

Coarse-Graining and Long-Range Response in Generated Systems

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ABSTRACT: We study when a reduced description preserves the response to an applied source. For systems defined by continuously differentiable state updates and observations a bounded surjective linear coarse map gives differentiable descent when the projected maps are constant on each fibre. Their common source response is then preserved on the invertible domain. For approximate descent an exact error identity separates defects in the update and source coupling from defects in the readout and direct term. Resolvent and contour estimates carry these errors and the displacement of the reference branch to visible poles and their residues at shifted critical roots. In a reciprocal Hilbert-space realisation an isolated finite-dimensional critical subspace with a positive crossing matrix has a positive-semidefinite finite-rank residue when the endpoint operator is self-adjoint and the complementary inverse is bounded. Source overlap determines its visible directions. Arithmetic and reflecting-heat examples separate stationary response preservation from memoryless dynamical closure. In the two-mode example pole displacement decays geometrically with the retained memory length. The residue error includes an additional factor of that length. For a matched vacuum covariance the residue is the zero-mass atom of its positive spectral measure. This identification uses the same physical projection and source pairing together with the continuation and subtraction conventions and endpoint matching.

KEYWORDS: Coarse-graining; Model reduction; Linear response theory; Pole stability; Spectral residues; Memory kernels

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1 Introduction

A macroscopic description replaces many microscopic variables by a few collective quantities. Agreement on an equilibrium value leaves open how those quantities respond to an applied disturbance. Unresolved variables can return the disturbance to the retained state. They can also alter its source coupling and observed output. Near a collective critical mode these effects can move a response pole or substantially change its strength. The physical question is which microscopic distinctions can be discarded while preserving that response.

Two microscopic states may give the same present observation and respond differently to a later intervention. A coarse state must therefore retain the distinctions visible under the allowed future operations. We formulate this requirement through a sourced state

equation and an observation map. Differentiating both follows a disturbance from its entry into the system to its measured effect. Joint differentiable descent preserves this derivative. Approximate descent introduces errors whose amplification depends on the collective modes.

For specified generating operations and source interventions two states are equivalent when every allowed future experiment gives the same output. This relation defines the coarsest observation-sufficient quotient. Its universal property is set-theoretic. Differentiable response requires additional topological structure. For a bounded surjective linear map between Banach spaces constancy of the projected update and observation on each fibre gives a unique differentiable coarse realisation. The operations and observations determine which distinctions must be retained. Regularity determines whether the resulting description supports linear response.

The future-output equivalence extends Moore’s state distinction for finite sequential machines [1]. Ho–Kalman realisation reconstructs a minimal linear state space from input–output data [2]. Here the full and coarse descriptions are specified state equations that may be nonlinear on Banach spaces. Fibre consistency gives exact descent. An identity for the response difference measures its failure under approximation.

The derivative has four parts. The source map carries the disturbance into the state. The inverse linearised state equation sums its repeated transmission through coupled variables. The readout converts the state change into an output. The direct term records the change in that output at fixed state. Theorem 2.2 proves preservation of their combination under joint descent. Source-dependent coarse coordinates contribute compensating derivatives to the source map and direct term. The implicit state equation determines the same response for any numerical iteration that solves it.

For approximate descent the response difference separates into one defect for each of these four maps. The comparison map may remove state coordinates. The estimates include the displacement of the reference branch and extend to contour integrals of the response. Schur–Feshbach reduction [3] gives the complementary description by exact block elimination. It changes the source and readout together with the direct term and effective kernel. All four enter the observed residue. In an evolution equation the eliminated coordinates carry an initial-state contribution and memory of earlier states and sources [4]. Both the block inverse and the memory mechanism are classical. The additional result here compares the complete response within one source–observation system through irreversible reduction and reference-branch displacement. The four-defect identity of Theorem 4.1 and the displaced-root estimates of Proposition 4.2 carry errors in the four maps to visible poles and their residues.

Hydrodynamics illustrates the same physical distinction. Conservation laws identify the slow sectors near local equilibrium. Constitutive data determine the response inside those sectors. Eliminated variables leave a delayed influence in the constitutive relations. Scale separation can justify replacing this memory by a local approximation.

Projection-based reduction of linear dynamical systems compares input–output transfer functions through global error norms for asymptotically stable systems (ref. [5] Sections 2.2–2.4). Our comparison concerns a neighbourhood of an isolated critical response.

It controls the root and residue using local resolvent bounds. The two sets of response maps may be derivatives of nonlinear state equations at different reference states. Their comparison retains that state displacement and the slope of the critical zero.

Analytic Fredholm theory describes the isolated singularities of the constructed response (ref. [6] Chapter VII). A simple crossing of a kernel eigenvalue through one gives a resolvent pole. Source and readout overlaps determine whether the pole is observed. In a reciprocal Hilbert-space realisation a positive crossing matrix on an isolated critical subspace gives a positive finite-rank residue. Norm and Riesz-projection estimates control its change under approximation (ref. [7] Chapter 5). Resolvent bounds retain the sensitivity of nonnormal kernels [8].

An arithmetic merge from ref. [9] and a two-mode reflecting-heat system distinguish static response preservation from memoryless dynamical closure. Their finite identities show how a hidden state can be removed from a stationary susceptibility while remaining visible to future observations. The reflecting-heat example gives explicit errors for finite memory. Its pole displacement decays geometrically with the retained memory length. The residue error contains an additional factor of that length. This factor comes from differentiating the memory remainder.

Gaussian averaging supplies a continuous family of operations. We prove its characterisation within the class of centred and orthogonally invariant convolution semigroups that are weakly continuous and have finite nonzero variance and square-root self-similarity. This is a special case of stable-law theory [10]. External averaging changes the external momentum multiplier. Averaging relative coordinates acts inside the composite response. An information-losing quotient instead changes which state distinctions are represented. The reflecting-window example relates the averaging law to calibrated correlation endpoints and a specified boundary condition.

Three uniqueness statements occur in the construction. The allowed future operations determine the observation-sufficient quotient up to its universal property. Composition and square-root self-similarity select the Gaussian law within the stated class. A positive spectral representation identifies the residue of the matched Euclidean correlator with a unique measure atom. Each statement concerns its own mathematical object.

Quantum field theory gives a physical realisation of the response. Vacuum positivity and covariance assign spectral content to a centred connected correlator. Renormalised operator mixing and physical symmetry quotients specify its channels. Ward identities and functional flows constrain and compute their responses. Matching the physical projection and source pairing together with the continuation and subtraction conventions identifies a Fredholm residue with a Källén–Lehmann atom. The reciprocal response theorem already gives residue positivity at the operator level. Spectral matching identifies the states that carry this weight. The complete covariant stress tensor provides a restrictive example. Its null-shell TT atom is excluded by the local tensor-measure argument. A directly reduced two-helicity covariance gives a positive null-shell example on a flat background. Its interpretation as gravity uses the matter couplings and constraints of an effective action together with the scattering consistency conditions [11, 12].

The tensor application in ref. [13] treats matter coupling and long-range exchange.

The compact construction of ref. [14] provides the gauge and full matter carriers with their common transport. It also constructs a Lorentz event carrier and calibrated phase clocks. Its compensation law transports nonlinear gauge–matter responses across Gaussian resolutions. Its Yang–Mills response completion gives a gapped colour-invariant example. The companion application [15] evaluates effective couplings and mass scales with their matching relations using the Coarse-Graining Genesis Framework [16]. Appendix G.6 of that work gives a coupled matter–amplitude evolution with a phase-clock readout and finite-interval propagation bounds. The model illustrates regular auxiliary elimination along an evolving background. These applications connect response preservation to explicit source algebras and observables.

For time-dependent generated spectra ref. [17] controls the discarded nonlinear source through interaction estimates and stronger-norm bounds on a common existence interval. The resulting state estimates supply concrete data for the local response comparison developed here.

2 Generated operations, sufficient coarse states, and response

The choice of a coarse state expresses which microscopic distinctions matter for the specified physical questions. Agreement of present observations is sufficient only if subsequent allowed operations never reveal a difference. This requirement first determines a set quotient. Its regular realisation makes it possible to differentiate the response to a source. In a physical realisation observations are the specified physical outputs of the system and need not involve a measuring apparatus.

2.1 Operations and the observation-sufficient quotient

Let X be a common invariant state domain and let $F_a : X \rightarrow X$ be the allowed deterministic operations. The label a includes each allowed source intervention not just the unforced update. A finite word w specifies an ordered composition F_w . The empty word acts as the identity. Let $\mathcal{O}$ be the declared observation family. Its members may have different output spaces but equality of their outputs is fixed before taking a quotient. These data come from a specified protocol. For finite sequential machines distinguishing states by their future outputs is the viewpoint of Moore’s experiments [1]. The same construction applies to infinite state and output sets. Differentiating the response then requires a regular realisation of the quotient.

In a generated physical realisation the operations and allowed sources together with the observations belong to one fixed construction. The arithmetic operations of Ref. [9] and the spectral products and covariant transport of Ref. [14] give companion realisations. Their continuous extensions use the specified common domains and convergence norms. For time evolution Ref. [17] gives model-specific interaction hypotheses and uniform bounds under which finite constructions and their controlled limits define the same observation. The quotient applies to these constructions through their fixed operations and observations.

For example observe only x in a state (x, u) whose next observed coordinate is $x' = ax + bu$ with $b \neq 0$. States with the same x and different u agree at present but give

different outputs after one update. Observation-sufficient equivalence tests whether such a distinction can appear after any allowed sequence of operations.

Theorem 2.1 (Coarsest observation-sufficient quotient). Define

$$x \sim y \iff O(F_w x) = O(F_w y) \text{ for every } O \in \mathcal{O} \text{ and } w. \quad (2.1)$$

Then $\sim$ is an equivalence relation and every allowed operation and observation descends to $q : \mathbf{X} \rightarrow \mathbf{X}/\sim$. If another map C admits exact descended operations and observations on its image there is a unique surjection $\pi : C(\mathbf{X}) \rightarrow \mathbf{X}/\sim$ with $q = \pi C$. It intertwines the descended operations and observations. Thus the quotient is unique up to its canonical bijection among coarsest exact observation-sufficient descriptions.

Proof. Equality of all specified outputs is reflexive and symmetric and transitive. If $x \sim y$ append F_a at the input of any word to obtain $F_a x \sim F_a y$. The empty word gives $O(x) = O(y)$ so both operations and observations descend. If $Cx = Cy$ induction along a word using the descended C -operations gives $CF_w x = CF_w y$. The descended observations then give $x \sim y$. Therefore $\pi(Cx) = [x]$ is well defined and surjective. This formula proves uniqueness and both intertwining claims. $\square$

The quotient is relative to the chosen operations and observations. Enlarging either family can distinguish previously equivalent states and therefore refine the quotient. The theorem concerns deterministic operations on a common invariant domain. Its set-theoretic conclusion leaves the dimension and regularity of the quotient to be determined from the particular system.

A coarse state therefore encodes future-response sufficiency. Two states are identified only when no allowed sequence can bring their difference back to an observed output. The quotient is thus the smallest state description that remains sufficient for the declared source interventions rather than a compression chosen solely from the present configuration.

2.2 Regular descent and its limits

For a fixed coarse map C exact descent on its actual image has a simple test. No linear structure is needed for the equivalence

$$CF = \bar{F}C \text{ for a unique } \bar{F} : C(\mathbf{X}) \rightarrow C(\mathbf{X}) \iff Cx = Cy \implies CF(x) = CF(y). \quad (2.2)$$

Indeed the only possible definition is $\bar{F}(Cx) = CF(x)$ and the right-hand condition is exactly its well-definedness. The fixed maps C and F determine the descended map uniquely on the actual image $C(\mathbf{X})$.

Fibre consistency makes the coarse output a function of the coarse input. Linear response requires differentiability of this function. The Banach-space statement fixes the regularity needed for this derivative.

Regular Banach descent. Let $\mathcal{X}, \mathcal{Y}, \mathcal{J}, \mathcal{Z}$ be Banach spaces and let $C : \mathcal{X} \rightarrow \mathcal{Y}$ be a bounded linear surjection. Let $U \subset \mathcal{X}$ and $V \subset \mathcal{J}$ be open and let $F : U \times V \rightarrow \mathcal{X}$ be continuously Fréchet differentiable. Assume fibre consistency for every $h \in V$. Then the unique map $\bar{F} : C(U) \times V \rightarrow \mathcal{Y}$ is C^1 on the open set $C(U) \times V$ and

$$\bar{F}(Cx, h) = CF(x, h), \quad D_y \bar{F}(Cx, h)C = CD_x F(x, h), \quad D_h \bar{F}(Cx, h) = CD_h F(x, h). \quad (2.3)$$

A C^1 observation $O : U \times V \rightarrow \mathcal{Z}$ descends with the same regularity if it is constant on each C -fibre at every h .

The quotient-norm proof is given in Appendix C. Bounded surjectivity makes $\text{Ker } C$ closed and identifies $\mathcal{X}/\text{Ker } C$ with $\mathcal{Y}$ by a bounded isomorphism. The quotient argument applies without a bounded linear section or a complemented kernel. Repeated iteration additionally requires the relevant invariant domains. A full-system fixed point projects to a coarse fixed point. A coarse fixed point permits a nonzero displacement $F(x) - x$ in $\text{Ker } C$ since it imposes only $CF(x) = Cx$.

A Gaussian heat map $\mathcal{H}_\tau$ on $L^2(\mathbb{R}^d)$ has the strictly positive Fourier multiplier $e^{-\tau|q|^2}$. It is injective but its multiplier tends to zero at large $|q|$. Unit vectors supported at successively higher frequencies therefore have images tending to zero. The inverse on the range is unbounded and that range is nonclosed for $\tau > 0$ in the inherited L^2 norm. It is not a surjection onto that L^2 space. Giving its range the norm $\|\mathcal{H}_\tau f\|_{\text{tr}} = \|f\|_2$ makes it a Banach isomorphic copy of the original space rather than an information-losing quotient. A genuine projection or observation quotient must still satisfy Eq. (2.2). Nonlinear descent can fail even after heat smoothing. On $[0, \pi]$ let $C = P_0$ be the constant-mode projection and $F(u) = H_t(u^2)$ on the finite cosine core. Then $C(a \cos nx) = C0 = 0$ whereas $CF(a \cos nx) = a^2/2 \neq CF(0)$ for $a \neq 0$ and $n \geq 1$. Here a is a function amplitude.

2.3 Linear response of the sourced state equation

An applied source can change both the state and the observation at fixed state. These two contributions explain the direct term in the response. Likewise eliminating a state can change how the source enters and how the remaining state is observed. Differentiating the update and observation together keeps all these contributions in the same calculation.

On a regular state space specify an actual update $F(X, h; z)$ and observation $O(X, h; z)$ with finite-dimensional source and output spaces. The parameter z is an analytic response coordinate. η denotes an independent control parameter. Neither is intrinsically a momentum or a clock. The symbols B, S, R, D denote the four derivatives of this rule at a declared fixed-point branch $X_*(z)$ and $h = 0$

$$B = D_X F, \quad S = D_h F, \quad R = D_X O, \quad D = D_h O. \quad (2.4)$$

Here D acts directly between the source and output spaces. We write $\mathcal{C}$ for the measured response and reserve C in the general descent results for the coarse map. The QFT realisation writes the measured matrix as $C(Q^2)$ and its kernel as $\mathcal{B}$.

Same-rule response. Suppose F, O are C^1 near the reference solution and $I - B$ is boundedly invertible. The local fixed-point branch is differentiable in h and

$$\mathcal{C} = D_h[O(X(h), h)]_{h=0} = D + R(I - B)^{-1}S. \quad (2.5)$$

More generally if a C^1 residual equation $\mathcal{E}(X, h; z) = 0$ defines the branch and $L = D_X \mathcal{E}$ is boundedly invertible then

$$\mathcal{C} = O_h - O_X L^{-1} \mathcal{E}_h. \quad (2.6)$$

The Banach implicit-function theorem gives the local branch. Differentiating its defining equation gives $(I - B)X_h = S$ or $LX_h = -\mathcal{E}_h$. Differentiating the observation gives $\mathcal{C} = D + RX_h$ proving both formulae.

The linearised equation $\delta X = B\delta X + S\delta h$ describes the source-induced change and its internal feedback. When $\|B\| < 1$ successive substitution converges in operator norm and gives

$$\mathcal{C} = D + \sum_{n=0}^{\infty} RB^n S = D + RS + RBS + RB^2S + \dots. \quad (2.7)$$

Here S injects the disturbance. Each B transmits it through the coupled state and R reads out the result. The direct term D records the source dependence of the observation at fixed state. In a stationary problem these are repeated feedback contributions. They represent successive evolution steps only when the update has that physical meaning. Away from the norm-convergent regime Eq. (2.5) still defines the response wherever the inverse exists. Eliminating variables must preserve the source and readout together with the internal feedback to preserve this sum.

For the residual convention put $S = -\mathcal{E}_h$. The displayed formula uses the residual derivatives and makes no choice of numerical iteration. Convergence of a solver and stability of the actual update are thus separate questions. Their common source rule gives mutually consistent derivatives. Reciprocity and positivity are properties of the resulting response representation.

Theorem 2.2 (Preservation of the complete visible response). Suppose $F, O, \bar{F}, \bar{O}$ are jointly C^1 in state and source and $CF(X, h) = \bar{F}(CX, h)$ and $O(X, h) = \bar{O}(CX, h)$ hold in compatible open neighbourhoods with fixed bounded linear C and corresponding reference branches. If both $I - B$ and $I - \bar{B}$ are invertible then

$$CB = \bar{B}C, \quad \bar{S} = CS, \quad R = \bar{R}C, \quad D = \bar{D}, \quad \mathcal{C} = \bar{\mathcal{C}}. \quad (2.8)$$

If both responses have meromorphic extensions to the same connected domain from a common nonempty open invertible set their visible poles and output Laurent coefficients agree there.

Proof. Differentiate the two neighbourhood identities. Multiplying $(I - \bar{B})C = C(I - B)$ by the inverses gives $(I - \bar{B})^{-1}C = C(I - B)^{-1}$. Substitution in Eq. (2.5) gives equality. The identity theorem gives the meromorphic assertion. $\square$

Joint descent preserves the source-to-observation response. Its dependence on the readout is visible in a two-component example. For $B(z) = \text{diag}(0, 1 - z)$ and $S = (0, 1)^\top$ with $R = (0, 1)$ and $C = (1, 0)$ the update descends to $\bar{B} = 0$ but the full response is $1/z$. It cannot be recovered from the retained coordinate because the readout does not descend.

State- and source-dependent coarse maps. Let $F, O, C = C(X, h)$ and the descended maps $\bar{F}, \bar{O}$ be jointly C^1 in state and source on compatible open neighbourhoods of the reference states and source with the following compositions defined. Suppose $C(F(X, h), h) = \bar{F}(C(X, h), h)$ and $O(X, h) = \bar{O}(C(X, h), h)$ hold near an actual sourced fixed point. Set $T = D_X C$ and $C_h = D_h C$ there. Then

$$\begin{aligned} TB &= \bar{B}T, \quad \bar{S} = TS + (I - \bar{B})C_h, \\ R &= \bar{R}T, \quad D = \bar{D} + \bar{R}C_h. \end{aligned} \tag{2.9}$$

On the common invertible domain the full responses still agree. At a fixed point the derivatives of C on both sides are evaluated at the same full state. The state derivative gives the first identity. The source derivative gives $TS + C_h = \bar{B}C_h + \bar{S}$. The observation chain rule gives the second line. Inserting all four identities into Eq. (2.5) cancels the C_h terms.

These formulae concern fixed-source stationary response. Time-varying windows or sources require the actual adjacent-time coarse maps and their history. Discarding C_h would already change the static response.

This also explains the distinction between static elimination and dynamical reduction. In a stationary calculation an unresolved variable can adjust to the current source and appear as an effective stiffness. During an actual update it carries its own initial data and relaxation law. After it is removed the retained variables remember the past through the induced kernel.

2.4 Gaussian operations and their representations

Gaussian averaging is one of the allowed operations when its spatial domain has been specified. It is distinct from the sufficient quotient. It changes the resolution by giving less weight to rapidly varying components. Interaction-induced collective feedback instead enters through the sourced state equation and its derivative. Keeping these roles explicit separates attenuation of an existing response from a change in the dynamics that produces it. The Euclidean Fourier representation separates external averaging from averaging of internal relative coordinates. For $\tau > 0$ define the normalised heat kernel and its action by

$$g_\tau(x) = \frac{1}{(4\pi\tau)^{d/2}} e^{-|x|^2/(4\tau)}, \quad (\mathcal{H}_\tau f)(x) = (g_\tau * f)(x). \tag{2.10}$$

We use $\widehat{f}(q) = \int e^{iq \cdot x} f(x) d^d x$ so that $\widehat{\partial_\mu f}(q) = -iq_\mu \widehat{f}(q)$. In this convention

$$\widehat{\mathcal{H}_\tau f}(q) = e^{-\tau|q|^2} \widehat{f}(q), \quad \mathcal{H}_{\tau_1} \mathcal{H}_{\tau_2} = \mathcal{H}_{\tau_1 + \tau_2}. \tag{2.11}$$

For a translationally invariant two-point function smearing each insertion with $\mathcal{H}_{\tau_Q/2}$ gives the external representation

$$\mathcal{G}_{\tau_Q}^{\text{ext}} C(Q^2) := e^{-\tau_Q Q^2} C(Q^2). \tag{2.12}$$

If a Gaussian is instead written as $\exp[-|x|^2/(2\sigma^2)]$ its Fourier multiplier is $\exp[-\sigma^2|q|^2/2]$ so that $\tau = \sigma^2/2$. Equivalently the heat-kernel parameter used here is one half of the one-coordinate variance. This convention is stated once because one-insertion and two-insertion averages together with relative-coordinate averaging otherwise differ by factors of two.

Independent averaging steps add their displacements. Their laws therefore convolve. Centring and orthogonal invariance express the absence of drift and a preferred direction. Finite nonzero variance sets a resolution length. We require the full displacement law to retain its shape when that length grows as the square root of the scale parameter. Within this composition class the following proposition determines the Gaussian semigroup.

Proposition 2.3 (Conditional uniqueness of the Gaussian semigroup). Let $\{\mu_\tau\}_{\tau \geq 0}$ be a weakly continuous family of probability measures on $\mathbb{R}^d$ with $\mu_0 = \delta_0$ and convolution law $\mu_{\tau+s} = \mu_\tau * \mu_s$. Assume the measures are centred and orthogonally invariant with finite nonzero second moment at $\tau = 1$. Assume exact square-root self-similarity $(D_{\sqrt{\tau}})_\# \mu_1 = \mu_\tau$. Then there is a constant $D > 0$ such that

$$\hat{\mu}_\tau(q) = e^{-D\tau|q|^2}. \quad (2.13)$$

Thus the semigroup is Gaussian up to the choice of the scale unit D .

The self-contained proof in Appendix C uses the characteristic semigroup and its quadratic homogeneity. For the stable-law background see Chapter 3 of Ref. [10]. A related Gaussian derivation from different axioms appears in Section 2 of Ref. [9]. The conclusion belongs to this probabilistic Euclidean class. An FRG regulator is a separate scale-dependent modification of the inverse propagator.

There is a complementary entropy characterisation at fixed covariance. If a density f has the same mean and positive covariance as the Gaussian g and the entropy integrals exist then $\int f \log(f/g) \geq 0$ and $\int f \log g = \int g \log g$ imply $-\int f \log f \leq -\int g \log g$ with equality only for $f = g$ almost everywhere. This entropy characterisation concerns densities at fixed moments. Proposition 2.3 instead characterises the whole semigroup through composition and exact self-similarity.

Theorem 2.4 (Tensor lift and momentum-routing form). Let $F \in \mathcal{S}'((\mathbb{R}^d)^n)$ and apply $\mathcal{H}_{a_i\tau}$ to its i th coordinate where $a_i > 0$. In Fourier space the tensor-product action is

$$\hat{F}_\tau(p_1, \dots, p_n) = \exp\left[-\tau \sum_{i=1}^n a_i |p_i|^2\right] \hat{F}(p_1, \dots, p_n). \quad (2.14)$$

Suppose in addition that translation invariance has been used to factor out the single overall momentum-conservation distribution. Let $\mathbf{E} = (\mathbb{R}^d)^n$ and define $\Pi : \mathbf{E} \rightarrow \mathbb{R}^d$ by $\Pi(p_1, \dots, p_n) = \sum_i p_i$ and set $r = d(n-1)$. Choose linear maps

$$A : \mathbb{R}^d \rightarrow \mathbf{E}, \quad L : \mathbb{R}^r \rightarrow \mathbf{E}, \quad \Pi A = I_d, \quad \text{Im } L = \text{Ker } \Pi, \quad (2.15)$$

with L injective. Then $T(P, \xi) = AP + L\xi$ is a linear coordinate isomorphism from $\mathbb{R}^d \times \mathbb{R}^r$ onto E and $L\xi$ parametrises the entire momentum-conserving subspace. Let $\widehat{F}_{\text{full}}$ denote the coordinate representation of $\widehat{F}$ under T defined for test functions Φ by

$$\langle \widehat{F}_{\text{full}}, \Phi \rangle := |\det T|^{-1} \langle \widehat{F}, \Phi \circ T^{-1} \rangle. \quad (2.16)$$

In these adapted variables assume

$$\widehat{F}_{\text{full}}(P, \xi) = (2\pi)^d \delta^{(d)}(P) \widehat{F}_{\text{red}}(\xi), \quad \widehat{F}_{\text{red}} \in \mathcal{S}'(\mathbb{R}^r), \quad (2.17)$$

in the distributional sense that

$$\langle \widehat{F}_{\text{full}}, \Phi \rangle = (2\pi)^d \langle \widehat{F}_{\text{red}}, \Phi(0, \cdot) \rangle \quad \text{for every } \Phi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^r), \quad (2.18)$$

where $\delta^{(d)}$ is normalised by $\int \delta^{(d)}(P) f(P) dP = f(0)$. This convention includes the Jacobian associated with the declared adapted coordinates. The equality then fixes the normalisation of $\widehat{F}_{\text{red}}$. The induced action on the reduced distribution is then

$$\widehat{F}_{\text{red},\tau}(\xi) = \exp[-\tau \xi^\top M_L \xi] \widehat{F}_{\text{red}}(\xi), \quad M_L = L^\top \text{diag}(a_1 I_d, \dots, a_n I_d) L \succeq 0. \quad (2.19)$$

If V is a routing subspace on which $M_L \succeq cI$ for some $c > 0$ then the multiplier is at most $e^{-\tau c q_0^2}$ on $\{\xi \in V : \|\xi\| \geq q_0\}$. Directions in $\text{Ker } M_L$ are not suppressed by this statement.

The translation-invariant n -point convention removes the overall momentum delta before introducing independent routing variables. The resulting tensor-product distribution defines Eq. (2.19) through its test-function pairing. Its test-function proof including the normalisation in Eq. (2.18) is given in Appendix C.

For two equally weighted coordinates the routing can be displayed without matrix notation. With

$$P_c := p_1 + p_2, \quad q_r := p_1 - p_2, \quad p_{1,2} = \frac{1}{2}(P_c \pm q_r), \quad (2.20)$$

one has the exact identity

$$|p_1|^2 + |p_2|^2 = \frac{1}{2}(|P_c|^2 + |q_r|^2). \quad (2.21)$$

Thus an equal-weight tensor Gaussian remains diagonal in centre and relative momentum with multiplier $\exp[-a\tau(|P_c|^2 + |q_r|^2)/2]$. For unequal weights the same substitution also produces the mixed term $(a_1 - a_2)P_c \cdot q_r/2$. It cannot be discarded unless the weights or the kinematics remove it.

For a two-point function with $p_1 = Q$ and $p_2 = -Q$ and $a_1 = a_2 = 1/2$ Eq. (2.14) reduces to Eq. (2.12). For an internal transfer Eq. (2.19) generally contains a different quadratic form and may contain mixed momentum terms. We denote that separately specified action by

$$\mathcal{G}_{\tau_q}^{\text{rel}} A(\xi) := e^{-\tau_q \xi^\top M_L \xi} A(\xi). \quad (2.22)$$

Equation (2.22) for an internal loop momentum follows from the declared coordinate action and routing. These data must be supplied separately from any external smearing of the composite insertion.

2.5 Gaussian momentum-transfer criterion

Let $A(q)$ denote a projected contribution as a function or distribution of a specified set of transferred momenta. Once the coordinate action and the routing have supplied a positive-semidefinite matrix M set

$$A_\tau(q) := \mathcal{G}_\tau^{\text{rel}} A(q) = e^{-\tau q^\top M q} A(q). \quad (2.23)$$

This definition quantifies the Gaussian action on a specified contribution. The corresponding graph and the scale dependence of the unfiltered amplitude remain part of the effective dynamics.

Theorem 2.5 (Fixed-transfer suppression). For $\delta > 0$ let $\Omega_\delta = \{q : q^\top M q \geq \delta^2\}$. The following statements hold.

