subsequence has coefficient limits v_ℓ at every index. Fatou's inequality gives $\sum_\ell 2^\ell \|v_\ell\|_\infty \leq 1$. For every member of the sequence the tail after degree R has supremum norm at most 2^{-R-1} . Passing first to the limit in the finite coefficient sum and then letting R tend to infinity identifies the uniform limit with $\sum_\ell v_\ell t^\ell$. It belongs to $\mathcal{C}$. The coefficient representation is unique because its series is analytic for $|t| < 2$. The set is nonempty and has diameter at most two in the weaker path norm.

Integration and the matrix bound give

$$\begin{aligned} \|\mathcal{P}v\|_{\mathcal{T}_2} &\leq \frac{1}{2} + \sum_{\ell \geq 0} \frac{2^{\ell+1}}{\ell+1} \|Bv_\ell\|_\infty \\ &\leq \frac{1}{2} + 2 \|B\|_{\infty \rightarrow \infty} \|v\|_{\mathcal{T}_2} \leq \frac{3}{4}. \end{aligned} \quad (6.14)$$

In the weaker path norm

$$\|\mathcal{P}v - \mathcal{P}w\|_{C_t \ell^\infty} \leq \frac{1}{8} \|v - w\|_{C_t \ell^\infty}. \quad (6.15)$$

Thus $\mathcal{P}$ has a unique fixed point in $\mathcal{C}$ and the fixed-point equation differentiates to (6.10). Uniqueness also follows in the full continuous trajectory space by Gronwall's inequality. The choices $a_x = 2$ and $B_x = 1$ satisfy the required contraction and diameter bounds.

Finite updates. For a recorded polynomial v write $\mathcal{P}v = \sum_{\ell=0}^{R+1} w_\ell t^\ell$. Padding the supplied p by zero coefficients gives the exact formulas

$$w_0 = p_0, \quad w_\ell = p_\ell + \frac{1}{\ell} B v_{\ell-1} \quad (1 \leq \ell \leq R+1). \quad (6.16)$$

Since $R \geq N$ the only possible omitted coefficient is w_{R+1} . Its path contribution is at most $(3/4)2^{-R-1}$ by (6.14). Quantizing the retained coefficients contributes at most $(R+1)2^{-\beta}$ in the weaker path norm. Consequently the implemented update satisfies

$$\begin{aligned} \|\text{Rec}_x \text{Step}(x, z, b) - \mathcal{P} \text{Rec}_x z\|_{C_t \ell^\infty} \\ \leq \frac{3}{4} 2^{-R-1} + (R+1)2^{-\beta} < 2^{-b}. \end{aligned} \quad (6.17)$$

For the last inequality use $R \geq b+3$ and $R+1 \leq 2^R$ for $R \geq 1$. Discarding coefficients and truncating each scalar towards zero cannot increase their maximum norms. Thus the updated $\mathcal{T}_2$ norm is at most $3/4$ and the record remains in the specified set. The zero initial record belongs to it and terminal evaluation by summation has zero error.

Integer lengths and cost. The compiler verifies (6.9) by rational comparisons. The powers 2^ℓ have $O(N+1)$ bits. A product of all supplied denominators has bit length at most their total input length. It therefore supplies a polynomial-length common denominator for these checks. From (6.13) we have $R, \beta = O(n+b)$. A recorded coefficient has magnitude at most one so its dyadic numerator has $O(\beta+1)$ bits. Membership in the record set is decided by the exact weighted coefficient sum.

Computing (6.16) uses $O(d^2(R+1))$ rational operations. A common denominator for any coordinate divides the product of the input denominators 2^β and the relevant integer ℓ . Its bit length is $O(n+\beta+\log(R+1))$. The corresponding numerators and partial sums also have polynomial bit lengths by the input norm bounds and the polynomial number of terms. Dyadic truncation is integer division. Each update returns to the same dyadic grid so denominators from different iterations do not accumulate. The record length is as stated. Initialization and terminal summation have smaller polynomial costs. This completes the verification of Definition 6.1. $\square$

For requested accuracy 2^{-m} Theorem 6.2 uses $b = m+4$ and $N_{\text{it}} = 2(m+4)$ in this realization. The finite coefficient rule (6.16) allows an independent checker to evaluate the difference between each recorded update and its exact Picard image. The sum of the maximum norms of those difference coefficients bounds the path error in that step.

6.3 The spectral part of an update

In a spectral implementation one part of the update error comes from discarding high modes. The cutoff and coefficient precision must control this error uniformly over the admissible records. Suppose the exact update of any admissible record has a tail bound

$$\|(I - P_K) \mathcal{P}_x \text{Rec}_x z\|_{\mathfrak{X}_x} \leq 2^{\ell_x} e^{-\delta_x K}, \quad 0 < \delta_{0,x} \leq \delta_x, \quad (6.18)$$

where the integer $\ell_x \geq 0$ and rational $\delta_{0,x}$ are supplied by the compiler. The integer cutoff

$$K_x(b) = \left\lceil \frac{b + \ell_x + 3}{\delta_{0,x}} \right\rceil \quad (6.19)$$

makes this contribution at most 2^{-b-3} . If ℓ_x and $\delta_{0,x}^{-1}$ have polynomial value bounds and $\text{rank } P_K \leq C_x(1+K)^d$ with fixed d and polynomially bounded C_x then the retained dimension is polynomial in $n+b$. Mode enumeration and its arithmetic must have corresponding cost bounds.

Let D_K denote this dimension. For a weighted Hilbert norm diagonal in an orthonormal retained basis let $W_x(K)$ be the largest weighted norm of a retained basis vector. A coefficient error at most $2^{-\beta}$ in each coordinate contributes at most $W_x(K)\sqrt{D_K}2^{-\beta}$. A supplied integer bound

$$W_x(K)\sqrt{D_K} \leq 2^{\chi_x(K)} \quad (6.20)$$

therefore permits $\beta = b + \chi_x(K) + 3$. Together with polynomial bounds for $\chi_x(K_x(b))$ and for the time representation and coefficient evaluation these estimates control the spectral contribution to the step cost. Their uniformity over exact images of admissible records is needed because each implemented update approximates one of those images.

6.4 Joining local calculations

When local computations are joined an error from one interval becomes part of the initial error for the next. The precision must therefore account for its amplification by all subsequent stability factors.

Proposition 6.4 (Finite composition). Suppose L successive computations have state errors satisfying

$$e_{j+1} \leq A_j e_j + \delta, \quad 0 \leq A_j \leq 2^{\beta_j}, \quad \beta_j \geq 0, \quad \sum_{j < L} \beta_j \leq B_{\text{prop}}. \quad (6.21)$$

Then

$$e_L \leq 2^{B_{\text{prop}}}(e_0 + L\delta). \quad (6.22)$$

If $e_0 \leq \delta$ then choosing

$$\delta = 2^{-b_{\text{loc}}}, \quad b_{\text{loc}} = m + \lceil B_{\text{prop}} \rceil + \lceil \log_2(L+1) \rceil + 2 \quad (6.23)$$

makes $e_L \leq 2^{-m-2}$.

Suppose a bound $L_{\text{max}} \geq L$ and B_{prop} are supplied in polynomial time with polynomial value bounds in the original input length. Assume the schedule has polynomial total cost. Every local call is required to satisfy Definition 6.1 on its actual initial states with common cost polynomials. All local input and transfer records must have polynomial length in the original input and precision and initialization and transfers must meet their allocated accuracy and cost bounds. Then the composed computation has polynomial total bit cost using L_{max} in (6.23) when necessary.

Proof. Expanding the recurrence gives

$$e_L \leq \left(\prod_{i=0}^{L-1} A_i \right) e_0 + \delta \sum_{j=0}^{L-1} \prod_{i=j+1}^{L-1} A_i. \quad (6.24)$$

Every product including an empty one is at most $2^{B_{\text{prop}}}$. This proves (6.22) and the precision assertion. The chosen local precision is polynomially bounded. Theorem 6.2 therefore applies with uniform polynomial cost at each of at most L_{max} calls. Adding the scheduling costs together with initialization and transfer costs proves the last claim. $\square$

The local stability factors A_j may exceed one. Truncation and quantization as well as transfer contribute to δ . A restart must also place the transferred state in the domain where the next local estimate applies.

6.5 Finite checking of quadratic ledgers

The state-error estimate makes a computed path usable as an approximation to the solution. A quadratic ledger makes the changes in a chosen observation traceable along that path. For a projection the discarded part can be recorded explicitly. For an injected contribution its interaction with the old state must also be included. The finite check therefore starts from the exact one-step identity before replacing signed terms by rational upper bounds.

Consider stages $r = 0, \dots, N$ with rational vectors $x_r \in \mathbb{Q}^{d_r}$ and rational symmetric matrices $Q_r \in \mathbb{Q}^{d_r \times d_r}$. For $r < N$ let $T_r \in \mathbb{Q}^{d_{r+1} \times d_r}$ and $j_r \in \mathbb{Q}^{d_{r+1}}$ be the declared transport and injection. The checker derives the discrepancy

$$e_r = x_{r+1} - T_r x_r - j_r \quad (6.25)$$

from the submitted states rather than trusting a separate error field. Put

$$\Delta_r = Q_r - T_r^\top Q_{r+1} T_r, \quad (6.26)$$

$$D_r = x_r^\top \Delta_r x_r, \quad (6.27)$$

$$J_r = 2(T_r x_r)^\top Q_{r+1} j_r + j_r^\top Q_{r+1} j_r, \quad (6.28)$$

$$A_r = 2(T_r x_r + j_r)^\top Q_{r+1} e_r + e_r^\top Q_{r+1} e_r. \quad (6.29)$$

All four quantities are exact rationals. The entries J_r and A_r can have either sign. The record may supply rational upper bounds ι_r and ϵ_r . The checker compares these bounds with the exact entries.

Positive semidefiniteness is also part of the finite evidence. We use a rational factorization certificate

$$\Pi_r^\top \Delta_r \Pi_r = L_r S_r L_r^\top, \quad (6.30)$$

where Π_r is a permutation matrix and L_r is rational lower triangular and S_r is rational diagonal with nonnegative entries. The same format may certify $Q_r \geq 0$ when positivity is not already part of fixed compiled data. Exact multiplication and comparison verify the certificate. A zero diagonal entry is allowed. No positive-definiteness assumption is hidden in the format.

Theorem 6.5 (Sound finite quadratic ledger). Suppose a finite rational ledger passes the dimension and symmetry checks as well as the update checks. Suppose it also passes the positivity and defect-factorization checks together with the majorant checks just described. Then

$$x_N^\top Q_N x_N + \sum_{r < N} D_r \leq x_0^\top Q_0 x_0 + \sum_{r < N} \iota_r + \sum_{r < N} \epsilon_r, \quad D_r \geq 0. \quad (6.31)$$

The checker is total on finite submissions after imposing explicit bounds on dimensions and field lengths as well as the complete record length. Its bit cost is polynomial in that complete length. If a construction produces such ledger records with polynomial length and polynomial bit cost then adjoining the checker preserves those polynomial bounds.

