Dynamics of Generated Spectra from Minimal Change: Certified Function Classes and Finite Windows

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Abstract: Nonlinear spectral evolution requires control of the frequencies present initially and those created by interaction. We study how spectral bounds can be retained with the construction of a state and reused in later operations and observations. A declared source and its native operations determine a generated class under fixed admissibility rules. Finite constructions pass to limits through a common stronger bound and convergence in a weaker norm. For a nonnegative self-adjoint generator with compact resolvent we prove a Galerkin error estimate on one local interval common to all cutoffs. The same spectral-weight gap controls the initial tail and the discarded nonlinear source. Stability and a common observation modulus then give the least cutoff certified by the estimate. Conditions on the interaction support determine which exponential reserves propagate without loss. Single-mode tests make the analytic threshold sharp in the Neumann and Stokes realizations. Compatible approximations of the source and time integrals preserve generated presentations along the local flow. Quadratic observations carry a complementary account of retained energy and transport defect together with signed injection and approximation increments. The cross terms are kept throughout composition and passage to limits. History-resolved Neumann readouts give response bounds without a multiplicity factor and identify the invariance condition for closed evolution of the observed values. The realizations have explicit constants. The Neumann source is derived within the paper under the stated composition and scaling assumptions with the endpoint and reflecting conditions.

Keywords: Nonlinear evolution equations; Spectral Galerkin methods; Uniform error bounds; Analytic regularity; Spectral truncation; A posteriori error estimates

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1 Introduction

Diffusion damps high frequencies while nonlinear multiplication can create them. A finite spectral window must therefore control both the tail of the initial state and the frequencies generated outside the window during the evolution. A bound obtained from heat flow is useful for this purpose only if its behaviour under the interaction is known. We study how such estimates can be retained through a construction and passed to a limit for reuse when the resulting state is evolved or observed.

The same state can be obtained by different constructions. Expanding a product of cosine modes can give a sharper bound than estimating its factored expression. The function and its norms remain unchanged but the bounds established by the two calculations may justify different subsequent estimates and spectral cutoffs.

A history is the finite derivation by which a state has been obtained from its declared inputs. Along that derivation each operation carries forward the norm bounds justified at the preceding steps. We call such a bound a certificate and use reserve for the part of the regularity that remains available to a later operation or approximation. The underlying norms are standard. The terminology records which estimates have actually been established by the construction. A presentation retains the state together with its derivation and established bounds. This distinction matters when an estimate must be reused without reconstructing it from the completed value.

Histories can also enter the state itself. The source associated with the spectral construction of arithmetic [1] assigns amplitudes to ordered expressions before their values are read. Two history states can have the same readout at one time and different readouts after the parent–child response has acted. For a bounded linear response invariance of the readout kernel characterizes when the reduced values have a closed evolution. This is a different issue from two derivations supplying different bounds for one function. Here compression can remove information needed to determine the subsequent observation.

A quadratic observation raises a related accounting question. A decrease in its retained value may come from contraction or projection. It may also come from interaction with a new contribution. We distinguish these effects by keeping a quadratic ledger. Compatible transport separates retained quantity from positive defect. Injection and approximation contribute signed increments containing their full cross terms. The one-step identities then telescope along the construction. A residual estimate controls the distance to the equation's solution while the ledger explains the changes in the specified quadratic quantity along that path.

A spectral reduction retains a value and leaves an unresolved component. Its subsequent use depends on how that component is controlled. The presentation keeps the construction and its tail bounds. A quadratic ledger records the effect of the reduction on the chosen observation. When eliminated variables can feed back into the retained equation their contribution must also be represented by an exact memory term or a certified approximation to it.

This is an intensional description of an otherwise familiar analytic argument. One keeps the limiting value together with the estimates that survived its construction. The

propagation of bounds through operations is related to abstract interpretation [2]. Here finite constructions pass to a limit through a common stronger bound and convergence in a weaker norm. For a fixed source and fixed operations the admissibility rules select values whose limiting presentations form the generatively regular class. We refer to this arrangement as a generated framework for spectral dynamics.