(i) If $A \in L^\infty(\Omega_\delta)$ then

$$\|A_\tau\|_{L^\infty(\Omega_\delta)} \leq e^{-\tau\delta^2} \|A\|_{L^\infty(\Omega_\delta)} \xrightarrow{\tau \rightarrow \infty} 0. \quad (2.24)$$

(ii) If A is a tempered distribution and $\chi \in C_c^\infty(\mathbb{R}^d)$ has support in Ω_δ then

$$\langle A_\tau, \chi \rangle = \langle A, e^{-\tau q^\top M q} \chi \rangle \xrightarrow{\tau \rightarrow \infty} 0. \quad (2.25)$$

(iii) Suppose a scale-dependent contribution has the form $A_\tau^{\text{full}} = e^{-\tau q^\top M q} \tilde{A}_\tau$ and

$$\|\tilde{A}_\tau\|_{L^\infty(\Omega_\delta)} \leq C e^{a\tau} \quad \text{with} \quad a < \delta^2. \quad (2.26)$$

Then $A_\tau^{\text{full}} \rightarrow 0$ in $L^\infty(\Omega_\delta)$.

The proof is given in Appendix C. The L^∞ bounds use essential suprema. For bounded continuous representatives they also give ordinary uniform bounds on the indicated region. Part (ii) is the appropriate formulation for amplitudes containing threshold distributions or other non-function singularities. Part (iii) records the additional hypothesis required when the amplitude itself runs with the averaging scale. Without such a bound growth of $\tilde{A}_\tau$ can offset the Gaussian multiplier. The convergence is not uniform in a shrinking neighbourhood of the origin. Indeed for $q = u/\sqrt{\tau}$

$$e^{-\tau q^\top M q} = e^{-u^\top M u}, \quad (2.27)$$

so the soft scaling region together with $\text{Ker } M$ remains in the problem. Its integrated weight also depends on the volume of that region. For example if $M \succ 0$ and A is bounded and continuous at the origin the substitution $q = u/\sqrt{\tau}$ gives

$$\begin{aligned} I_\tau &:= \int_{\mathbb{R}^d} e^{-\tau q^\top M q} A(q) d^d q \\ &= \tau^{-d/2} \int_{\mathbb{R}^d} e^{-u^\top M u} A(u/\sqrt{\tau}) d^d u, \\ \lim_{\tau \rightarrow \infty} \tau^{d/2} I_\tau &= \frac{\pi^{d/2}}{\sqrt{\det M}} A(0). \end{aligned} \quad (2.28)$$

The Gaussian supplies an integrable majorant so the last step follows by dominated convergence. A regular soft amplitude thus carries a phase-volume factor $\tau^{-d/2}$ even though the multiplier equals one at zero. Singular amplitudes and undamped directions in $\text{Ker } M$ retain their own scaling and integration measures. Integrating the damped variables into an effective action can also leave local operators and derivative corrections together with running couplings in the retained variables.

The multiplier describes dephasing in the transferred momentum. Contributions with a resolved nonzero transfer acquire rapidly varying phases under spatial averaging whereas back-to-back internal momenta with zero net transfer do not dephase. This is a kinematic retention statement. The size and topology of the contribution that survives in the soft region are still determined by the response dynamics.

Momentum routing specifies which combinations of momenta are damped. Graph topology specifies how interactions connect the internal lines. Bubbles and crossed ladders together with vertex corrections can all contribute in the soft region. Their relative weights follow from the interaction vertices and propagators of the response equation.

Preservation of an isolated pole. Let a projected Euclidean form factor be meromorphic near $Q^2 = 0$ and let $w(Q^2)$ be analytic there with $w(0) = 1$ and no zero in a neighbourhood of the origin. Then multiplication by w preserves both the order and residue of a possible simple pole at $Q^2 = 0$ and cannot create such a pole from a regular form factor. In particular $w(Q^2) = e^{-\tau_Q Q^2}$ preserves

$$Z_0 = \lim_{Q^2 \rightarrow 0} Q^2 C(Q^2). \quad (2.29)$$

This statement concerns $\mathcal{G}_{\tau_Q}^{\text{ext}}$ acting on a fixed correlator. Interactions integrated along an RG flow can change the correlator through the effective action. The derivative of the sourced effective action measures this dynamical change. The positive spectral measure continues to be defined from the original Wightman correlator. In particular multiplication turns a contact polynomial into an analytic non-polynomial short-range term. The physical spectral decomposition and its contact subtraction are therefore defined before the diagnostic filter is applied.

2.6 An additive-source realisation

The response operator should be identified from the linearisation of the self-consistent dynamics rather than from the topology of a selected set of graphs. An additive-source specialisation of Eq. (2.5) uses a Banach response space $\mathcal{X}_{\mathfrak{C}}$ and maps S_X, R_X of the types in Eq. (2.4). Let $\mathfrak{F}_{\tau_q}(\cdot; z, \eta) : \mathcal{X}_{\mathfrak{C}} \rightarrow \mathcal{X}_{\mathfrak{C}}$ and consider near a reference solution the sourced equation

$$X = \mathfrak{F}_{\tau_q}(X; z, \eta) + S_{X, \tau_Q}(z)J. \quad (2.30)$$

Here $J \in \mathbb{C}^m$ while $S_{X, \tau_Q} : \mathbb{C}^m \rightarrow \mathcal{X}_{\mathfrak{C}}$ and $R_{X, \tau_Q} : \mathcal{X}_{\mathfrak{C}} \rightarrow \mathbb{C}^m$ may include external Gaussian factors. The relative scale τ_q if present belongs to the self-consistent map itself. The analytic term C_{reg} is understood in the same external convention as the source and readout.

For a local solution branch $X(J; z, \eta)$ its measured output is a map into the external operator coordinates

$$\begin{aligned}\mathcal{Y}(J; z, \eta) &= \mathcal{Y}_{\text{reg}}(J; z, \eta) \\ &\quad + \mathbf{R}_{X, \tau_Q}(z)X(J; z, \eta), \\ C_{\tau_Q, \tau_q} &:= D_J \mathcal{Y}|_{J=0}, \quad C_{\text{reg}} := D_J \mathcal{Y}_{\text{reg}}|_{J=0}.\end{aligned}\tag{2.31}$$

Thus the correlator or susceptibility is the Jacobian from the declared physical source coordinates to the declared readout coordinates. A critical direction contributes when the source excites it and the readout detects it. When the sources couple to physical insertions and the output is their normalised expectation value differentiation gives the connected two-point matrix with the specified source-pairing convention and local contact terms. The source perturbs the state equation. The state adjusts and the output records that adjustment together with its direct source dependence.

Proposition 2.6 (Response factorisation). Suppose $\mathfrak{F}_{\tau_q}$ is continuously Fréchet differentiable in X on a neighbourhood of a solution X_* of Eq. (2.30) with $J = 0$ and suppose $\mathcal{Y}_{\text{reg}}$ is Fréchet differentiable in J at the origin. Set

$$\mathcal{B}_{\tau_q}(z; \eta) := D_X \mathfrak{F}_{\tau_q}(X_*; z, \eta).\tag{2.32}$$

If $I - \mathcal{B}_{\tau_q}$ is invertible the Banach-space implicit-function theorem gives a differentiable local solution branch and its measured response has the form

$$C_{\tau_Q, \tau_q}(z; \eta) = C_{\text{reg}}(z; \eta) + \mathbf{R}_{X, \tau_Q}(z)[I - \mathcal{B}_{\tau_q}(z; \eta)]^{-1} \mathbf{S}_{X, \tau_Q}(z).\tag{2.33}$$

At a critical point the same identity holds as a meromorphic continuation on a domain satisfying the analytic Fredholm hypotheses.

Thus an external Gaussian changes the source and readout maps but not the spectrum of the fixed response operator in Eq. (2.32). For a source–source matrix with the same scalar filter on both insertions one may write on the Euclidean axis

$$\begin{aligned}\mathbf{S}_{X, \tau_Q}(z) &= \mathbf{S}_{X, 0}(z)W_{\tau_Q}(z), \\ \mathbf{R}_{X, \tau_Q}(z) &= W_{\tau_Q}(z)\mathbf{R}_{X, 0}(z), \\ C_{\text{reg}, \tau_Q}(z) &= W_{\tau_Q}(z)C_{\text{reg}, 0}(z)W_{\tau_Q}(z), \\ W_{\tau_Q}(z) &= e^{-\tau_Q z/2}I_m.\end{aligned}\tag{2.34}$$

Since $W_{\tau_Q}(0) = I_m$ this factorisation also makes the preservation of an existing residue manifest. Internal averaging affects that spectrum through a specified τ_q -dependence of $\mathfrak{F}_{\tau_q}$. This dependence must be derived from the declared Gaussian action.

3 Channels, elimination, and visible critical structure

The sourced state equation of Section 2 supplies a response space and the four maps B, S, R, D . A channel is a specified source-to-observation problem on this space. A bounded

projector P_X selects a retained subspace and $Q_X = I_X - P_X$ its complement. The retained sector is closed only when the update does not send it into the complement. We use $\mathcal{B} = B$ and $\mathbf{S}_X = S$ together with $\mathbf{R}_X = R$ and $C_{\text{reg}} = D$ to keep the notation consistent with the QFT realisation.

3.1 Reference-state symmetry and exact elimination

The relevant symmetry is the symmetry of the reference state as well as of the update. A transformation that moves the reference state relates two different linear-response problems. It need not split either one into invariant channels.

Equivariant linearisation at the reference state. Let a group act by bounded linear isomorphisms U_g on the state space and suppose $F(U_g X) = U_g F(X)$. At X_*

$$DF(U_g X_*) U_g = U_g DF(X_*). \quad (3.1)$$

Consequently $DF(X_*)$ commutes with U_g for the stabilizer $G_{X_*} = \{g : U_g X_* = X_*\}$. The same fixed-background restriction applies to source and observation intertwiners when their equivariance holds. Differentiate the equivariance identity with respect to X . For $g \in G_{X_*}$ the two Jacobians have the same base point. Differentiating the sourced equivariance and observation identities gives their intertwining relations. A broken symmetry instead relates different backgrounds and need not block-diagonalise one of their Jacobians.

Symmetry provides invariant sectors only after these fixed-background conditions and the relevant bounded projectors have been established. The source and readout maps determine which of these sectors is observable. Exact finite systems and finite approximations to infinite systems both admit block elimination but only the latter need cutoff convergence to identify an infinite target.

Proposition 3.1 (Response-closed channel). The spectrum of the restriction $\mathcal{B}|_{\text{Ran } P_X}$ is represented by $P_X \mathcal{B} P_X$ only if

$$Q_X \mathcal{B} P_X = 0. \quad (3.2)$$

If Eq. (3.2) fails and $I_X - Q_X \mathcal{B} Q_X$ is invertible in the domain under consideration elimination of the complementary sector gives the exact effective response operator

$$\mathcal{B}_{\text{eff}} = P_X \mathcal{B} P_X + P_X \mathcal{B} Q_X (I_X - Q_X \mathcal{B} Q_X)^{-1} Q_X \mathcal{B} P_X. \quad (3.3)$$

Writing $D_Q := I_X - Q_X \mathcal{B} Q_X$ the associated source and readout maps are

$$\mathbf{S}_{X,\text{eff}} = P_X \mathbf{S}_X + P_X \mathcal{B} Q_X D_Q^{-1} Q_X \mathbf{S}_X, \quad (3.4)$$

$$\mathbf{R}_{X,\text{eff}} = \mathbf{R}_X P_X + \mathbf{R}_X Q_X D_Q^{-1} Q_X \mathcal{B} P_X. \quad (3.5)$$

If the unreduced response contains the analytic term C_{reg} of Eq. (2.33) its reduced value is

$$C_{\text{reg,eff}} = C_{\text{reg}} + \mathbf{R}_X Q_X D_Q^{-1} Q_X \mathbf{S}_X. \quad (3.6)$$

The full response is therefore $C_{\text{reg,eff}} + \mathbf{R}_{X,\text{eff}}(I_X - \mathcal{B}_{\text{eff}})^{-1} \mathbf{S}_{X,\text{eff}}$. Consequently the spectrum of $P_X \mathcal{B} P_X$ alone has no channel interpretation when the omitted sector remains coupled.

If $\mathcal{B}(z)$ is a compact analytic family and $D_Q(z)^{-1}$ exists throughout a neighbourhood then $\mathcal{B}_{\text{eff}}(z)$ is a compact analytic family on $\text{Ran } P_X$. On a connected neighbourhood where $I_X - \mathcal{B}_{\text{eff}}$ is invertible at one point analytic Fredholm theory gives a meromorphic inverse. With analytic source and readout maps together with the background term and no continuum threshold in that neighbourhood the effective response is meromorphic as well.

Proof. Write $I - \mathcal{B}$ in $P \oplus Q$ block form and solve the Q equation before the P equation. This gives Eqs. (3.3)–(3.6). Compactness of $\mathcal{B}_{\text{eff}}$ follows because bounded left and right multiplication preserve compactness while analyticity follows from analyticity of the inverse map on the open set of bounded invertible operators. $\square$

This is the Feshbach–Schur block elimination adapted to response maps between the stated spaces [3]. A channel that fails Eq. (3.2) can be enlarged to include the coupled states or described by the energy-dependent operator Eq. (3.3). The second term accounts for excursions into the eliminated sector and their return to the retained one. The source and readout maps together with the direct term account for the other ways that sector contributes to the observation. Thus a retained channel can acquire collective feedback through other channels even when its restricted update is noncritical. Whether the feedback reaches criticality is determined by the full effective operator including these excursions.

If the source lies in the retained sector and the readout annihilates its complement then $Q_X S_X = 0$ and $R_X Q_X = 0$. The general block formulae then reduce to

$$S_{X,\text{eff}} = S_X, \quad R_{X,\text{eff}} = R_X, \quad C_{\text{reg},\text{eff}} = C_{\text{reg}}. \quad (3.7)$$

These simplifications require both compatibility identities. A general source and observation can couple to Q_X in which case all four unsimplified terms must be retained.

For differentiation with respect to a parameter a the inverse identity is $\partial_a D_Q^{-1} = D_Q^{-1} Q_X (\partial_a \mathcal{B}) Q_X D_Q^{-1}$ for fixed projectors. The product rule then contributes derivatives of both off-diagonal blocks and of the source and readout maps together with the direct term. Derivatives along a reference branch include its state displacement. A moving projector additionally contributes its own derivatives. Appendix C records the fixed-projector formula. Frequency dependence generated by this elimination carries the delayed influence of complementary states.

An eliminated dynamical variable with independent initial data generally leaves both memory and an initial-state contribution. A local auxiliary field instead obeys an algebraic equation. If its derivative with respect to that field is invertible the implicit-function theorem determines the field locally from the retained fields and their derivatives together with the sources. Substitution into the common quadratic action gives the corresponding Schur correction. In the auxiliary-radius realisation of ref. [15] Appendix G.6 this operation is performed along a coupled matter–amplitude FLRW solution. The radius is first eliminated from the common action and the lapse and shift constraints are then removed from its quadratic variation. The resulting physical kinetic and gradient matrices are positive on a certified finite interval. This example supplies an actual algebraic complementary

block. The time-dependent background equations provide the propagation and constraint conditions used to interpret its reduced response.

3.2 Isolated Fredholm response

Near an isolated critical mode an internally transmitted disturbance nearly reproduces itself. An eigenvalue approaching one measures this feedback in the linearised equation. For it to appear in an observation the source must excite the corresponding direction and the readout must detect it. Feedback criticality and visibility enter the residue through separate factors. The reciprocal and spectral realisations then determine the sign and physical interpretation of the residue.

The operator $I - \mathcal{B}$ is the restoring operator of the coupled channel and its inverse is the corresponding susceptibility. When an eigenvalue of $\mathcal{B}$ reaches one a collective direction reproduces its disturbance after a complete feedback cycle so the restoring coefficient along that direction vanishes. The condition $\partial_z \lambda \neq 0$ specifies how this softening is crossed in the declared response coordinate. The source and readout overlaps then decide whether the softened direction is visible.

Theorem 3.2 (Kernel criterion for an isolated algebraic pole). Fix $\eta = \eta_c$ and let U be a connected neighbourhood of $z = 0$. Suppose that $\mathcal{B}(z; \eta_c)$ is a compact analytic operator family on U that $I - \mathcal{B}(z; \eta_c)$ is invertible at least at one point of U and that U contains no continuum threshold or other non-meromorphic singularity. Suppose also that the response-space source $\mathbf{S}_X(z)$ and readout $\mathbf{R}_X(z)$ together with the matrix background $C_{\text{reg}}(z)$ in Eq. (2.33) are analytic on U . Suppress the fixed parameter η_c and consider that matrix response. Then

- (i) if $1 \notin \text{spec } \mathcal{B}(0)$ the response has no pole at $z = 0$ generated by this kernel.
- (ii) if $\mathcal{B}(z)$ has an isolated algebraically simple eigenvalue $\lambda(z)$ with

$$\lambda(0) = 1, \quad \partial_z \lambda(0) \neq 0, \quad (3.8)$$

then the resolvent has a simple pole. The measured response contains that pole precisely when both $\mathbf{R}_X(0)|r\rangle \neq 0$ and $\langle \ell | \mathbf{S}_X(0) \neq 0$.

- (iii) with $\langle \ell | r \rangle = 1$ its algebraic residue matrix is

$$Z_{\text{alg}} = - \frac{\mathbf{R}_X(0)|r\rangle \langle \ell | \mathbf{S}_X(0)}{\partial_z \lambda(0)}. \quad (3.9)$$

The algebraic residue includes the sign of the crossing and both overlaps. Matching the response to a positive physical spectral representation gives its interpretation as a Lorentzian mass atom.

The proof in Appendix C is an application of the analytic Fredholm theorem and the Riesz projection onto the critical eigenspace. On the simple branch choose local analytic eigenvectors with $\langle \ell(z) | r(z) \rangle = 1$. The singular contribution is

$$\frac{\mathbf{R}_X(z)|r(z)\rangle \langle \ell(z) | \mathbf{S}_X(z)}{1 - \lambda(z)}.$$

The two numerator factors measure excitation and detection of the collective direction. Spectral isolation permits the response to be resolved into this collective mode and an analytic complementary part. The bounded complementary inverse controls the response of the remaining directions while the critical mode becomes soft. Expanding $1 - \lambda(z) = -\lambda'(0)z + O(z^2)$ gives the residue in Eq. (3.9). Its magnitude therefore depends both on the coupling to the mode and on how rapidly its inverse response vanishes. This is a local statement in z . Its interpretation as a frequency or momentum singularity uses the specified physical realisation. Condition (3.8) is transversality in the declared complex response coordinate z . For a real eigenvalue branch along a real control parameter $\partial_\eta \lambda(0; \eta_c) \neq 0$ certifies a first-order transverse crossing of one. Higher-order control crossings are possible. For example $\lambda(z; \eta) = 1 - z + \eta^3$ crosses at $\eta = 0$ despite a vanishing first control derivative. Higher-order crossings require their own analysis and do not change the simple-pole criterion in z . When $\eta = k$ is an RG scale a crossing at intermediate k is a flow diagnostic. A physical-state claim requires the pole conditions for the exact endpoint response. If a continuum threshold starts at $z = 0$ the meromorphic-neighbourhood hypothesis of Theorem 3.2 fails in general. A limiting-absorption estimate or direct spectral analysis treats this threshold.

3.3 Reciprocal response and a finite-rank positive residue

The Fredholm result accommodates nonnormal kernels and independent source and read-out maps. A reciprocal Hilbert-space realisation permits an additional positive statement before any Lorentzian representation. The critical subspace determines the singular part. The invertible complement contributes analytic terms while the derivative of the response operator fixes the sign of the residue. Source overlap then determines which critical directions are visible.

Theorem 3.3 (Positive residue for reciprocal response). Let $L(z)$ be analytic in operator norm near zero on a Hilbert space $\mathcal{H}$. Suppose $L(0)$ is self-adjoint with a nonzero finite-dimensional kernel and orthogonal projector P . Suppose $QL(0)Q|_{Q\mathcal{H}}$ has a bounded inverse where $Q = I - P$. If

$$G_0 = PL'(0)P|_{P\mathcal{H}} \succ 0, \quad (3.10)$$

then on a sufficiently small punctured neighbourhood

$$L(z)^{-1} = \frac{PG_0^{-1}P}{z} + A(z), \quad A \text{ analytic at } 0. \quad (3.11)$$

Let $S(z) : \mathbb{C}^m \rightarrow \mathcal{H}$ and $R(z) : \mathcal{H} \rightarrow \mathbb{C}^m$ together with $D(z)$ be analytic with $R(0) = S(0)^*$ in the declared source-output pairing. The response $D + RL^{-1}S$ has residue

$$Z_{\text{native}} = S(0)^* PG_0^{-1} PS(0) \succeq 0, \quad \text{rank } Z_{\text{native}} = \text{rank}(PS(0)). \quad (3.12)$$

A pole is visible precisely when $PS(0) \neq 0$.

Proof. Self-adjointness makes both off-diagonal blocks of $L(0)$ zero. The inverse of the Q block persists analytically. The Schur complement on $P\mathcal{H}$ is $zG_0 + O(z^2)$ since the

product of the off-diagonal blocks is $O(z^2)$. Its inverse is $z^{-1}G_0^{-1}$ plus an analytic remainder. Block inversion proves Eq. (3.11). The off-diagonal factors cancel the possible z^{-1} singularity outside the P block. Taking the residue gives Eq. (3.12). Its factorisation as $(G_0^{-1/2}PS(0))^*(G_0^{-1/2}PS(0))$ proves positivity and rank. $\square$

For a source direction j the residue obeys $j^*Z_{\text{native}}j = \|G_0^{-1/2}PS(0)j\|^2$. It is positive exactly when that source excites the critical subspace. Reciprocity makes readout the adjoint of excitation in the same pairing. The two mode overlaps therefore combine into a squared norm. The positive crossing matrix supplies the weight with which that norm enters the pole. Dynamical stability is determined separately by the actual evolution. The sign of an interaction energy also depends on how physical sources couple to this response.

The theorem applies to the specified Hilbert pairing with reciprocity and an isolated critical subspace whose crossing matrix is positive. For $L = I - \mathcal{B}$ its crossing matrix is $-P\mathcal{B}'(0)P$ in the response variable z . The complementary inverse controls spectral accumulation. For example $\text{diag}(0, 1, 1/2, \dots)$ has no bounded complementary inverse and adding zI produces singularities accumulating at zero. For unbounded operators the result applies after a justified common-domain or closed-form analytic reduction. Nonreciprocal updates retain independent source and readout maps.

For a finite-dimensional real affine response $L(z) = H + zK$ under the hypotheses of the theorem with $H = H^\top$ and reciprocal source pairing a basis of the critical space gives a direct residue formula. For rational matrices this basis can be obtained by exact elimination. If N is a full kernel basis of H and $N^\top KN \succ 0$ the residue is

$$Z = S^\top N(N^\top KN)^{-1}N^\top S. \quad (3.13)$$

To see this without orthogonalising N complete it to an invertible basis and apply the same Schur argument by congruence. This expression is invariant under $N \mapsto NT$ for invertible T and has rank $\text{rank}(N^\top S)$. The software of Ref. [16] provides supplementary rational finite-matrix evaluations of this formula. A small positive eigenvalue instead gives a regular response at zero since $\lim_{z \downarrow 0} z/(\epsilon + z) = 0$ for every $\epsilon > 0$. The residue here refers to the specified analytic parameter z . An interpretation as a spatial propagation pole or a vacuum spectral atom additionally requires a representation of that parameter and of the same source-contracted response. Iteration stability is determined by the update spectrum independently of the sign of this residue.

3.4 Quantum-field-theory realisation

For the local-operator realisation work with a local unitary Poincaré-invariant quantum field theory admitting a Euclidean continuation for the physical correlation functions considered below. Definition A.1 also specifies the directly reduced helicity realisation whose physical Hilbert space and source covariance enter the same response factorisation. The invariant-mass spectral assertions use a Poincaré-invariant positive-energy vacuum unique in the chosen physical representation as specified in Appendix A. Whenever a response is identified with a physical two-point function its insertions are centred $\Theta_i = O_i - \langle O_i \rangle I$ and its correlation matrix is connected. Subtracting these one-point products precedes and is

distinct from the subtraction of local contact terms. Gauge theories are understood in a renormalised BRST formulation without an uncanceled gauge anomaly. Let $\mathcal{V}_{\text{phys}}$ be the renormalised operator space after passage to the physical quotient described in Appendix B. The quotient is algebraic. There is no canonical Hilbert norm on a space of local composite operators so every infinite-dimensional boundedness statement below is made relative to a specified channel norm for which $\mathcal{V}_{\text{phys}}$ has been completed to a Banach space and all stated maps extend continuously. We use the same symbol for that completion. All boundedness statements refer to this channel norm. A bounded projector $\mathcal{P}_{\mathfrak{C}}$ selects a Lorentz and internal-symmetry sector with the closed Banach subspace $\mathcal{H}_{\mathfrak{C}} = \text{Ran } \mathcal{P}_{\mathfrak{C}}$. For a finite operator block the choice of norm is immaterial because all norms are equivalent. A finite block approximates this exact space through specified projection and comparison maps.

The operator space specifies which insertions are probed. The response space carries the variables used to solve their sourced equations. The realisation maps connect these descriptions while the regulator and truncation data record how a concrete calculation is performed.

Definition 3.4 (Physical channel). A physical operator channel is the tuple

$$\mathfrak{C}_{\text{phys}} = (\mathsf{T}, |\Omega\rangle, \mathcal{V}_{\text{phys}}, \mathcal{P}_{\mathfrak{C}}, \mathcal{S}, \mathcal{E}, \mathcal{S}_{\mathcal{O}}, \mathcal{R}_{\mathcal{O}}), \quad (3.14)$$

Here T specifies the theory and $|\Omega\rangle$ the state or vacuum. The data $\mathcal{S}$ specify the exact symmetries and anomalies and $\mathcal{E}$ the external kinematics. For a general state the response variables must respect its unbroken symmetries. The vacuum identification $z = Q^2$ and the invariant-mass atom test do not extend to an ensemble without a separate spectral argument. The bounded maps

$$\mathcal{S}_{\mathcal{O}}(z) : \mathbb{C}^m \longrightarrow \mathcal{H}_{\mathfrak{C}}, \quad \mathcal{R}_{\mathcal{O}}(z) : \mathcal{H}_{\mathfrak{C}} \longrightarrow \mathbb{C}^m \quad (3.15)$$

specify the physical operator insertions and their output coordinates. To use these insertions in a response equation one must also specify how they are realised on the response space. Let $\mathcal{X}_{\mathfrak{C}}$ be a Banach response space and let $P_X \in \mathcal{L}(\mathcal{X}_{\mathfrak{C}})$ be a bounded projector and let

$$\iota_{\mathfrak{C}}(z) : \mathcal{H}_{\mathfrak{C}} \longrightarrow \mathcal{X}_{\mathfrak{C}}, \quad \pi_{\mathfrak{C}}(z) : \mathcal{X}_{\mathfrak{C}} \longrightarrow \mathcal{H}_{\mathfrak{C}} \quad (3.16)$$

be bounded source-realisation and readout-realisation maps. They need not be mutual inverses. Compatibility with the selected response block means

$$P_X \iota_{\mathfrak{C}} = \iota_{\mathfrak{C}}, \quad \pi_{\mathfrak{C}} P_X = \pi_{\mathfrak{C}}. \quad (3.17)$$

The maps that enter a response equation are then

$$\mathcal{S}_X := \iota_{\mathfrak{C}} \mathcal{S}_{\mathcal{O}} : \mathbb{C}^m \longrightarrow \mathcal{X}_{\mathfrak{C}}, \quad \mathcal{R}_X := \mathcal{R}_{\mathcal{O}} \pi_{\mathfrak{C}} : \mathcal{X}_{\mathfrak{C}} \longrightarrow \mathbb{C}^m. \quad (3.18)$$

Their construction specifies how the physical model excites and observes the response state. In particular a critical response eigenspace can fail to couple to either map.

The operator part of the tuple is an exact channel when $\text{Ran } \mathcal{P}_{\mathfrak{C}}$ is stable under the renormalisation map and the unbroken symmetries recorded in $\mathcal{S}$. A finite-dimensional projected space $P_N \mathcal{V}_{\text{phys}}$ is a computational block. It is exact only when closure under renormalisation and symmetry together with dynamical response has been proved. Otherwise conclusions drawn from it carry the truncation label. The classification applies to the full block and its correlation matrix. Here boundedness is taken in the Banach completion and channel norm declared as part of the response data below. The algebraic BRST/EOM quotient precedes this completion. Choosing a completion does not alter which representatives define physical cohomology classes.

For later reference the response and approximation data are recorded as

$$\mathfrak{I} = (\mathcal{X}_{\mathfrak{C}}, P_X, \iota_{\mathfrak{C}}, \pi_{\mathfrak{C}}, \mathcal{H}_{\tau}, \mathcal{G}_{\tau_Q}^{\text{ext}}, \mathcal{G}_{\tau_q}^{\text{rel}}, \mathfrak{F}_{\tau_q}, \mathcal{B}(z; \eta), \mathcal{R}_{\text{ren}}, \mathcal{T}_N), \quad (3.19)$$

Here $\mathcal{H}_{\tau}$ is the one-coordinate heat semigroup. The maps $\mathcal{G}^{\text{ext}}$ and $\mathcal{G}^{\text{rel}}$ are its induced representations from Section 2. The map $\mathfrak{F}_{\tau_q}$ specifies the self-consistent response and $\mathcal{B}$ its linearisation. The data $\mathcal{R}_{\text{ren}}$ fix renormalisation and the regulator. The truncation $\mathcal{T}_N$ approximates an infinite target with $N = \infty$ denoting the exact hierarchy. A finite system defined by an exact operation protocol has no such limiting requirement.