Proof. The factorization (6.30) proves $\Delta_r \geq 0$ and hence $D_r \geq 0$. Direct expansion of (6.25) gives the exact identity

$$x_{r+1}^\top Q_{r+1} x_{r+1} + D_r = x_r^\top Q_r x_r + J_r + A_r. \quad (6.32)$$

Replacing J_r and A_r by their checked upper bounds and summing over $r < N$ proves (6.31).

For the cost assertion the checker first reads bounded dimension and length fields and rejects a record outside the declared envelope. It then performs a polynomial number of rational matrix products and dot products together with comparisons in the submitted dimensions. All identities being checked have fixed algebraic degree in the submitted rational entries. A common denominator formed from the input denominators has bit length at most their total encoded length and raising it only to those fixed degrees preserves a linear bit bound. The numerators are sums of only polynomially many fixed-degree products and consequently have polynomial bit length in the complete record. Schoolbook integer arithmetic therefore gives a polynomial bit bound. A polynomial-time producer adds only a polynomial amount of data and work. $\square$

The positive-defect relation specifies which transports are compatible with the declared quadratic observation. It is proved for the model or verified by the matrix factorization in the submitted record. Together with residual soundness the ledger gives two statements about the same path. The residual bounds its distance from the prescribed evolution and the finite identities account for the changes in the observation. The latter identities alone concern the submitted updates so the residual test remains the link to the governing equation.

For the rational cosine coordinates of the Neumann calculation the quadratic form

$$q_K(c) = 2c_0^2 + \sum_{j=1}^K c_j^2 \quad (6.33)$$

is $2/\pi$ times the $L^2(0, \pi)$ norm squared of $\sum_{j=0}^K c_j \cos(jx)$. Its Gram matrix is rational and diagonal. Every coordinate cutoff P_L with $0 \leq L \leq K$ commutes with this matrix and therefore has the exact finite split

$$q_K(c) = q_K(P_L c) + q_K((I - P_L)c). \quad (6.34)$$

Thus a submitted Neumann trajectory may be accompanied by exact rational window ledgers. The present trajectory format does not require this optional evidence for acceptance. Theorem 3.1 continues to rest on its initial and residual checks together with the tail checks.

6.6 Certified finite response operators

A source response may be specified by a convergent sum of finite operators. A calculation retains only finitely many terms and represents their coefficients by rationals. Its residual can then be checked exactly but the omitted terms and coefficient errors still act on the computed path. Bounding that action converts the finite residual into an error estimate for the full response. Approximating the observation requires a further readout estimate.

Let $\mathbf{L} \geq 0$ be a bounded self-adjoint operator on a Hilbert space with a declared orthonormal coordinate system. Let P retain a finite coordinate set and let $\widehat{\mathbf{L}} = P\mathbf{L}P$ be a rational symmetric matrix extended by zero. Suppose verified rational bounds are available for

$$\|\mathbf{L}\| \leq \Lambda, \quad \|\mathbf{L} - \widehat{\mathbf{L}}\| \leq \Delta, \quad \|C\| \leq 1, \quad \|(C - \widehat{C})P\| \leq \eta_C, \quad (6.35)$$

where $\widehat{C}$ is a rational readout on the retained coordinates. These are model estimates or consequences of checked source enclosures rather than unverified numerical fields in a submitted record.

Consider a continuous piecewise rational polynomial $w : [0, S] \rightarrow P\mathcal{H}$ where S is positive rational. Rational coefficient sums give upper bounds for its Hilbert norm and integrated residual. On each interval the sum of the coefficient norms bounds the polynomial in normalized local time and the interval length bounds its integral. One-norm bounds for the coefficient vectors keep these tests rational. The initial discrepancy is supplied by an enclosure of the source state including its omitted tail. This supplies the initial-state test in the same form as the operator and readout enclosures.

Proposition 6.6 (Finite response acceptance). Suppose the structural checks and (6.35) are verified together with rational bounds

$$\|f_0 - w(0)\| \leq E_0, \quad \sup_{\tau \leq S} \|w(\tau)\| \leq W, \quad \int_0^S \|w' + \widehat{\mathbf{L}}w\| \, d\tau \leq D. \quad (6.36)$$

Put $E = E_0 + D + S\Delta W$. For $f(\tau) = e^{-\tau\mathbf{L}}f_0$ and every $\tau \in [0, S]$ we have

$$\|f(\tau) - w(\tau)\| \leq E, \quad (6.37)$$

$$\|Cf(\tau) - \widehat{C}w(\tau)\| \leq E + \eta_C W, \quad (6.38)$$

$$\left| \langle f(\tau), \mathbf{L}f(\tau) \rangle - \langle w(\tau), \widehat{\mathbf{L}}w(\tau) \rangle \right| \leq \Lambda(2W + E)E + \Delta W^2. \quad (6.39)$$

Once the source enclosures have been verified the rational matrix and polynomial tests together with the acceptance tests have polynomial bit cost in their complete explicit encoding length with the preliminary length bounds used for the other finite checkers.

Proof. The full residual is $w' + \mathsf{L}w = (w' + \widehat{\mathsf{L}}w) + (\mathsf{L} - \widehat{\mathsf{L}})w$. Contractivity of $e^{-\tau\mathsf{L}}$ and variation of constants give the first estimate. The decomposition $C(f - w) + (C - \widehat{C})w$ gives the second. For the third estimate add and subtract $\langle w, \mathsf{L}w \rangle$. The part with the same operator is at most $\Lambda(\|f\| + \|w\|)\|f - w\|$ and the remaining part is at most $\Delta\|w\|^2$. Use $\|f\| \leq W + E$. Residual expansion and coefficient sums together with the displayed comparisons involve polynomially many rational operations of fixed algebraic degree. Their integer lengths and bit costs are polynomial in the complete submitted length as in the proof of Theorem 6.5. $\square$

The history source in [1] illustrates the enclosures. With $I_\beta(e) = e^{-\beta n(e)^2} p(e)$ and $b_e = |e\rangle - |e_L\rangle - |e_R\rangle$ its finite parent response is $\mathsf{L}_{\beta,N} = \sum_{n(e) \leq N} I_\beta(e) b_e b_e^\top$. Here $\sum_{n(e)=n} p(e) = 1$ and $\|b_e\|^2 \leq 5$. Consequently the operator tail is at most $5 \sum_{n > N} e^{-\beta n^2}$. If nonnegative rational weights $\widehat{I}(e)$ have checked enclosures with $\sum_{n(e) \leq N} |I_\beta(e) - \widehat{I}(e)| \leq \epsilon_I$ then

$$\widehat{\mathsf{L}}_{\beta,N} = \sum_{n(e) \leq N} \widehat{I}(e) b_e b_e^\top \geq 0, \quad \|\mathsf{L}_\beta - \widehat{\mathsf{L}}_{\beta,N}\| \leq 5 \sum_{n > N} e^{-\beta n^2} + 5\epsilon_I. \quad (6.40)$$

The rank-one construction preserves positivity exactly. Gaussian series and their tails supply rational upper bounds when β is effectively specified with a positive lower bound. Square roots in the source readout $C_p|e\rangle = \sqrt{p(e)}|n(e)\rangle$ can likewise be enclosed by rational intervals. The Frobenius norm bounds the resulting finite matrix error. A rounded readout need not be an exact coisometry and η_C accounts for that change.

The quadratic response estimate compares both the state and the operator. A rational ledger for the finite states can be checked by Theorem 6.5. The response estimate then connects its terminal observation to the full operator.

The Gaussian root mass supplies the analytic tail estimate. Constructing its rational enclosures and enumerating the retained histories are separate computational tasks charged to the construction. Once those enclosures have been verified checking the explicit rational data has the polynomial cost just proved. The Neumann construction and acceptance format in sections 3–5 use their own cosine representation and model bounds.

6.7 Observations and effective presentations

Let O_x be defined on the exact terminal state and reconstructed outputs with a common nondecreasing modulus

$$|O_x(v) - O_x(w)| \leq \omega_x(\|v - w\|_{X_x}), \quad \omega_x(h) \longrightarrow 0 \quad (h \downarrow 0). \quad (6.41)$$

Suppose an integer $m_x(q)$ can be computed in polynomial time and has polynomial value bound in $n + q$ and satisfies $\omega_x(2^{-m_x(q)}) \leq 2^{-q-1}$. If evaluating O_x on a finite state record to error 2^{-q-1} also has polynomial cost then the state algorithm at precision $m_x(q)$ followed by this evaluation has error at most 2^{-q} and polynomial cost. This follows by the triangle inequality. A Lipschitz bound 2^{d_x} permits $m_x(q) = q + d_x + 1$. The two Neumann observations have $d_x = 0$.

For spectral probability states Appendix C derives a common entropy modulus from an exponential certificate and a trace bound. Coefficient transforms have the spectral-norm

bounds given in Appendix D. Discrete outputs require a successful readout margin as well as state accuracy as formulated in Appendix E.

An individual accepted record describes a finite approximation. To present the exact limit effectively one also needs a procedure generating approximations at increasing precision together with a procedure to evaluate them and a computable modulus of convergence. Here a finite history lists the permitted construction operations and their input sources and $\text{Val}(H)$ denotes its value. Fix a stronger normed space Y^+ continuously embedded in a complete weaker space Y^- and source and resource rules admitting the histories under consideration. Assume Y^- has an effectively indexed dense set whose mutual distances are uniformly computable.

Definition 6.7 (Effective certified presentation). An effective certified presentation supplies a sequence of finite histories H_j and a common verified bound in Y^+ together with the following computational data.

- (i) a total procedure returning H_j from j .
- (ii) a uniform procedure returning a named approximation to $\text{Val}(H_j)$ with Y^- error at most 2^{-q} from (j, q) .
- (iii) a computable Cauchy modulus $N(q)$ such that

$$\|\text{Val}(H_j) - \text{Val}(H_k)\|_{Y^-} \leq 2^{-q} \quad (j, k \geq N(q)); \quad (6.42)$$

- (iv) computable verified upper bounds for the common stronger certificate and the realization and operation constants used in these approximations.

The resource indices and parameters as well as scalar data have effective names. Relative inputs also have effective names. When they are supplied by specified oracles the procedures are relative to those oracles. The histories and their limit presentation obey the declared source and resource rules. The completed value is the limit in Y^- .

In the usual Cauchy representation a name provides approximations with a prescribed error. An effective sequence with a computable convergence modulus supplies such a name [9]. The additional information retained here is the construction history together with a verified common bound in Y^+ and bounds for the realization and operation constants. The stronger bound controls spectral tails when a corresponding embedding estimate is available while the operation bounds supply quantitative data for subsequent admitted constructions. For fixed source and resource rules values admitting such a presentation are called effectively generatively regular.