The word generated has two distinct uses here. A spectral datum may itself have a specified source construction as in the Neumann realization derived from the minimal-change assumptions. Once a spectral datum has been fixed a state may separately have a generated presentation built from admitted operations and certified limits. The analytic estimates then depend on the datum and its interaction bounds. The source construction records how that datum was obtained.

The source system and admissibility policy select a class within the ambient Hilbert or function space. An intrinsic history uses only the model's declared modes and coefficients. Its operators and interactions also belong to the source language as do its limiting procedures. A completed finite calculation already has provenance relative to its procedure. Membership in the class studied here also requires the quantitative bounds to survive composition and completion. These bounds determine how far the construction can be continued. Effectivity and finite certification require further data.

The approximation problem requires uniform control even for a bounded family of states. Each Hilbert-space state is the limit of its spectral truncations. An ambient norm bound need not give one cutoff for the family. A stronger spectral bound does. For a nonlinear flow it must hold along the exact solution and all its Galerkin approximations. We obtain a common interval and a uniform error estimate from stronger data bounds and common time profiles for the nonlinearity. One gap between spectral weights controls both the initial truncation and the nonlinear projection residual. A common observation modulus then gives an error bound for the reported quantities and selects the least spectral cutoff meeting that bound.

The linear datum is a nonnegative self-adjoint generator with compact resolvent. Its spectrum is supplemented by the operations to be used. Scalar correlations carry a Markov semigroup and pointwise multiplication. Constrained states carry a projection and a bilinear state interaction. The Neumann interval provides the scalar example. Its cosine product exhibits the creation of new frequencies directly. The periodic Stokes example includes the Leray projection and an interaction that loses one derivative. For a physical application the equation and its constraints supply the model data together with the observations. The admissible class then describes the states for which the available estimates justify the declared operations and measurements. It is therefore finer than a bare list of function-space memberships while its analytic content remains expressed in standard spectral norms.

These ingredients have different logical roles. The spectral datum and interaction tensor specify the analytic setting. The source language and admissibility policy specify which constructions belong to the class. A resource triple or a stronger–weaker resource pair determines the estimate being requested. An observation tolerance fixes the required accuracy. The compatible cost cone and critical reserve threshold are derived from the

interaction. Persistence of a presentation and the certified cutoff are further conclusions. The Neumann and Stokes systems test the same conclusions in two different realizations.

Interaction weights. The relation between submultiplicative weights and convolution algebras is classical [3, section 4]. We apply the coefficient argument with subadditivity imposed only on the support of the spectral interaction tensor. The exponential weight can then be transferred to the inputs. The remaining coefficient sum is controlled by the polynomial estimate for the absolute majorant. Necessary restrictions also depend on the size of the interaction coefficients. For weights exponential in a power of the frequency single-mode tests give a sharp threshold for preserving the same positive reserve at both inputs and the output. These tests require comparable input frequencies together with asymptotic saturation and quantitative non-degeneracy. The Neumann and Stokes examples verify these conditions. A Gaussian reserve supplied by heat flow incurs a loss under products while an analytic reserve is preserved by the frequency triangle law. The threshold therefore identifies which estimates can be reused at successive nonlinear steps with the same exponential parameter.

Limits and local evolution. We use the analytic-semigroup method of Pazy and Henry [4, 5] in the weighted spaces. Foias and Temam established Gevrey regularity for periodic Navier–Stokes solutions [6]. Bae and Biswas treat analytic nonlinearities under a derivative-loss condition relative to fractional dissipation [7, section 2.2]. Uniform control of spectral Galerkin flows is also established in the semilinear approximation literature. For a normal generator and smooth nonlinearities on a Hilbert scale Wulff and Evans obtain a common local interval together with regularity bounds uniform in the cutoff and projection-error estimates [8, Lemmas 4.1–4.2]. We apply semigroup smoothing to the derivative loss in the interaction estimate with constants uniform over compatible approximations of the data and nonlinearity. The weaker difference estimate controls convergence of the resulting solutions while the common stronger bound remains available for estimating their spectral tails. Retaining the approximating histories also gives a presentation of the limit when their semigroup and time-integral constructions are admitted.