The spectral and response variables have distinct roles. The invariant $s \geq 0$ labels Lorentzian spectral mass. The variable $Q^2 \geq 0$ is the squared external Euclidean momentum of a projected source–source correlator. The scale k labels a Wilsonian or functional RG trajectory. Within a diagram or kernel $q \in \mathbb{R}^d$ denotes the Euclidean momentum transferred between a fast subgraph and the external channel. In this QFT realisation z is the complexified external invariant with $z = Q^2$ on the Euclidean axis and $z = -p^2 - i0$ at the Lorentzian physical boundary. Finally η denotes a control parameter such as a coupling or background or RG scale. A derivative with respect to z is therefore not a flow derivative with respect to η or k .

3.5 Operator blocks and the physical quotient

Begin with the renormalised local operators of fixed Lorentz representation with specified internal quantum numbers and ghost number. Their exact completion can be infinite dimensional. A finite set selected by a canonical-dimension bound or a numerical basis is denoted $\mathcal{V}_N$ and is treated as a truncation unless a separate closure argument applies. Renormalisation takes the matrix form

$$[\mathcal{O}_i]_R(\mu) = Z_{ij}(\mu, \mathcal{R}) \mathcal{O}_j^{\text{bare}}. \quad (3.20)$$

In a gauge theory this set may contain gauge-invariant representatives and BRST-exact operators together with equation-of-motion operators and total derivatives. The triangular renormalisation system retains these directions before passage to physical matrix elements [18].

Let $\bar{\mathcal{V}} = \mathcal{V}/\mathcal{I}_{\text{EOM}}$ be the on-shell local operator space and assume that the renormalised BRST differential $\mathfrak{s}_B$ descends to it. For physical separated-point matrix elements the

relevant ghost-number-zero cohomology is

$$\mathcal{V}_{\text{phys}} = H^0(\mathfrak{s}_B, \bar{\mathcal{V}}) = \frac{\text{Ker}(\mathfrak{s}_B : \bar{\mathcal{V}}^0 \longrightarrow \bar{\mathcal{V}}^1)}{\text{Ran}(\mathfrak{s}_B : \bar{\mathcal{V}}^{-1} \longrightarrow \bar{\mathcal{V}}^0)}, \quad (3.21)$$

with contact terms retained separately. The cohomological treatment of local operator renormalisation is developed in Section 8.6 of Ref. [19]. The on-shell gluonic matrix elements at zero momentum transfer studied in Ref. [20] illustrate the infrared care needed when taking exceptional limits. Here $\mathcal{I}_{\text{EOM}}$ is represented quantum mechanically by the renormalised field-redefinition insertions $\mathcal{E}_F^R$ of Eq. (B.3) including the measure divergence and operator-mixing counterterms. Total derivatives require separate treatment. They are redundant for a constant-source integrated insertion or at zero insertion momentum only when the boundary term vanishes and the indicated limit is well defined. For a nonconstant source integration by parts retains a source-gradient insertion. At nonzero insertion momentum the descendants must remain in the mixing block. Equation (3.21) is the local cohomological construction reviewed in Appendix B. Anomaly insertions and descendants together with contact contributions remain in the appropriate mixing and source terms.

For non-gauge theories the BRST quotient is absent but operator mixing and EOM redundancies remain. For the scalar improvement $T_{\mu\nu} \mapsto T_{\mu\nu} + (\partial_\mu \partial_\nu - \eta_{\mu\nu} \square)X$ with X a renormalised Lorentz scalar the separated-point TT projection is unchanged whereas the spin-zero response and the poles visible in an individual representative can change. If the improvement descendant is retained in a closed operator block a finite nonsingular constant basis change preserves the pole structure of the full correlation matrix and the rank of each isolated residue. The full matrix and its residue transform by $C \mapsto MCM^\dagger$ with M constant and nonsingular.

This definition accommodates mixing between scalar operators such as mass terms and interaction densities together with total derivatives and between tensor operators with the same physical spin. It also prevents unlike objects from being entered into a common table. A running coupling is a coordinate on theory space. A local composite operator is an insertion. They are related through source derivatives and beta functions but are not the same classification object.

Renormalisation closure does not by itself imply closure under the dynamical response. The operator projector $\mathcal{P}_{\mathfrak{E}}$ acts on $\mathcal{V}_{\text{phys}}$ whereas the compatible response projector P_X from Eq. (3.17) acts on $\mathcal{X}_{\mathfrak{E}}$. Put $Q_X = I_X - P_X$. The two projectors are related by the realisation maps but are not identified.

3.6 Symmetry-allowance criterion

The original QED Ward identity and its Ward–Takahashi generalisation relate the fermion propagator and current vertex [21, 22]. More generally a symmetry identity relates current insertions to variations of the other fields. For a current J_A^μ and renormalised insertions

Φ_r the position-space identity allowing for an anomaly has the schematic form

$$\begin{aligned} \partial_\mu^x \langle T J_A^\mu(x) \Phi_1(x_1) \cdots \Phi_n(x_n) \rangle = & -i \sum_{r=1}^n \delta^{(d)}(x - x_r) \langle T \Phi_1 \cdots (\delta_A \Phi_r) \cdots \Phi_n \rangle \\ & + \langle T \mathcal{A}_A(x) \Phi_1 \cdots \Phi_n \rangle, \end{aligned} \quad (3.22)$$

where $\mathcal{A}_A$ is an anomaly insertion and the displayed delta functions are contact terms. Translation Ward identities for $T_{\mu\nu}$ have the same logical structure with derivatives acting on the other insertions. After Fourier transformation these identities fix divergences and selected normalisations. They do not determine the unconstrained transverse form factors.

For a gauge theory physical local insertions are represented by BRST cohomology classes rather than by the condition of BRST closure alone. The corresponding constraints on gauge-fixed Green functions are the Slavnov–Taylor identities [23, 24]. Writing the BRST differential as s_B ghost-number-zero closure $s_B \mathcal{O} = 0$ is supplemented by the identification $\mathcal{O} \sim \mathcal{O} + s_B \Psi$ after passage to the equation-of-motion quotient. A total derivative dY is also quotiented for a constant-source integrated functional with vanishing boundary term or at zero insertion momentum when that limit is well defined. A nonconstant source leaves a source-gradient insertion after integration by parts and descendants are retained at nonzero insertion momentum. A colour component $F_{\mu\nu}^a$ transforms covariantly and is not by itself a gauge-invariant scalar observable whereas $\text{tr} F_{\mu\nu} F^{\mu\nu}$ defines a scalar gauge-invariant class.

Scope of a symmetry identity. Fix a renormalised physical operator block with its state and projector. Assume the relevant Ward or Slavnov–Taylor identity is anomaly free in that block. After the contact and anomaly terms have been kept explicitly the identity has the following consequences

- (i) set symmetry-forbidden projected components to zero up to specified contact terms.
- (ii) determine longitudinal components in terms of lower-point functions.
- (iii) fix a zero-transfer form-factor normalisation when stable external states carry a nonzero conserved charge and LSZ reduction applies and the diagonal matrix element has a continuous zero-transfer limit.
- (iv) leave the remaining physical transverse components symmetry allowed.

The state and dynamical response determine vacuum spectral weight and the presence of an isolated pole. After Fourier transformation and projection with LSZ amputation in part (iii) Eq. (3.22) together with the stated representation and regularity assumptions gives (i)–(iii). Part (iv) is the complement of those constraints in the physical operator block.

This leaves form factors not fixed by the specified symmetry to the dynamical calculation. Symmetry thus organises which disturbances can couple to which outputs and fixes selected normalisations while the state and interactions determine the remaining spectral weight. Comparing channels requires both parts of this information. Composite scalar and

fermion-bilinear channels may contain bound-state poles even though no conserved-current normalisation fixes their residues.

The Bethe–Salpeter construction describes bound states through a two-particle kernel [25]. In a sourced channel this kernel enters the inhomogeneous response equation

$$\Gamma_{\mathfrak{C}}(z) = \Gamma_{\mathfrak{C}}^{(0)}(z) + \mathcal{B}_{\mathfrak{C}}(z; \eta) \Gamma_{\mathfrak{C}}(z), \quad \mathcal{B}_{\mathfrak{C}}(z; \eta) = \mathcal{K}_{2\text{PI}, \mathfrak{C}}(z; \eta) G_{0, \mathfrak{C}}(z; \eta). \quad (3.23)$$

The dimensionless eigenvalue problem is the product of the irreducible kernel $\mathcal{K}_{2\text{PI}}$ with the two-particle propagation operator G_0 . For an eigenvalue-based solution of the homogeneous bound-state equation see Ref. [26]. A concrete application must show that its 2PI Bethe–Salpeter or Dyson–Schwinger or FRG construction represents the Jacobian in Eq. (2.32) exactly or with a stated defect.

A vacuum spectral representation gives the residue a particle interpretation. Resolving the centred insertions into physical intermediate states writes their Wightman matrix as a positive measure of excitation amplitudes. This is the intermediate-state construction of the Källén–Lehmann representation [27, 28]. Compression onto positive-norm polarisations preserves that positivity. Euclidean continuation then expresses the same source response as a mass spectral integral with local subtractions recorded separately. Matching these insertions and conventions to the response equation allows its Laurent coefficient to be read as the weight of the zero-mass atom. The matching conditions identify both coefficients through the same infrared limit.

Theorem 3.5 (Fredholm–KL bridge). Let the hypotheses of Theorem 3.2(ii)–(iii) hold and set $z = Q^2$ on the Euclidean axis. Assume in addition that

- (i) at the physical endpoint and with $\tau_Q = 0$ the matrix in Eq. (2.33) is exactly the projected unfiltered connected Euclidean covariance of a physical spectral realisation in Definition A.1 with the vacuum and spectral assumptions of Appendix A.
- (ii) the tensor or spin part of $\mathcal{P}_{\mathfrak{C}}$ uses the physical-polarisation compression in Assumption A.2 including its separately defined zero-mass helicity endpoint and the external basis and vacuum centring together with the source pairing and continuation convention and subtraction polynomial $P(Q^2)$ agree in the response and spectral constructions.
- (iii) after subtraction of $P(Q^2)$ the direct term and the complementary Fredholm resolvent contribution are analytic at $Q^2 = 0$.
- (iv) the continuum part of the physical matrix-valued spectral measure is locally finite and has no atom at zero and satisfies the ultraviolet integrability hypotheses of Proposition A.4.

Then the algebraic residue is the physical zero-mass atom matrix

$$Z_{\text{alg}} = \lim_{Q^2 \downarrow 0} Q^2 [C(Q^2) - P(Q^2)] = Z_0 \succeq 0. \quad (3.24)$$

Consequently a nonzero observable Fredholm pole satisfying these matching hypotheses is a physical zero-mass spectral atom. Conversely if an exact calculation that purports to satisfy (i)–(iv) produces a non-Hermitian or indefinite Z_{alg} the response and spectral constructions violate at least one of the matching or positivity assumptions.

Proof. The Fredholm decomposition gives $C(Q^2) = Z_{\text{alg}}/Q^2 + A(Q^2)$ with A analytic at the origin under assumption (iii). Hence $\lim_{Q^2 \downarrow 0} Q^2[C(Q^2) - P(Q^2)] = Z_{\text{alg}}$. The same connected matrix is by assumptions (i)–(ii) the correlator entering the subtracted Källén–Lehmann representation. Assumption (iv) permits Proposition A.4 to identify the same limit with Z_0 . Positivity of the Wightman measure gives $Z_0 \succeq 0$. $\square$

The bridge applies to the isolated meromorphic endpoint specified in its hypotheses. At the level of the spectral measure a zero-mass atom may also coexist with continuum weight accumulating at $s = 0$. Proposition A.4 still extracts the atomic coefficient under its local-finiteness and integrability hypotheses while the threshold density determines an additional non-meromorphic contribution and its own long-distance tail. At such an endpoint the atom is defined by the measure limit and the threshold contribution is treated through its boundary behaviour. The isolated Fredholm pole describes a meromorphic response. The spectral atom also has meaning at a continuum threshold.

Spatial response of the pole. The spatial meaning can be made explicit in a translation-invariant realisation with three spatial dimensions. Suppose a fixed physical source contraction j has static pole contribution $Z_j/|\mathbf{q}|^2$ where $Z_j = j^* Z_0 j > 0$ is independent of $\mathbf{q}$ and $Q^2 = |\mathbf{q}|^2$ at zero frequency. For the external Gaussian representation already defined put $r = |\mathbf{x}| > 0$ and $\tau_Q > 0$. The pole alone gives

$$\begin{aligned} K_{\tau_Q}^{\text{pole}}(r) &= Z_j \int \frac{d^3 q}{(2\pi)^3} \frac{e^{-i\mathbf{q}\cdot\mathbf{x}} e^{-\tau_Q |\mathbf{q}|^2}}{|\mathbf{q}|^2} = Z_j \int_{\tau_Q}^{\infty} \frac{e^{-r^2/(4u)}}{(4\pi u)^{3/2}} du \\ &= \frac{Z_j}{4\pi r} \operatorname{erf}\left(\frac{r}{2\sqrt{\tau_Q}}\right). \end{aligned} \quad (3.25)$$

Here $\operatorname{erf}(v) = (2/\sqrt{\pi}) \int_0^v e^{-t^2} dt$. The integral representation follows by writing

$$\frac{e^{-\tau_Q |\mathbf{q}|^2}}{|\mathbf{q}|^2} = \int_{\tau_Q}^{\infty} e^{-u |\mathbf{q}|^2} du.$$

The Gaussian Fourier integral and the substitution $r/(2\sqrt{u})$ give the remaining equalities. At distances large compared with $\sqrt{\tau_Q}$ this contribution approaches $Z_j/(4\pi r)$. External smoothing resolves its short-distance behaviour while preserving its long-distance coefficient. For a tensor source with a momentum-dependent numerator that numerator remains inside the transform. The complete spatial response also includes the transformed regular and continuous-spectrum terms. Dominance of the pole requires bounds on those terms. Equation (3.25) is a response kernel. A force interpretation requires the corresponding matter coupling and interaction energy.

A realised helicity response. The bridge can be evaluated explicitly for the two physical helicities of a free gauge tensor on Minkowski space. Let ϕ_λ with $\lambda = \pm 2$ denote unit-normalised transverse traceless amplitudes with reduced action

$$S_2 = \frac{Z}{2} \sum_{\lambda=\pm 2} \int [(\partial_t \phi_\lambda)^2 - |\nabla \phi_\lambda|^2] d^4x, \quad Z > 0.$$

The one-particle space is $L^2(H_0^+, d\Omega_0) \otimes \mathbb{C}^2$ with the unitary massless helicity representations $+2$ and -2 . Its symmetric Fock space has the Poincaré-invariant free vacuum and centred insertions $\Theta_\lambda = g\phi_\lambda$ paired with physical helicity test sections. Their positive-frequency measure and continued covariance are

$$\begin{aligned} \widehat{W}_{\lambda\lambda'}(p) &= \frac{2\pi g^2}{Z} \theta(p^0) \delta(p^2) \delta_{\lambda\lambda'}, \\ C_{\lambda\lambda'}(Q^2) &= \frac{g^2}{ZQ^2} \delta_{\lambda\lambda'}. \end{aligned} \tag{3.26}$$

The Fourier convention absorbs the displayed 2π in the mass-shell measure. At fixed $p = |p| > 0$ the reflected time covariance is $g^2 e^{-p(t_i+t_j)}/(2Zp)$ times the helicity identity and is a positive Gram matrix. This establishes the directly reduced realisation of Definition A.1(ii) including its continuation and null-shell source pairing. The subtraction polynomial and direct term together with the complementary contribution all vanish. Thus the two helicity channels separately satisfy the simple-eigenvalue and realisation hypotheses of Theorem 3.5. Their direct sum gives the displayed two-dimensional residue. Equivalently Theorem 3.3 has $L(z) = ZzI_2$ and $P = I_2$ with $S = gI_2$ giving the same matrix $Z_0 = (g^2/Z)I_2$. The two constructions use the same vacuum and source normalisation together with the helicity endpoint. This response is the physical quotient of the quadratic macroscopic tensor action. Its covariant Einstein–matter completion [29] supplies a common matter source and constraint propagation while the displayed translation-invariant spectral measure describes the flat-background linear response. The coupling of this tensor response to stationary matter and the separate potential and force estimates form a companion application in Ref. [13]. The response and spectral-identification theorems above are proved independently.

Finite truncations. An exact finite operation protocol can satisfy the native response and positive-residue theorems without an infinite-cutoff limit. Its interpretation as a QFT spectral atom still needs a corresponding physical representation. By contrast a truncation approximating an infinite target must establish that limiting identification. A finite response truncation that satisfies the algebraic pole and overlap tests produces a candidate residue $Z_{\text{alg},N}$. Even if this matrix is Hermitian and positive semidefinite it represents a physical spectral atom only after the realisation maps and kernels together with their z -derivatives converge along with the source and readout maps and subtracted projected correlators so that the critical root remains isolated and simple and its slope stays bounded away from zero and the hypotheses of Theorem 3.5 hold in the limit. Proposition C.1 gives the corresponding convergence result. This distinction is necessary because positivity of a finite matrix does not identify the limiting correlator or its spectral measure.

Ladder diagrams provide a possible approximation to $\mathcal{K}_{2\text{PI}}$. A complete response may also contain crossed ladders and vertex corrections together with ghost contributions and higher-particle sectors. Likewise the sign of a closed fermion loop does not by itself determine the sign of a projected response eigenvalue. Vertices and group factors together with propagators and the kernel convention enter the result.

3.7 Classification and scope of the conclusions

The classification returns four complementary entries

$$\mathfrak{R}(\mathfrak{C}_{\text{phys}}; \mathfrak{J}) = (R_{\text{kin}}, R_{\text{sym}}, R_{\text{dyn}}, R_{\text{scope}}). \quad (3.27)$$

Here R_{kin} records momentum suppression and soft scaling. The entry R_{sym} records the components fixed or forbidden by an exact symmetry and any anomaly or breaking. The entry R_{dyn} records the spectral conclusion established for the response. The entry R_{scope} specifies its response-kernel or topological realisation. A symmetry-allowed channel can have Gaussian decay at fixed transfer while retaining soft spectral weight determined by its dynamics.

Averaging fixes the momentum regions that retain support. Symmetry restricts the corresponding source components. The full response kernel determines collective criticality and the physical realisation assigns its spectral meaning. These are complementary properties of the same channel.

For an isolated simple eigenvalue criticality means $1 \in \text{spec } \mathcal{B}(0)$ and transversality means $\partial_z \lambda(0) \neq 0$. Both nonzero source–readout overlaps make the resulting pole visible. Theorem 3.3 gives a positive residue for the reciprocal Hilbert-space realisation. Theorem 3.5 identifies the coefficient with a Lorentzian atom when the unfiltered covariance and spectral conventions match. A continuum threshold instead requires its boundary analysis. Noncriticality is established for the exact operator or through a controlled approximation to it. An exact finite operation protocol is treated directly as the specified system.

Compatibility with a physical realisation. For a QFT channel the operator quotient and projector $\mathcal{P}_{\mathfrak{C}}$ first determine the source and readout $\mathcal{S}_{\mathcal{O}}, \mathcal{R}_{\mathcal{O}}$. Their realisation on the Banach response space uses P_X and $\iota_{\mathfrak{C}}, \pi_{\mathfrak{C}}$ subject to Eqs. (3.17)–(3.18). The kernel is the Jacobian of the same sourced state equation. An approximate coarse equation also requires comparison at actual reference solutions. If the coarse solution differs from the image of the full solution its displacement must be bounded. State derivatives of the kernel and source and readout maps then convert that bound into response errors. Residues must be compared at the two actual critical roots including the variation of their slopes.

If $\text{Ran } P_X$ is not invariant the four Schur–Feshbach terms of Proposition 3.1 retain the omitted feedback. On the resulting Fredholm domain the simple zero and the two overlaps determine the observable pole. For a finite approximation to an infinite target operator-norm and derivative convergence together with Riesz-projection convergence transfer this conclusion to the limiting response.

Its spectral interpretation concerns the same unfiltered connected endpoint correlator. Common vacuum assumptions and centring together with the physical projection and continuation and source pairing and subtraction polynomial are precisely the matching hypotheses of Theorem 3.5. They identify the residue with a physical atom. Theorem 3.3 instead establishes positivity directly for the reciprocal Hilbert-space response at the level of the reciprocal response operator.

Two exclusions complement these positive conclusions. External Gaussian filtering preserves an existing zero-mass residue and cannot create a pole from a regular form factor as shown around Eq. (2.29). If an exact symmetry sets the complete separated-point block to zero uniqueness of its spectral decomposition gives $Z_0 = 0$. Contact polynomials contribute only local terms. Neither statement evaluates an unevaluated symmetry-allowed response kernel.

3.8 Four-axis classification

For a channel block $\mathfrak{C}_{\text{phys}}$ and a computational realisation $\mathfrak{I}$ the result $\mathfrak{R}(\mathfrak{C}_{\text{phys}}; \mathfrak{I})$ of Eq. (3.27) is read as follows.

Theorem 2.5 gives decay on support separated from the zero set of the routed quadratic form. The soft region $q = O(\tau^{-1/2})$ and directions in $\text{Ker } M$ retain their own scaling. The same channel can exhibit both behaviours in different momentum regions.

An exact selection rule sets a forbidden component to zero. A Ward or Slavnov–Taylor identity can instead fix a component or its normalisation while leaving other form factors dynamical. Anomalies and explicit breaking modify the operator identity. Breaking by the state changes its matrix elements.

The dynamical statement specifies the kernel and external variable whose crossing is tested. Transversality and source visibility determine the observable pole. Spectral matching determines its physical atomic weight. Estimates at a continuum threshold use the measure limit or a limiting-absorption construction.

The response-kernel mechanism uses the Jacobian of the sourced equations. An internal topological zero mode is fixed instead by an index or cohomological construction. The native positive-residue theorem also applies to an isolated finite-dimensional critical subspace of an analytic Hilbert-space response operator.

These entries answer different questions. In particular neither “allowed” nor “constrained” entails spectral criticality and fixed-transfer suppression is compatible with a nontrivial soft response.

3.9 Representative channels

Table 1 applies the four-axis language to a representative set of channels. The dynamical column specifies the operator calculation that determines the corresponding spectrum.

Several distinctions in the table are essential. The translation Ward identity alone does not fix the transverse-traceless stress-tensor form factor. With complete-tensor positivity and exact Lorentz covariance together with conservation and the invariant-vacuum hypotheses of Proposition A.3 however the zero-mass TT atom is excluded. The spin-two spectrum at $s > 0$ still requires a dynamical calculation. This distinction concerns the QFT

Table 1. Representative renormalised channels. Under the hypotheses of Theorem 2.5 fixed-transfer decay holds away from the zero set of the routed Gaussian quadratic form. The soft and degenerate regions have the scaling of Eq. (2.27).

Channel	R_{sym}	R_{kin}	R_{dyn}	R_{scope}
$T_{\mu\nu}^{\text{TT}}$	conserved. Full tensor covariance and positivity	fixed-transfer decay	$s > 0$ spin-two kernel. $s = 0$ TT atom excluded	QFT response kernel. $d = 4$ vacuum
conserved J_μ	charge normalisation conditional. Transverse weight unfixed	fixed-transfer decay	vector kernel calculation required	response kernel
$[\bar{\psi}\psi]_R$ block	no current normalisation	fixed-transfer decay	scalar bound-state kernel calculation required	response kernel
longitudinal nonsinglet J_μ^5 block	exact anomaly-free chiral identity. State dependent	fixed-transfer decay	Goldstone test with nonzero symmetry-breaking expectation. Kernel otherwise	Ward identity / response
longitudinal J_μ^5 block with anomaly or explicit breaking	anomaly or explicit breaking	fixed-transfer decay	pseudoscalar kernel calculation required	response kernel
$[\text{tr } F^2]_R$ block	physical BRST cohomology class	fixed-transfer decay	scalar glueball kernel calculation required	response kernel
$[H^\dagger H]_R$ $[(H^\dagger H)^2]_R$	gauge invariant. No charge normalisation	fixed-transfer decay	mixed scalar kernel calculation required	response kernel
internal gauge zero modes	topology and background dependent	not applicable	index or harmonic analysis required	internal topology

Charge normalisation assumes stable external states and LSZ reduction with a continuous diagonal zero-transfer limit. The stress-tensor row assumes positivity and exact covariance of the complete tensor before TT compression. Proposition A.3 classifies its actual zero-mass shell. The axial rows refer to longitudinal current–scalar or pseudoscalar mixing blocks. Identification of a physical zero-mass atom uses the centred connected vacuum correlator and the matching and positivity and threshold hypotheses of Theorem 3.5.

representation. It does not exclude a native tensor response or a discrete transfer residue that has not been identified with that complete Wightman tensor measure. A conserved vector current illustrates the same distinction. An exactly normalised charge can coexist with a gapped current–current spectral function.

The scalar bilinear is not assigned a zero source. Depending on the theory and state its kernel can contain a bound-state pole. The axial rows concern longitudinal mixing blocks. Transverse axial response requires a separate projection and kernel calculation. An anomaly-free nonsinglet current in a chirally symmetric Lagrangian obeys a different identity from a singlet anomalous current or a current with explicit mass breaking. In the exact case a Goldstone pole additionally requires a nonzero symmetry-breaking vacuum

expectation in the current Ward identity. The operator name alone does not decide the result.

The notation $\text{tr } F^2$ denotes the scalar 0^{++} gauge-invariant composite channel. Its response measures colour-singlet intensity fluctuations. Gauge-field zero modes on a compact internal space instead belong to the kernel of a background differential operator. The compact geometry and gauge and matter representations of ref. [14] give a concrete realisation of these different spectral questions. A scalar bound-state pole is determined by its source response. Harmonic analysis determines the constant kernel of the scalar Laplacian.

A generated colour-singlet response. The Yang–Mills construction of ref. [14] Section 8 fixes a common source law with stationary mean intensity q_L on the condensed branch $L > L_c$ and response operator

$$\mathcal{M}_{\text{YM}}^2 = m_{\text{YM,rad}}^2 \oplus (\Delta_1^\perp + \delta_{q_L} I), \quad \Delta_{\text{YM}}^2 = \min\{m_{\text{YM,rad}}^2, 4/L^2 + \delta_{q_L}\} > 0.$$

The coexact one-form Laplacian $\Delta_1^\perp$ has bottom $4/L^2$ and $\delta_{q_L} > 0$ is the source-induced restoring shift. Its equilibrium response completion has the second-quantised generator $K_{\text{YM}} = d\Gamma(\mathcal{M}_{\text{YM}}^2)$. The vacuum is the only zero mode and the centred colour-invariant cyclic space has spectrum bounded below by Δ_{YM}^2 . For a centred source vector v the response $\langle v, (K_{\text{YM}} + z)^{-1} v \rangle$ therefore has a positive spectral measure supported away from zero. Write $\mathcal{Q} = \|F_B\|^2$ for the curvature intensity of the generated connection sample B and put $\sigma_L^2 = \text{Var}_{q_L} \|B\|^2$. The source-mean radial coefficient is the normalised derivative $b_{\mathcal{Q}} = \sigma_L \partial_q \mathbb{E}_q \mathcal{Q}|_{q=q_L} > 0$. It gives

$$C_{\text{rad}}(z) = \frac{b_{\mathcal{Q}}^2}{m_{\text{YM,rad}}^2 + z}, \quad d\rho_{\text{rad}}(\mu) = b_{\mathcal{Q}}^2 \delta_{m_{\text{YM,rad}}^2}(d\mu).$$

Here z is the Euclidean resolvent variable. This readout detects the spectral bottom when the radial block is the lower block. The fixed-source curvature polynomial retains its original products and higher Hermite components in the same completion. The displayed response measures the variation of its mean. In the condensed quadratic field realisation of that reference Section 10 and Appendix C a Wick intensity also acquires a linear radial term. Its two-point function then has an isolated positive-mass atom and a continuum beginning at the two-particle threshold. This model shows how a generated spectrum and specified source together with a nonzero readout together determine an observable pole. The mass-squared generator and the external invariant-mass measure are identified by the particular field realisation.

Finally $\|DH\|^2$ and four-fermion monomials are local composite operators. Whether they are independent representatives depends on integration by parts and field redefinitions together with EOM relations in the chosen operator basis. They cannot be excluded merely because one is a kinetic insertion and the other is a four-point interaction.

3.10 Symmetry permission and dynamical existence

For a symmetry-allowed block the principal output is a well-posed dynamical question. If a symmetry forbids a projected block no kernel calculation is needed for that block. If

the block is allowed the classification identifies the response operator and quantum-number sector together with the relevant soft kinematics in which a pole could occur. The existence question is thereby reduced to a specified spectral problem.

The distinction can be phrased directly in spectral language. For a Hermitian operator block symmetry controls linear relations among the matrix elements of the positive spectral measure $d\rho_{ij}(s)$. It does not ordinarily fix whether ρ_{ij} has an atom at $s = 0$. Such an atom can be forced when additional state information closes a Ward identity as in the Goldstone theorem. A critical response kernel yields an observable pole under the hypotheses and source–readout visibility conditions of Theorem 3.2. Under the additional matching hypotheses of Theorem 3.5 its residue is the physical zero-mass atom matrix. Otherwise its weight is a dynamical observable.

This observation eliminates a false dichotomy between “protected” and “unprotected” operators. Protection can refer to current conservation or gauge-parameter independence or a nonrenormalisation statement or membership in physical BRST cohomology. These notions are related but not equivalent. The classification therefore records the actual identity and its consequence instead of assigning a universal binary attribute.