An effective certified presentation therefore computes a named approximation to its completed value with Y^- error at most 2^{-q} for every $q \geq 1$. In (6.42) take $j = N(q+1)$ and let k tend to infinity. The difference from the completed value is at most 2^{-q-1} . Evaluate H_j to error 2^{-q-1} and add the two bounds.

This definition adds a generating program and a computable modulus to the certified limits of the companion analysis [12]. Polynomial cost requires the additional bounds on running time and precision established above.

Effective presentation of the Neumann solution. Fix an input (2.2) and a rational $t \in [0, T]$. For source and resource rules admitting the rational cosine histories and the resource pair in Table 2 the histories H_m obtained by evaluating the trajectory records of Theorem 4.1 at t have values $w_m(t)$ and give an effective relative generated presentation of $u(t)$. The common stronger Hilbert bound is six and one Cauchy modulus in the weaker Hilbert norm is $N(q) = \max\{1, q\}$. Indeed each piece of a record has a finite Horner presentation. At rational t selecting its interval and evaluating the time polynomial are exact rational operations. They leave a finite rational cosine sum. The node arithmetic and the input data retain their declared sources. The coefficient norm bound and (2.7) give the stronger bound $3\sqrt{\pi} < 6$. The same comparison and (4.5) give weaker Hilbert error less than $\sqrt{\pi} 2^{-m-2}$. Thus for $j, k \geq q$ we have

$$\|w_j(t) - w_k(t)\|_{Y^-} < \sqrt{\pi}(2^{-j-2} + 2^{-k-2}) < 2^{-q}. \quad (6.43)$$

The coefficients and their weighted finite distances together with the fixed normalization constants are computable. Rational cosine polynomials give an effective dense set in Y^- with uniformly computable distances obtained from their finite weighted L^2 sums. The rational construction and the explicit bounds therefore supply all four parts of Definition 6.7.

7 Numerical examples and certificate verification

The computations use exact integer and rational arithmetic to construct and verify trajectories for the input family of section 2. The verifier evaluates the finite hypotheses of Theorem 3.1 directly on the recorded coefficients. Input normalization and rational parsing are shared with the construction program. The two programs also share admissibility tests and parameter bounds. The acceptance proof is independent of the Taylor recurrence.

7.1 Construction and residual verification

With the guaranteed parameter choice the construction uses (4.1) and the integer recurrence (5.2). It truncates nonconstant time coefficients towards zero and takes the exact sum as the next initial node. The coefficient bound identifies Taylor rows that round to zero so they need not be computed. At each computed node the producer checks the bound $9/4$ in $\mathcal{A}_4$. For

$$\ell > \left\lfloor \frac{b+4}{3} \right\rfloor, \quad \frac{11}{4} 8^{-\ell} < 2^{-b}. \quad (7.1)$$

Lemma 4.2 therefore shows that every coordinate of $c_{i,\ell}$ truncates to zero. The recurrence need only be computed through the smaller of r and the displayed threshold. The remaining rows of the degree- r record are filled with zero. The resulting record is identical to that obtained from the full recurrence.

The practical calculations use smaller parameters within the same record bounds. These parameters were fixed before construction and the completed trajectories were accepted or rejected by the exact residual test.

For guaranteed records the checker also reports the stronger initial and residual comparisons with ξ . These directly test the construction bounds.

Case	Nonzero coefficients or full coefficient rule
E1-baseline	$a_0 = 1/4, a_1 = 1/16, a_2 = 1/64$
E2-constant	$a_0 = 1/2$
E3-signed	$(a_0, a_1, a_2, a_3) = (3/8, -1/32, 1/128, -1/1024)$
E4-boundary	$a_0 = 31/32, a_1 = 1/128$
E5-high-tail	$a_0 = 1/4, a_{20} = 1/(2 \cdot 4^{20})$
E6-den8	$(a_0, a_1, a_2) = (1/3, 1/(4(2^8 + 1)), -1/(16(2^8 + 1)))$
E6-den128	Replace $2^8 + 1$ in E6-den8 by $2^{128} + 1$
E7-sparse	$a_0 = 1/4, a_6 = 4^{-7}$ with last index six
E7-dense	$a_j = (-1)^j/(8 \cdot 4^j)$ for $0 \leq j \leq 6$

Table 5. The nine initial data in the experiment matrix.

Case	Input bytes	R_*	E_{cert}	Record bytes	Time (s)
E1-baseline	163	1.134×10^{-14}	4.070×10^{-4}	21858	0.205/0.026
E2-constant	137	2.239×10^{-16}	4.070×10^{-4}	15029	0.010/0.012
E3-signed	181	1.403×10^{-14}	4.070×10^{-4}	22814	0.150/0.029
E4-boundary	153	2.642×10^{-15}	4.070×10^{-4}	17840	0.145/0.012
E5-high-tail	351	1.980×10^{-16}	4.070×10^{-4}	14979	0.011/0.010
E6-den8	168	3.581×10^{-15}	4.070×10^{-4}	18594	0.154/0.015
E6-den128	240	2.313×10^{-16}	4.070×10^{-4}	15056	0.009/0.008
E7-sparse	203	3.570×10^{-15}	4.070×10^{-4}	16324	0.047/0.010
E7-dense	227	2.386×10^{-14}	4.070×10^{-4}	26693	0.154/0.045

Table 6. Practical runs at $m = 8$ using (7.2). The last column gives construction/checking times. Residual and error displays are rounded upward. The acceptance tests use exact rationals.

7.2 Inputs and accuracy requests

The nine rational initial tables test signed coefficients and high modes within the declared family. They also vary the denominator lengths and the patterns of zero coefficients. Unlisted coefficients are zero. Each stored input contains the complete table through its specified last index.

The boundary case has $\mathcal{A}_4$ norm exactly one. The high-tail case has support beyond the cutoff in every reported practical run. The two denominator cases keep the number of modes fixed while changing the input bit length. The last two cases use the same dense table length with different patterns of zero coefficients. To check certificates at practical parameter choices each input was run at $m = 1, 4, 8, 12$ with

$$K = m + 3, \quad J = 16, \quad r = m + 8, \quad b = 64. \quad (7.2)$$

All 36 records were accepted. The parameter choice of Theorem 4.1 was also used for the constant datum at $m = 1, 2$ and the mixed-sign datum at $m = 1$. All three guaranteed records were accepted and satisfied the stronger $\mathcal{A}_4$ comparisons with ξ .

Table 6 displays the $m = 8$ slice. Its common spatial allowance is 5/12288 while the residual column distinguishes the individual records. Table 7 summarizes all four

m	Record bytes	Checking time (s)	Max. time/spatial ratio
1	[4833, 9786]	[0.0032, 0.0097]	6.96×10^{-15}
4	[8450, 17031]	[0.0052, 0.0194]	1.67×10^{-13}
8	[14979, 26693]	[0.0079, 0.0455]	5.33×10^{-12}
12	[23556, 36402]	[0.0114, 0.0570]	1.39×10^{-10}

Table 7. Ranges over the nine practical records at each precision with the parameters in (7.2). Byte counts are exact file lengths. Time endpoints are rounded outwards to four decimal places and the maximal ratios upwards to three significant figures.

Case	m	J	r	b	E_{cert}	Record bytes	Time (s)
E2-constant	1	1024	75	96	5.209×10^{-2}	1819895	2.320/0.824
E2-constant	2	2048	99	122	2.605×10^{-2}	5641558	12.846/2.672
E3-signed	1	1024	75	96	5.209×10^{-2}	2219525	47.129/1.291

Table 8. Complete runs with the parameters proved in Theorem 4.1. Here $K = m + 3$. Byte counts are exact record file lengths. The last column gives construction/checking times.

m	K	r	R_*	E_{cert}	Record bytes	Time (s)
4	7	8	7.864×10^{-13}	6.511×10^{-3}	3942	0.006/0.005
6	9	10	5.840×10^{-15}	1.628×10^{-3}	4854	0.040/0.010
8	11	12	6.425×10^{-15}	4.070×10^{-4}	5640	0.019/0.012

Table 9. Baseline records with $J = 4$ and $b = 64$. The final column gives construction/checking times. The accompanying files contain the exact coefficients and error bounds.

precisions. The last column is the largest ratio of the time and rounding contribution $(d_0 + TR_*)/(1 - 5T)$ to the common spatial allowance $(5/3)2^{-(K+1)}$ over the nine inputs. The ratios are computed from the exact rational error bounds of the recorded trajectories.

The spatial allowance dominates the certificate at every reported precision. This term uses one bound for the whole input family. For constant initial data the exact solution and every Galerkin solution remain in the zeroth mode so the spatial error is zero. The positive allowance in the common test is an overestimate for this input. Reducing the time residual improves the second term of the certificate while leaving this allowance unchanged.

The guaranteed runs in Table 8 use the parameters that prove success throughout the input family. At $m = 1$ they give $K = 4$ and $J = 1024$ with $r = 75$ and $b = 96$. The practical records at this precision use the same cutoff with $J = 16$ and $r = 9$ at $b = 64$. The residual inequality certifies each completed practical record. For constant input $a_0 = 1/2$ the exact terminal value is $16/31$. Its comparison with the reconstructed mean gives an additional check in every constant-input run.

7.3 A baseline and its error allocation

For the baseline datum in Table 5 we also report calculations on four time intervals with degree $m + 4$ and 64 fractional bits at $m = 4, 6, 8$. The baseline calculation in Table 9 and

the finer partition in Table 6 give two parameter choices for the same $m = 8$ target. Both pass the checker. Their common spatial allowance accounts for almost all of the certified error so the additional time coefficients have little effect on the displayed total.

The reported times are representative single runs with Python 3.12.14 under Windows 11 with interpreter startup excluded. They describe the present implementation on the listed inputs and precisions. The error bounds themselves are exact rational quantities. The polynomial cost bound is the one established in section 5.

7.4 Direct Picard records

The Picard calculations use the rational matrix and polynomial inputs of Proposition 6.3. The recorded dyadic iterates allow each update error to be checked directly against the exact Picard image.

For a recorded iterate $v^{(j)}$ the checker computes the exact coefficient array of $\mathcal{P}v^{(j)}$ using (6.16). With zero padding through degree $R + 1$ set

$$\delta_j = \sum_{\ell=0}^{R+1} \left\| v_\ell^{(j+1)} - (\mathcal{P}v^{(j)})_\ell \right\|_\infty, \quad (7.3)$$

$$E_0 = 1, \quad E_{j+1} = \frac{1}{8}E_j + \delta_j.$$

The weighted coefficient sum checks the common strong bound. The unweighted sum δ_j bounds the path error of the update. Combining this bound with (6.15) gives the recursion for the distance to the exact fixed point. Since the first iterate is zero the initial bound $E_0 = 1$ follows from the fixed point's membership in $\mathcal{C}$. Here the checked discrepancy is from an exact Picard update whereas the Neumann checker uses the differential residual of one stored path. Terminal summation adds no error. Thus $E_{N_{\text{it}}}$ is an exact rational error bound for both the final path and terminal vector.