Time-dependent coefficients enter through growth and difference bounds for the complete nonlinear expression. Their integrability must control the derivative-recovery singularity in the Duhamel integral. We check the profiles after composition since products of integrable coefficient functions need not remain integrable. Heat-prepared Neumann data and a quadratic equation verify these conditions and the compatibility of the approximations through the finite time sums representing the Picard integrals. Finite Fourier data give a second verification for the constrained Stokes equation.

Residuals and observations. Residual stability is central to reduced-basis certification [9, Chapter 4]. Adjoint methods use residuals to estimate a selected quantity of interest [10]. For a prescribed observation tolerance the residual estimate must hold along the entire Galerkin family on one time interval. In the present setting the full-space residual is the negative of the discarded nonlinear and forcing terms. Lowering both source orders by the derivative loss leaves the ratio of the stronger and weaker weights unchanged. One ratio

therefore bounds the nonlinear residual and the initial projection error. Local stability and a common observation modulus give an error bound uniform over the cutoffs.

Adaptive spectral methods also use regularity to select modes. Canuto and collaborators prove exponential convergence and linear computational complexity for a dynamically marked Galerkin method for elliptic problems on Gevrey approximation classes under an optimal-linear-solver assumption [11]. Our threshold is the least spectral cutoff satisfying a specified observation bound from the nonlinear analysis. Both the error estimate and the set over which it is minimized are explicit. The companion computation paper [12] treats the construction and independent verification of finite records for an encoded Neumann equation family. Its bounds include the bit cost. It also develops information and arithmetic readouts from certified states.

Moving a spectral window changes the error budget in two ways. The discarded terms contribute to the residual while a hard switch can alter the reported state and hence the initial error of the next interval. For smooth selectors and memory truncation the residual and any error in the reported state enter the total error budget together with the bounds for the restarted data.

The Neumann source has a self-contained derivation from a minimal orthogonal representation and independent composition with square-root scaling. Reflecting closure acts between two correlation-length readouts. Covariance fixes the heat-time normalization. This source is shared with the arithmetic construction in [1]. The Stokes datum is supplied independently. On the history source a recursive conditional law distributes each root's Gaussian mass over its expressions. The normalization bounds both weighted state mass and the parent–child response including the error from omitted parent relations. Root frequency and parent–child variation measure different properties. The response has arbitrarily small positive Rayleigh quotients even at large root degree. An initially invisible history difference makes the resulting obstruction to a closed readout equation explicit.

For quadratic observations the same distinction requires keeping mixed terms between retained and unresolved components. Exact ledgers pass to limits under response continuity. Closed unbounded forms give lower-semicontinuous inequalities. Together with residual and readout bounds these accounts retain the information needed to certify a later calculation. Turning them into finite acceptance tests requires effective constructions together with constants and convergence moduli as in the companion paper. Analytic mass control and the cost of storing and updating the coordinates remain separate estimates.

2 Spectral data and a running example

On the interval $[0, \pi]$ heat flow and multiplication can be compared on the same Neumann basis. The normalized modes are

$$\phi_0 = \pi^{-1/2}, \qquad \phi_n = \sqrt{2/\pi}\cos(nx), \qquad A\phi_n = n^2\phi_n.$$

After time t heat flow multiplies the nth coefficient by e^{-tn^2} while $2\cos(mx)\cos(nx) = \cos((m+n)x) + \cos((m-n)x)$. Thus heat flow changes the amplitude of a nonconstant mode whereas squaring that mode introduces twice its frequency. A bound on the input

coefficients must account for these new output frequencies. We express this requirement through a spectral weight and the coefficients of the product.