3.11 External basis covariance, response coordinates, and schemes

Two different transformations must be kept separate. First let $O = (O_1, \dots, O_m)^T$ be the external physical operator basis and let

$$O' = MO, \quad M \in GL(m, \mathbb{C}), \quad (3.28)$$

be a finite change of renormalised basis. The Wightman spectral matrix and Euclidean correlator together with any isolated residue matrix transform by congruence

$$d\rho' = M d\rho M^\dagger, \quad C' = MCM^\dagger, \quad Z' = MZM^\dagger. \quad (3.29)$$

This is an external change of source and readout coordinates. The same physical source has coordinates $j' = (M^\dagger)^{-1}j$ hence $j'^\dagger Z' j' = j^\dagger Z j$. The pole strength measured by that source is unchanged even when its individual matrix components change.

Second let $T : \mathcal{X}_{\mathfrak{e}} \rightarrow \mathcal{X}'_{\mathfrak{e}}$ be a bounded isomorphism that changes only the internal coordinates of the response state. Then

$$\begin{aligned} \mathcal{B}_T &= T\mathcal{B}T^{-1}, & P_{X,T} &= TP_XT^{-1}, \\ \mathcal{S}_{X,T} &= T\mathcal{S}_X, & \mathcal{R}_{X,T} &= \mathcal{R}_XT^{-1}. \end{aligned} \quad (3.30)$$

It leaves the output matrix unchanged. If an external basis change M and an internal coordinate change T are performed together the complete transformation law is

$$\begin{aligned} \mathcal{B}' &= T\mathcal{B}T^{-1}, \\ P'_X &= TP_XT^{-1}, \\ \mathcal{S}'_X &= T\mathcal{S}_XM^\dagger, \\ \mathcal{R}'_X &= M\mathcal{R}_XT^{-1}, \\ C'_{\text{reg}} &= MC_{\text{reg}}M^\dagger. \end{aligned} \quad (3.31)$$

Therefore

$$C' = C'_{\text{reg}} + R'_X(I - \mathcal{B}')^{-1}S'_X = MCM^\dagger, \quad (3.32)$$

as required by the Wightman construction. The adjoint on the source leg follows the convention $C_{ij} = \langle O_i O_j^\dagger \rangle$. In a real bilinear convention it is replaced consistently by the corresponding transpose.

Proposition 3.6 (External and internal covariance). A finite nonsingular external transformation M and bounded internal isomorphism T preserve the location and order of full-matrix poles. They preserve the rank and positive-semidefinite character of each isolated residue. The symmetry subspaces transform by the isomorphism induced by M and retain their dimensions. The internal response spectrum is unchanged. At an isolated critical eigenvalue vanishing and nonvanishing of both source and observable overlaps are invariant. Individual residue components transform with the chosen basis.

The proof is given in Appendix B. For a scale-dependent finite transformation $M(\mu)$ the anomalous-dimension matrix obeys

$$\gamma' = M\gamma M^{-1} - \left(\mu \frac{dM}{d\mu} \right) M^{-1}. \quad (3.33)$$

The extra term is why a running operator basis cannot be treated as a constant similarity transformation.

Exact physical pole positions are independent of a consistent renormalisation scheme. The transformation changes intermediate Wilson coefficients and contact terms together with operator normalisations and mixing-matrix entries. At finite truncation even the inferred kernel eigenvalues and crossing scales generally depend on the regulator and basis size together with the projection prescription. Claims of scheme independence must therefore be demonstrated by the exact transformation law or replaced by a convergence and regulator-variation study.

For Higgs-sector applications the construction identifies the scalar mixing block and the kernel governing its correlations. Ultraviolet boundary conditions for the mass and quartic coupling remain independent renormalisation or phenomenological inputs.

4 Coarse-response errors, residual states, and evolution

Exact observation-sufficient descent preserves the full response. When a chosen coarse description is only approximate the first object to compare is that response even if the state dimensions differ. The comparison below needs no inverse coarse map. Its four defects distinguish unresolved-state effects from branch displacement and source-coupling errors.

4.1 A response error identity without an inverse coarse map

The two state spaces may have different dimensions so their resolvents cannot in general be subtracted directly. The map C places the update defect in a common space while the common source and output spaces allow the observable responses to be compared. The identity below propagates each of the four defects to that output.

Theorem 4.1 (Irreversible coarse-response error). Let B, S, R, D and $\bar{B}, \bar{S}, \bar{R}, \bar{D}$ be bounded response data on Banach state spaces $\mathcal{X}, \mathcal{Y}$ with the same source and output spaces. Let $C : \mathcal{X} \rightarrow \mathcal{Y}$ be any bounded linear comparison map. Set $G = (I - B)^{-1}$ and $\bar{G} = (I - \bar{B})^{-1}$ where both inverses exist and define

$$\begin{aligned} E_B &= \bar{B}C - CB, & E_S &= \bar{S} - CS, \\ E_R &= \bar{R}C - R, & E_D &= \bar{D} - D. \end{aligned} \quad (4.1)$$

Then the complete responses obey the exact identity

$$\bar{\mathcal{C}} - \mathcal{C} = E_D + \bar{R}\bar{G}E_S + \bar{R}\bar{G}E_BGS + E_RGS. \quad (4.2)$$

In particular writing $e_i = \|E_i\|$ in the specified operator norms

$$\|\bar{\mathcal{C}} - \mathcal{C}\| \leq e_D + \|\bar{R}\|\|\bar{G}\|e_S + \|\bar{R}\|\|\bar{G}\|e_B\|G\|\|S\| + e_R\|G\|\|S\|. \quad (4.3)$$

Neither injectivity nor surjectivity of C is required.

Proof. From $(I - \bar{B})C = C(I - B) - E_B$ obtain $\bar{G}C - CG = \bar{G}E_BG$. Inserting $\bar{S} = CS + E_S$ into $\bar{D} + \bar{R}\bar{G}\bar{S} - (D + RGS)$ and using $\bar{R}C = R + E_R$ gives Eq. (4.2). Submultiplicativity gives the norm bound. $\square$

The update defect is multiplied by both resolvents whereas the source and readout defects each carry one. Thus near criticality the response error depends on where a defect enters as well as on its norm. For example in a scalar reciprocal response with $S = R = 1$ and $D = 0$ and $B = 1 - \epsilon$ for $0 < \epsilon < 1$ the susceptibility is ϵ^{-1} . Changing B to $B + \delta b$ changes it by

$$\frac{1}{\epsilon - \delta b} - \frac{1}{\epsilon} = \frac{\delta b}{\epsilon(\epsilon - \delta b)}.$$

For $0 < |\delta b| \ll \epsilon$ the relative error is approximately $|\delta b|/\epsilon$. Thus the distance from criticality sets the required accuracy of the reduced feedback. Comparing the displaced roots and their residues describes this sensitivity even when no uniform pointwise comparison across the poles is possible. Directional amplification also matters for nonnormal updates. With the Euclidean norm on $\mathbb{R}^2$ and $g \in \mathbb{R}$ take

$$B = \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}, \quad S = e_2, \quad R = e_1^T, \quad D = 0.$$

Here $B^2 = 0$ and both eigenvalues vanish but $R(I - B)^{-1}S = R(I + B)S = g$. The source excites the second coordinate which the update transfers into the observed first coordinate with gain g . This finite response can be large at a fixed distance from the kernel spectrum. The resolvent norms in the error estimate account for such amplification as well as for proximity to a pole. This role of the resolvent is central to the pseudospectral analysis of nonnormal operators [8]. For the source-dependent coarse map in Eq. (2.9) apply this theorem with $C = T$ and the corrected data

$$\bar{S}_{\text{corr}} = \bar{S} - (I - \bar{B})C_h, \quad \bar{D}_{\text{corr}} = \bar{D} + \bar{R}C_h. \quad (4.4)$$

They give exactly the same coarse response since the two corrections cancel in $\bar{D}_{\text{corr}} + \bar{R}\bar{G}\bar{S}_{\text{corr}}$.

All comparisons are at the actual reference states. For a fixed linear C let $\delta = \|\bar{X}_* - CX_*\|$ be independently bounded and let $e_i^{(0)}$ compare the data at X_* and the transported point CX_* . If the coarse data have state-Lipschitz constants L_i on the intervening neighbourhood the resulting bounds are

$$\begin{aligned} e_B &\leq e_B^{(0)} + L_B\|C\|\delta, & e_S &\leq e_S^{(0)} + L_S\delta, \\ e_R &\leq e_R^{(0)} + L_R\|C\|\delta, & e_D &\leq e_D^{(0)} + L_D\delta. \end{aligned} \tag{4.5}$$

For a moving map its T, C_h errors must also enter the corrected defects. Near criticality neither a small stationary residual nor the norm of C alone bounds δ . Resolvent growth in Eq. (4.3) records the associated sensitivity rather than removing it.

Contour control of the total output residue. Suppose both responses extend meromorphically to a neighbourhood of a simple closed positively oriented rectifiable contour Γ and its interior with finitely many poles inside and none on Γ . If Eq. (4.3) is bounded by b_Γ on Γ then

$$\left\| \sum_{\Gamma} \text{Res } \bar{\mathcal{C}} - \sum_{\Gamma} \text{Res } \mathcal{C} \right\| \leq \frac{\text{length}(\Gamma)}{2\pi} b_\Gamma. \tag{4.6}$$

Indeed integration of Eq. (4.2) followed by the matrix-valued residue theorem and the contour norm bound gives Eq. (4.6).

This controls the sum of output residues inside the contour. A branch cut inside the contour would contribute its jump integral and invalidate a pole-only interpretation. Comparison of individual roots uses an isolated spectral branch with a simple zero and slope bounded away from zero together with control of its Riesz projection. These conditions distinguish a root comparison from a bound on the sum of residues inside a contour.

Quadratic observations. Many physical readouts are intensities or energies. On a common Hilbert space let $Q \geq 0$ be bounded and self-adjoint and let $\hat{Q}$ approximate the observation operator. Adding and subtracting $\langle y, Qy \rangle$ gives

$$|\langle x, Qx \rangle - \langle y, \hat{Q}y \rangle| \leq \|Q\|(\|x\| + \|y\|)\|x - y\| + \|Q - \hat{Q}\| \|y\|^2.$$

Thus a state-error bound and an observation-operator bound determine the accuracy of the quadratic output. For an unbounded closed positive form the corresponding comparison uses convergence in its form norm. A bounded linear step T that admits a source j also changes the quadratic output through the mixed term

$$\langle Tx + j, Q(Tx + j) \rangle - \langle Tx, QT x \rangle = 2 \text{Re} \langle Tx, Qj \rangle + \langle j, Qj \rangle.$$

The mixed term records the relative direction of the incoming source and the state already present. The finite quadratic identities in ref. [17] Section 5 retain these terms through successive operations. Combined with a residual bound for the governing evolution these identities relate the disposition of a measured quantity among retained and unresolved channels to the accuracy of the computed evolution.

4.2 Residual states and exact memory

When fibre consistency fails the next coarse state depends on the unresolved coordinate. Its elimination produces memory as in the projection-operator formulation of Mori [4]. For the bounded discrete linear system considered here the terms follow directly from the block equations

$$\begin{aligned} x_{n+1} &= A_{11}x_n + A_{12}u_n + S_1h_n, \\ u_{n+1} &= A_{21}x_n + A_{22}u_n + S_2h_n. \end{aligned} \tag{4.7}$$

Induction gives

$$u_n = A_{22}^n u_0 + \sum_{j=0}^{n-1} A_{22}^{n-1-j} (A_{21}x_j + S_2h_j). \tag{4.8}$$

Substitution into the first equation produces the initial-residual term $A_{12}A_{22}^n u_0$ and two convolution memories acting on the retained state and source respectively. An observation $R_1x_n + R_2u_n + Dh_n$ retains the same hidden history. Static Schur elimination represents these contributions in the stationary or transform domain.

Projection-operator descriptions of transport exhibit the same mechanism. Eliminating microscopic coordinates leaves a generalized force with memory and a local constitutive coefficient requires a further Markovian or scale-separation approximation [4]. Here the static Schur identity closes the stationary response while the actual update retains the delayed influence of the eliminated coordinates.

Each factor $A_{12}A_{22}^{n-1-j}A_{21}$ describes a disturbance entering the hidden sector and evolving there before returning to the retained one. The analogous factor with S_2 records a source applied directly to that sector. Memory therefore preserves the delayed physical influence of eliminated variables while the initial term preserves the influence of their preparation. A closed description can retain these effects through additional state coordinates or through the history terms.

If $\|A_{22}^k\| \leq M\rho^k$ with $0 < \rho < 1$ discarding lags $k \geq m$ incurs on a history with bounds X_H, H_H at most

$$\frac{\|A_{12}\|M\rho^m}{1-\rho}(\|A_{21}\|X_H + \|S_2\|H_H) \tag{4.9}$$

in one retained update. The initial term remains separately bounded by $\|A_{12}\|M\rho^n\|u_0\|$. Replace A_{12} by R_2 for the readout tail. Summing a geometric series proves these bounds. Propagation of these local update and readout errors uses a stability estimate for the full trajectory. If an independently verified augmented evolution bound gives $e_{n+1} \leq qe_n + \epsilon_n$ then induction gives $e_n \leq q^n e_0 + \sum_{j<n} q^{n-1-j} \epsilon_j$. In particular $0 \leq q < 1$ and bounded forcing errors give a uniform trajectory bound. For nonlinear dynamics the corresponding propagation bound must be established from the nonlinear evolution.

Resolution of a generated gauge-matter response. The common connection of ref. [14] supplies a nonlinear example in which the unresolved response is determined explicitly. In its fixed coefficient realization put $U = (A, \psi)$ and $\mathcal{O}_{\text{gm}}(U) = (F_A, \nabla_A^\rho \psi)$. The same coefficient heat map G_t acts on the connection and matter fields. Their products

produce the difference

$$C_t(U) = G_t \mathcal{O}_{\text{gm}}(U) - \mathcal{O}_{\text{gm}}(G_t U), \quad C_{s+t}(U) = G_s C_t(U) + C_s(G_t U).$$

The second identity follows by inserting and subtracting $G_s \mathcal{O}_{\text{gm}}(G_t U)$. With unresolved response $\mathcal{R}$ define

$$\mathcal{T}_t(U, \mathcal{R}) = (G_t U, G_t \mathcal{R} + C_t(U)).$$

The composition identity gives $\mathcal{T}_s \mathcal{T}_t = \mathcal{T}_{s+t}$ and

$$\mathcal{O}_{\text{gm}}(U_t) + \mathcal{R}_t = G_t (\mathcal{O}_{\text{gm}}(U) + \mathcal{R}).$$

For $A, \psi \in H^\eta$ and $\mathcal{R} \in H^{\eta-1}$ with $\eta > 5/2$ the modal product estimates in that reference Appendix A make this a strongly continuous locally Lipschitz semigroup. Starting with $\mathcal{R} = 0$ the residual at every later resolution is fixed by the initial coupled fields. Differentiation carries source variations through the same identity. The augmented state therefore preserves both the resolved gauge-matter fields and the response carried by their unresolved mode products. The parameter t orders coefficient resolution on the fixed internal carrier.

For semilinear evolution of generated spectra common stronger bounds and a weaker difference norm give estimates on the interval where the Duhamel map is contractive [17]. The residual includes the discarded nonlinear source as well as the initial discrepancy. Finite response operators add their own approximation and readout errors to this comparison. The state and source-variation estimates propagate local response and memory errors over the interval on which the stability bounds hold.

The arithmetic and two-mode examples below distinguish a faithful environment retaining all full-system states from an environment sufficient for one observation protocol. Minimal dimension is proved there within a specified finite-dimensional linear class. The set quotient fixes observational equivalence. The distinct exponential rates and the Hankel rank determine the minimal linear realisation in these examples.

4.3 Actual updates and stationary solvers

Internal coordinate similarity and external source-basis congruence re-express a response. Equation preconditioning changes its solver representation. Actual coarse compression removes state directions. For a residual $\mathcal{E}(X, h) = 0$ let M be a differentiable bounded isomorphism from the state space to the residual space with bounded inverse and introduce the solver $F_{\text{solve}} = X - M^{-1} \mathcal{E}$. At a solution derivatives of M multiply $\mathcal{E} = 0$ so

$$I - B_{\text{solve}} = M^{-1} L, \quad S_{\text{solve}} = -M^{-1} \mathcal{E}_h, \quad (I - B_{\text{solve}})^{-1} S_{\text{solve}} = -L^{-1} \mathcal{E}_h. \quad (4.10)$$

This proves invariance of the complete static response after the source is transformed consistently. It does not preserve the solver spectrum or its convergence rate. For $\mathcal{E}(X, h) = X - h$ the choice $M = I$ gives derivative zero while $M = I/3$ gives derivative -2 and a divergent iteration. Both stationary responses are one. Numerical convergence therefore establishes no physical stability unless the iteration has separately been identified with the actual update. The update index and Gaussian scale together with the RG coordinate and physical time have their respective definitions.

4.4 Invertible sector comparisons and displaced branches

When the comparison map has a bounded inverse on a specified subspace one can also transport the response operator and its spectral data between the two sectors. This additional hypothesis is useful for coordinate changes and convergent approximations. General irreversible maps remain covered by Theorem 4.1. Let $\mathcal{C}_{\tau_q} : \mathcal{X}_{\mathfrak{C}} \rightarrow \mathcal{X}_{\mathfrak{C}, \tau_q}$ be the stated coarse-graining map and define

$$E_{\tau_q}(X; z) := \mathcal{C}_{\tau_q} \mathfrak{F}(X; z) - \mathfrak{F}_{\tau_q}(\mathcal{C}_{\tau_q} X; z). \quad (4.11)$$

If $X_*(z) = \mathfrak{F}(X_*(z); z)$ and $\bar{X}_*(z) := \mathcal{C}_{\tau_q} X_*(z)$ then

$$E_{\tau_q}(X_*(z); z) = \bar{X}_*(z) - \mathfrak{F}_{\tau_q}(\bar{X}_*(z); z). \quad (4.12)$$

Thus $E_{\tau_q}(X_*; z)$ is the residual of the transported point in the coarse fixed-point equation. Its derivative compares Jacobians at that same transported point. It does not by itself compare with a Jacobian evaluated at a different coarse solution. If $E_{\tau_q} = 0$ on a neighbourhood of the reference solution differentiation gives the exact intertwining relation

$$\mathcal{C}_{\tau_q} D_X \mathfrak{F}(X_*; z) = D_X \mathfrak{F}_{\tau_q}(\bar{X}_*; z) \mathcal{C}_{\tau_q}. \quad (4.13)$$

For an approximate closure both the base-point residual and its derivative must therefore be accounted for.

The following stability result uses exact matching of the reference branches. With this matching the derivative defect compares the two response Jacobians at actual solutions without a separate state-displacement term.

The estimates proceed from the derivative defect to the critical spectral subspace then to the location of a simple root and the residue at that root. Source and readout errors enter when passing from the internal mode to the measured response. Resolvent factors record how these errors can be amplified near criticality.

Proposition 4.2 (Stability under a closure defect). Let $X_*(z) = \mathfrak{F}(X_*(z); z)$ be a reference branch on U and set $\bar{X}_*(z) := \mathcal{C}_{\tau_q} X_*(z)$ and assume the pointwise reference-branch matching condition

$$E_{\tau_q}(X_*(z); z) = 0, \quad z \in U. \quad (4.14)$$

Then $\bar{X}_*(z)$ is an actual coarse fixed point. Define the response Jacobians on the two matched branches by

$$\mathcal{B}(z) := D_X \mathfrak{F}(X_*(z); z), \quad \mathcal{B}_{\tau_q}(z) := D_X \mathfrak{F}_{\tau_q}(\bar{X}_*(z); z). \quad (4.15)$$

Let $Y \subset \mathcal{X}_{\mathfrak{C}}$ and $\tilde{Y} \subset \mathcal{X}_{\mathfrak{C}, \tau_q}$ be comparison spaces invariant under $\mathcal{B}(z)$ and $\mathcal{B}_{\tau_q}(z)$ respectively and suppose $\mathcal{C}_{\tau_q} : Y \rightarrow \tilde{Y}$ is a bounded isomorphism. Assume $\text{Ran } \mathbf{S}_X(z) \subset Y$ and $\mathbf{R}_X(z)|_Y : Y \rightarrow \mathbb{C}^m$ is bounded. Suppose the response family and both source and readout families are analytic in z . In the remainder of the proposition $\mathcal{B}(z)$ and $\mathcal{B}_{\tau_q}(z)$ denote their restrictions to Y and $\tilde{Y}$ respectively. Define

$$\begin{aligned} \mathcal{B}_{\text{ind}}(z) &:= \mathcal{C}_{\tau_q} \mathcal{B}(z) \mathcal{C}_{\tau_q}^{-1}, \\ \Delta \mathcal{B}(z) &:= \mathcal{B}_{\tau_q}(z) - \mathcal{B}_{\text{ind}}(z) \\ &= -[D_X E_{\tau_q}(X_*(z); z)|_Y] \mathcal{C}_{\tau_q}^{-1}. \end{aligned} \quad (4.16)$$

The consistently transported source and readout are

$$\mathbf{S}_{\text{ind}}(z) := \mathcal{C}_{\tau_q} \mathbf{S}_X(z) : \mathbb{C}^m \longrightarrow \tilde{Y}, \quad \mathbf{R}_{\text{ind}}(z) := (\mathbf{R}_X(z)|_Y) \mathcal{C}_{\tau_q}^{-1} : \tilde{Y} \longrightarrow \mathbb{C}^m. \quad (4.17)$$

They preserve the unfiltered response represented on the comparison sector

$$\mathbf{R}_{\text{ind}}(I_{\tilde{Y}} - \mathcal{B}_{\text{ind}})^{-1} \mathbf{S}_{\text{ind}} = (\mathbf{R}_X|_Y)[I_Y - (\mathcal{B}|_Y)]^{-1} \mathbf{S}_X \quad (4.18)$$

wherever the two inverses exist. Let

$$\tilde{\mathbf{S}}_X(z) : \mathbb{C}^m \rightarrow \tilde{Y}, \quad \tilde{\mathbf{R}}_X(z) : \tilde{Y} \rightarrow \mathbb{C}^m \quad (4.19)$$

be the source and readout of the coarse response evaluated on $\bar{X}_*(z)$.

Let $K \Subset U$ and let a positively oriented contour γ in the spectral ζ plane enclose exactly one algebraically simple eigenvalue of $\mathcal{B}_{\text{ind}}(z)$ and no other spectrum for every $z \in K$. Put

$$\begin{aligned} \epsilon &:= \sup_{z \in K} \|\Delta \mathcal{B}(z)\| \leq \sup_{z \in K} \|D_X E_{\tau_q}(X_*(z); z)|_Y\| \|\mathcal{C}_{\tau_q}^{-1}\|, \\ K_\gamma &:= \sup_{z \in K, \zeta \in \gamma} \|(\zeta - \mathcal{B}_{\text{ind}}(z))^{-1}\|, \\ L_\gamma &:= \text{length}(\gamma). \end{aligned} \quad (4.20)$$

If $K_\gamma \epsilon < 1$ the contour remains in the resolvent set of the coarse family and its Riesz projection obeys

$$\|\Pi_{\tau_q}(z) - \Pi_{\text{ind}}(z)\| \leq d_\Pi := \frac{L_\gamma}{2\pi} \frac{K_\gamma^2 \epsilon}{1 - K_\gamma \epsilon}. \quad (4.21)$$

Writing $B_* := \sup_{z \in K} \|\mathcal{B}_{\text{ind}}(z)\|$ and $\Pi_* := \sup_{z \in K} \|\Pi_{\text{ind}}(z)\|$ the enclosed eigenvalues satisfy

$$|\lambda_{\tau_q}(z) - \lambda_{\text{ind}}(z)| \leq d_\lambda := \epsilon(\Pi_* + d_\Pi) + 2B_* d_\Pi. \quad (4.22)$$

Suppose $\lambda_{\text{ind}}(z_*) = 1$ and $\alpha := |\partial_z \lambda_{\text{ind}}(z_*)| > 0$. For sufficiently small $d_\lambda > 0$ assume the disk $D_* := \{z : |z - z_*| \leq 4d_\lambda/\alpha\}$ lies in K that z_* is the only zero of $\lambda_{\text{ind}} - 1$ in D_* and that

$$|\lambda_{\text{ind}}(z) - 1| \geq \frac{\alpha}{2} |z - z_*|, \quad z \in \partial D_*. \quad (4.23)$$

Then $\lambda_{\tau_q} - 1$ has exactly one zero $\tilde{z}_*$ in D_* and

$$\delta_z := |\tilde{z}_* - z_*| \leq \frac{4d_\lambda}{\alpha}. \quad (4.24)$$

When $d_\lambda = 0$ take $\tilde{z}_* = z_*$.

For $z \in D_*$ set

$$\begin{aligned} \mathcal{N}_{\text{ind}}(z) &:= \mathbf{R}_{\text{ind}}(z) \Pi_{\text{ind}}(z) \mathbf{S}_{\text{ind}}(z), \\ \tilde{\mathcal{N}}(z) &:= \tilde{\mathbf{R}}_X(z) \Pi_{\tau_q}(z) \tilde{\mathbf{S}}_X(z). \end{aligned} \quad (4.25)$$

Introduce the same-point map discrepancies and uniform norms

$$\begin{aligned} d_R &:= \sup_{D_*} \|\tilde{\mathbf{R}}_X - \mathbf{R}_{\text{ind}}\|, & d_S &:= \sup_{D_*} \|\tilde{\mathbf{S}}_X - \mathbf{S}_{\text{ind}}\|, \\ R_* &:= \sup_{D_*} \|\mathbf{R}_{\text{ind}}\|, & \Pi_*^D &:= \sup_{D_*} \|\Pi_{\text{ind}}\|, & \tilde{\Pi}_* &:= \sup_{D_*} \|\Pi_{\tau_q}\|, & \tilde{S}_* &:= \sup_{D_*} \|\tilde{\mathbf{S}}_X\|, \end{aligned} \quad (4.26)$$

and define

$$d_{\text{num}}^{(0)} := d_R \tilde{\Pi}_* \tilde{S}_* + R_* d_{\Pi} \tilde{S}_* + R_* \Pi_*^D d_S. \quad (4.27)$$

Analyticity on D_* gives finite constants

$$L_{\text{num}} := \sup_{z \in D_*} \|\partial_z \tilde{\mathcal{N}}(z)\|, \quad d_{\lambda'}^{(0)} := \sup_{z \in D_*} |\partial_z \lambda_{\tau_q}(z) - \partial_z \lambda_{\text{ind}}(z)|, \quad L_{\lambda'} := \sup_{z \in D_*} |\partial_z^2 \lambda_{\tau_q}(z)|. \quad (4.28)$$

In particular L_{num} may be bounded by the product rule from the separate z -derivative bounds for $\tilde{R}_X$ and Π_{τ_q} together with $\tilde{S}_X$. Put

$$D_{\text{num}} := d_{\text{num}}^{(0)} + L_{\text{num}} \delta_z, \quad D_{\lambda'} := d_{\lambda'}^{(0)} + L_{\lambda'} \delta_z. \quad (4.29)$$

If $D_{\lambda'} < \alpha/2$ the residues at their respective critical roots

$$Z_{\text{alg,ind}} := -\frac{\mathcal{N}_{\text{ind}}(z_*)}{\partial_z \lambda_{\text{ind}}(z_*)}, \quad \tilde{Z}_{\text{alg}} := -\frac{\tilde{\mathcal{N}}(\tilde{z}_*)}{\partial_z \lambda_{\tau_q}(\tilde{z}_*)}, \quad (4.30)$$

obey

$$\|\tilde{Z}_{\text{alg}} - Z_{\text{alg,ind}}\| \leq \frac{2D_{\text{num}}}{\alpha} + \frac{2\|\mathcal{N}_{\text{ind}}(z_*)\|}{\alpha^2} D_{\lambda'}. \quad (4.31)$$

Thus the response estimate uses reference-branch matching and a bounded comparison inverse together with nonnormal resolvent control and transversality. The residue estimate additionally transports the numerator and eigenvalue slope between the two actual critical roots.

The proof is given in Appendix C. Condition (4.14) is substantive. If it fails an actual coarse branch $\tilde{X}_*(z)$ and an independent bound

$$d_X := \sup_{z \in K} \|\tilde{X}_*(z) - \mathcal{C}_{\tau_q} X_*(z)\| \quad (4.32)$$

must first be established. Provided the reevaluated families preserve the same comparison spaces re-evaluate $\mathcal{B}_{\tau_q}$ and every coarse source or readout map at $\tilde{X}_*$. If $D_X \mathfrak{F}_{\tau_q}$ is L_B -Lipschitz in X the kernel bound then obeys

$$\epsilon \leq \sup_{z \in K} \|D_X E_{\tau_q}(X_*(z); z)|_Y\| \|\mathcal{C}_{\tau_q}^{-1}\| + L_B d_X. \quad (4.33)$$

Base-point-dependent source and readout maps acquire their corresponding Lipschitz displacement terms. At a critical response the residual $\|E_{\tau_q}(X_*; z)\|$ alone does not supply such a displacement bound because $I - \mathcal{B}_{\tau_q}$ need not be invertible.

The word “flow” is used for several inequivalent operations in field theory. The Gaussian maps $\mathcal{H}_\tau$ form a heat semigroup ordered by resolution. A Wilsonian or functional RG trajectory orders effective descriptions by a cutoff scale k . Physical time parametrises actual evolution through distinguishable calibrated clock readings. Hamiltonian evolution and open-system dynamical maps provide field-theoretic realisations of that evolution. In the generated event carrier of ref. [14] Section 10 a timelike translation changes a matter phase at a rate fixed by its energy. Reading the two phase quadratures calibrates elapsed time. Gaussian variance orders resolution while this calibrated phase orders the physical evolution.