The eight inputs include constant data and polynomial forcing in an uncoupled two-component system. They also include nilpotent couplings in dimensions two and four. The remaining cases use a dense five-dimensional matrix with mixed signs and longer rational denominators. The inputs also test data on the two norm boundaries and a scalar equation with a degree-30 forcing polynomial. To check the Picard updates and their accumulated error bounds each was computed at $m = 1, 6, 12, 20$. All 32 records passed the independent checks. Table 10 gives the $m = 12$ slice. The complete input files specify every rational coefficient.

For nilpotent matrices the matrix-power expansion terminates and gives an exact polynomial solution. In the two-component case $B_{12} = 1/8$ is the only nonzero matrix entry $p_0 = (1/4, -1/8)$ and $p_1 = (1/16, 1/32)$. The exact terminal vector is $(153/512, -3/32)$. Its comparison with the recorded output checks the final error independently of the step budget. In all 16 runs with zero or nilpotent coupling a separate exact polynomial calculation also bounds the error over the full time interval. Each comparison lies within the independently checked path budget.

The complete execution record including the input copy and all iterates has length $O(n + (N_{\text{it}} + 1)d(R + 1)(\beta + 1))$. The fixed-record bound in Proposition 6.3 concerns one

Case	d	N	R	$E_{N_{it}}$	Record bytes	Time (s)
P1-constant	1	0	19	1.263×10^{-29}	4755	0.131/0.254
P2-forcing	2	3	19	1.263×10^{-29}	9304	0.289/0.563
P3-nilpotent2	2	1	19	1.263×10^{-29}	8720	0.284/0.465
P4-nilpotent4	4	2	19	3.270×10^{-12}	17463	0.770/1.115
P5-dense5	5	4	19	6.047×10^{-12}	22338	1.044/1.541
P6-den96	2	1	19	1.348×10^{-11}	9584	0.279/0.541
P7-boundary	2	1	19	8.405×10^{-12}	10533	0.335/0.511
P8-highdegree	1	30	30	1.076×10^{-12}	9158	0.193/0.512

Table 10. Direct Picard records at $m = 12$. Error displays are rounded upward from the rational bounds in (7.3). The last column gives construction/checking times.

iterate. The additional factor here counts the stored iteration history. As for the Neumann records the encoded length and the dimensions of each coefficient array are bounded before integer conversion.

7.5 Validation of the recorded computations

All 42 Neumann records and 32 Picard records were checked again from their stored coefficients and separately supplied inputs. The Neumann checks use the expanded differential residual. The Picard checks use the exact update discrepancies and their accumulated error bounds. Neither check requires rerunning the corresponding construction program.

The validation was designed around the hypotheses used in the two soundness arguments. For the Neumann records it covered input quantization and data above the computational cutoff. It also tested continuity at time nodes together with the norm and residual inequalities. It also included a polynomial trajectory obtained independently of the Taylor recurrence. The same residual criterion accepted that trajectory. For the Picard records the checks covered signed truncation and the stronger coefficient bound as well as step discrepancies and terminal summation. Altered inputs and parameters as well as coefficient arrays were rejected when they violated the corresponding conditions. In total 22 Neumann and 15 Picard validation cases passed their prescribed outcomes.

The accompanying computational material contains the complete inputs and trajectory records together with rational verification reports. It also documents the encoding checks and reproduction procedure together with the tests of outward rounding used in the tables.

8 Conclusion

For the stated Neumann family a finite trajectory can be checked against the original equation without reconstructing the calculation that produced it. The residual criterion combines exact rational tests with the spectral tail estimate to certify accuracy throughout the prescribed interval. Its proof applies to any coefficients satisfying the record conditions. The Taylor construction supplies an accepted record for every admissible input and requested precision. Its convergence rate gives an effective presentation of the solution.

The quadratic ledger records a complementary property of that trajectory. Compatible transport divides a declared positive quantity into retained and defect parts. Injection and finite-arithmetic error contribute signed increments that include their cross terms with the state. The rational checker verifies these relations and sums them along the construction. Together the two certificates state how accurately the path follows the equation and how the chosen quadratic observation changes along it. In cosine coordinates a spectral projection gives an exact diagonal split. More general observations retain their rational Gram matrices and mixed terms.

For a response specified by a convergent source construction the same principle also requires certified operator enclosures. The omitted operator and coefficient rounding act on the computed state while readout approximation affects the reported observation. Their bounds therefore enter alongside the finite-equation residual. In the normalized history source Gaussian mass at each root controls the operator tail. A component invisible to the initial value readout can affect its later response. Retaining that component or its contribution after elimination preserves the equation being certified.

The bit-cost proof follows the actual quantized restart values and controls intermediate denominators and integers before counting operations. It gives polynomial bounds for construction and checking as well as record length in the full encoded input length and the requested number of accuracy bits. The coupled rational linear system gives a second realization through quantized Picard iteration. Its coefficient bound controls truncation and preserves the invariant set. For the general ledger checker the cost is polynomial in the complete explicit record length. A normalized history law can simplify analytic bounds while leaving many coordinates to enumerate. Storing and updating them also have a cost. Those operations remain part of construction cost.

The reported Neumann calculations also distinguish the size of a bound from the error actually incurred. In several runs the common spatial allowance dominates the certificate. It remains positive for constant data even though their spatial truncation error vanishes. An input-dependent tail estimate can remove this contribution. Refining the time partition alone leaves it unchanged. The residual test supports both the prescribed uniform construction and practical parameter choices with uniform production proved for the former.

The stored evidence can be checked later or by another party without recovering the producer’s derivation. For another equation family the same argument requires nonlinear coefficient bounds and a complex-time domain together with control at the computed restart values. Under composition stability propagates state errors and compatible ledgers preserve the accounting of the selected quadratic quantities.

A Integer formulas and parameter estimates

The cosine product formula makes the Neumann residual check a finite rational calculation. Parameter estimates bound the record fields before the checker forms large integers. The local residual calculation also gives the constants used in section 4.

A.1 Cosine convolution

For coefficient vectors v, w indexed by $0 \leq j \leq K$ the k th coefficient of $M(v, w) = 2P_K(vw)$ is

$$M(v, w)_k = \sum_{i,j=0}^K (\mathbf{1}_{i+j=k} + \mathbf{1}_{|i-j|=k}) v_i w_j, \quad 0 \leq k \leq K. \quad (\text{A.1})$$

When both indicators equal one both contributions are counted. For the zero mode $M(v, w)_0 = 2v_0 w_0 + \sum_{j=1}^K v_j w_j$. For $k > 0$ a pair with one zero input index has both indicators equal to one. Division by two therefore gives the ordinary constant-mode product. Formula (A.1) thus covers the constant mode and repeated frequencies with the same rule.

For a record piece $q(\tau) = \sum_{\ell=0}^r q_\ell \tau^\ell$ extend $q_\ell = 0$ outside $0 \leq \ell \leq r$. The residual coefficient at time degree ℓ is

$$(\mathcal{R})_\ell = h^{-1}(\ell+1)q_{\ell+1} + A_K q_\ell - \frac{1}{2} \sum_{a=0}^{\ell} M(q_a, q_{\ell-a}), \quad 0 \leq \ell \leq 2r. \quad (\text{A.2})$$

Applying this formula to the stored coefficients includes their rounding error in the residual checked for acceptance.

A.2 Computing logarithmic ceilings

For positive integers p, q the integer $c = \lceil \log_2(p/q) \rceil$ is characterized by

$$2^{c-1}q < p \leq 2^c q. \quad (\text{A.3})$$

The bit lengths of p and q determine c to within one unit and an integer comparison gives its exact value. Negative c is handled by moving the power of two to the other side. All arguments in (4.1) are rational so this procedure computes the parameters exactly.

Explicit bounds on the parameters give polynomial limits for the record fields. Let $D = m + 4 \geq 5$. From $32L_K = 512D^2$

$$g \leq 10 + 2 \log_2 D, \quad J < 64D^2, \quad \log_2 \xi^{-1} < m + 2D^2 + 4, \quad R_K < 4^D. \quad (\text{A.4})$$

Since $1 + h^{-1} \leq 2^{g+1}$ it follows that

$$r \leq m + 2D^2 + g + 12, \quad b \leq m + 2D^2 + 2D + g + 2 \lceil \log_2(r+2) \rceil + 12. \quad (\text{A.5})$$

These inequalities imply the orders in (5.5) and give polynomial bounds on the encoded parameters before the dyadic denominator 2^b is formed. The implemented bounds use the exact integer formulas for those parameters.

A.3 Constants in the local residual estimate

The derivative-tail constant used in (4.10) follows from the identity

$$\sum_{\ell=r+1}^{\infty} \ell q^\ell = \frac{q^{r+1}((r+1) - rq)}{(1-q)^2}, \quad 0 < q < 1, \quad (\text{A.6})$$

with $q = 1/8$. The truncated-series residual is therefore at most

$$h^{-1} \left(\frac{11(7r+8)}{196} + \frac{11hL_K}{28} \right) 8^{-r} \leq h^{-1}(r+2)8^{-r}. \quad (\text{A.7})$$

Quantizing the nonconstant coefficients changes the derivative by at most $r(r+1)\delta_b/2$ and the Galerkin vector field by at most $L_K r \delta_b$. Since $hL_K \leq 1/32$

$$h^{-1} \frac{r(r+1)}{2} \delta_b + L_K r \delta_b \leq h^{-1}(r+1)^2 \delta_b. \quad (\text{A.8})$$

Both calculations use absolute sums of time coefficients in $\mathcal{A}_4$ as required by the stronger residual estimate in Theorem 4.1.

B Finite record format and validation order

The implemented Neumann format specializes the finite trajectories of section 3 to a uniform dyadic partition. Each record identifies its encoding version and equation together with the final time and includes a copy of the initial coefficient table. Rational coefficients are reduced to fractions with positive denominator before comparison with the separately supplied input. The final time is the exact rational number $1/16$.

Recorded quantity	Required consistency
Encoding and equation with final time	Fixed version and equation with final time
Initial data	Complete normalized initial table equal to the external input
Accuracy m	Equal to the externally requested precision
Parameter choice	Declared parameter mode with the same residual test in either case
Cutoff and partition	K and J on the uniform grid $t_i = i/(16J)$
Degree and dyadic precision	r and b fixing the common shape and dyadic denominator
Coefficient arrays	Integer strings $z_{i,\ell,j}$ representing $z_{i,\ell,j}/2^b$
Observations	The terminal mean and value at zero

Table 11. Mathematical contents of a Neumann record and their consistency checks.

Table 11 specifies the trajectory certificate. An optional quadratic ledger retains the rational forms with their transports and injections. It also retains the positivity factorizations and increment majorants of section 6.5. Its separate checks certify the quadratic account while the trajectory’s initial and residual tests together with the tail tests continue to certify accuracy for the equation.