2.1 Generators, modes, and interactions

Definition 2.1 (Spectral datum)**.** Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let A be a nonnegative self-adjoint operator with compact resolvent. Put

$$B = A^{1/2}, \qquad P_K = \mathbf{1}_{[0,K]}(B), \qquad \mathcal{D}_{\mathrm{fin}} = \bigcup_{K<\infty} \operatorname{Ran} P_K. \tag{2.1}$$

The triple $\mathfrak{S} = (\mathcal{H}, A, \mathcal{D}_{\mathrm{fin}})$ is the spectral datum. Inner products are linear in the first argument.

An orthonormal eigenbasis satisfies

$$A\phi_j = \lambda_j \phi_j, \qquad \omega_j = \sqrt{\lambda_j}, \qquad P_K f = \sum_{\omega_j \le K} \langle f, \phi_j \rangle \phi_j. \tag{2.2}$$

Eigenvalues are repeated according to multiplicity. In infinite dimension $\lambda_j \to \infty$. In finite dimension the sequence terminates. K always denotes a frequency cutoff. We call the generator hypothesis C0. Finite spectral sums are dense in $\mathcal{H}$ and in the graph norm of every power of $I + A$ by convergence of the omitted coefficient series.

The additional structure depends on the unknown. Scalar functions carry pointwise multiplication while constrained states require a bilinear operation taking values in the constraint space. In the scalar case $\mathcal{H} = L^2(\mathcal{X}, \mu)$ with $\mathcal{X}$ compact and μ finite of full support. The operators e^{-tA} form a conservative symmetric Markov semigroup and m is pointwise multiplication on its declared domain. In particular

$$e^{-tA}\mathbf{1} = \mathbf{1}, \qquad 0 \le f \le 1 \implies 0 \le e^{-tA} f \le 1. \tag{2.3}$$

For states a bounded projection Π on an ambient Hilbert space defines the closed space $\mathcal{V} = \operatorname{Ran} \Pi$. The operator A acts on $\mathcal{V}$ and a bilinear map $\mathcal{B}$ on the finite core takes values in $\mathcal{V}$. The semigroup e^{-tA} is analytic and contractive on $\mathcal{V}$. The periodic Stokes operator is of this type.

For either operation $\mathcal{M} = \mathsf{m}$ or $\mathcal{B}$ fix the appropriate basis and set

$$\Gamma^{\mathcal{M}}_{ijk} = \langle \mathcal{M}(\phi_i, \phi_j), \phi_k \rangle. \tag{2.4}$$

Condition C2 is a polynomially weighted estimate on the absolute interaction coefficients with specified input orders and derivative loss. It bounds the output coefficient norm by the product of the two input norms. Conclusions for power weights also use the support condition

$$\Gamma^{\mathcal{M}}_{ijk} \neq 0 \implies \omega_k \le \omega_i + \omega_j. \tag{2.5}$$

This condition and the coefficient estimate are verified for the operation under consideration whether scalar or state-valued. Condition C3 means that the projection preserves the declared cores and the projected interaction has the stated C2 estimate and every ambient

functional-calculus operation used with Π either commutes with it or has a controlled commutator in the norms being used. Calculus performed directly on $\mathcal{V}$ needs no ambient commutator condition. We denote a chosen scalar or state datum together with its declared operations by $\mathfrak{G}$.

The spectral weight measures the size of high-frequency components. An evolution on the same state space can depend on a different operator so the weight and the response must be specified separately. For example let a state space have a basis indexed by finite histories e with positive root degree $n(e)$ and only finitely many histories at each root. The root operator $\widetilde{A}|e\rangle = n(e)^2|e\rangle$ satisfies C0 because below any finite energy there are finitely many basis vectors and its diagonal resolvent entries tend to zero. A bounded operator comparing parent and child histories can act on this same space with bounds measured using $\widetilde{A}$. Its matrix and spectral properties require their own estimates. Compact resolvent follows here from finiteness at each root. The growth of the fibre cardinalities still determines the number of coordinates below a cutoff. The pointwise and counting hypotheses used for a function realization are therefore verified separately from C0.