4.5 The Gaussian scale semigroup

Equation (2.11) gives

$$\mathcal{H}_0 = I, \quad \mathcal{H}_{\tau+s} = \mathcal{H}_\tau \mathcal{H}_s, \quad \partial_\tau \mathcal{H}_\tau = \Delta \mathcal{H}_\tau. \quad (4.34)$$

The parameter τ has dimensions of length squared and labels the amount of smoothing. Increasing τ orders resolution by stronger attenuation at each fixed nonzero wave number.

For every finite τ the Fourier multiplier $e^{-\tau|q|^2}$ is pointwise nonzero. Consequently $\mathcal{H}_\tau$ is injective on $L^2(\mathbb{R}^d)$. Its inverse on the range is unstable for $\tau > 0$.

Operational irreversibility. For every $\tau > 0$ the inverse of $\mathcal{H}_\tau$ on $\text{Ran } \mathcal{H}_\tau \subset L^2(\mathbb{R}^d)$ is unbounded. Hence arbitrarily small errors in coarse-grained high-frequency data can correspond to order-one errors before coarse-graining. Choose $f_n \in L^2$ with $\|f_n\|_2 = 1$ and Fourier support in the annulus $n \leq |q| \leq n+1$. Plancherel's theorem gives

$$\|\mathcal{H}_\tau f_n\|_2 \leq e^{-\tau n^2} \|f_n\|_2 \xrightarrow{n \rightarrow \infty} 0. \quad (4.35)$$

If the inverse were bounded then the inequality $1 = \|f_n\|_2 \leq \|\mathcal{H}_\tau^{-1}\| \|\mathcal{H}_\tau f_n\|_2$ would contradict Eq. (4.35).

This supplies the precise sense in which Gaussian smoothing is operationally irreversible at finite precision. Algebraically the map remains injective. The loss of recoverability is the instability of its inverse. A partial trace is a separate state-level operation.

This is the familiar obstruction in deblurring. Every finite Fourier mode survives while sufficiently fine structure falls below any fixed observational tolerance and inversion amplifies that uncertainty without bound.

4.6 Wilsonian integration and information loss

Wilsonian integration produces an effective functional for retained variables by integrating over the eliminated momentum modes [30]. At the level of quantum states discarding a factor of a specified Hilbert space acts by partial trace. Restricting observations can instead be described by a conditional expectation onto an observable algebra. For density matrices a completely positive trace-preserving reduction contracts their quantum relative entropy [31]. In an interacting field theory identifying such factors or algebras is part of the physical realisation of the reduction.

For a finite carrier with $A = A^*$ the random-phase channel

$$\mathcal{E}_v(\rho) = \mathbb{E}_{\theta \sim N(0,v)} [e^{-i\theta A} \rho e^{i\theta A}]$$

is completely positive and trace preserving. In an eigenbasis its multiplier is $e^{-v(a_i - a_j)^2/2}$. Random phases leave the populations unchanged and attenuate coherence between different eigenvalues. The squared eigenvalue difference determines the decay. Field-amplitude smoothing instead acts on each mode with $e^{-\tau a_i}$. Composition adds the variances.

For a Euclidean field representation reflection positivity tests whether positive-time insertions define nonnegative state norms after reflection [32, 33]. This question involves

the covariance across time slices. Let $A \geq 0$ be self-adjoint and let $m > 0$. Put $H = (A + m^2)^{1/2}$ and let B be a bounded time-independent operator on each time slice. For $C_B(t) = B e^{-|t|H} (2H)^{-1} B^*$ and $t_i \geq 0$

$$\sum_{i,j} \langle f_i, C_B(t_i + t_j) f_j \rangle = \left\| \sum_i e^{-t_i H} (2H)^{-1/2} B^* f_i \right\|^2 \geq 0.$$

Each positive-time insertion thus contributes a vector to a common Hilbert space and the reflected quadratic form is their Gram norm. The mass gap bounds the inverse square root. B need not commute with H . This proves reflection positivity of the declared free covariance. The time Gaussian $C(t) = e^{-t^2}$ fails the test at $t_1 = 1/4, t_2 = 3/4$ since its reflected matrix has determinant $e^{-5/2} - e^{-2} < 0$. Finite-dimensional evaluations of these covariances and of the random-phase channel use the software of Ref. [16].

The Gaussian averaging semigroup and an FRG regulator serve different roles. The former is the canonical averaging prescription within the composition class of Proposition 2.3 and is the one used in Theorem 2.5. The latter is a computational term chosen to interpolate between effective actions. Different admissible regulators can be used to estimate a channel but regulator variation measures the uncertainty of a finite approximation.

4.7 Composite-source functional flow

The dynamical kernel in Eq. (3.23) can be computed with several nonperturbative methods. For FRG calculations the operator block must be coupled to sources before the flow is differentiated. Let h_i couple to the renormalised composite insertions and introduce

$$Z_k[J, h] = \int \mathcal{D}\phi \exp \left[-S[\phi] - \Delta S_k[\phi] + J \cdot \phi + \sum_i h_i \cdot [\mathcal{O}_i]_R \right], \quad (4.36)$$

where $\Delta S_k = \frac{1}{2} \phi R_k \phi$ is written schematically with the superfield generalisation understood. The modified Legendre transform is taken in J while retaining h as an external source [34, 35]. For bosonic variables write $W_k = \log Z_k$ and $\varphi = \delta_J W_k$ and take R_k independent of h . On a twice differentiable local Legendre branch

$$\begin{aligned} \Gamma_k[\varphi, h] &= J \cdot \varphi - W_k[J, h] - \frac{1}{2} \varphi \cdot R_k \varphi, \\ \Gamma_{k,\varphi} + R_k \varphi &= J, \quad W_{k,h} = -\Gamma_{k,h}. \end{aligned}$$

At fixed J where $L_k := \Gamma_{k,\varphi} + R_k$ has a bounded inverse on the chosen response spaces differentiation gives

$$\begin{aligned} \varphi_h &= -L_k^{-1} \Gamma_{k,\varphi h}, \\ W_{k,hh} &= -\Gamma_{k,hh} + \Gamma_{k,h\varphi} L_k^{-1} \Gamma_{k,\varphi h}. \end{aligned} \quad (4.37)$$

The first term is the change of the composite expectation at fixed mean field. The second is the response transmitted by adjustment of that field. In the notation $D + RL^{-1}S$ the same action gives $D = -\Gamma_{k,hh}$ and $S = -\Gamma_{k,\varphi h}$ with $R = -\Gamma_{k,h\varphi}$. Thus the source and readout maps together with the direct response are fixed together by the generating functional.

The choice of state variables determines where a collective singularity appears in this identity. For an even composite in a symmetric phase for example the mixed derivative with an odd fundamental mean field vanishes at the symmetric state and $-\Gamma_{k,hh}$ carries the composite response. Including composite expectation values or propagators among the response variables exposes their adjustment in the inverse operator as in composite-field and 2PI formulations [35, 36]. Application of the isolated-pole theorem then uses the analyticity of the direct and complementary terms in that particular representation.

At fixed composite sources the effective action obeys the Wetterich equation [34]

$$\partial_t \Gamma_k[\varphi, h] = \frac{1}{2} \text{STr} \left[(\Gamma_k^{(2)}[\varphi, h] + R_k)^{-1} \partial_t R_k \right], \quad t = \ln(k/k_0). \quad (4.38)$$

Functional derivatives with respect to h_i generate flows for composite insertions including their mixing and contact terms. A formulation with explicit composite fields and scale-dependent bosonisation is given in Ref. [35]. Derivatives with respect to both h_i and the fundamental fields generate the vertices entering a projected response equation. After its self-consistent variables X have been specified that equation defines a map $\mathfrak{F}_k(X; z)$ and the exact response operator is its Jacobian $D_X \mathfrak{F}_k$ as in Eq. (2.32). In the 2PI formalism the self-energy and the Bethe–Salpeter kernel are obtained from successive functional derivatives of the same effective action [36]. The renormalisation of the resulting scalar two- and higher-point functions is treated in Ref. [37]. This gives an equivalent response representation only when the source pairing and propagators agree together with the renormalisation conditions and continuation convention with those used to define $\mathfrak{F}_k$. A selected list of diagrams does not by itself establish that equivalence.

Equation (4.38) is exact but not closed on a finite collection of vertices. A practical computation specifies an operator basis and field expansion together with momentum resolution and a regulator. Standard truncation schemes and their applications are reviewed in Ref. [38]. The resulting $\mathcal{B}_{k,N,\mathcal{R}}$ is a truncated response operator. Its omitted-vertex and channel-closure errors contribute to the derivative of the closure defect in Eq. (4.11). An eigenvalue reaching one in that operator is evidence within the stated implementation. Stability under increasing N and improving momentum resolution together with variation of R_k is required before extrapolating to the exact channel. The retained vertices and their closure specify which contributions enter each finite approximation.

The parameter $t = \ln(k/k_0)$ in Eq. (4.38) is an RG scale coordinate and orders features along a family of effective descriptions. Lorentzian formation is described through a real-time evolution law with specified initial data.

5 Generated examples and independent response tests

The examples follow a disturbance from its source through the unresolved variables to the observed response. An arithmetic merge first shows how those variables change a static susceptibility and retain a relaxation history. A reflecting-heat update then makes the collective feedback and critical mode together with the memory error explicit. In the field-theory examples the mass spectrum determines how the response extends through space while symmetry determines which source couplings are allowed.

5.1 An arithmetic fibre: merge, environment, and static response

This example separates preservation of a static source response from closure of the underlying update. The eliminated variables can be accounted for in the static response even though they must be retained or represented by memory for evolution.

The arithmetic merge construction of Ref. [9] on multiplicative and additive closure motivates the ordered fixed-sum fibre used here. Relative to the finite-fibre expression resolution in that reference the present fibre is its two-input additive value-level compression. It retains the ordered evaluated inputs while discarding their internal expression histories. The finite data follow directly from the ordered input basis. For $k \geq 2$ take the orthonormal basis $|m, k-m\rangle$ for $1 \leq m \leq k-1$ and define N_1, N_2 by their displayed degrees. The total degree is $N_+ = kI$ on this fibre. The cross-term of its quadratic cost is

$$K_+ = N_+^2 - N_1^2 - N_2^2 = 2N_1N_2 = \text{diag}\{2m(k-m) : 1 \leq m \leq k-1\}. \quad (5.1)$$

All these equalities follow on the basis. They specify the cost of the arithmetic operation.

Write $C_k|m, k-m\rangle = c_m$. Invariance under every permutation of this basis forces equal coefficients and $C_k C_k^* = 1$ forces $|c_m| = (k-1)^{-1/2}$. Requiring positive coefficients fixes the target phase. This proves locally the naturality of the normalised merge

$$C_k = \frac{(1, \dots, 1)}{\sqrt{k-1}}. \quad (5.2)$$

A faithful orthogonal dilation $|m, k-m\rangle \mapsto |k\rangle \otimes f_m$ requires $k-1$ orthonormal environment vectors. That lower bound is attained by retaining the ordered input label and all such minimal orthogonal dilations are unitarily equivalent. This dimension counts the records required to distinguish all ordered inputs. A future observation protocol specifies which of those distinctions its outputs retain.

Since $N_+^2 = k^2I$ the heat family obeys the exact descent relation $C_k e^{-tN_+^2} = e^{-tk^2} C_k$. The connected cost need not descend through the same merge. For $k=4$ it gives

$$K_+ = \text{diag}(6, 8, 6), \quad C = \frac{(1, 1, 1)}{\sqrt{3}}, \quad C e^{-tK_+} (1, -1, 0)^\top = \frac{e^{-6t} - e^{-8t}}{\sqrt{3}} \neq 0 \quad (t > 0). \quad (5.3)$$

The input vector lies in $\text{Ker } C$ so Eq. (2.2) fails. The compressed family $f(t) = C e^{-tK_+} C^* = (2e^{-6t} + e^{-8t})/3$ satisfies $f''(0) = 136/3 \neq 400/9 = f'(0)^2$ whereas a differentiable scalar semigroup with $f(0) = 1$ has $f''(0) = f'(0)^2$.

To define an actual static source response use real amplitudes and the declared cost and source pairing

$$\Phi(X, J) = \frac{1}{2} X^\top K_+ X - J C X, \quad K_+ X = C^* J, \quad O(X) = C X. \quad (5.4)$$

Positive definiteness gives a unique minimiser. Its derivative with respect to the source gives

$$\mathcal{C}(4) = C K_+^{-1} C^* = \frac{1}{3} \left(\frac{1}{6} + \frac{1}{8} + \frac{1}{6} \right) = \frac{11}{72}, \quad \mathcal{S}_+(4) = \mathcal{C}(4)^{-1} = \frac{72}{11}. \quad (5.5)$$

The Schur stiffness $\mathcal{S}_+(4)$ differs from $CK_+C^* = 20/3$. More generally

$$C_k K_+^{-1} C_k^* = \frac{1}{k-1} \sum_{m=1}^{k-1} \frac{1}{2m(k-m)} = \frac{H_{k-1}}{k(k-1)}, \quad H_j = \sum_{m=1}^j \frac{1}{m}.$$

The identity $1/[m(k-m)] = (1/m + 1/(k-m))/k$ proves the sum. Minimising the quadratic cost at fixed $C_k X$ gives the reciprocal as the Schur stiffness proving the required finite version of the exact arithmetic Schur theorem. Static elimination and the failure of memoryless heat descent are therefore compatible results. The effective stiffness measures the response after unresolved amplitudes have adjusted to the applied source. Their distinct relaxation rates still affect transient observations so this static coefficient alone does not determine the coarse evolution.

For a concrete environment calculation at $k = 4$ put

$$v = (1, 1, 1)/\sqrt{3}, \quad b = (1, -2, 1)/\sqrt{6}, \quad a = (1, 0, -1)/\sqrt{2}.$$

In this basis

$$K_+ = \begin{pmatrix} 20/3 & -2\sqrt{2}/3 & 0 \\ -2\sqrt{2}/3 & 22/3 & 0 \\ 0 & 0 & 6 \end{pmatrix}. \quad (5.6)$$

The a coordinate is invisible to $C = v^*$ under the fixed heat family and affine source injections along v . The observational equivalence is equality of $(X_1 + X_3, X_2)$. The two distinct exponentials e^{-6t}, e^{-8t} separate these two coordinates. They give a two-state realisation. A one-state linear realisation cannot produce both rates. Equivalently its Hankel determinant would vanish whereas $f(0)f''(0) - f'(0)^2 = 8/9 > 0$. Thus two states or one retained coordinate and one residual are minimal in the linear class for this restricted protocol. Allowing all fibre permutations changes the quotient. Permuting a before heat action can make it visible and the observation rows $Ce^{-tK_+}P$ span all three coordinates for fixed $t > 0$. The full three-state information must then be retained. The minimal observation space thus depends on the allowed protocol whereas the faithful orthogonal environment preserves all input labels.

In the two visible coordinates (x, u) the heat equation has $\dot{x} = -20x/3 + (2\sqrt{2}/3)u$ and $\dot{u} = (2\sqrt{2}/3)x - 22u/3$. Hence

$$u(t) = e^{-22t/3}u(0) + \frac{2\sqrt{2}}{3} \int_0^t e^{-22(t-s)/3}x(s) ds.$$

The retained equation contains the initial residual and the memory kernel $(8/9)e^{-22t/3}$. The parameter t orders the arithmetic heat operation. Equation (5.4) specifies its cost and source pairing.

5.2 A reflecting-window protocol with a visible critical mode

The mechanism combines spatial smoothing with an inhomogeneous modulation. Smoothing attenuates a nonconstant mode. Modulation couples it to the constant mode allowing

a disturbance to return through the unresolved coordinate. We construct this update on a reflecting window and follow the resulting feedback through its static response and relaxation together with the induced memory.

The interval can be connected to the same Gaussian operation used in Section 2. The finite-window construction in Section 3 of Ref. [9] uses a calibrated local information level and a global peak-equivalent capacity. The endpoint calculation fixes the window on which the boundary closure is imposed. The entropy calculation uses the continuous-density form of Rényi's entropy family (Ref. [39] Eq. (1.21)) with natural logarithms and a fixed Lebesgue reference measure. The endpoint rules are specified separately.

Calibration of the Gaussian correlation window. Fix a displacement unit and Lebesgue measure in that unit. For $b > 0$ the one-dimensional heat kernel of Eq. (2.10) gives the unit-peak profile

$$K_b(d) = \frac{g_{b^2/2}(d)}{g_{b^2/2}(0)} = e^{-d^2/(2b^2)}. \quad (5.7)$$

Let $\mathcal{I} : (0, 1] \rightarrow [0, \infty)$ be continuous and additive under products and calibrated by $\mathcal{I}(e^{-1}) = 1$. Define the local endpoint by unit information loss and the global endpoint by complete peak-equivalent capacity

$$\ell_{UV} := \inf\{d > 0 : \mathcal{I}(K_b(d)) \geq 1\}, \quad L_{IR} := \frac{\int_{\mathbb{R}} K_b(d) dd}{K_b(0)}. \quad (5.8)$$

Then these endpoints are unique and

$$\ell_{UV} = \sqrt{2} b, \quad L_{IR} = \sqrt{2\pi} b, \quad \Delta_b = L_{IR} - \ell_{UV} = b(\sqrt{2\pi} - \sqrt{2}) > 0. \quad (5.9)$$

For $p_b = K_b / \int_{\mathbb{R}} K_b$ the same normalization gives $L_{IR} = \exp H_{\infty}^R(p_b)$ where $H_{\infty}^R(p_b) = -\log \|p_b\|_{\infty}$ is the infinite-order Rényi entropy. The function $u \mapsto \mathcal{I}(e^{-u})$ is continuous and additive on $[0, \infty)$ hence linear. The unit calibration makes it u . Thus $\mathcal{I}(K_b(d)) = d^2/(2b^2)$ strictly increasing for $d > 0$ which gives the unique local endpoint. The Gaussian integral gives $\int_{\mathbb{R}} K_b = \sqrt{2\pi} b$ proving the global endpoint and window length. More explicitly for $\alpha > 1$

$$\int_{\mathbb{R}} p_b^{\alpha} dd = \frac{(\sqrt{2\pi} b)^{1-\alpha}}{\sqrt{\alpha}}, \quad H_{\alpha}^R(p_b) = \log(\sqrt{2\pi} b) + \frac{\log \alpha}{2(\alpha - 1)}.$$

Taking $\alpha \rightarrow \infty$ proves the asserted entropy characterization.

These endpoint rules use the full signed displacement line for capacity. There is no extra factor of one half. The resulting global carrier $[0, L_{IR}]$ represents the peak-equivalent mass at unit height. Resolving its part beyond unit local information gives the active window $[\ell_{UV}, L_{IR}]$. In the information coordinate its endpoints are 1 and π . The declared calibration and capacity rule fix this length uniquely. A different information unit or endpoint rule would be different data. In particular Shannon entropy would give the different Gaussian width $\sqrt{2\pi e} b$. Fixing the reference length unit above is essential when using a differential entropy as a width.

Reflecting closure and the operational scale. Translate the active window by $x = d - \ell_{\text{UV}}$ to $[0, \Delta_b]$. Among separated homogeneous self-adjoint boundary conditions for the local Laplacian preserving constants selects Neumann closure. An endpoint condition $af + cf' = 0$ with $(a, c) \neq (0, 0)$ must have $a = 0$ because $f = 1$ is in the domain. Consequently $f' = 0$ at each endpoint giving zero diffusive flux there. Within this boundary class preservation of a uniform profile therefore fixes the closure. The Neumann heat operator fixes the constant function and conserves mass by self-adjointness.

For the common-source realisation set $\Delta = \Delta_b$. The calculations below also hold for an independently specified $\Delta > 0$ with the same reflecting closure. Our convention is $H_\tau = e^{-\tau A}$. The normalisation $e^{-tA/2}$ in Ref. [9] has $\tau = t/2$. Its endpoint scale convention $t(d) = d^2/2$ therefore gives $\tau_{\text{UV}} = b^2/2$ and $\tau_{\text{IR}} = \pi b^2/2$. The protocol may choose either or any declared positive τ . These parameters specify the averaging scale.

Let $\mathcal{H} = L^2(0, \Delta)$. The nonnegative Neumann operator is $A = -\partial_x^2$ with domain $\{f \in H^2(0, \Delta) : f'(0) = f'(\Delta) = 0\}$. Its eigenfunctions and eigenvalues are

$$e_0 = \Delta^{-1/2}, \quad e_n = \sqrt{2/\Delta} \cos(n\pi x/\Delta), \quad E_n = (n\pi/\Delta)^2 \quad (n \geq 1), \quad E_0 = 0. \quad (5.10)$$

Solving $-e'' = Ee$ with the two boundary conditions gives these modes. Even extension to a 2Δ -periodic function and Fourier completeness give a complete orthonormal basis. Integration by parts gives $\langle f, Af \rangle = \|f'\|_2^2$. Equivalently the cosine diagonal realisation is self-adjoint with domain $\sum E_n^2 |f_n|^2 < \infty$. Thus $H_\tau = e^{-\tau A}$ is a bounded positive self-adjoint contraction with $H_\tau e_n = e^{-\tau E_n} e_n$. Its kernel is

$$h_\tau(x, y) = \sum_{j \in \mathbb{Z}} [g_\tau(x - y + 2j\Delta) + g_\tau(x + y + 2j\Delta)],$$

the even periodisation of the line Gaussian. Fourier coefficients give the displayed Neumann eigenvalues and multiplier. This specifies the reflection operation. Neither a finite interval nor reflection truncates its infinite spectrum. Line increments define the heat law. Reflection imposes the boundary closure and a finite spectral projection selects retained modes.

Fix $\tau > 0$ and $\varepsilon \in \mathbb{R}$ with $0 < |\varepsilon| < 1$. Let $q(x) = \cos(\pi x/\Delta)$ and let P_2 project onto $\text{span}\{e_0, e_1\}$. Compose the declared operations

$$T = P_2 H_\tau M_{1+\varepsilon q} H_\tau P_2 = \begin{pmatrix} 1 & \gamma \\ \gamma & r^2 \end{pmatrix}, \quad r = e^{-\tau E_1}, \quad \gamma = \frac{\varepsilon r}{\sqrt{2}}. \quad (5.11)$$

The multiplication identities $\langle e_0, qe_0 \rangle = \langle e_1, qe_1 \rangle = 0$ and $\langle e_0, qe_1 \rangle = 1/\sqrt{2}$ follow by elementary cosine integration. Heat action contributes the two diagonal factors $(1, r)$. The multiplier is bounded between $1 - |\varepsilon|$ and $1 + |\varepsilon|$ so $\langle v, Tv \rangle \geq (1 - |\varepsilon|) \|H_\tau v\|^2 > 0$ on this two-mode space. The finite projection completes the specified two-mode operation. Its amplitudes are Hilbert vectors with positivity understood in the quadratic-form sense just proved. The factor r measures attenuation of the nonconstant profile in each heat step while γ measures its coupling to the uniform profile through the spatial modulation.

Define the actual discrete update and observation by

$$X_{n+1} = \eta T X_n + e_0 h_n, \quad O_n = e_0^* X_n, \quad \eta \geq 0. \quad (5.12)$$

In the common-source specialization Δ is fixed by b through Eq. (5.9). Otherwise it is a declared window parameter. The remaining τ, ε, η and the source pairing specify the protocol. The window length fixes E_1 and the heat scale fixes r . The modulation fixes γ and η sets the overall feedback strength per update. These relations connect the Gaussian calibration and boundary closure to the observed response.

Failure of one-state descent and minimal residual. Write $X_n = (x_n, u_n)$ and set $a = \eta$ and $\beta = \eta\gamma$ with $d = \eta r^2$. For $\beta \neq 0$ the projection onto x fails fibre consistency because $x_{n+1} = ax_n + \beta u_n + h_n$. Keeping u closes the actual system and

$$\begin{aligned} u_n &= d^n u_0 + \beta \sum_{j=0}^{n-1} d^{n-1-j} x_j, \\ x_{n+1} &= ax_n + h_n + \beta d^n u_0 + \beta^2 \sum_{j=0}^{n-1} d^{n-1-j} x_j. \end{aligned} \quad (5.13)$$

These identities follow by induction including the empty sum at $n = 0$. If $0 \leq d < 1$ dropping lags $k \geq m$ gives the effective tail bound

$$|\text{discarded memory}| \leq \frac{\eta^2 \gamma^2 d^m}{1-d} \sup_{j < n} |x_j|, \quad |\text{initial residual}| = |\beta| d^n |u_0|. \quad (5.14)$$

The history bound is essential. Critical forcing does not have a uniformly bounded history merely because $d < 1$.

For zero initial state put $m_j = e_0^*(\eta T)^j e_0$. Direct multiplication gives $m_0 = 1$ and $m_1 = \eta$ with $m_2 = \eta^2(1 + \gamma^2)$ and $m_0 m_2 - m_1^2 = \eta^2 \gamma^2 > 0$ when $\eta > 0$. In an n -state linear time-invariant realisation a Hankel matrix factors as an observability matrix times a reachability matrix so has rank at most n . This is the rank bound underlying linear minimal-realisation methods [2]. Two states are therefore necessary and sufficient for this transfer function. The one residual coordinate is minimal in that class. At $\eta = 0$ the update vanishes and the Hankel determinant is zero. Independently observing x and its next unforced update recovers $u = (x_{n+1} - ax_n)/\beta$ so the observation-sufficient quotient for $\beta \neq 0$ separates both amplitude coordinates.

Actual stability, crossing, and response. The two eigenvalues obtained from the characteristic polynomial are

$$\lambda_{\pm} = \frac{1 + r^2 \pm \sqrt{(1 - r^2)^2 + 4\gamma^2}}{2}, \quad \eta_* = \lambda_+^{-1}. \quad (5.15)$$

Since $r^2 - \gamma^2 = r^2(1 - \varepsilon^2/2) > 0$ and the characteristic polynomial is negative at r^2 and at 1 one has $0 < \lambda_- < r^2 < 1 < \lambda_+$. Therefore $\|\eta T\| = \eta \lambda_+ < 1$ for $0 \leq \eta < \eta_*$. In the unmodulated limit $\varepsilon = 0$ the uniform mode has $\lambda_+ = 1$ and the other mode decays by r^2

per application of T . For nonzero modulation the two modes mix raising λ_+ above one and lowering the critical gain η_* . The critical direction is therefore a collective profile of the full update. Reflection supplies the constant mode. Modulation and feedback determine when its coupled response becomes critical. The actual update is a strict contraction there. Constant forcing has a unique attracting fixed point. At η_* one eigenvalue reaches one with control derivative $\partial_\eta(\eta\lambda_+) = \lambda_+ > 0$. The other is strictly below one and $d_* := \eta_* r^2 < 1$. Above η_* the largest mode grows. These stability properties belong to the specified update.

For exponential input $h_n = w^n h$ define the particular amplitude by $X_n = w^n X$. The transfer equation is $(wI - \eta T)X = e_0 h$. Put $w = 1 + z$. This discrete transfer variable measures the frequency of the input sequence. Differentiating the sourced update gives

$$\mathcal{C}_\eta(z) = e_0^*[(1+z)I - \eta T]^{-1}e_0 = \frac{1+z - \eta r^2}{(1+z - \eta)(1+z - \eta r^2) - \eta^2 \gamma^2}. \quad (5.16)$$

Schur elimination gives the inverse scalar denominator $1+z - \eta - \beta^2/(1+z - d)$ retaining the transformed hidden-state propagator. At η_*

$$\mathcal{C}_{\eta_*}(z) = \frac{Z}{z} + \frac{1-Z}{z+1-\eta_*\lambda_-}, \quad Z = \frac{1-\eta_* r^2}{1-\eta_*\lambda_-} \in (0,1). \quad (5.17)$$

The inequalities for λ_- prove the stated range of Z . Here $L(z) = (1+z)I - \eta_* T$. Its complementary inverse is bounded by $(1-\eta_*\lambda_-)^{-1}$ at zero and $L'(0) = I$. Hence $G_0 = I$ on the one-dimensional critical space. The e_0 source sees that space since an eigenvector with zero first component would require $\gamma = 0$. This verifies every hypothesis of Theorem 3.3 from the actual operations. The variable z is the transfer coordinate of this discrete update. In the QFT realisation below invariant mass enters through the corresponding spectral representation. Here e_0 is the coarse uniform level and e_1 is the first spatial contrast. Heat attenuates the contrast. The inhomogeneous multiplier converts level into contrast and back while the gain η closes the feedback loop. The critical threshold therefore belongs to a collective profile of the full update. At criticality the unforced fixed points form the critical line. Constant nonzero forcing accumulates along its visible direction so there is no unique forced stationary state at that endpoint.

The same decomposition displays critical slowing in the actual discrete update. Let $P_+ = (T - \lambda_- I)/(\lambda_+ - \lambda_-)$ be the orthogonal projector at λ_+ and $\mu_\pm = \eta\lambda_\pm$. Then $e_0^* P_+ e_0 = Z$ so zero initial state and constant input $h \neq 0$ give for $0 < \eta < \eta_*$

$$\frac{x_n}{h} = Z \frac{1 - \mu_+^n}{1 - \mu_+} + (1 - Z) \frac{1 - \mu_-^n}{1 - \mu_-}. \quad (5.18)$$

This follows from $X_n = \sum_{j=0}^{n-1} (\eta T)^j e_0 h$ and the two orthogonal spectral projectors. The slow contribution has relaxation length $(-\log \mu_+)^{-1} \sim (1 - \mu_+)^{-1}$ as $\eta \uparrow \eta_*$ in update steps while its stationary susceptibility is $Z/(1 - \mu_+)$. At criticality the first term becomes Zn as the source accumulates in the visible neutral direction. Thus the transfer-pole residue is also the weight of the slow response. A physical relaxation time additionally requires a calibration of these update steps.