Let J_g, r_g, b_g be the guaranteed parameters computed from the external m . The header fields for K, J, r, b in Table 11 are restricted by the polynomial bounds

$$\begin{aligned} 1 &\leq K \leq K_{\max} = 2(m+4), \\ 1 &\leq J \leq J_{\max} = J_g, \quad J \text{ a power of two}, \\ 1 &\leq r \leq r_{\max} = \max\{m+8, r_g\}, \\ 1 &\leq b \leq b_{\max} = \max\{64, b_g\}. \end{aligned} \quad (\text{B.1})$$

The array has exactly $J(r + 1)(K + 1)$ entries. Each coefficient string has at most $b + 3$ characters. After bounded conversion its absolute integer value must be at most $3 \cdot 2^b$. The checker therefore bounds header values and string lengths before converting coefficients for the magnitude test.

If n_{can} denotes the byte length of the canonical external input then a record file is rejected before parsing when its byte length exceeds

$$8192 + 2n_{\text{can}} + J_{\text{max}}(r_{\text{max}} + 1)(K_{\text{max}} + 1)(b_{\text{max}} + 7). \quad (\text{B.2})$$

This is a polynomial in the external input length and m . The constant and per-entry allowance cover the fixed serialization specified in the accompanying computational material. Header integers also have bounded string length before conversion. Malformed nesting and types are rejected. These preliminary checks limit the storage allocated when the coefficient integers are constructed.

The verifier next checks the canonical input copy with the precision and array shape. It builds the rational coefficients and checks exact joins and $\|q_i\|_{t,4} \leq 3$. It then expands (A.2) and computes d_0 and R_* and tests (3.5). The observation centers are computed from (3.8). Supplied descriptive error fields play no role in acceptance.

Every output of the guaranteed construction lies within these bounds. Its parameters are among the values used to define the maxima and Theorem 4.1 supplies the coefficient magnitude bounds. Practical records inside the envelope use the same mathematical test. Acceptance requires both the finite format and the analytic inequalities. It therefore depends on the submitted representation as well as on the trajectory.

Picard execution records. The second format records the construction in Proposition 6.3. Its external input is the dense rational matrix B and coefficient list for p in (6.9). The input shape determines d and N . For the external precision m the verifier independently computes $b = m + 4$ and $N_{\text{it}} = 2(m + 4)$. It computes R, β from (6.13). The record contains the canonical input copy and these parameters together with the $N_{\text{it}} + 1$ dyadic coefficient arrays and the terminal vector. Each array has $(R + 1)d$ integer entries with common denominator 2^β .

The file and field bounds are polynomial in the external input length and m as given by the execution length in section 7.4. These bounds are enforced before coefficient conversion or allocation. The verifier checks the input copy and zero initial array. It also checks the exact stronger norm bound for every iterate. It then computes each Picard image by (6.16). For a nonnegative image coefficient a proposed dyadic value $z2^{-\beta}$ is the truncation towards zero exactly when that coefficient lies in $[z2^{-\beta}, (z + 1)2^{-\beta})$. For a nonpositive coefficient the corresponding interval is $((z - 1)2^{-\beta}, z2^{-\beta}]$. These rational comparisons verify the truncation directly from the stored coefficients. The omitted degree $R + 1$ coefficient is included in the step error of (7.3). Finally the checker verifies the exact terminal sum and the requested final error bound.

C Information under spectral reduction

A small unresolved spectral mass can be spread over many modes and carry appreciable entropy. An exponential spectral moment together with a trace bound controls this loss uniformly. Combining the tail bound with a finite-dimensional entropy estimate gives a continuity modulus on the certified states for use in (6.41). The observation takes an already certified state bound as its input and returns a certified entropy error.

Let A be a nonnegative self-adjoint operator with compact resolvent on a Hilbert space $\mathcal{H}$. Unsubscripted vector norms refer to this Hilbert space. Write Λ_n for the distinct eigenvalues of A . Their multiplicities are m_n and their orthogonal projections are Π_n . For a unit vector f put $q_n = \|\Pi_n f\|^2$. Uniformly distributing this mass within each eigenspace gives the basis-independent shell probability state

$$\rho_f = \bigoplus_n \frac{q_n}{m_n} I_{m_n}. \quad (\text{C.1})$$

Write $p = (p_j)_j$ for its eigenvalue distribution with multiplicity and λ_j for the corresponding eigenvalues of A repeated with the same multiplicities. Thus $H(p) = H((q_n)_n) + \sum_n q_n \log m_n$. The observable studied here is the entropy of the shell-averaged state ρ_f . It generally differs from the pure state $|f\rangle\langle f|$. All logarithms in this appendix are natural with $0 \log 0 = 0$.

C.1 The information discarded by a window

Suppose $H(p) < \infty$. Let $\varepsilon = \sum_{\lambda_j > K^2} p_j$ and for $0 < \varepsilon < 1$ let p_R, p_T be the conditional laws on the retained and unresolved labels. The entropy chain rule [13, Chapter 2] gives

$$H(p) = h_2(\varepsilon) + (1 - \varepsilon)H(p_R) + \varepsilon H(p_T), \quad (\text{C.2})$$

where $h_2(r) = -r \log r - (1 - r) \log(1 - r)$. Replacing the unresolved labels by one flag loses exactly $\varepsilon H(p_T)$. Terms of zero mass are omitted at the endpoints. The flagged law has the same retained masses and one mass ε so its entropy consists of the first two terms of (C.2). The retained capacity $\log D_K$ where $D_K = \text{rank } \mathbf{1}_{[0, K^2]}(A) > 0$ is nondecreasing with K . The conditional retained entropy need not be monotone. Changing the cutoff changes the partition of the same law into retained and unresolved labels. A family of smooth spectral selectors gives probability weights when their squared multipliers sum to one at each spectral value.

For a discrete random variable a deterministic recoverable encoding $X \mapsto (Y, M)$ with $H(X) < \infty$ satisfies $H(X) = H(Y) + H(M | Y)$. For a finite encoding of a decomposed trajectory recoverability requires that (Y, M) determine the unresolved initial state and its driven evolution as well as the retained part.

C.2 A certificate for the entropy tail

For $0 < \theta \leq 1$ and $\alpha \geq 0$ with $s \in \mathbb{R}$ write $\|f\|_{\theta, \alpha, s; A}$ for the Hilbert spectral norm of $e^{\alpha A^\theta} (I + A)^{s/2} f$. If $\hat{f}_j$ are the coefficients in an orthonormal eigenbasis then the certificates

used here have the form

$$\|f\|_{\theta, \alpha, s; A}^2 = \sum_j e^{2\alpha\lambda_j^\theta} (1 + \lambda_j)^s |\widehat{f}_j|^2 \leq M^2, \quad \alpha > 0, \quad s \geq 0. \quad (\text{C.3})$$

The weight is constant on each eigenspace so shell averaging preserves the weighted sum. The resulting exponential moment controls both the unresolved mass and its entropy.

Proposition C.1 (Entropy tails from a spectral certificate). Suppose $\|f\| = 1$ and (C.3) holds. Assume $Z_c = \text{Tr } e^{-cA^\theta} < \infty$ for some $c > 0$. Set $a_K = K^{2\theta}$ and

$$T(K) = cM^2 \sup_{x > a_K} x e^{-2\alpha x} + M^2 e^{-2\alpha a_K} \log^+ Z_c. \quad (\text{C.4})$$

Then $H(p) < \infty$ and the flagged law $p^{[K]}$ satisfies

$$\varepsilon \leq M^2 e^{-2\alpha a_K}, \quad 0 \leq H(p) - H(p^{[K]}) = \varepsilon H(p_T) \leq T(K) \longrightarrow 0. \quad (\text{C.5})$$

The conditional term is zero if $\varepsilon = 0$. Polynomial spectral counting with multiplicities implies the trace assumption for every $c > 0$.

Proof. Since $s \geq 0$ the certificate bounds $\sum_j e^{2\alpha\lambda_j^\theta} p_j$ by M^2 . This gives the tail-mass estimate and

$$\sum_{\lambda_j > K^2} \lambda_j^\theta p_j \leq M^2 \sup_{x > a_K} x e^{-2\alpha x}. \quad (\text{C.6})$$

For $\varepsilon > 0$ compare p_T with the probability law proportional to $e^{-c\lambda_j^\theta}$ on the unresolved labels. Writing its normalizing sum as $Z_{c,T}$ the log-sum inequality gives

$$\varepsilon H(p_T) \leq c \sum_{\lambda_j > K^2} \lambda_j^\theta p_j + \varepsilon \log Z_{c,T}. \quad (\text{C.7})$$

For an infinite set of labels apply the log-sum inequality to normalized restrictions to finite sets and let those sets increase to the full support. The finite spectral moment gives the same bound in the limit. On the whole spectrum this argument bounds $H(p)$ by c times its spectral moment plus $\log Z_c$. This proves finiteness. Since $Z_{c,T} \leq Z_c$ the preceding estimates prove (C.5). Both terms in $T(K)$ tend to zero. Grouping frequencies into unit shells proves summability of $e^{-c\lambda_j^\theta}$ under polynomial counting. $\square$

A mass ε spread uniformly over N unresolved labels contributes $\varepsilon \log N$ to the discarded entropy. Controlling the mass alone therefore does not bound the discarded entropy uniformly in N .

C.3 A common observation modulus

A common certificate makes the preceding tail bound uniform for two states. Merging each tail into one flag then leaves finite probability laws whose distance can be estimated from the state norm.

Assume the trace condition of Proposition C.1. On unit vectors with the common certificate (C.3) put $S(f) = H(p(f))$ and

$$\begin{aligned}\omega_d(\eta) &= \max_{0 \leq x \leq \min\{\eta, 1\}} \{h_2(x) + x \log d\}, \\ \omega_S(\eta) &= \inf_K \{2T(K) + \omega_{D_K+1}(\eta)\}.\end{aligned}\tag{C.8}$$

Then $|S(f) - S(g)| \leq \omega_S(\|f - g\|)$ and $\omega_S(\eta) \rightarrow 0$ as $\eta \downarrow 0$. This common modulus follows by combining the uniform tail bound with a finite-dimensional entropy estimate. Cauchy–Schwarz applied to the differences of squared shell norms gives

$$\frac{1}{2} \sum_n |\|\Pi_n f\|^2 - \|\Pi_n g\|^2| \leq \frac{1}{2} \|f - g\| (\|f\| + \|g\|) = \|f - g\|.\tag{C.9}$$

Uniform lifting within each shell preserves total variation and merging the tail cannot increase it. For two laws on d labels a maximal coupling with disagreement probability δ gives $|H(p) - H(q)| \leq h_2(\delta) + \delta \log d$ by the conditional entropy bound. Apply this to the flagged laws and add the two bounds $T(K)$. Taking the infimum proves the estimate. To prove that the modulus vanishes first choose K to make both tail bounds small. At this fixed finite dimension the remaining term tends to zero with η .

Along a trajectory a positive lower bound for α and an upper bound for M with the other spectral data fixed give a common modulus. When both the exact and reconstructed readout states belong to this unit-vector class then their Hilbert-space error controls the entropy difference. Applying this estimate in (6.41) also requires a bound comparing the norms of X_x and $\mathcal{H}$. If the states are normalized before readout then their spectral certificates and state-error estimate must be verified after normalization. A common positive lower bound on the unnormalized norms controls the additional error from this step. For polynomial bit cost the precision selection and finite evaluation must also satisfy the computability bounds in section 6.

C.4 Conditional information in a history source

A root value identifies an evaluation result but leaves its expression undetermined. The normalized source assigns probabilities to those expressions through its recursive conditional law rather than by uniform shell averaging. The information lost on reading only the root is measured by the conditional entropy within its fibre. Let p_n be that law on the finite fibre $\mathcal{F}_n$ as in [1] and set

$$q_\beta(n) = Z_\beta^{-1} e^{-\beta n^2}, \quad Z_\beta = \sum_{n \geq 1} e^{-\beta n^2}, \quad P_\beta(e) = q_\beta(n(e)) p_{n(e)}(e).\tag{C.10}$$

Summing probabilities within each fibre gives the root law q_β . The full entropy separates into the entropy of that law and the mean information needed to distinguish expressions with the same value.

$$H(P_\beta) = H(q_\beta) + \sum_{n \geq 1} q_\beta(n) H(p_n).\tag{C.11}$$

Indeed insert $\log P_\beta(e) = \log q_\beta(n) + \log p_n(e)$ in the entropy sum. Finiteness follows from the source fibre bound $|\mathcal{F}_n| < 8^n$ proved in [1]. We obtain $H(p_n) \leq n \log 8$ while $H(q_\beta) = \log Z_\beta + \beta \sum_n q_\beta(n) n^2$. All these Gaussian moments are finite. In particular the unresolved conditional information beyond root N is bounded by the explicit tail

$$\sum_{n>N} q_\beta(n) H(p_n) \leq \frac{\log 8}{Z_\beta} \sum_{n>N} n e^{-\beta n^2}. \quad (\text{C.12})$$

Gaussian sums and a positive lower bound on Z_β give certified finite approximations for effectively specified $\beta > 0$.

Equation (C.11) is a statement about a classical probability law on histories. It differs both from the entropy of a pure amplitude state and from the uniform shell law in (C.1). The trace condition for an arbitrary history amplitude must also retain its multiplicities. For the root operator $\tilde{A}|e\rangle = n(e)^2|e\rangle$ the Gaussian trace is $\sum_n |\mathcal{F}_n| e^{-cn^2} < \infty$ for $c > 0$ by the same fibre bound. Polynomial mode counting is unnecessary for this trace but its history multiplicities cannot be omitted. Conditional source normalization and spectral trace counting serve different purposes.

D Coefficient transforms and arithmetic propagation

Generating functions and Dirichlet series provide scalar observations of a spectral coefficient sequence. A bound on that sequence determines both a convergence domain and the sensitivity of the readout to coefficient errors. On a Neumann interval the frequency scale and basis normalization must be taken into account in this coefficient bound. This coefficient readout starts from an established spectral certificate and transfers that certificate to the displayed generating functions and Dirichlet series.

D.1 Frequency normalization and analytic domains

Let $a = (a_n)_{n \geq 0}$ be a coefficient sequence satisfying

$$M_{\theta, \alpha, s}(a)^2 = \sum_{n \geq 0} e^{2\alpha n^{2\theta}} (1 + n^2)^s |a_n|^2 < \infty, \quad 0 < \theta \leq 1, \quad \alpha \geq 0, \quad s \in \mathbb{R}. \quad (\text{D.1})$$

Source-weighted coherent readout supplies such a coefficient sequence directly from history amplitudes. If $\mathcal{F}_n$ denotes the history fibre over n and $a_n = \sum_{e \in \mathcal{F}_n} \sqrt{p_n(e)} f_e$ and $\sum_{e \in \mathcal{F}_n} p_n(e) = 1$ then Cauchy–Schwarz on each fibre gives

$$\sum_n e^{2\alpha n^{2\theta}} (1 + n^2)^s |a_n|^2 \leq \sum_e e^{2\alpha n(e)^{2\theta}} (1 + n(e)^2)^s |f_e|^2. \quad (\text{D.2})$$

Thus a history spectral certificate passes to the value coefficients without a multiplicity factor. This readout concerns amplitudes. Probability pushforward instead adds the probabilities in each fibre. Subsequent coefficient transforms use the resulting value sequence and its retained bound.

For the Neumann operator $A_{\text{Neu}} = -\partial_x^2$ on $(0, L)$ the normalized eigenfunctions are $\phi_0 = L^{-1/2}$ and $\phi_n = (2/L)^{1/2} \cos(n\pi x/L)$ for $n \geq 1$. Take $a_n = \langle f, \phi_n \rangle$ and $c_L = \pi/L$.

The frequency is $c_L n$ so a physical reserve α_{phys} gives the coefficient reserve $\alpha = \alpha_{\text{phys}} c_L^{2\theta}$ and

$$M_{\theta, \alpha_{\text{phys}} c_L^{2\theta}, s}(a) \asymp_{L, s} \|f\|_{\theta, \alpha_{\text{phys}}, s; A_{\text{Neu}}} . \quad (\text{D.3})$$

The norm on the right is the Hilbert spectral norm defined in Appendix C. The exponential factors coincide. Only the comparison of $1 + n^2$ and $1 + c_L^2 n^2$ contributes constants. For $L = \pi$ the eigenfunctions coincide with the normalized basis in (2.6). For unnormalized cosine coefficients f_n the conversion is $a_0 = \sqrt{L} f_0$ and $a_n = \sqrt{L/2} f_n$ for $n \geq 1$. In coefficient coordinates physical heat time τ corresponds to the parameter $t = \tau c_L^2$.

Bounding each weighted term by the full sum gives the coefficient estimate

$$|a_n| \leq M_{\theta, \alpha, s}(a) e^{-\alpha n^{2\theta}} (1 + n^2)^{-s/2} \quad (\text{D.4})$$

used to control the two series

$$F_a(z) = \sum_{n \geq 0} a_n z^n, \quad \Theta_a(t, z) = \sum_{n \geq 0} a_n e^{-tn^2} z^n. \quad (\text{D.5})$$

The coefficient estimate determines the analytic domains directly. Under (D.1) the following statements hold.

- (i) If $\alpha > 0$ and $\theta > 1/2$ then F_a is entire.
- (ii) If $\alpha > 0$ and $\theta = 1/2$ then its radius is at least e^α . The series is absolutely convergent on that boundary when $s > 1/2$.
- (iii) If $\alpha = 0$ or if $\alpha > 0$ and $\theta < 1/2$ then the radius is at least one and this universal lower bound is sharp.
- (iv) For every $t > 0$ the function $\Theta_a(t, \cdot)$ is entire.

The root test applied to (D.4) gives the asserted lower bounds for the radius. At the critical boundary the exponential factors cancel and Cauchy–Schwarz gives

$$\sum_{n \geq 0} |a_n| e^{\alpha n} \leq M_{1/2, \alpha, s}(a) \left(\sum_{n \geq 0} (1 + n^2)^{-s} \right)^{1/2}. \quad (\text{D.6})$$

For sharpness in (iii) use $a_n = (1 + n)^{-m}$ with $m > \max\{0, s + 1/2\}$ when $\alpha = 0$ and multiply this sequence by $e^{-\alpha n^{2\theta}}$ in the subcritical case. Finally e^{-tn^2} suppresses every fixed exponential growth in n and every polynomial growth allowed by the coefficient bound. This gives locally uniform convergence of the heat transform on $\mathbb{C}$.

D.2 Bohr and Dirichlet readouts

For the Dirichlet readout prime factorization converts the positive integer index into a monomial. Write $n = \prod_p p^{v_p(n)}$ for $n \geq 1$. The Bohr correspondence identifies these exponents with degrees [14, section 2.2]. Define

$$\mathcal{B}_a^+(w) = \sum_{n \geq 1} a_n \prod_p w_p^{v_p(n)}, \quad \mathcal{D}_a(z) = \sum_{n \geq 1} a_n n^{-z}. \quad (\text{D.7})$$

Each monomial has finite prime support. The pair $(a_0, \mathcal{B}_a^+)$ retains the full coefficient sequence. The spectral zero mode a_0 is separate from the constant monomial. The latter has coefficient a_1 .

Cauchy–Schwarz gives the absolute majorant

$$\sum_{n \geq 1} |a_n| \prod_p |w_p|^{v_p(n)} \leq M_{\theta, \alpha, s}(a) \left(\sum_{n \geq 1} e^{-2\alpha n^{2\theta}} (1+n^2)^{-s} \prod_p |w_p|^{2v_p(n)} \right)^{1/2}. \quad (\text{D.8})$$

The Bohr series is absolutely convergent wherever this bound is finite. Applied to the difference of two coefficient sequences the same estimate bounds the readout difference by the weighted coefficient distance times the displayed square-root factor. A uniform bound on that factor over a chosen set therefore gives a Lipschitz observation modulus in this coefficient norm. Substitution $w_p = p^{-z}$ replaces its final sum by

$$\sum_{n \geq 1} e^{-2\alpha n^{2\theta}} (1+n^2)^{-s} n^{-2\operatorname{Re} z}. \quad (\text{D.9})$$

For $\alpha > 0$ this sum converges locally uniformly on $\mathbb{C}$ and defines an entire Dirichlet function. When $\alpha = 0$ it converges locally uniformly for $\operatorname{Re} z > 1/2 - s$. Absolute convergence on these domains justifies

$$\mathcal{B}_a^+((p^{-z})_p) = \mathcal{D}_a(z). \quad (\text{D.10})$$

A uniform bound for the majorant on a boundary or a restricted set supplies the same observation estimate there. Applying (6.41) with a weaker state-error norm requires a readout estimate in that norm. An implemented readout must also control truncation and finite evaluation. Polynomial-time evaluation requires bounds on their precision and bit cost for the chosen representations of the coefficients and evaluation points.

D.3 A capacity recurrence

The product expansion used in Proposition 6.4 also applies to a scalar recurrence with contraction factors and additive injections. A bound on each injection must be combined with control of their accumulated contribution relative to the terminal capacity.

Let $X_{n+1} = \rho_n X_n + \iota_n$ where $0 \leq \rho_n \leq 1$ and let $C_n > 0$. Fix $a \geq 1$ and $K < \infty$ with $C_\Sigma < \infty$. Suppose that

$$|\iota_n| \leq K C_n^a, \quad \sum_{j=0}^{N-1} \left(\prod_{i=j+1}^{N-1} \rho_i \right) C_j^a \leq C_\Sigma C_N^a \quad (N \geq 1). \quad (\text{D.11})$$

Then with empty products equal to one we have

$$|X_N| \leq \left(\prod_{i=0}^{N-1} \rho_i \right) |X_0| + K C_\Sigma C_N^a. \quad (\text{D.12})$$

Expand the recurrence as in (6.24). Its product weights are nonnegative so taking absolute values and applying (D.11) gives the result.