The error from dropping the residual altogether can also be computed. With $C = (1, 0)$ and $B = \eta T - zI$ together with $\bar{B} = a - z$ and the inherited scalar source and readout the four defects are $E_B = (0, -\beta)$ and $E_S = E_R = E_D = 0$. On the common inverse domain Theorem 4.1 gives

$$\bar{\mathcal{C}} - \mathcal{C} = -\frac{\beta^2}{(w-a)[(w-a)(w-d) - \beta^2]}, \quad w = 1 + z. \quad (5.19)$$

Direct subtraction of $1/(w-a)$ and Eq. (5.16) verifies the same identity. Thus update closure failure alone produces a nonzero output error even with compatible source and readout maps and direct terms.

Schur reduction and finite-memory error. To expose terms hidden by a one-leg probe allow two source columns $S = I_2$ and two readouts $R = I_2$ with zero full-system direct term. Let $w = 1 + z$ and $H = (w-d)^{-1}$ with $L_{\text{eff}} = w - a - \beta^2 H$. Eliminating the second state gives

$$S_{\text{eff}} = (1, \beta H), \quad R_{\text{eff}} = (1, \beta H)^\top, \quad D_{\text{eff}} = \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}, \quad \mathcal{C} = D_{\text{eff}} + R_{\text{eff}} L_{\text{eff}}^{-1} S_{\text{eff}}. \quad (5.20)$$

Solving the two linear equations proves the formula including the off-diagonal observations and the nonzero direct term. In the time domain the second source also enters the hidden memory as in Eq. (4.8).

For a finite-memory approximation use $H_m(w) = \sum_{k=0}^{m-1} d^k w^{-k-1}$ in the complete source-response tuple. On $|w| \geq w_0 > d$

$$|H - H_m| \leq \frac{d^m}{w_0^m (w_0 - d)} =: \delta_m. \quad (5.21)$$

Thus the effective-kernel error is bounded by $\beta^2 \delta_m$. Both source and readout errors are bounded by $|\beta| \delta_m$ and the direct-term error by δ_m . Use $B_{\text{eff}} = I - L_{\text{eff}}$ to apply Theorem 4.1. On a compact invertible domain with $\sup |L_{\text{eff}}^{-1}| \leq M$ the condition $M\beta^2 \delta_m < 1$ gives $\sup |L_{\text{eff},m}^{-1}| \leq M/(1 - M\beta^2 \delta_m)$ by the Neumann series. Substitution gives an explicit convergent complete-response error bound. A small contour around the critical root excluding the second pole and $w = d$ then gives the total-residue bound of Eq. (4.6). All tail bounds have supplied geometric majorants.

The pole displacement and residue error can be resolved more sharply at the critical parameter. The following consequence also distinguishes approximation of the memory kernel from approximation of its derivative.

Corollary 5.1 (Pole and residue under finite-memory truncation). Fix the two-mode model at $\eta = \eta_*$ and fix $d = \eta_* r^2 \in (0, 1)$ as $m \rightarrow \infty$. All $O(\cdot)$ constants below may depend on this fixed model including d . Let $m \geq 1$ be an integer and write $a = \eta_*$ and $\beta = \eta_* \gamma \neq 0$. In the transfer variable w set

$$L(w) = w - a - \frac{\beta^2}{w - d}, \quad L_m(w) = w - a - \beta^2 H_m(w), \quad Z = \frac{1}{L'(1)}. \quad (5.22)$$

Here $L(1) = 0$ and Z is the scalar residue in Eq. (5.17). For sufficiently large m the function L_m has exactly one zero w_m in a fixed sufficiently small neighbourhood of 1. It is real and simple. The residue $Z_m = \text{Res}_{w=w_m} L_m(w)^{-1}$ satisfies

$$w_m - 1 = -\frac{Z\beta^2}{1-d} d^m + O(md^{2m}), \quad (5.23)$$

$$Z_m - Z = \frac{Z^2\beta^2}{1-d} md^m + O(d^m). \quad (5.24)$$

For the two-source response in Eq. (5.20) replace H by H_m in all four terms and denote its residue at w_m by Z_m . With $v = (1, \beta/(1-d))^\top$ and $Z = Zvv^\top$

$$\|Z_m - Z\| = O(md^m). \quad (5.25)$$

The constants depend on the fixed value of d . The control parameter remains η_* and the residues are compared at their respective transfer poles.

Proof. The geometric sum gives the exact identity

$$L_m(w) - L(w) = \frac{\beta^2 d^m w^{-m}}{w-d}, \quad L_m(1) = \frac{\beta^2 d^m}{1-d}. \quad (5.26)$$

Choose a closed disk about 1 that excludes d and zero together with the second zero of L and on which $|w| > d$. The defect converges uniformly there. Rouché's theorem gives exactly one zero w_m counted with multiplicity for large m . Reality of the coefficients makes it real. To obtain the sharper rate put $t = d^m$. On $|w-1| = Kt$ with any fixed $K > Z\beta^2/(1-d)$ one has $w^{-m} = 1 + O(mt)$

$$L(w) = Z^{-1}(w-1) + O(t^2), \quad L_m(w) - L(w) = \frac{\beta^2}{1-d} t + O(mt^2).$$

Rouché's theorem on this smaller circle yields $w_m - 1 = O(t)$. Substitution in $L_m(w_m) = 0$ then proves Eq. (5.23).

Differentiating the defect at 1 gives

$$L'_m(1) - L'(1) = -\beta^2 d^m \left(\frac{m}{1-d} + \frac{1}{(1-d)^2} \right). \quad (5.27)$$

On $|w-1| \leq Kt$ the second derivative of the defect is $O(m^2t)$ whereas L'' is bounded. Thus transport from 1 to w_m gives

$$L'_m(w_m) = Z^{-1} - \frac{\beta^2}{1-d} mt + O(t + m^2t^2).$$

Since $m^2t \rightarrow 0$ taking the reciprocal proves Eq. (5.24). For the full matrix response $H_m(w_m) = H(1) + O(t)$ by the geometric remainder and the root estimate. The direct term is analytic near w_m and the residue equals $Z_m v_m v_m^\top$ with $v_m = (1, \beta H_m(w_m))^\top = v + O(t)$. Equation (5.25) follows. $\square$

The nonzero leading coefficient in Eq. (5.24) shows that the factor m is present in the asymptotic scalar residue error. This is an explicit instance of the derivative and displaced-root terms in Proposition 4.2. A memory defect of order d^m moves the pole by that order while its derivative changes the residue by order md^m . Locating the critical response and predicting its strength therefore require different accuracies in the same finite-memory model. The extra derivative weights long lags more strongly which accounts for the factor m from differentiating the tail.

Ref. [16] supplies supplementary finite-matrix implementations of this reduction. The response formula retains all four terms with the row functional continued analytically in z . An elementary matter example is the sixteen-dimensional operator $D = \begin{pmatrix} 0 & rI_8 \\ rI_8 & 0 \end{pmatrix}$ with $r > 0$ and one right coordinate designated neutral. Its charged compression has dimension fifteen. The operator sends the neutral coordinate into its left partner so the neutral line is not invariant. The charged matrix is therefore a compression. On domains where the complementary block of the sourced inverse is invertible Proposition 3.1 makes direct inversion and complete elimination of that coordinate identical. Excursions from the retained left partner into the neutral coordinate and back enter the effective kernel. Sources and observations with neutral components also contribute to the effective source and readout maps together with the direct term. The actual full and charged carriers in ref. [14] obey the same operator-descent condition described in Appendix B.

Replicated matter and its quadratic source. For the companion compact-matter construction consider a field-theoretic realisation with fixed family multiplicity. Its one-family carrier $\mathcal{E}_1$ is generated by the internal representation and covariant transport of Ref. [14]. The common family module in the companion application Ref. [15] is taken to be unitarily identified with $\mathcal{E}_g = \mathbb{C}^g \otimes \mathcal{E}_1$. At fixed common bosonic background with vanishing Higgs field the regulated fermion integrals of the families are independent. With the same source variation of the domain and measure together with the spectral projectors and regulator in every copy

$$K_g[h] = I_g \otimes K_1[h], \quad W_g[h] = gW_1[h], \quad \partial_a \partial_b W_g = g(\langle S_{1,a} S_{1,b} \rangle_c - \langle S_{1,ab} \rangle). \quad (5.28)$$

The generating functionals are evaluated at zero fermion sources. Source-independent constants are immaterial. Their product makes the logarithms add giving the second identity for the fermion contribution at that background. The third follows by twice differentiating the same Euclidean integral including the measure identification and moving-domain contributions. The common physical TT compression therefore has the same factor g . This is the quadratic response used in the companion compact-endpoint calculation of ref. [15]. Within its replication family equality of the matter and colour normalisations of the same screening coefficient selects $g = 3$. The one-generation charge moments fix the torsion readout and the shell and three-position conditions select $R = 5$ in the declared integer response-level family. This bandwidth specifies the source domain. Numerical mode enumeration and quadrature evaluate the response on that domain. Higgs vertices and a displaced matter background are incorporated by differentiating their full action. This example fixes the direct term as well as the propagated source terms in the classification.

5.3 Scalar feedback and the fold value $4/27$

For comparison with the transfer pole consider the explicitly specified scalar feedback

$$\Pi_d = \frac{\Pi_0}{(1 - V\Pi_d)^2}, \quad F_x(y) = \frac{x}{(1 - y)^2}, \quad y = V\Pi_d, \quad x = V\Pi_0, \quad (5.29)$$

on $0 \leq y < 1$ with $x \geq 0$. Its fixed points satisfy $x = y(1 - y)^2$. The derivative of this cubic is $(1 - y)(1 - 3y)$ so it increases up to $y = 1/3$ and then decreases to zero. Thus

$$x_c = \frac{4}{27}, \quad y_c = \frac{1}{3}, \quad (5.30)$$

and $0 < x < x_c$ has exactly two admissible branches $y_- \in (0, 1/3)$ and $y_+ \in (1/3, 1)$. At $x = 0$ the only admissible fixed point is zero. At x_c the branches meet and above it there is no fixed point in this domain. At a fixed point $F'_x(y) = 2y/(1 - y)$ giving local attraction of y_- and repulsion of y_+ . If $a < 1/3$ and $0 \leq x \leq a(1 - a)^2$ then $F_x([0, a]) \subset [0, a]$ and $\sup_{[0, a]} F'_x \leq 2a/(1 - a) < 1$. The contraction theorem gives a unique fixed point in this invariant interval and convergence to it. In particular the declared initialization $y_0 = 0$ selects y_- for $0 < x < x_c$.

Implicit differentiation of the same rule gives

$$\frac{dy}{dx} = \frac{1}{(1 - y)(1 - 3y)}. \quad (5.31)$$

Writing $y = 1/3 + u$ gives $x = 4/27 - u^2 + u^3$. The susceptibility therefore diverges with inverse square-root magnitude as $x \uparrow x_c$ along either branch with opposite signs. At the end-point $F'_x = 1$ so the strict contraction proof no longer applies. This control-parameter fold has an inverse-square-root susceptibility. Replacing the iteration by a preconditioned solver leaves the implicit derivative unchanged when sources are transformed as in Eq. (4.10) while solver stability can change. Even where iteration is the declared update here a physical time or a physical branch-selection law requires its own model identification. Applying this feedback to an arithmetic or TT construction uses a matching of the state and source normalisation together with the parameter maps. These maps determine whether the same fold mechanism is present.

Field-theory cross-checks

The following previously established channels test the additional QFT representation and individual parts of the general response theory. They examine spectral weight and symmetry constraints in physical representations complementing the operation-based examples above.

5.4 A free field: preservation of an existing pole

For a canonically normalised free scalar field of mass m

$$C_\phi(Q^2) = \frac{1}{Q^2 + m^2}. \quad (5.32)$$

Applying the external representation Eq. (2.12) gives

$$C_{\phi,\tau}(Q^2) = \frac{e^{-\tau Q^2}}{Q^2 + m^2}. \quad (5.33)$$

The multiplier is entire and nonzero at every finite Q^2 . It therefore leaves the pole location and order unchanged. For $m = 0$ it also leaves the residue unchanged

$$\lim_{Q^2 \rightarrow 0} Q^2 C_{\phi,\tau}(Q^2) = 1. \quad (5.34)$$

This verifies the pole-preservation statement surrounding Eq. (2.29). Pole generation belongs to the dynamical step.

5.5 A free composite field: a bubble continuum

Let ϕ be a free real scalar of mass $m > 0$ and choose the normal-ordered composite

$$\mathcal{O}(x) = \frac{1}{\sqrt{2}} : \phi^2(x) :. \quad (5.35)$$

Wick's contraction rule [40] gives the connected Euclidean two-point function

$$C_{\phi^2}(Q^2) = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 + m^2)((p + Q)^2 + m^2)}. \quad (5.36)$$

After a momentum-independent subtraction dimensional regularisation gives

$$C_{\phi^2,R}(Q^2) - C_{\phi^2,R}(0) = -\frac{1}{16\pi^2} \int_0^1 dx \ln \left[1 + \frac{x(1-x)Q^2}{m^2} \right]. \quad (5.37)$$

Analytic continuation has a branch point at $s = 4m^2$. With the spectral convention of Appendix A the nonlocal part has

$$\rho_{\phi^2}(s) = \frac{1}{16\pi^2} \sqrt{1 - \frac{4m^2}{s}} \theta(s - 4m^2), \quad (5.38)$$

with the normalisation in Eq. (5.35). There is no isolated one-particle atom. This example separates a positive threshold density from an isolated pole. In the massless limit the continuum begins at $s = 0$. Its spatial response can be evaluated at separated points where the subtraction at zero momentum is unnecessary. Write $r = |\mathbf{x}| > 0$. The four-dimensional massless propagator is $G(t, \mathbf{x}) = [4\pi^2(t^2 + r^2)]^{-1}$ and Wick contraction with the normalisation in Eq. (5.35) gives $\langle \mathcal{O}(t, \mathbf{x}) \mathcal{O}(0) \rangle_c = G(t, \mathbf{x})^2$. The response to a time-independent source is the zero-frequency kernel obtained by integration over Euclidean time

$$K_{\text{cont}}(r) = \frac{1}{16\pi^4} \int_{-\infty}^{\infty} \frac{dt}{(t^2 + r^2)^2} = \frac{1}{32\pi^3 r^3}, \quad r > 0. \quad (5.39)$$

The substitution $t = r \tan \theta$ reduces the integral to $r^{-3} \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta$. Equivalently the constant massless density in Eq. (5.38) gives the same result by integrating the static Yukawa kernels $\int_0^\infty ds e^{-\sqrt{s}r} / (64\pi^3 r)$. Local subtraction terms are supported at $r = 0$ and

leave this expression unchanged. The threshold continuum therefore gives a long-distance algebraic response. A continuum reaching zero mass superposes all small masses and produces the algebraic r^{-3} tail in this example. The zero-mass atom of Eq. (3.25) selects an isolated propagation channel and supplies the Coulombic $1/r$ term. The threshold tail is determined by the full spectral integral whereas the isolated collective pole admits the local Fredholm decomposition.

5.6 An unbroken conserved current without a massless pole

Consider a free massive Dirac field and its exactly conserved global current $J_\mu = \bar{\psi}\gamma_\mu\psi$. The Ward identity makes the separated-point two-point function transverse

$$\langle J_\mu(Q)J_\nu(-Q) \rangle = (Q_\mu Q_\nu - Q^2 \delta_{\mu\nu})\Pi_V(Q^2) + \text{contact terms.} \quad (5.40)$$

It also fixes the charge form factor of a one-particle fermion at zero transfer. Nevertheless the vacuum spectral density of Π_V starts at the fermion–antifermion threshold

$$\rho_V(s) \propto \left(1 + \frac{2m^2}{s}\right) \sqrt{1 - \frac{4m^2}{s}} \theta(s - 4m^2), \quad (5.41)$$

and contains no massless atom. The example directly separates charge normalisation from the existence of a long-range vacuum channel.

5.7 A Goldstone channel: symmetry plus state information

Let an exact continuous global symmetry have conserved current J_A^μ and let a local Lorentz-scalar field Φ satisfy

$$v_A \equiv \langle \Omega | \delta_A \Phi(0) | \Omega \rangle \neq 0. \quad (5.42)$$

The nonzero expectation value specifies a vacuum that breaks the symmetry of the action. Apply Eq. (3.22) to the mixed correlator. After subtracting polynomial contact contributions from that correlator retain the order-parameter contact term on the right-hand side of the Ward identity. With the Fourier convention of Section 2 this gives

$$p_\mu G_A^\mu(p) = v_A, \quad G_A^\mu(p) = \int d^d x e^{ipx} \langle \Omega | T J_A^\mu(x) \Phi(0) | \Omega \rangle. \quad (5.43)$$

Indeed the Fourier transform of the divergence is $-ip_\mu G_A^\mu$ and that of its Ward contact term is $-iv_A$. For a Lorentz-invariant vacuum there is no independent vector so Lorentz covariance gives $G_A^\mu(p) = p^\mu F_A(p^2)$. Consequently modulo the contact terms already removed the longitudinal form factor has the singular part

$$F_A(p^2) = \frac{v_A}{p^2 + i0}, \quad (5.44)$$

which is the massless Goldstone pole [41]. The conclusion required both the exact identity and Eq. (5.42). With $v_A = 0$ the same Ward identity does not force the pole. For an operator carrying Lorentz indices additional covariants may occur and the one-form-factor argument must be replaced by the corresponding tensor decomposition.

5.8 A separable response kernel

Let the response operator on $\mathcal{X}_{\mathfrak{C}}$ be rank one

$$\mathcal{B}(z) = \lambda(z)|r\rangle\langle\ell|, \quad \langle\ell|r\rangle = 1, \quad (5.45)$$

with λ analytic. The resolvent is obtained exactly

$$[I - \mathcal{B}(z)]^{-1} = I + \frac{\lambda(z)}{1 - \lambda(z)}|r\rangle\langle\ell|. \quad (5.46)$$

Suppose $\lambda(0) = 1$ and $\partial_z \lambda(0) \neq 0$. For the response-space maps $S_X : \mathbb{C}^m \rightarrow \mathcal{X}_{\mathfrak{C}}$ and $R_X : \mathcal{X}_{\mathfrak{C}} \rightarrow \mathbb{C}^m$ the singular part is

$$C_{\text{sing}}(z) = -\frac{R_X|r\rangle\langle\ell|S_X}{\partial_z \lambda(0)z} + O(1). \quad (5.47)$$

The pole disappears from the measured channel if either matrix factor vanishes even though the kernel still has eigenvalue one. This solvable example contains every algebraic condition in Theorem 3.2. Only when the response is the unfiltered physical correlator and the hypotheses of Theorem 3.5 hold does the displayed residue equal the Hermitian positive-semidefinite atom Z_0 . The two source factors determine the weight carried by the zero of the denominator.

5.9 Spectral extraction and approximation control

In lattice and FRG calculations together with Dyson–Schwinger and Bethe–Salpeter approaches the extraction of a spectral pole involves both the definition of the renormalised channel and control of its finite approximation.

Renormalised channel. The action and vacuum or ensemble specify the system with a chosen spacetime signature and dimension. The operator basis and mixing matrix together with the EOM/BRST quotient and spin/internal projector specify the physical channel. The operator source–readout maps and response space together with its projector and realisation maps specify S_X and R_X . In a gauge theory the Slavnov–Taylor residuals and the treatment of ghost and gauge-fixing sectors are part of these data. Closure is tested by Eq. (3.2). If it fails the calculation requires an enlarged block or Eq. (3.3).

Spectral observable. The spectral observable is the projected connected Euclidean correlation matrix of centred insertions. Removed one-point products and subtracted contact polynomials have distinct origins and are recorded separately. For the Poincaré-invariant vacuum spectral branch of Appendix A a candidate massless atom is tested through

$$Z_0 = \lim_{Q^2 \rightarrow 0} Q^2 [C(Q^2) - P(Q^2)] \quad (5.48)$$

or the corresponding positive residue matrix. The exchange of this limit with the spectral integral uses the continuum integrability condition of Proposition A.4. The spectral bridge requires this unfiltered projected matrix to be the same response used in the Fredholm factorisation with common vacuum centring and source pairing together with the

same continuation and contact subtraction. An ensemble calculation must instead supply a spectral representation and diagnostic appropriate to that state. For a spectral-FRG calculation the spectral representation and its continuation must be specified in the truncation. For example Ref. [42] computes two- and four-point spectral functions in scalar ϕ^4 theory in three spacetime dimensions directly in a Lorentzian formulation.

Finite-volume scaling. In a massive theory the large-volume behaviour of an isolated stable particle is treated in Ref. [43] and that of two-particle scattering levels in Ref. [44]. Applying a level-scaling test requires the finite-volume assumptions and spectral separation appropriate to the state being extrapolated. The massive-theory estimates do not supply a zero-mass extrapolation theorem. A pole claim requires a sequence of volumes and resolutions showing the scaling of the candidate energy and matrix element together with their separation from multiparticle levels. When the infinite-volume continuum also reaches $s = 0$ the cumulative weight near zero distinguishes atomic concentration from continuous threshold weight. In the isolated-state setting the result must survive the continuum and infinite-volume extrapolations.

Kernel test. In the two-particle realisation the dimensionless operator $\mathcal{B} = \mathcal{K}_{2\text{PI}}G_0$ is constructed in the same projected basis. Following its relevant eigenvalue branches in the complex external invariant z gives $\lambda(0)$ and $\partial_z\lambda(0)$. The associated left and right eigenvectors together with the source and readout give both matrix factors in Eq. (5.47). Onset along a control-parameter family also calls for a separate evaluation of $\partial_\eta\lambda(0;\eta_c)$ with higher-order analysis if it vanishes. At intermediate FRG scale this remains a flow diagnostic. A physical pole is a property of the endpoint response.

For a nonnormal discretised operator sensitivity is assessed through the right and left eigenvector residuals and the condition number

$$\kappa_N = \frac{\|r_N\| \|\ell_N\|}{|\langle \ell_N, r_N \rangle|}. \quad (5.49)$$

The smallest singular value of $I - \mathcal{B}_N$ or an equivalent resolvent bound should accompany the eigenvalue. A small value of $|1 - \lambda_N(0)|$ without overlap and conditioning and convergence information is not a pole determination.

Approximation analysis. Approximation errors are tested by controlled enlargement of the operator and momentum basis together with regulator variation and quadrature refinement and changes of finite-volume parameters. Stability under these changes tests sensitivity to the approximation. The observed spread is not a certified truncation bound. Certification requires a remainder estimate relative to the limiting operator. Once the assumptions and normalisation of Proposition 2.3 are fixed the Gaussian averaging prescription is held fixed during this analysis while FRG regulator variation probes the truncation uncertainty.

For a continuum Fredholm claim all approximants are realised on a common response space. On a compact set K in the z plane a sufficient operator-norm convergence condition

of the form

$$\sup_{z \in K} \|\tilde{\mathcal{B}}_N(z) - \mathcal{B}(z)\| \longrightarrow 0, \quad (5.50)$$

together with convergence of $\partial_z \mathcal{B}_N$ and $\mathbf{S}_{X,N}$ and $\mathbf{R}_{X,N}$ supplies the required approximation control. This condition is directly checkable when a derivation supplies an explicit remainder bound. Alternatively one may bound the nested increments by a summable sequence and supply the computable tail estimate $\sum_{n \geq N} \varepsilon_n(K) \leq r_N(K) \rightarrow 0$ from Eqs. (C.28)–(C.29). This effective tail controls the continuum error once a consistency argument identifies the limit with the exact response Jacobian.

For a coarse-grained closure the reference-state comparison starts with $E_{\tau_q}(X_*; z)$. If it is nonzero an independent fixed-point displacement estimate is needed. Lipschitz constants for $D_X \mathfrak{F}_{\tau_q}$ and every base-point-dependent source or readout then convert that displacement into kernel and map-error terms. In the invertible-sector comparison the remaining data include $\|D_X E_{\tau_q}\|$ and $\|\mathcal{C}_{\tau_q}^{-1}\|$ together with the resolvent constants in Eqs. (4.20)–(4.31) and the transported-map discrepancies $\|\tilde{\mathbf{S}}_X - \mathbf{S}_{\text{ind}}\|$ and $\|\tilde{\mathbf{R}}_X - \mathbf{R}_{\text{ind}}\|$. A residue comparison concerns the reference root z_* and the perturbed root $\tilde{z}_*$. Equation (4.24) and the analytic-variation bounds transport the Riesz numerator and $\partial_z \lambda$ between these points. This step is what permits the comparison of residues at their actual generally different roots. If γ is a contour in the spectral ζ plane that encloses the eigenvalue one and remains in the resolvent set the relevant Riesz projections are

$$\Pi_{\gamma,N}(z) = \frac{1}{2\pi i} \oint_{\gamma} (\zeta - \tilde{\mathcal{B}}_N(z))^{-1} d\zeta. \quad (5.51)$$

Their convergence is required alongside that of the eigenvalues. Isolated-eigenvalue approximation and its error bounds are treated in Chapter 5 of Ref. [7]. Nonnormal resolvent sensitivity is the subject of Ref. [8]. When Eq. (5.50) is unavailable checks for spectral pollution and pseudospectral sensitivity remain necessary diagnostics but do not by themselves certify convergence to the exact operator.

Numerical results and reproducibility. The action and normalisation conventions together with the regulator and basis determine the finite calculation. Numerical tolerances and eigenvector residuals specify the accuracy of its evaluation. The software in Ref. [16] provides supplementary finite-matrix response calculations. The companion compact-spectrum application Ref. [15] uses common spectral data to evaluate effective couplings and mass scales. Its spectral-tail bounds and endpoint sensitivities distinguish numerical approximation errors from changes in the scale prescription. For a scalar observable realised by a Wick polynomial the one-particle and multiparticle contributions depend on its centred expansion and source pairing. Its use in Appendix A requires the positive vacuum representation and subtraction convention of that same observable. For a channel classification the result comprises the four quantities in Eq. (3.27) subject to the compatibility assumptions in Section 3.7. A finite-truncation estimate must be distinguished from a limiting result supported by the convergence bounds above.

The analytic examples distinguish pole preservation from continuous spectral weight and charge normalisation. They also exhibit the Goldstone mechanism and source visibility

at a kernel zero. Their closed-form responses allow individual parts of a numerical calculation to be compared with exact results. The closed-form identities supply comparison values for the dynamical and convergence analysis of interacting channels.

6 Conclusions

A reduced description preserves collective response when it retains the effects of eliminated variables on feedback and source coupling together with their contribution to the observation. Hidden dynamical variables also leave preparation and memory terms in the reduced evolution. Joint differentiable descent preserves the direct and propagated response. Departures from descent obey the four-defect identity of Theorem 4.1. Its comparison map may remove state coordinates. The two resolvents in the update-defect term account for the enhanced sensitivity near criticality. The displaced-root estimates then carry branch and map errors to a comparison of observed residues.

The arithmetic and reflecting-heat examples separate stationary susceptibility from dynamical closure. A hidden state can be eliminated from the former and remain necessary for future observations. Retaining that state gives a local update. Eliminating it gives an exact memory equation. A finite-memory approximation introduces an error that can be estimated explicitly in the two-mode system. The pole displacement decays geometrically with the retained memory length. Differentiating the memory remainder adds a factor of that length to the residue error. The generated gauge-matter example gives a nonlinear counterpart. Its residual state carries the mode products discarded by coefficient smoothing. The compensation identity preserves their response through successive resolutions.

The transfer pole of the two-mode system measures the slow-mode weight in the response to constant forcing. Its relaxation length diverges at the critical update. In three spatial dimensions a constant contracted zero-mass residue instead gives a Gaussian-smoothed Coulomb kernel. Continuous spectral weight in the massless free composite gives a faster algebraic tail. The decay of the response thus records the source coupling to the low-mass spectrum.

An isolated critical residue depends on the crossing slope and both source overlaps together with the complementary resolvent. For the reciprocal Hilbert-space response of Theorem 3.3 a positive crossing matrix gives a positive-semidefinite finite-rank residue under the stated endpoint and isolation hypotheses. Theorem 3.5 identifies this residue with a zero-mass atom when the response matches the physical vacuum covariance. The free two-helicity realisation has positive null-shell weight. The complete stress tensor obeys the exclusion in Proposition A.3 for its positive covariant conserved measure.

The generated colour-singlet example has a positive spectral bottom and a visible massive radial mode. Its source-mean derivative fixes the weight. Its field realisation supplies the mass interpretation. For finite evolutions the measured response depends jointly on the state residual and the operator and readout errors. Quadratic observations retain the cross terms between an admitted source and the state already present. These examples specify the source and approximation data entering the general comparison.

The estimates apply locally to specified reference branches with isolated critical roots and controlled resolvents. Within this domain the same four response maps connect generated operations to finite-memory models and quantum-field-theory channels. The quotient determines which state distinctions can be removed. The response derivative follows the source through the retained state. Schur elimination includes the feedback of omitted variables. The spectral analysis identifies the collective modes visible in the resulting observation.

A Response parameters and physical spectral measures

The spectral identification in Section 3 matches the analytic response to the unfiltered connected vacuum covariance in a positive physical representation. The projection and source pairing are fixed together with the continuation and subtraction prescription and the endpoint conventions of Theorem 3.5. We treat local physical insertions and covariances constructed directly on the physical helicity space. For each realisation we specify the positive measure and its continuation together with the zero-mass atom. Under those exact matching hypotheses these data identify a Laurent residue of the response with a vacuum spectral weight.

A.1 Response parameters and their representations

The internal Neumann eigenvalue E_n determines heat attenuation in Section 5.2. The transfer parameter there is $z = w - 1$ with w the discrete input frequency variable. The control η changes the actual update and is not z . The arithmetic heat parameter orders a cost semigroup. A physical clock requires a calibrated readout of the evolution. Theorem 3.3 certifies a Laurent coefficient in the analytic parameter actually specified. For a finite self-adjoint A with orthonormal eigenvectors v_j direct diagonalisation gives

$$S^*(wI - A)^{-1}S = \sum_j \frac{S^*v_jv_j^*S}{w - a_j}.$$

Each residue is positive. This measure describes the spectrum of the transfer operator. A Lorentzian realisation instead uses the invariant mass $s = p^2$ and its Euclidean continuation Q^2 . A spatial Fourier representation gives the spatial Green function. These identifications depend on the representation of the response parameter.

The positive measure is obtained from connected Wightman two-point functions of centred physical operators. Time-ordered correlators and commutators follow from the Wightman measure through their respective boundary prescriptions. Positivity belongs to the Wightman measure and to its physical channel compressions. The invariant-mass representation has its origins in Refs. [27, 28]. The vacuum framework used below is that of Ref. [45]. For tensor channels we state the physical-polarisation compression separately and check its positivity.

A.2 Physical realisations and their positive measures

Definition A.1 (Physical spectral realisation). A physical spectral realisation consists of a positive-energy unitary representation of translations with an invariant vacuum. Centred source insertions define the connected covariance through one of the following constructions.

- (i) Renormalised local physical operators act on the positive-metric vacuum Hilbert space. Their tensor Wightman measure is compressed onto the chosen physical polarisations.
- (ii) The covariance is constructed on the reduced physical helicity space. The states carry the unitary Poincaré representation of those helicities and sources are test sections of the corresponding polarisation bundle. Positivity and the spectrum condition hold for this reduced covariance and its continuation is defined on the same physical space.

In both cases the realisation specifies the invariant-mass disintegration and physical source pairing together with the Euclidean continuation and local subtraction terms. The vacuum projection and integrability assumptions stated below apply to that same covariance.

A change of helicity frame is unitary on the source fibres. It preserves both the positive measure and the source-contracted covariance. For a directly constructed helicity covariance the null-shell compression in Assumption A.2 is this fibre pairing. The following local-operator construction supplies the first realisation. Eq. (3.26) supplies the second.

For the invariant-mass spectral representation work in a positive-metric physical representation with a Poincaré-invariant vacuum $|\Omega\rangle$ and the positive-energy spectrum condition. We assume that the joint spectral projection of the momentum operators at zero is $E_P(\{0\}) = |\Omega\rangle\langle\Omega|$. The vacuum is thus the unique zero-momentum state within the chosen representation. For the local realisation let the operators O_i with $i = 1, \dots, n$ form a finite block of renormalised local physical operators with $O_i^\dagger$ included when necessary and set

$$v_i = \langle\Omega|O_i|\Omega\rangle, \quad \Theta_i = O_i - v_i I. \quad (\text{A.1})$$

The two-point matrix used throughout this appendix is connected

$$\begin{aligned} W_{ij}(x) &= \langle\Omega|\Theta_i(x)\Theta_j^\dagger(0)|\Omega\rangle \\ &= \langle\Omega|O_i(x)O_j^\dagger(0)|\Omega\rangle - v_i \bar{v}_j. \end{aligned} \quad (\text{A.2})$$

The omitted constant has Fourier transform $(2\pi)^d v_i \bar{v}_j \delta^{(d)}(p)$. This vacuum contribution is supported at zero four-momentum. A massless particle shell has nonzero null momentum whereas a contact term is polynomial in momentum. Since $E_P(\{0\})\Theta_j^\dagger|\Omega\rangle = 0$ the connected measure has no atom at $p = 0$. Its possible $\delta_0(ds)$ term instead arises from disintegration on the nonzero future null shell.

Thermal states and other ensembles generally have spectral representations that depend separately on frequency and spatial momentum. Their response analysis uses these variables in place of the vacuum invariant-mass representation.

Inserting the spectral resolution of the four-momentum operator yields a matrix-valued measure supported in the closed forward cone. For tensor operators write this Fourier-space

Wightman measure as $d\Sigma_{ij,AB}(p)$ where A, B collect covariant tensor indices. Positivity is first a statement on the positive-metric physical Hilbert space for every smooth compactly supported physical test section $f_i^A(p)$

$$\sum_{i,j} \int_{V_+} \overline{f_i^A(p)} d\Sigma_{ij,AB}(p) f_j^B(p) \geq 0. \quad (\text{A.3})$$

Assumption A.2 (Positive physical-polarisation compression). For $s > 0$ a tensor or spin channel a is represented by a measurable orthonormal family $\{\varepsilon_{a,r}(p)\}_{r=1}^{N_a}$ in the positive-norm physical polarisation space. Its channel measure is obtained from the compression

$$d\Sigma_{ij;rs}^{(a)}(p) := \overline{\varepsilon_{a,r}^A(p)} d\Sigma_{ij,AB}(p) \varepsilon_{a,s}^B(p), \quad (\text{A.4})$$

followed by Lorentz-covariant disintegration with respect to $s = p^2$ and a positive contraction in the polarisation indices. Here a positive contraction means $X \mapsto \text{tr}_{\text{pol}}[H_a(p)X]$ for a specified measurable $0 \preceq H_a(p) \preceq I$ that transforms covariantly under a change of polarisation frame. $H_a = I$ gives the full irreducible block. If $s = 0$ is included the endpoint compression is defined separately on the null shell by an orthonormal family spanning the specified physical helicities. The Euclidean tensor projector used in the channel is required after analytic continuation to induce this same helicity compression. The zero-mass limit is defined for the compressed forms. The null-shell helicity compression determines this endpoint independently of the full massive-spin projector. For a scalar channel the compression is the identity. A measurable change of physical polarisation frame is unitary on each polarisation fibre and does not affect positivity.

Under Assumption A.2 denote the invariant-mass part of the compressed spectral measure by $d\rho_{ij}^{(a)}(s)$ on $s \geq 0$. A Hilbert-space compression and a positive contraction preserve Eq. (A.3). Hence for every $c \in \mathbb{C}^n$ and every nonnegative Borel test function f

$$\sum_{i,j} \bar{c}_i \int_0^\infty f(s) d\rho_{ij}^{(a)}(s) c_j \geq 0. \quad (\text{A.5})$$

Thus $d\rho^{(a)}$ is positive semidefinite as a matrix-valued measure. The physical channel compression is essential to this positivity statement. The Fourier transform of a commutator contains the difference of positive- and negative-energy boundary values and is not itself a nonnegative scalar measure on all four-momenta.

A.3 The zero-mass stress-tensor channel

For a complete covariant stress tensor positivity before any channel compression imposes an additional restriction at the null shell. This is stronger input than positivity of a projected response alone.

Proposition A.3 (Null-shell TT exclusion for a complete positive covariant stress tensor). In four-dimensional Minkowski spacetime work in a positive-metric representation with positive energy with a Poincaré-invariant vacuum $|\Omega\rangle$ satisfying $E_P(\{0\}) = |\Omega\rangle\langle\Omega|$ so that

the vacuum is the unique zero-momentum state in the chosen representation. Let $T_{\mu\nu}$ be a Hermitian symmetric conserved tensor that transforms exactly as a Lorentz tensor on the physical Hilbert space. Assume that its centred connected Wightman measure $d\Sigma_{\mu\nu,\rho\sigma}$ is locally finite and positive on all symmetric tensor test functions before TT compression. On the nonzero future null shell $H_0^+ = \{p : p^0 > 0, p^2 = 0\}$ it then has the form

$$d\Sigma_{\mu\nu,\rho\sigma}(p)|_{H_0^+} = c p_\mu p_\nu p_\rho p_\sigma d\Omega_0(p), \quad c \geq 0, \quad (\text{A.6})$$

where $d\Omega_0$ is a fixed invariant measure on that shell. Its contraction with every conserved source is zero. Since the connected measure has no atom at $p = 0$ every conserved TT source matrix has zero invariant-mass atom at $s = 0$.

Proof. The nonzero null shell is a single Lorentz orbit. Its restricted measure has a co-variant matrix density $H(p)$ relative to $d\Omega_0$. To obtain this density average compensated Lorentz translates against a smooth compactly supported Haar weight of integral one. Covariance leaves the measure unchanged. Since the orbit map is a submersion this convolution smooths a distributional section along the orbit and hence gives the asserted density. Positivity of the measure gives $H(p) \succeq 0$. This argument is confined to one orbit and does not require a density in the mass variable s .

At a reference null momentum ℓ factor $H(\ell) = VV^\dagger$ with V of full column rank. The case $H = 0$ is immediate. If $D(g)$ is the symmetric-tensor representation of an element of the stabiliser of ℓ covariance gives $D(g)H(\ell)D(g)^\dagger = H(\ell)$. Thus $D(g)V = VU(g)$ and a left inverse of V shows $U(g)U(g)^\dagger = I$. Null rotations are unipotent in D so their restrictions U are both unitary and unipotent and hence equal to the identity. Every column of V is therefore fixed by all null rotations.

For the component calculation raise the tensor indices and choose a null tetrad (ℓ, n, e_1, e_2) with $\ell \cdot n = 1$ and $e_i \cdot e_j = -\delta_{ij}$. The null-rotation generators satisfy $N_k \ell = 0$ together with $N_k n = e_k$ and $N_k e_i = \delta_{ki} \ell$. Write a symmetric tensor with juxtaposition denoting tensor product as

$$\begin{aligned} u = & a\ell\ell + bnn + c_1(\ell n + n\ell) + \sum_i d_i(\ell e_i + e_i \ell) \\ & + \sum_i f_i(ne_i + e_i n) + \sum_{i,j} g_{ij}e_i e_j, \quad g_{ij} = g_{ji}. \end{aligned}$$

Applying N_k to both indices and setting every coefficient to zero gives $b = f_i = d_i = 0$ and $g_{ij} = -c_1 \delta_{ij}$. Consequently $u = a\ell\ell + c_1 \eta$ where $\eta = \ell n + n\ell - \sum_i e_i e_i$. Conservation of the measure places every column of V in the transverse subspace. $\ell_\mu u^{\mu\nu} = c_1 \ell^\nu = 0$ then forces $c_1 = 0$. The range of $H(\ell)$ is therefore spanned by $\ell\ell$. Positivity and Lorentz transport give Eq. (A.6) with $c \geq 0$. The conservation identity $p_\mu J^{\mu\nu} = 0$ makes the source contraction vanish. The absence of a connected atom at $p = 0$ proved above completes the invariant-mass endpoint statement. $\square$

The exclusion classifies the actual null-shell measure of the complete positive exactly covariant tensor. Its positive-mass spectrum remains a dynamical question. A generated

tensor response or a discrete transfer residue falls within this classification when it admits the stated Wightman representation and satisfies the proposition's hypotheses. The general response theorems also apply to other representations.

A.4 Time-ordered and Euclidean correlators

Choose a spacelike subtraction point $Q_*^2 > 0$ and an integer $n_{\text{sub}} \geq 0$ large enough for ultraviolet convergence. The projected Euclidean form factors then have the matrix-valued subtracted dispersion representation

$$C_{ij}^{(a)}(Q^2) = P_{ij}^{(a)}(Q^2) + (Q_*^2 - Q^2)^{n_{\text{sub}}} \int_0^\infty \frac{d\rho_{ij}^{(a)}(s)}{(s + Q^2)(s + Q_*^2)^{n_{\text{sub}}}}. \quad (\text{A.7})$$

Here $P_{ij}^{(a)}$ is the finite subtraction polynomial including the allowed local contact terms. For $n_{\text{sub}} = 0$ the dispersive term in Eq. (A.7) reduces to the ordinary Stieltjes transform. A scheme-dependent local polynomial may still be present. Changing n_{sub} or Q_*^2 redistributes analytic terms between the integral and the polynomial but does not alter the spectral measure or an isolated pole residue. The Euclidean axioms of Ref. [32] with the reconstruction result corrected and extended in Ref. [33] provide a Hilbert-space interpretation when the full reconstruction hypotheses hold. Reflection positivity is one of these hypotheses rather than a sufficient condition by itself.

Equation (A.7) keeps distinct the external Euclidean variable Q^2 and the Lorentzian spectral variable s . A Gaussian factor in external momentum multiplies the Euclidean correlator. It is not by that operation alone a redefinition of $d\rho(s)$ or a new Wightman theory.

A.5 Zero-mass atoms and the Euclidean diagnostic

Decompose the spectral measure in a channel as

$$d\rho(s) = Z_0 \delta_0(ds) + d\rho_c(s), \quad \rho_c(\{0\}) = 0, \quad (\text{A.8})$$

where Z_0 is a positive-semidefinite matrix. The atom contributes Z_0/Q^2 to the singular part of Eq. (A.7). The remaining terms from its subtraction factor are analytic at the origin.

Proposition A.4 (Euclidean extraction of the atom). For every vector c let $d\nu_c(s) = c^\dagger d\rho_c(s) c / (s + Q_*^2)^{n_{\text{sub}}}$. Suppose ν_c is locally finite at $s = 0$ with no atom there and satisfies $\int_1^\infty s^{-1} d\nu_c(s) < \infty$. Then entrywise and as a quadratic form

$$Z_0 = \lim_{Q^2 \downarrow 0} Q^2 [C(Q^2) - P(Q^2)]. \quad (\text{A.9})$$

Proof. The atom term in Eq. (A.7) multiplied by Q^2 tends to Z_0 . For a quadratic form c the remaining term is

$$Q^2(Q_*^2 - Q^2)^{n_{\text{sub}}} \int_{(0,\infty)} \frac{d\nu_c(s)}{Q^2 + s}. \quad (\text{A.10})$$

Apart from the bounded prefactor $(Q_*^2 - Q^2)^{n_{\text{sub}}}$ the integrand is $Q^2/(Q^2 + s)$. On $(0, 1]$ it is bounded by one and converges pointwise to zero. Local finiteness and the absence of an atom at zero permit dominated convergence. On $[1, \infty)$ it is bounded by Q^2/s and the assumed weighted ultraviolet bound makes the tail vanish. Polarisation of the quadratic-form result gives the entrywise limit. The polynomial term is removed before the limit and the result follows. $\square$

Proposition A.4 applies to the unfiltered connected vacuum correlator of the centred insertions. Theorem 3.5 connects it to a response residue when the vacuum centring and operator projection agree together with the source pairing and the continuation and subtraction prescriptions for the endpoint correlator.

If the continuum falls outside the hypotheses of Proposition A.4 the endpoint requires a separate spectral analysis. Even when the limit exists a branch cut can begin at $s = 0$ and coexist with an atom. The decomposition (A.8) distinguishes the atomic weight from the threshold continuum through their spectral measures.

The connection with long-range response is direct. For $d > 2$ away from coincident points

$$\int \frac{d^d Q}{(2\pi)^d} \frac{e^{iQ \cdot x}}{Q^2} = \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}} \frac{1}{|x|^{d-2}}. \quad (\text{A.11})$$

Thus a nonzero Z_0 produces a power-law tail whereas the subtraction polynomial contributes only distributions supported at coincident points. In $d = 2$ the corresponding Green function is logarithmic. A continuum beginning at $s = 0$ can also generate long-distance power laws but it is not an isolated atom and must be analysed with its threshold density.

A.6 Action of the Gaussian multiplier

Smearing both insertions with $\mathcal{H}_{\tau_Q/2}$ gives the external diagnostic map

$$C_{\tau_Q}(Q^2) = \mathcal{G}_{\tau_Q}^{\text{ext}} C(Q^2) = e^{-\tau_Q Q^2} C(Q^2). \quad (\text{A.12})$$

If $C(Q^2) = Z_0/Q^2 + C_{\text{rem}}(Q^2)$ near the origin with $Q^2 C_{\text{rem}}(Q^2) \rightarrow 0$ then

$$C_{\tau_Q}(Q^2) = \frac{Z_0}{Q^2} + C_{\text{rem}}(Q^2) - \tau_Q Z_0 + o(1). \quad (\text{A.13})$$

The atom residue is unchanged. Conversely multiplication of a regular form factor by an entire function cannot generate a pole. This proves the pole-preservation statement in Section 2.

The multiplier acts on a fixed correlator. Its positive KL measure is defined by the original Wightman covariance. Gaussian filtering generally changes the Stieltjes form. In particular $e^{-\tau_Q Q^2} P(Q^2)$ is no longer a contact polynomial. In position space it is a short-range smearing of a contact distribution. For this reason the physical contact subtraction and spectral measure are defined from the unfiltered correlator before Eq. (A.12) is applied. Pole preservation may be checked on the filtered object whereas physical positivity must be checked on the original Wightman matrix.

The scale dependence of a full interacting correlator also enters through the effective vertices and response operator generated by integrating fluctuations.

A.7 Non-Hermitian and gauge-theory operators

For a non-Hermitian operator O positivity is formulated with its centred insertion $\Theta = O - \langle O \rangle I$ and $\langle \Theta(x) \Theta^\dagger(0) \rangle$ or for a mixing block with the enlarged connected matrix containing Θ and $\Theta^\dagger$.

In a covariantly gauge-fixed formulation the state space before the physical quotient has indefinite metric and elementary gauge-potential or ghost correlators need not possess positive spectral measures. The positivity used in Eq. (A.5) applies to physical BRST cohomology classes and to the physical-polarisation compression of Assumption A.2. Slavnov–Taylor identities control the gauge dependence and mixing of their representatives. Gauge invariance and the weight of a spectral atom are then assessed in the physical cohomology class and polarisation channel.

B Compatible operations, observations, and operator realisations

For the channel reduction in Section 3 invariance of a retained subspace and descent through a projection impose different block identities. These identities determine the renormalised-operator realisation and basis covariance of Proposition 3.6.

B.1 Projection, restriction, and quotient compatibility

For a linear update B and projector P put $Q = I - P$. Invariance of the retained subspace requires $QBP = 0$ whereas descent through the coarse map $X \mapsto PX$ requires $PBQ = 0$. The first prevents escape from an initially retained state. The second prevents hidden states from changing the next observation. Indeed $PX = PY$ means $X - Y \in \text{Ran } Q$ so $PBX = PBY$ for all such pairs exactly when $PBQ = 0$. A readout descends through P exactly when $RQ = 0$. Neither a dimension cutoff nor smoothing proves either identity.

In the finite Neumann protocol the spectrum and bounded heat action are proved directly in Section 5.2. Multiplication and projection are explicit bounded operations. Their finite composition defines the system studied rather than asserting nonlinear closure for arbitrary omitted modes. For a gauge or other equivalence relation one must likewise show that the update preserves equivalent states and that observations agree on them before passing to the quotient. The physical operator construction below implements this statement in its BRST and separated-point setting.

The charged matter quotient of ref. [14] Sections 4 and 5 gives an explicit finite example. If N is its neutral line an operator A on the sixteen-dimensional full module descends to the fifteen-dimensional charged quotient precisely when $AN \subset N$. For self-adjoint A this is equivalent to $[A, P_N] = 0$. The reduced gauge action satisfies this condition. The full joint algebra contains matrix units that connect the neutral and charged directions so its response is computed on the full module or by complete Schur elimination. The compressed matrix $P_{N^\perp} A P_{N^\perp}$ is the retained block in that calculation. Source and readout maps together with the direct term retain the effect of the complementary direction as in Eqs. (3.4)–(3.6).

Internal coordinates and external source coordinates have different covariance laws even without QFT. For a bounded state isomorphism T and a nonsingular source–output basis matrix M set

$$B' = TBT^{-1}, \quad S' = TSM^*, \quad R' = MRT^{-1}, \quad D' = MDM^*.$$

Substitution in $D + R(I - B)^{-1}S$ proves $\mathcal{C}' = M\mathcal{C}M^*$. Full visible pole locations and orders and the rank of a residue are unchanged. Positive residues remain positive. Pure internal similarity leaves the output unchanged. This argument supplies general response covariance. The proof additionally identifies Wightman and renormalised-operator data in a QFT realisation.

B.2 Mixing and composite sources

For operators with common exact quantum numbers renormalisation is a matrix problem

$$O_R = ZO_B, \quad \mu \frac{dO_R}{d\mu} = -\gamma O_R, \quad \gamma = -\left(\mu \frac{dZ}{d\mu}\right) Z^{-1}, \quad (\text{B.1})$$

with the sign convention used in Eq. (3.33). If the source term is $h_R^T O_R = h_B^T O_B$ then $h_B = Z^T h_R$. Two or more composite insertions generally require additional local counterterms polynomial in the sources. Those counterterms are the source-space origin of the contact polynomial $P_{ij}(Q^2)$ in Eq. (A.7).

Lorentz representation and internal charges together with ghost number and canonical dimension can be used to construct finite computational spaces $\mathcal{V}_N$. Membership in such a space does not imply invariance under mixing. In an effective field theory increasing the dimension bound enlarges the space. Conclusions at finite N are truncation dependent unless a power-counting or operator-norm estimate controls the omitted sector.

B.3 BRST, EOM, and descendant sectors

In an anomaly-free gauge theory first form the on-shell space $\bar{\mathcal{V}} = \mathcal{V}/\mathcal{I}_{\text{EOM}}$. The BRST differential $\mathfrak{s}_B$ preserves the EOM ideal and therefore induces a nilpotent differential on $\bar{\mathcal{V}}$ under the standard regularity assumptions. A convenient off-shell basis used to determine counterterms contains three types of representatives

$$O_G \in \text{Ker } \mathfrak{s}_B, \quad O_A = \mathfrak{s}_B \Psi, \quad O_E = \mathcal{E}_F^R. \quad (\text{B.2})$$

Here $\mathcal{E}_F^R$ is the renormalised insertion associated with a local change of integration variables. For a regulated bosonic integral with flat measure its defining identity is

$$\mathcal{E}_F = F^a \frac{\delta S}{\delta \phi^a} - \frac{\delta F^a}{\delta \phi^a}, \quad \langle \mathcal{E}_F \mathcal{O} \rangle = \left\langle F^a \frac{\delta \mathcal{O}}{\delta \phi^a} \right\rangle. \quad (\text{B.3})$$

Repeated field labels include spacetime integration. The identity follows by integrating the variation of $F^a \mathcal{O} e^{-S}$. For a general measure the second term is its field-space divergence with the graded signs for fermionic variables. Renormalisation includes the Jacobian and operator-mixing counterterms so that this Schwinger–Dyson identity holds for renormalised

insertions. With local variation support its right-hand side has contact support in the other insertion coordinates. These are the EOM representatives used in $\mathcal{I}_{\text{EOM}}$. Derivatives of source-dependent insertions and the associated local source counterterms are retained in the direct response. The stress-tensor application uses the same renormalised representative in Ref. [13]. Under the standard assumptions for local composite-operator renormalisation the renormalisation matrix can be chosen triangular between physical cohomology classes and BRST-exact and equation-of-motion sectors [18]. Section 8.6 of Ref. [19] gives the local BRST cohomological formulation. BRST-exact insertions vanish between physical cohomology classes while EOM insertions reduce to contact terms in separated-point correlators. After the mixing problem is closed these statements justify taking the ghost-number-zero cohomology in Eq. (3.21). The corresponding counterterms remain in the determination of Z .

This cohomological quotient is an algebraic construction. Functional-analytic claims made with bounded channel projectors or source-readout maps use the explicit Banach completion and norm included in the channel data of Definition 3.4. The required maps must extend continuously to that completion. This specified completion fixes the operator norm.

Infrared singularities can obstruct decoupling at exceptional kinematics. The on-shell gluonic matrix elements at zero momentum transfer in Ref. [20] provide a concrete example. The classification therefore forms the quotient at nonexceptional kinematics and retains all contact terms before taking a zero-momentum limit only after the projected renormalised correlator has been constructed.

Total derivatives occupy a different position. When the boundary term vanishes integration by parts gives

$$\int d^d x h \partial_\mu V^\mu = - \int d^d x (\partial_\mu h) V^\mu. \quad (\text{B.4})$$

Thus a total derivative is redundant for a constant-source integrated insertion or at zero insertion momentum when that limit is well defined. A nonconstant source generally leaves the source-gradient term on the right-hand side and does not make the insertion redundant. For a local insertion carrying momentum q however

$$[\partial_\mu V^\mu](q) = -i q_\mu V^\mu(q), \quad (\text{B.5})$$

and the descendant can contribute away from $q = 0$ or mix with other local operators. It is retained whenever the external kinematics resolve it.

B.4 Symmetry blocks and anomalies

Let $R(g)$ be the action of an exact global or spacetime symmetry on the renormalised operator space. A projector $\mathcal{P}$ defines a closed channel only if its range is stable under the mixing matrix and the unbroken symmetry action. When the regulator temporarily violates a symmetry modified Ward or Slavnov–Taylor identities must be solved so that the physical $k \rightarrow 0$ block obeys the unmodified identity.

An anomalous current is not placed in the same block as an exactly conserved current. Its divergence contains a renormalised local insertion and that insertion participates in the

mixing problem. Similarly explicit mass breaking modifies the operator identity. Spontaneous breaking by the state changes its matrix elements while preserving the identity.

For the stress tensor improvement transformations of the form

$$T_{\mu\nu} \mapsto T_{\mu\nu} + (\partial_\mu \partial_\nu - \eta_{\mu\nu} \square)X \quad (\text{B.6})$$

with X a renormalised Lorentz scalar preserve conservation. A physical TT polarisation satisfies $p_\mu \epsilon^{\mu\nu} = 0$ and $\eta_{\mu\nu} \epsilon^{\mu\nu} = 0$ and hence

$$\epsilon^{\mu\nu}(-p_\mu p_\nu + \eta_{\mu\nu} p^2) = 0.$$

The improvement therefore vanishes under TT compression including at the massless end-point when the physical helicity compression of Appendix A exists. Contact terms are treated using the same subtraction convention. Spin-zero contributions and the overlaps of an individual representative with physical states need not be invariant.

Let $D_X = (\partial_\mu \partial_\nu - \eta_{\mu\nu} \square)X$ be retained among the representatives of a closed block. When $(T, D_X, \dots) \mapsto (T + D_X, D_X, \dots)$ is a finite nonsingular constant basis change on that block the covariance statement of Proposition 3.6 preserves the pole locations and orders of the full correlation matrix and the ranks of its isolated residues. No such invariance is asserted for a single entry after the descendant sector has been discarded.

B.5 Proof of external and internal covariance

Proof of Proposition 3.6. Let $O' = MO$ with constant finite and nonsingular M . In the same reference vacuum the centred insertions obey $\Theta' = M\Theta$. The connected Wightman matrices therefore satisfy

$$W'(x) = MW(x)M^\dagger. \quad (\text{B.7})$$

Uniqueness of the distributional spectral decomposition then gives $d\rho' = M d\rho M^\dagger$ and analytic continuation gives $C' = MCM^\dagger$ including the transformed contact polynomial.

Suppose near $z = z_0$ the correlator matrix has a Laurent expansion

$$C(z) = \sum_{r=-p}^{\infty} C_r(z - z_0)^r, \quad C_{-p} \neq 0. \quad (\text{B.8})$$

Then $C'(z) = \sum_r MC_r M^\dagger (z - z_0)^r$. Because M and $M^\dagger$ are invertible $MC_{-p}M^\dagger \neq 0$. The pole location and order are unchanged. For a simple spectral pole $Z' = MZM^\dagger$ has the same rank as Z and

$$c^\dagger Z' c = (M^\dagger c)^\dagger Z (M^\dagger c) \geq 0 \quad (\text{B.9})$$

whenever Z is positive semidefinite.

A linear symmetry constraint $DC = 0$ is carried to $(DM^{-1})C' = 0$ after multiplication on the right by $M^\dagger$. Thus the dimensions of its forbidden and constrained subspaces together with its allowed subspace are unchanged.

Now let $T : \mathcal{X}_\mathcal{E} \rightarrow \mathcal{X}'_\mathcal{E}$ be a bounded response-state isomorphism. Direct substitution gives

$$\mathbf{R}_{X,T}(I - \mathcal{B}_T)^{-1} \mathbf{S}_{X,T} = \mathbf{R}_X T^{-1} T (I - \mathcal{B})^{-1} T^{-1} T \mathbf{S}_X = \mathbf{R}_X (I - \mathcal{B})^{-1} \mathbf{S}_X. \quad (\text{B.10})$$

Thus T changes internal coordinates by similarity but does not transform the physical output matrix. The response projector transforms as $P_{X,T} = TP_X T^{-1}$ so $Q_{X,T} \mathcal{B}_T P_{X,T} = T(Q_X \mathcal{B} P_X) T^{-1}$. Response closure is invariant. Combining T with the external basis change M instead gives

$$\begin{aligned} R'_X (I - \mathcal{B}')^{-1} S'_X \\ = MR_X T^{-1} T (I - \mathcal{B})^{-1} T^{-1} T S_X M^\dagger = MR_X (I - \mathcal{B})^{-1} S_X M^\dagger, \end{aligned} \quad (\text{B.11})$$

and the same congruence holds for C_{reg} .