A sufficient condition is $\rho_n \leq r \leq 1$ and $C_{n+1} \geq dC_n$ where $d > 0$ and $r < d^a$. Indeed

$$\sum_{j=0}^{N-1} \left(\prod_{i=j+1}^{N-1} \rho_i \right) C_j^a \leq C_N^a d^{-a} \sum_{k=0}^{N-1} (rd^{-a})^k \leq \frac{C_N^a}{d^a - r}. \quad (\text{D.13})$$

E Certified relations and robust readout

The computation estimates control the accuracy and cost of finite approximations. To obtain an accepted finite witness from a terminal approximation we require a decoder that succeeds throughout a known state-error ball. Polynomial bounds on the required precision and decoding cost let the computation produce a witness on each yes input. A polynomial clock and exact final verification then decide the associated promise problem. This appendix is a downstream robust-readout module. Its hypotheses combine a polynomial-time verifier with a uniformly realizable certified computation. The latter supplies the additional structure that produces a witness from the input.

E.1 Finite encodings

Let $\phi_0 = \pi^{-1/2}$ be the normalized zero mode in (2.6). For a binary word s of length N put

$$\mathbf{E}(s) = \frac{2 \text{val}_2(s) + 1}{2^{N+1}} \phi_0. \quad (\text{E.1})$$

The coefficient is stored as an explicit numerator and denominator. Leading zeros remain part of the word.

Encoding and recognition of the canonical image together with decoding in (E.1) have polynomial bit cost. The encoded length is $\Theta(N + 1)$. For every $\alpha, r \geq 0$ and $\theta > 0$

$$\left\| e^{\alpha A_N^\theta} (I + A_N)^{r/2} \mathbf{E}(s) \right\|_2 \leq 1. \quad (\text{E.2})$$

With ϕ_0 among the declared source modes the encoding has a finite rational construction. Polynomial checking costs and witness lengths are preserved by transport through this encoding. The numerator is positive and odd and is smaller than the denominator so the fraction is reduced. Subtracting one from the exponent of its power-of-two denominator recovers N . Subtracting one from the numerator and dividing by two recovers the integer value. This is padded to N binary digits. Conversely these tests characterize the canonical image. The empty word has coefficient $1/2$. All operations use $O(N + 1)$ -bit integers and polynomial time. The spectral weight equals one on ϕ_0 and its coefficient lies in $(0, 1)$. Decoding two words and applying their original checker proves the last assertion.

E.2 A success margin for terminal observations

Fix a polynomial-time verifier $\mathbf{V}(x, y)$ and a polynomial witness-length bound $s_{\mathbf{V}}$. Write

$$Y_{\mathbf{V}} = \{x : \exists y, |y| \leq s_{\mathbf{V}}(|x|), \mathbf{V}(x, y) = 1\}. \quad (\text{E.3})$$

This is the usual finite checking-relation description of NP [15, section 1]. On a declared domain Ω the associated promise problem has yes set $\Omega \cap Y_V$ and no set $\Omega \setminus Y_V$. We use the convention of [16, section 1.2].

The encoding gives a finite representation of a supplied witness. To produce one by evolution we require a terminal state whose whole readout ball decodes to accepted witnesses. The implementation and the radius of this ball must be determined from the input.

Definition E.1 (Robust successful readout). An equation implementation has robust successful readout for V on Ω if the following holds for every $x \in \Omega \cap Y_V$. It satisfies the computation bounds of Definition 6.1 or their composition form in Proposition 6.4. A nonnegative integer r_x is computable in polynomial time and has polynomial value bound in $|x|$ and a fixed decoder satisfies

$$\|\langle a \rangle_x - u_x(T_x)\|_{X_x} \leq 2^{-r_x} \implies |\text{Dec}(x, a)| \leq s_V(|x|), \quad V(x, \text{Dec}(x, a)) = 1 \quad (\text{E.4})$$

for every valid finite state record a in the readout ball. The decoder has polynomial bit cost in $|x| + |a|$ and retains the declared finite provenance. Compilation and initialization together with all resource bounds use x and the fixed implementation without an accepted witness as additional input.

Corollary E.2 (Resolution of a certified relation). Under Definition E.1 there is one deterministic algorithm producing an accepted witness in polynomial time for every $x \in \Omega \cap Y_V$. The corresponding promise problem has a polynomial-time decision algorithm. If Ω is polynomial-time decidable then $\Omega \cap Y_V$ is an ordinary polynomial-time language.

Proof. On a yes input request state precision $m = \max\{1, r_x + 1\}$ and run the computation of Theorem 6.2 or its finite composition and decode the terminal record. Its error is smaller than 2^{-r_x} so (E.4) gives an accepted witness. Every stage has polynomial cost by the stated bounds.

The uniform bounds give a fixed polynomial clock dominating this successful computation on all yes inputs. On every finite input run the clocked procedure and check the proposed witness length. Evaluate V exactly. Accept only if the verification succeeds. A timeout or malformed output causes rejection. The procedure accepts every yes input. A no input in the promise domain has no accepted witness so it is rejected. This proves the promise claim. If Ω is decidable in polynomial time then test membership first to obtain the ordinary language claim. $\square$

E.3 Polynomial residuals and Boolean readout

Boolean circuits give an explicit readout margin. Sufficiently small polynomial residuals force exact gate identities after rounding. Let C be a finite acyclic circuit with AND and NOT gates. Each gate has at most two inputs and constant wires are specified where needed. A circuit without free inputs can be evaluated directly. Otherwise give coordinates z_i to its free input wires and gate outputs and let D be their number. Constant wires are

substituted as zero or one. Form the list of residuals

$$z_i(1 - z_i), \quad z_k - z_i z_j, \quad z_k + z_i - 1, \quad z_{\text{out}} - 1, \quad (\text{E.5})$$

for Booleanity and AND gates as well as NOT gates and acceptance respectively. If this list is $(r_\ell)_{\ell=1}^Q$ define

$$E_C(z) = \sum_{\ell=1}^Q r_\ell(z)^2, \quad z \in [0, 1]^D. \quad (\text{E.6})$$

The output residual is included even when it is constant so $Q \geq 1$.

Proposition E.3 (Residual compilation and exact readout). The map $C \mapsto E_C$ is a polynomial-time compilation into sparse integer polynomials of degree at most four. The circuit has an accepting input if and only if E_C has a zero in $[0, 1]^D$. Moreover if $E_C(z) \leq 1/256$ then coordinatewise Boolean rounding satisfies all gate and output equations. For rational coordinates the evaluation of E_C and rounding together with verification of the resulting finite word have polynomial bit cost in the complete circuit and coordinate encodings.

Proof. Every residual has degree at most two and a bounded number of integer monomials. The squared list has $O(Q)$ monomial occurrences. Collecting equal monomials yields integer coefficients of $O(\log(Q+1))$ bits. The degree remains at most four. An accepting Boolean input extends through the circuit to a zero of E_C . Conversely a zero satisfies Booleanity and every gate equation as well as acceptance.

For the quantitative assertion we bound each rounded gate residual by a number strictly below one. Since that residual is an integer it must vanish. Put $\delta = 1/16$. If $E_C(z) \leq \delta^2$ then every residual has absolute value at most δ . For $d_i = \min\{z_i, 1 - z_i\}$ Booleanity gives $d_i(1 - d_i) \leq \delta$. Since $d_i \leq 1/2$ we obtain $d_i \leq 2\delta$. Let b_i be the closest Boolean coordinate. At an AND gate

$$|b_k - b_i b_j| \leq |b_k - z_k| + |z_k - z_i z_j| + |z_i z_j - b_i b_j| \leq 2\delta + \delta + 4\delta < 1. \quad (\text{E.7})$$

The left side is an integer and hence zero. The analogous integer residuals for NOT and output are bounded by 5δ and 3δ respectively and vanish as well. Fixed wires have zero rounding error and the bound on d_i excludes ties.

The polynomial degree is fixed. Products of rational coordinates therefore have bit lengths bounded by a fixed multiple of their input lengths. Forming and summing $O(Q)$ terms have polynomial bit cost. So do comparison with $1/256$ and rounding as well as evaluation of the circuit. $\square$

The exact readout remains stable under a quantitative perturbation. On the cube $[0, 1]^D$ we have

$$|E_C(z) - E_C(w)| \leq 6Q \|z - w\|_\infty. \quad (\text{E.8})$$

Suppose z_* belongs to this cube and $E_C(z_*) \leq 1/1024$. If a finite rational vector $\tilde{z}$ satisfies

$$\|\tilde{z} - z_*\|_\infty \leq \frac{1}{6144Q}, \quad (\text{E.9})$$

then clipping it to the cube and rounding its coordinates to Boolean values produces an accepting input. Requesting $\lceil \log_2(6144Q) \rceil$ accuracy bits suffices for this margin. The decoder has polynomial bit cost. The gradient bounds give the stated uniform perturbation estimate for the energy. Clipping the approximate vector keeps the comparison inside the cube without increasing its error. For an AND residual $|r| \leq 1$ and $\|\nabla r\|_1 \leq 3$ on the cube including when its input wires coincide. Hence $\|\nabla(r^2)\|_1 \leq 6$. The corresponding bounds are four for a NOT residual and at most $1/2$ for a Booleanity residual. The output residual has bound two. Constant-wire substitutions preserve these bounds. Summation and integration along the segment from w to z prove (E.8).

Coordinatewise clipping is nonexpansive relative to every point of the cube. Thus

$$E_C(\text{clip}(\tilde{z})) \leq \frac{1}{1024} + \frac{6Q}{6144Q} = \frac{1}{512} < \frac{1}{256}. \quad (\text{E.10})$$

Proposition E.3 gives exact acceptance after rounding. Clipping and rounding are finite rational comparisons with polynomial cost.

Each rational coordinate has the finite zero-mode presentation $z_i\phi_0$. The residual polynomial is evaluated by finite rational additions and multiplications of the coordinate coefficients. The product z_iz_j is represented by $(z_iz_j)\phi_0$. Pointwise multiplication of the two coordinate states instead introduces the factor ϕ_0^2 . On the cube the coordinate states have direct-sum Hilbert norm at most $\sqrt{D}$. The norm of their difference is the Euclidean coordinate distance. This bounds the coordinate supremum distance used in the preceding perturbation estimate. That estimate supplies the decoder and precision for Definition E.1 provided the compiled trajectory reaches a terminal state with the stated bound on $E_C(z_*)$. Both the uniform computation bounds and this estimate at the terminal state are required.