Similarity preserves the response spectrum and algebraic multiplicities. At a simple critical eigenvalue $|r'\rangle = T|r\rangle$ and $\langle \ell'| = \langle \ell| T^{-1}$ whence

$$R'_X |r'\rangle = MR_X |r\rangle, \quad \langle \ell'| S'_X = \langle \ell| S_X M^\dagger. \quad (\text{B.12})$$

Invertibility of M preserves vanishing or nonvanishing of both overlaps and Eq. (3.9) gives $Z'_{\text{alg}} = M Z_{\text{alg}} M^\dagger$. These observations prove Proposition 3.6. $\square$

B.6 Scale-dependent bases and finite schemes

For $O' = M(\mu)O$ differentiation of Eq. (3.28) and use of Eq. (B.1) give

$$\mu \frac{dO'}{d\mu} = - \left[M \gamma M^{-1} - \left(\mu \frac{dM}{d\mu} \right) M^{-1} \right] O', \quad (\text{B.13})$$

which proves Eq. (3.33). Physical poles of the exact correlator remain unchanged by a finite nonsingular scheme transformation but the residue coordinates and Wilson coefficients change together with the anomalous-dimension matrix. At finite truncation two implementations need not be related by an exact similarity transformation because the transformed operator basis can have components outside $P_N \mathcal{V}_{\text{phys}}$ and its realised response can leave the retained range of a response projector $P_{X,N}$. The off-block response operators

$$\Delta_N^{\text{out}} = (I_X - P_{X,N}) \mathcal{B} P_{X,N}, \quad \Delta_N^{\text{in}} = P_{X,N} \mathcal{B} (I_X - P_{X,N}) \quad (\text{B.14})$$

measure the direct coupling omitted by the compression. If they do not vanish either the Schur complement Eq. (3.3) or a convergence argument such as Proposition C.1 is required. Regulator and basis variation measure the sensitivity of the finite approximation. Exact covariance follows from the full transformed response.

C Descent, Gaussian, kernel, and error proofs

The proofs establish differentiability through a Banach quotient and Gaussian averaging estimates together with control of isolated response residues. In the last step the resolvent identity first controls the spectral projection. Root displacement and variation of the source and readout then enter the residue estimate of Proposition 4.2.

C.1 Regular descent through a Banach quotient

Proof of regular Banach descent. Put $K = \text{Ker } C$ and let $q : \mathcal{X} \rightarrow \mathcal{X}/K$ be the quotient map with

$$\|qu\|_{\mathcal{X}/K} = \inf_{k \in K} \|u + k\|_{\mathcal{X}}.$$

Since K is closed $\mathcal{X}/K$ is Banach. The map

$$\widehat{C} : \mathcal{X}/K \longrightarrow \mathcal{Y}, \quad \widehat{C}(qx) = Cx,$$

is a bounded bijection and hence a bounded isomorphism. The quotient map q is open since $q(B_{\mathcal{X}}(0, \rho)) = B_{\mathcal{X}/K}(0, \rho)$ for every $\rho > 0$. Thus $q(U)$ is open and so is $C(U) = \widehat{C}(q(U))$. Fibre consistency defines a unique map $\widetilde{F} : q(U) \times V \rightarrow \mathcal{Y}$ by

$$\widetilde{F}(qx, h) = CF(x, h), \quad \bar{F}(y, h) = \widetilde{F}(\widehat{C}^{-1}y, h).$$

Fix $(x, h) \in U \times V$ and choose $\rho > 0$ with $B_{\mathcal{X}}(x, \rho) \subset U$. If $k \in K$ then $CF(x + tk, h) = CF(x, h)$ for all sufficiently small t . Differentiation at $t = 0$ gives

$$CD_x F(x, h)k = 0.$$

It follows that

$$\widetilde{A}_{x,h}(qu) := CD_x F(x, h)u$$

is a well-defined bounded operator on $\mathcal{X}/K$. Indeed taking the infimum over representatives gives

$$\|\widetilde{A}_{x,h}\| \leq \|C\| \|D_x F(x, h)\|.$$

Let $\eta \in \mathcal{X}/K$ satisfy $\|\eta\| < \rho/2$. By the definition of the quotient norm there is a representative u with $qu = \eta$ and $\|u\| \leq 2\|\eta\| < \rho$. For j small enough that $h + j \in V$ fibre consistency and the joint Fréchet expansion at (x, h) give

$$\begin{aligned} & \widetilde{F}(qx + \eta, h + j) - \widetilde{F}(qx, h) - \widetilde{A}_{x,h}\eta - CD_h F(x, h)j \\ &= C[F(x + u, h + j) - F(x, h) - D_x F(x, h)u - D_h F(x, h)j]. \end{aligned} \quad (\text{C.1})$$

The right-hand side is $o(\|u\| + \|j\|) = o(\|\eta\| + \|j\|)$. This estimate is uniform over the chosen near-minimal representatives because $\|u\| \leq 2\|\eta\|$. Its left-hand side depends only on η . Hence $\widetilde{F}$ is Fréchet differentiable at (qx, h) .

It remains to verify continuity of the derivative. If $(\zeta, h') \rightarrow (qx, h)$ in $q(U) \times V$ choose u_{ζ} with $q(x + u_{\zeta}) = \zeta$ and $\|u_{\zeta}\| \leq 2\|\zeta - qx\|$. For ζ sufficiently close to qx the point $x + u_{\zeta}$ belongs to U . Both $CD_x F(x + u_{\zeta}, h')$ and $CD_x F(x, h)$ annihilate K and therefore their induced quotient operators satisfy

$$\|\widetilde{A}_{x+u_{\zeta}, h'} - \widetilde{A}_{x,h}\| \leq \|C\| \|D_x F(x + u_{\zeta}, h') - D_x F(x, h)\| \longrightarrow 0.$$

The same argument applied to $CD_h F$ proves continuity of the source derivative. Thus $\widetilde{F}$ and consequently $\bar{F}$ are C^1 . Composition with $\widehat{C}^{-1}$ gives precisely Eq. (2.3). The induced derivatives do not depend on a representative either by the construction above or by uniqueness of the Fréchet derivative. Replacing CF by a fibre-constant C^1 observation O proves the final assertion in the same way. $\square$

This proof also identifies the bounded isomorphism $\widehat{C} : \mathcal{X} / \text{Ker } C \rightarrow \mathcal{Y}$ supplied by the quotient norm. Its inverse acts on equivalence classes in the Banach quotient. The error identity of Theorem 4.1 applies more generally to any bounded linear comparison map.

C.2 Differentiating exact block elimination

For fixed complementary projectors abbreviate the blocks by B_{ij} and put $H = (I - B_{22})^{-1}$. A dot denotes a total derivative along the declared parameter and reference branch. Then $\dot{H} = H\dot{B}_{22}H$ and the four Schur derivatives are

$$\begin{aligned}\dot{B}_{\text{eff}} &= \dot{B}_{11} + \dot{B}_{12}HB_{21} + B_{12}\dot{H}B_{21} + B_{12}H\dot{B}_{21}, \\ \dot{S}_{\text{eff}} &= \dot{S}_1 + \dot{B}_{12}HS_2 + B_{12}\dot{H}S_2 + B_{12}H\dot{S}_2, \\ \dot{R}_{\text{eff}} &= \dot{R}_1 + \dot{R}_2HB_{21} + R_2\dot{H}B_{21} + R_2H\dot{B}_{21}, \\ \dot{D}_{\text{eff}} &= \dot{D} + \dot{R}_2HS_2 + R_2\dot{H}S_2 + R_2H\dot{S}_2.\end{aligned}\tag{C.2}$$

They follow by differentiating $(I - B_{22})H = I$ and the four product formulae. If the projectors depend on the parameter first differentiate the projected blocks themselves including every projector derivative. Together with $\dot{B} = \partial_a B + D_X B \dot{X}_*$ and the corresponding identities for S, R, D these formulae retain the actual reference-branch dependence. The product rule applies to all four maps including their source and direct-response contributions.

C.3 Uniqueness of the Gaussian semigroup

Proof of Proposition 2.3. Write

$$\varphi_\tau(q) = \int_{\mathbb{R}^d} e^{iq \cdot x} d\mu_\tau(x).\tag{C.3}$$

Weak continuity and convolution give $\varphi_{\tau+s}(q) = \varphi_\tau(q)\varphi_s(q)$ and $\varphi_\tau(q) \rightarrow 1$ as $\tau \downarrow 0$. Orthogonal invariance includes $x \mapsto -x$ so φ_τ is real. Moreover $\varphi_\tau(q) = \varphi_{\tau/(2n)}(q)^{2n} \geq 0$. It is in fact strictly positive. If $\varphi_\tau(q) = 0$ for some $\tau > 0$ then

$$0 = \varphi_\tau(q) = \varphi_{\tau/n}(q)^n$$

would imply $\varphi_{\tau/n}(q) = 0$ for every positive integer n in contradiction with $\varphi_{\tau/n}(q) \rightarrow \varphi_0(q) = 1$. The continuous real logarithm is therefore defined and the scalar semigroup has the form

$$\varphi_\tau(q) = e^{-\tau\psi(q)}\tag{C.4}$$

with $\psi(q) \geq 0$ continuous.

Square-root self-similarity also gives $\varphi_\tau(q) = \varphi_1(\sqrt{\tau}q)$. Comparison with Eq. (C.4) yields $\psi(rq) = r^2\psi(q)$ for $r > 0$. Orthogonal invariance then gives $\psi(q) = D|q|^2$. Finite nonzero covariance makes D finite and strictly positive. Hence

$$\widehat{\mu}_\tau(q) = e^{-D\tau|q|^2},\tag{C.5}$$

which proves Proposition 2.3. $\square$

The exact square-root scaling fixes the homogeneity degree at two. Removing the second-moment assumption alone does not change that conclusion. The stated covariance condition fixes a finite nonzero scale. Without orthogonal invariance the Gaussian covariance need not be isotropic. Non-Gaussian stable semigroups require a different scaling exponent $\tau^{1/\alpha}$ with $0 < \alpha < 2$ and relaxation of the finite-second-moment assumption. Dropping exact self-similarity permits symmetric jump components including ones of finite variance. The retained orthogonal invariance excludes a nonzero drift.

C.4 Tensor lift and suppression away from zero transfer

Proof of Theorems 2.4 and 2.5. The Fourier transform converts convolution in the i th coordinate into multiplication by $e^{-a_i \tau |p_i|^2}$. These smooth multipliers and all their derivatives are polynomially bounded so they act continuously on tempered distributions. Since the coordinate convolutions commute their tensor product gives

$$\prod_{i=1}^n e^{-a_i \tau |p_i|^2} = \exp \left[-\tau \sum_{i=1}^n a_i |p_i|^2 \right]. \quad (\text{C.6})$$

For the routed statement $\widehat{F}_{\text{red}}$ is part of the data after the overall delta distribution has been factored out. Write $\mathbf{W} = \text{diag}(a_1 I_d, \dots, a_n I_d)$ and let $T(P, \boldsymbol{\xi}) = AP + L\boldsymbol{\xi}$ be the adapted linear coordinate isomorphism of Theorem 2.4. With $g_\tau(p) = e^{-\tau p^\top \mathbf{W} p}$ the distributional identity in Eq. (2.17) means that for every $\Phi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^r)$

$$\langle \widehat{F}_{\text{full}}, \Phi \rangle = (2\pi)^d \langle \widehat{F}_{\text{red}}, \Phi(0, \cdot) \rangle, \quad \int_{\mathbb{R}^d} \delta^{(d)}(P) f(P) dP = f(0). \quad (\text{C.7})$$

Multiplication by the smooth Gaussian is therefore computed without restricting an arbitrary distribution to $P = 0$

$$\begin{aligned} \langle g_\tau(T) \widehat{F}_{\text{full}}, \Phi \rangle &= (2\pi)^d \left\langle \widehat{F}_{\text{red}}, g_\tau(T(0, \cdot)) \Phi(0, \cdot) \right\rangle \\ &= (2\pi)^d \left\langle \widehat{F}_{\text{red}}, e^{-\tau \boldsymbol{\xi}^\top L^\top \mathbf{W} L \boldsymbol{\xi}} \Phi(0, \boldsymbol{\xi}) \right\rangle. \end{aligned} \quad (\text{C.8})$$

Since $T(0, \boldsymbol{\xi}) = L\boldsymbol{\xi}$ parametrises all of $\text{Ker } \Pi$ this is exactly the reduced action in Eq. (2.19) with $M_L = L^\top \text{diag}(a_1 I_d, \dots, a_n I_d) L$. This matrix is positive semidefinite. On a subspace V on which $M_L \succeq cI$ one has $\boldsymbol{\xi}^\top M_L \boldsymbol{\xi} \geq c \|\boldsymbol{\xi}\|^2$. This proves Theorem 2.4 including its bound and the absence of a conclusion on $\text{Ker } M_L$.

For Theorem 2.5(i) $q^\top M q \geq \delta^2$ on Ω_δ and hence

$$\|e^{-\tau q^\top M q} A\|_{L^\infty(\Omega_\delta)} \leq e^{-\tau \delta^2} \|A\|_{L^\infty(\Omega_\delta)}. \quad (\text{C.9})$$

For part (ii) put $\chi_\tau(q) = e^{-\tau q^\top M q} \chi(q)$. Each derivative of χ_τ is a finite sum of a polynomial in q and τ multiplied by $e^{-\tau q^\top M q}$ and a derivative of χ . The support is fixed inside Ω_δ so every Schwartz seminorm is bounded by a polynomial in τ times $e^{-\tau \delta^2}$ and tends to zero. Continuity of the tempered distribution A gives

$$\langle A_\tau, \chi \rangle = \langle A, \chi_\tau \rangle \longrightarrow 0. \quad (\text{C.10})$$

Finally Eq. (2.26) gives $\|A_\tau^{\text{full}}\| \leq C e^{-\tau(\delta^2 - a)}$ proving part (iii). $\square$

C.5 Response factorisation and the closure defect

Proof of Propositions 2.6 and 4.2. Response factorisation. Apply the implicit-function theorem on Banach spaces to $G(X, J) = X - \mathfrak{F}_{\tau_q}(X; z, \eta) - S_{X, \tau_q}(z)J$. The derivative $D_X G = I - \mathcal{B}$ is invertible by hypothesis so a unique differentiable solution branch exists locally. Differentiating Eq. (2.30) at $J = 0$ with $\delta X = D_J X \delta J$ and $\mathcal{B} = D_X \mathfrak{F}_{\tau_q}(X_*; z, \eta)$ one obtains

$$(I - \mathcal{B})\delta X = S_{X, \tau_q} \delta J. \quad (\text{C.11})$$

Where $I - \mathcal{B}$ is invertible composing its solution with the readout and differentiating the direct term in Eq. (2.31) gives Eq. (2.33). Under the hypotheses of Theorem 3.2 the same formula extends meromorphically. This proves Proposition 2.6.

Transport of the closure defect. Assume $\mathcal{C}_{\tau_q}$ is bounded and linear and that the two maps in Eq. (4.11) are Fréchet differentiable. The chain rule gives

$$D_X E_{\tau_q}(X; z) = \mathcal{C}_{\tau_q} D_X \mathfrak{F}(X; z) - D_X \mathfrak{F}_{\tau_q}(\mathcal{C}_{\tau_q} X; z) \mathcal{C}_{\tau_q}. \quad (\text{C.12})$$

Thus an exact commuting coarse-graining has the response intertwining Eq. (4.13). In an approximate closure $D_X E_{\tau_q}$ measures its failure at the same base point. Compactness and analyticity together with threshold isolation are additional properties of the realised response family.

For Proposition 4.2 Eq. (4.14) and Eq. (4.12) first give $\mathfrak{F}_{\tau_q}(\bar{X}_*; z) = \bar{X}_*$. Thus the derivative at $\bar{X}_*$ is the response Jacobian at an actual coarse solution rather than at a transported trial point. On the comparison spaces multiply Eq. (C.12) on the right by $\mathcal{C}_{\tau_q}^{-1}$ to obtain

$$[D_X E_{\tau_q}(X_*(z); z)|_Y] \mathcal{C}_{\tau_q}^{-1} = \mathcal{B}_{\text{ind}}(z) - \mathcal{B}_{\tau_q}(z) = -\Delta \mathcal{B}(z), \quad (\text{C.13})$$

This proves the first norm estimate.

Spectral projection and root displacement. For $R(\zeta, z) = (\zeta - \mathcal{B}_{\text{ind}}(z))^{-1}$ the condition $K_\gamma \epsilon < 1$ permits the Neumann expansion of the perturbed resolvent. The resolvent identity gives uniformly on $K \times \gamma$

$$\|R_{\tau_q}(\zeta, z) - R(\zeta, z)\| \leq \frac{K_\gamma^2 \epsilon}{1 - K_\gamma \epsilon}. \quad (\text{C.14})$$

Integrating $(2\pi i)^{-1} [R_{\tau_q}(\zeta, z) - R(\zeta, z)] d\zeta$ around γ proves Eq. (4.21). The same resolvent estimate holds for $\mathcal{B}_{\text{ind}} + t\Delta \mathcal{B}$ throughout $0 \leq t \leq 1$. The associated Riesz projections form a norm-continuous path of finite-rank idempotents and therefore have constant rank. Since the rank at $t = 0$ is one Π_{τ_q} also has rank one. The finite-rank trace identities

$$\lambda_{\text{ind}} = \text{tr}(\mathcal{B}_{\text{ind}} \Pi_{\text{ind}}), \quad \lambda_{\tau_q} = \text{tr}(\mathcal{B}_{\tau_q} \Pi_{\tau_q}) \quad (\text{C.15})$$

then imply Eq. (4.22). The term $\Delta \mathcal{B} \Pi_{\tau_q}$ has rank at most one while $\mathcal{B}_{\text{ind}}(\Pi_{\tau_q} - \Pi_{\text{ind}})$ has rank at most two.

On ∂D_* Eq. (4.23) gives $|\lambda_{\text{ind}}(z) - 1| \geq 2d_\lambda$ whereas Eq. (4.22) bounds the perturbation by d_λ . Rouché's theorem therefore gives exactly one zero $\tilde{z}_*$ of $\lambda_{\tau_q} - 1$ in D_* and proves Eq. (4.24).

Residue at the displaced root. At a common value of z insert and subtract the two intermediate products $R_{\text{ind}}\Pi_{\tau_q}\tilde{S}_X$ and $R_{\text{ind}}\Pi_{\text{ind}}\tilde{S}_X$. The triangle inequality and Eq. (4.21) together with the uniform constants in Eqs. (4.26)–(4.27) give

$$\sup_{z \in D_*} \|\tilde{\mathcal{N}}(z) - \mathcal{N}_{\text{ind}}(z)\| \leq d_{\text{num}}^{(0)}. \quad (\text{C.16})$$

The fundamental theorem of calculus along the line segment from z_* to $\tilde{z}_*$ gives

$$\|\tilde{\mathcal{N}}(\tilde{z}_*) - \tilde{\mathcal{N}}(z_*)\| \leq L_{\text{num}}\delta_z. \quad (\text{C.17})$$

Combining the last two estimates yields

$$\|\tilde{\mathcal{N}}(\tilde{z}_*) - \mathcal{N}_{\text{ind}}(z_*)\| \leq D_{\text{num}}. \quad (\text{C.18})$$

The same argument applied to $\partial_z \lambda_{\tau_q}$ gives

$$|\partial_z \lambda_{\tau_q}(\tilde{z}_*) - \partial_z \lambda_{\text{ind}}(z_*)| \leq d_{\lambda'}^{(0)} + L_{\lambda'}\delta_z = D_{\lambda'}. \quad (\text{C.19})$$

The bound $D_{\lambda'} < \alpha/2$ gives $|\partial_z \lambda_{\tau_q}(\tilde{z}_*)| \geq \alpha/2$. The elementary difference-of-quotients estimate now gives

$$\left\| -\frac{\tilde{\mathcal{N}}(\tilde{z}_*)}{\partial_z \lambda_{\tau_q}(\tilde{z}_*)} + \frac{\mathcal{N}_{\text{ind}}(z_*)}{\partial_z \lambda_{\text{ind}}(z_*)} \right\| \leq \frac{2D_{\text{num}}}{\alpha} + \frac{2\|\mathcal{N}_{\text{ind}}(z_*)\|D_{\lambda'}}{\alpha^2}, \quad (\text{C.20})$$

which is Eq. (4.31). This proves Proposition 4.2. $\square$

C.6 Fredholm resolvent near a simple spectral zero

Proof of Theorem 3.2. Under the hypotheses of Theorem 3.2 the analytic Fredholm theorem implies that $[I - \mathcal{B}(z)]^{-1}$ is a meromorphic operator-valued function on U . See Chapter VII of Ref. [6]. If $1 \notin \text{spec } \mathcal{B}(0)$ bounded invertibility is open in operator norm. The inverse is analytic near $z = 0$ and insertion into Eq. (2.33) proves Theorem 3.2(i).

Suppose now that $\lambda(0) = 1$ is algebraically simple. Analytic perturbation theory supplies local analytic continuations $\lambda(z)$ and $|r(z)\rangle$ together with $\langle \ell(z)|$ after reducing U if necessary satisfying

$$\mathcal{B}(z)|r(z)\rangle = \lambda(z)|r(z)\rangle, \quad \langle \ell(z)|\mathcal{B}(z) = \lambda(z)\langle \ell(z)|, \quad \langle \ell(z)|r(z)\rangle = 1. \quad (\text{C.21})$$

The associated Riesz projection is

$$\Pi_\lambda(z) = |r(z)\rangle\langle \ell(z)|. \quad (\text{C.22})$$

On the complementary invariant subspace $I - \mathcal{B}(z)$ remains invertible near the origin. Hence

$$[I - \mathcal{B}(z)]^{-1} = \frac{\Pi_\lambda(z)}{1 - \lambda(z)} + \mathcal{R}_\lambda(z), \quad (\text{C.23})$$

where $\mathcal{R}_\lambda$ is analytic. The simple-zero condition gives

$$1 - \lambda(z) = -\partial_z \lambda(0)z + O(z^2). \quad (\text{C.24})$$

Substitution in Eq. (2.33) yields

$$C_{\text{sing}}(z) = -\frac{R_X(0)|r(0)\rangle\langle\ell(0)|S_X(0)}{\partial_z\lambda(0)z}. \quad (\text{C.25})$$

The outer product is nonzero exactly when both displayed factors are nonzero. This proves part (ii) and Eq. (3.9).

Equation (C.25) is an algebraic matrix residue in the chosen response convention and proves part (iii). Its identification with a Wightman atom follows under the matching hypotheses of Theorem 3.5. For an approximate nonnormal kernel those hypotheses must be tested independently of the critical eigenvalue. $\square$

C.7 Uniform truncation and Riesz projections

Compactness of a finite matrix gives no convergence statement for the operator it approximates. The next proposition is a standard consequence of norm-resolvent control and analytic perturbation [6, 7]. Its final assertion specifies the additional simple-zero hypothesis needed to pass from spectral projections to response residues at moving roots.

Proposition C.1 (Convergence of an isolated spectral subspace). Let $K \Subset U$ be compact and let $\tilde{\mathcal{B}}_N(z)$ and $\mathcal{B}(z)$ act on one Banach space and be operator-norm continuous on a neighbourhood of K . Suppose

$$\sup_{z \in K} \|\tilde{\mathcal{B}}_N(z) - \mathcal{B}(z)\| \longrightarrow 0. \quad (\text{C.26})$$

Let γ be a closed contour in the spectral ζ plane which lies in the resolvent set of $\mathcal{B}(z)$ for every $z \in K$ and is uniformly separated from its spectrum. Then for all sufficiently large N the contour γ also lies in the resolvent set of $\tilde{\mathcal{B}}_N(z)$ and

$$\sup_{z \in K} \|\Pi_{\gamma,N}(z) - \Pi_{\gamma}(z)\| \longrightarrow 0. \quad (\text{C.27})$$

For convergence of critical roots and residues assume additionally that the kernel families are analytic on a common open neighbourhood of a closed disk $D \subset \text{int } K$. Let γ enclose exactly one algebraically simple eigenvalue $\lambda(z)$ and no other spectrum for every $z \in D$ and suppose $z_* \in \text{int } D$ is the only zero of $\lambda - 1$ in D with

$$\lambda(z_*) = 1, \quad \lambda'(z_*) \neq 0.$$

Assume the kernel derivatives and the analytic source and readout maps converge uniformly on D in their specified operator norms. Then for all sufficiently large N the corresponding branch $\lambda_N - 1$ has exactly one zero $z_{*,N}$ in D . This zero is simple and $z_{*,N} \rightarrow z_*$. The algebraic residues at their respective roots obey

$$-\frac{R_{X,N}(z_{*,N})\Pi_{\gamma,N}(z_{*,N})S_{X,N}(z_{*,N})}{\lambda'_{N}(z_{*,N})} \longrightarrow -\frac{R_X(z_*)\Pi_{\gamma}(z_*)S_X(z_*)}{\lambda'(z_*)}.$$

This is a residue of the selected critical response. Analytic direct terms do not contribute. Algebraic simplicity in the spectral variable alone does not give the required first-order zero in z .

Proof. Indeed the exact resolvents are uniformly bounded on the compact set $K \times \gamma$. Equation (C.26) and a Neumann series give existence and uniform convergence of the approximate resolvents. Contour integration then proves Eq. (C.27). The explicit norm-continuity hypothesis gives the asserted compact-set bound even for nonnormal kernels. For the additional assertion nearby projections have the same rank one. Analytic perturbation on these spectral subspaces gives uniform convergence of λ_N to λ and using the kernel-derivative convergence of λ'_N to λ' [6, 7]. Since $\lambda - 1$ is nonzero on ∂D Rouché's theorem gives exactly one approximate zero in D counted with multiplicity hence a simple zero. Applying the same argument to arbitrarily small disks about z_* gives $z_{*,N} \rightarrow z_*$. Uniform derivative convergence and continuity imply $\lambda'_N(z_{*,N}) \rightarrow \lambda'(z_*) \neq 0$. The denominators are therefore bounded away from zero for large N . Uniform convergence of the projections and source and readout maps gives convergence of the numerators at these moving roots proving the residue limit. This is the qualitative version of the cross-root comparison in Proposition 4.2 specifically Eqs. (4.24) and (4.31). $\square$

The norm hypothesis in Eq. (C.26) can be certified either by an explicit remainder estimate $\sup_{z \in K} \|\mathcal{B}(z) - \tilde{\mathcal{B}}_N(z)\| \leq r_N(K)$ with $r_N(K) \rightarrow 0$ or by the following a posteriori construction when all approximants act on the same Banach space.

A computable summable-increment criterion. Let $U \subset \mathbb{C}$ be open and let $\tilde{\mathcal{B}}_N : U \rightarrow \mathcal{K}(\mathcal{X}_{\mathfrak{C}})$ be analytic compact-operator families. Suppose that for every $K \Subset U$ there are certified bounds $\varepsilon_N(K) \geq 0$ and explicitly computable numbers $r_N(K) \geq 0$ with certified decay to zero such that

$$\begin{aligned} \sup_{z \in K} \|\tilde{\mathcal{B}}_{N+1}(z) - \tilde{\mathcal{B}}_N(z)\| &\leq \varepsilon_N(K), \\ \sum_{n=N}^{\infty} \varepsilon_n(K) &\leq r_N(K), \quad r_N(K) \longrightarrow 0. \end{aligned} \tag{C.28}$$

Then the sequence converges locally uniformly to an analytic compact family $\mathcal{B}_{\infty}$ with the computable tail estimate

$$\sup_{z \in K} \|\mathcal{B}_{\infty}(z) - \tilde{\mathcal{B}}_N(z)\| \leq \sum_{n=N}^{\infty} \varepsilon_n(K) \leq r_N(K). \tag{C.29}$$

If a consistency argument for the truncation identifies $\mathcal{B}_{\infty} = D_X \mathfrak{F}(X_*; z, \eta)$ this tail bounds the truncation error in Eq. (C.26). The same construction applies to the source and readout maps when their increments have corresponding effective tail bounds. Local uniform convergence of the analytic kernel families also yields convergence of their z derivatives on smaller compact subsets of U . The Banach-valued Weierstrass argument applies. Equation (C.28) makes the sequence Cauchy in $C(K, \mathcal{L}(\mathcal{X}_{\mathfrak{C}}))$. Summation gives Eq. (C.29) and norm closure preserves compactness. Local uniform convergence preserves analyticity while the Cauchy formula on a slightly larger compact set gives derivative convergence. The explicit majorants $r_N(K)$ make the tail effective. Summability by itself gives convergence but not a computable error bound.

For example certified computable constants $A(K) \geq 0$ and $0 < \theta(K) < 1$ with $\varepsilon_n(K) \leq A(K)\theta(K)^n$ give $r_N(K) = A(K)\theta(K)^N/[1 - \theta(K)]$. This geometric majorant supplies both the tail bound and its rate of decay.

If a threshold enters U the meromorphic decomposition Eq. (C.23) may fail. When operator-norm convergence is unavailable stable-looking eigenvalues of finite nonnormal matrices also require a spectral-pollution analysis. These cases call for an additional spectral argument beyond the isolated-pole theorem.

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Declarations

Conflict of interest. The author declares no competing interests.

Data and code availability. This work uses no empirical dataset. The analytic examples are specified by the equations and conventions in the manuscript and can be reproduced directly from them. Related finite-dimensional calculations use the Coarse-Graining Genesis Framework [16] available at <https://doi.org/10.5281/zenodo.22067005>.

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