E.4 Target residuals and finite budget descent

To obtain a terminal readout margin from a decay estimate relate acceptance to a nonnegative target residual $\mathcal{E}_x$. Suppose a decoder obeys

$$\mathcal{E}_x(\langle a \rangle_x) \leq \gamma_x \implies \mathbf{V}(x, \text{Dec}(x, a)) = 1, \quad (\text{E.11})$$

with the witness-length and cost bounds just stated. Choose nonnegative integer exponents U_x, G_x, D_x, s_x and let $\kappa_x > 0$. Assume $t \mapsto \mathcal{E}_x(u_x(t))$ is absolutely continuous and

$$\frac{d}{dt}\mathcal{E}_x(u_x(t)) \leq -\kappa_x\mathcal{E}_x(u_x(t)), \quad \mathcal{E}_x(u_x(0)) \leq 2^{U_x}, \quad \gamma_x \geq 2^{-G_x}. \quad (\text{E.12})$$

If $\kappa_x T_x \geq U_x + G_x + 2$ integration gives $\mathcal{E}_x(u_x(T_x)) \leq 2^{-G_x-2}$. Suppose $\mathcal{E}_x$ has Lipschitz constant at most 2^{D_x} on the closed radius 2^{-s_x} ball about that terminal state. Then

$$r_x = \max\{s_x, D_x + G_x + 2\} \quad (\text{E.13})$$

bounds the change in the target residual by 2^{-G_x-2} for every state record within 2^{-r_x} . Adding the terminal bound gives a target residual at most $2^{-G_x-1} \leq \gamma_x$. When the exponents are polynomially computable and polynomially bounded this precision gives

polynomial bit cost under the computation bounds. The terminal time must lie in the interval covered by the equation computation and its restart estimates.

The decay estimate controls approach to a successful state while the equation residual in (3.3) controls the accuracy of its finite approximation. A spectral gap supplies the decay rate when

$$-\frac{d}{dt}\mathcal{E}_x(u_x(t)) \geq \mathcal{Q}_x(u_x(t)), \quad \mathcal{Q}_x(u_x(t)) \geq 2\lambda_x\mathcal{E}_x(u_x(t)). \quad (\text{E.14})$$

Then $\kappa_x = 2\lambda_x$. For a nonnegative self-adjoint operator H_x with gap λ_x on $(\ker H_x)^\perp$ the second inequality holds for $\mathcal{E}_x = \|v\|^2/2$ and $\mathcal{Q}_x = \langle H_x v, v \rangle$ with $v \in D(H_x) \cap (\ker H_x)^\perp$. Applying this gap estimate to the initialized trajectory also requires the stated representation of its target residual and the first inequality.

The finite quadratic ledger of section 6.5 provides one way to retain this descent in a checkable record. When its quadratic forms represent the target residual then the positive defect records certified decrease while injection and approximation increments retain the terms that can offset that decrease. Rational majorants yielding a net contraction give precisely the type of budget recurrence used below. Thus the ledger preserves a proved descent mechanism. The contraction rate and the dominance of the defect over the increments remain quantitative properties of the particular implementation.

For finite updates truncation and rounding as well as state transfer introduce an additive error at each step. The geometric-sum estimate used in Theorem 6.2 applies to the target budget once the update error is small enough relative to the readout threshold.

Proposition E.4 (Finite descent with rounding). For each $x \in \Omega \cap Y_V$ suppose fixed initialization and update procedures produce admissible records a_j and rational budgets $\mathcal{B}_j \geq 0$ such that

$$\begin{aligned} \mathcal{E}_x(\langle a_j \rangle_x) &\leq \mathcal{B}_j, & \mathcal{B}_0 &\leq 2^{U_x}, \\ \mathcal{B}_{j+1} &\leq (1 - a_x^{-1})\mathcal{B}_j + \eta_x, & 0 &\leq \eta_x \leq \frac{2^{-G_x}}{4a_x}. \end{aligned} \quad (\text{E.15})$$

Assume (E.11) with $\gamma_x \geq 2^{-G_x}$. The integers $a_x \geq 1$ and $U_x, G_x \geq 0$ are computed in polynomial time and have polynomial value bounds. Records including budgets have a common polynomial length bound. Initialization and updates with their bounds as well as decoding have polynomial bit cost. These programs and polynomials are fixed for the relation and updates preserve the stated admissibility and certification.

Then the first record with $\mathcal{B}_j \leq 2^{-G_x}$ decodes to an accepted witness. It occurs after at most

$$N_x = a_x(U_x + G_x + 2) \quad (\text{E.16})$$

updates with polynomial total cost. The associated promise problem has a polynomial-time decision algorithm.

Proof. Iterating the budget inequality gives the following bound for $j \geq 1$.

$$\mathcal{B}_j \leq (1 - a_x^{-1})^j 2^{U_x} + a_x \eta_x \leq e^{-j/a_x} 2^{U_x} + a_x \eta_x. \quad (\text{E.17})$$

For $a_x = 1$ the geometric term is zero. At $j = N_x$ both displayed contributions are at most 2^{-G_x-2} so the stopping test has succeeded. At stopping (E.11) gives an accepted witness. The polynomial bounds on N_x and on record lengths and per-step costs cover all updates and rational stopping tests. The clock and final verification in the proof of Corollary E.2 give the promise decision algorithm. $\square$

A budget perturbation bounded by 2^{-b} meets the required upper bound on η_x when $b \geq G_x + \lceil \log_2 a_x \rceil + 2$. Before stopping (E.15) decreases the budget by more than $3 \cdot 2^{-G_x} / (4a_x)$.

Let $M_x \geq 1$ and $\tau_x > 0$. For $\mathcal{B}(z) = \sum_{\ell=1}^{M_x} d_\ell(z)$ with nonnegative contributions then updating a maximal contribution gives such a descent estimate if the total change satisfies

$$\mathcal{B}(z^+) \leq \mathcal{B}(z) - \tau_x \max_{\ell} d_\ell(z) + \eta_x. \quad (\text{E.18})$$

Since the maximum is at least $\mathcal{B}(z)/M_x$ the coefficient of $\mathcal{B}(z)$ in the resulting upper bound is $1 - \tau_x/M_x$. A polynomial-time computable integer $a_x \geq \max\{1, M_x/\tau_x\}$ with polynomial value bound supplies the nonnegative coefficient $1 - a_x^{-1}$ in (E.15). The estimate must account for changes to other contributions and the update cost must include selection. A linear scan finds the exact maximum of polynomially many explicit rational contributions of polynomial length.

E.5 A fixed framework of certified presentations

State and operator certificates together with quadratic certificates retain the accuracy and observational information used by a successful readout. For a relation presentation their finite precision and evaluation costs enter the implemented update and decoding bounds. Normalized source tails can certify an operator approximation. The representation length and number of updates together with the decoding margin are verified through the same computation interface as for the realizations above.

Definition E.5 (Certified computation framework). Fix source and resource rules admitting the finite encodings of (E.1). A framework $\mathfrak{F}$ specifies permitted generated procedures and their exact finite-word and rational operations. These include constants and the identity as well as encoding and decoding. They are closed under finite composition and polynomially bounded iteration together with polynomial clocks and exact finite tests. Operations retain their declared sources and are charged their bit costs. Programs and polynomial bounds are fixed for each presentation.

A certified relation presentation consists of a permitted polynomial-time verifier V with a polynomial witness bound and one of the following permitted realizations.

- (i) An equation implementation satisfying the computation and successful-readout assumptions of Corollary E.2 including its finite-composition alternative.
- (ii) An initialized finite-record implementation satisfying Proposition E.4.

The assumptions hold on all of Y_V with $\Omega = \{0, 1\}^*$ and one set of programs and polynomial bounds. Initialization uses the input and the fixed realization.

A certified relation presentation concerns the complete route from an encoded input to a robustly readable successful witness together with one set of polynomial resource bounds. It is therefore stronger than the polynomial-time evaluation of $(x, y) \mapsto V(x, y)$ alone.

Let $P_{\mathfrak{F}}$ be the languages decided on all inputs by permitted deterministic polynomial-time programs. The class of languages admitting certified relation presentations is

$$CR_{\mathfrak{F}} = \{Y_V : V \text{ admits a certified relation presentation in } \mathfrak{F}\}. \quad (\text{E.19})$$

Framework-relative closure. Within Definition E.5 every certified relation presentation gives a permitted deterministic decision procedure. Indeed Corollary E.2 and Proposition E.4 with their clocked decision procedures give $CR_{\mathfrak{F}} \subseteq P_{\mathfrak{F}}$.

Conversely let a permitted polynomial-time program D decide a language L with output in $\{0, 1\}$. For the empty word ϵ set

$$V_D(x, y) = 1 \iff y = \epsilon \text{ and } D(x) = 1. \quad (\text{E.20})$$

After evaluating $D(x)$ store ϵ with budget and constant target residual $\mathcal{E}_x(v) = 1 - D(x)$. The identity update together with the decoder returning ϵ and the threshold $\gamma_x = 1/2$ satisfy Proposition E.4 on L with $a_x = 1$ and $U_x = 0$. The remaining parameters are $G_x = 1$ and $\eta_x = 0$. These permitted polynomial-time procedures give a presentation with witness bound zero. This reverse inclusion uses a new verifier with an empty witness. Producing a witness for a specified verifier requires its certified realization. Thus the two classes have the same extension under the enriched presentation convention. This consistency property uses the uniform polynomial-time realization required in the definition. Standard NP requires verification of supplied witnesses.

Conditional comparison with the standard classes. Suppose that for every polynomially balanced polynomial-time verifier the computation retains a certified generated route in the sense of Definition E.5 from each yes input to an accepted witness in one fixed framework $\mathfrak{F}$. Then $P = NP$ in the standard bit model. It is enough that every language in standard NP have at least one verifier for which the computation retains such a route. Indeed the hypothesis and the framework-relative inclusion above give $NP \subseteq CR_{\mathfrak{F}} \subseteq P_{\mathfrak{F}}$. Every permitted program counted in $P_{\mathfrak{F}}$ has a standard deterministic polynomial-time implementation by Definition E.5 so $P_{\mathfrak{F}} \subseteq P$. Hence $NP \subseteq P$. The reverse standard inclusion is immediate.

For a declared class of actual computations or finitely specified models requiring the calculation to retain its generation and certification in this form therefore makes the corresponding search problems uniformly solvable. What is retained includes the route from input to witness and its resource bounds as required by the certified-presentation definition. The comparison with standard NP concerns whether such routes can be supplied uniformly for its verification problems.

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checks. The author is responsible for the scientific content and conclusions.

Declarations

Conflict of interest. The author declares no competing interests.

Data and code availability. The accompanying computational material contains the construction and verification programs together with complete inputs and trajectory records and exact rational reports for the computations in section 7. Reproduction instructions and complete encoding specifications are included.

Lean Formalization of Arithmetic and Basic Physical Interfaces [17] provides independent formal checks of the common source construction and its finite interfaces. The spectral estimates and finite-record accuracy and bit-cost arguments used here are proved in the article and its appendices. The library is archived at <https://doi.org/10.5281/zenodo.23254521>.

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