The datum is model-native only relative to a stated realization. For the Neumann problem the generator and multiplication come from the reflected interval together with its constant mode. For the Stokes problem the divergence-free space and Leray projection come from the constrained fluid equation together with the transport interaction. The designation records provenance. The estimates below rest on the explicit conditions imposed on these objects.

2.2 Weights and the estimates they record

We use an exponential weight to measure the cost of creating a frequency and a polynomial weight to measure the loss caused by differentiation. This separates the two effects of an operation on the available regularity.

Let $\Sigma_B = \mathrm{Spec}(B)$. A frequency cost is any finite-valued function

$$\Psi : \{0\} \cup \Sigma_B \longrightarrow [0,\infty), \qquad \Psi(0) = 0; \qquad \mathfrak{W}_0(\Sigma_B) \text{ denotes their set.} \tag{2.6}$$

For $\alpha \geq 0$ and $s \in \mathbb{R}$ define

$$\|f\|^2_{\Psi,\alpha,s;A} = \sum_j e^{2\alpha\Psi(\omega_j)}(1+\omega_j^2)^s |f_j|^2, \qquad \mathscr{G}^{\Psi}_{\alpha,s}(A) = \{f : \|f\|_{\Psi,\alpha,s;A} < \infty\}. \tag{2.7}$$

Finite spectral truncations converge in these Hilbert coefficient spaces. The power costs used throughout the paper are

$$\Psi_\theta(\omega) = \omega^{2\theta}, \quad \mathscr{G}^{(\theta)}_{\alpha,s} = \mathscr{G}^{\Psi_\theta}_{\alpha,s}, \quad \|f\|^2_{\theta,\alpha,s;A} = \sum_j e^{2\alpha\lambda_j^\theta}(1+\lambda_j)^s |f_j|^2, \quad 0 < \theta \leq 1. \tag{2.8}$$

At $\alpha = 0$ the notation is $\mathcal{H}^s_A$. For $s \geq 0$ this is $\mathrm{Dom}(I+A)^{s/2}$ and for $s < 0$ its usual negative-order completion. For power costs positive α or nonnegative s gives an embedding into $\mathcal{H}$. The analytic and Gaussian layers correspond respectively to $\theta = 1/2$ and $\theta = 1$. The parameter α measures the exponential reserve available for subsequent operations.

A certificate is an upper bound for one of these norms. A finite expansion can be bounded by its weighted coefficient sum. For an infinite expansion that method also requires control of the omitted sum. For a fixed value f the region of all valid bounds is

$$\mathsf{Cert}_{\mathfrak{S}}(f) = \{(\Psi, \alpha, s, M) : \|f\|_{\Psi,\alpha,s;A} \le M\}. \tag{2.9}$$

For a power cost the first entry is abbreviated to θ. Reducing α or s or increasing M preserves a valid certificate. The full region is determined by f. A construction may establish only some of its bounds. Retaining the construction records which bounds have been derived.

In particular heat flow turns an $\mathcal{H}^s_A$ bound into Gaussian reserve without increasing the norm bound

$$\|e^{-tA} f\|_{1,\alpha+t,s;A} = \|f\|_{1,\alpha,s;A}. \tag{2.10}$$

2.3 Spectral geometry and pointwise bounds

To pass from a spectral norm to pointwise values we need bounds on the eigenfunctions. We state the assumptions separately for function values and derivatives together with Hölder differences so that each conclusion uses only the relevant part of the spectral geometry.

Suppose the datum has an isometric realization in $L^2(\mathcal{X}, \mu; F)$ where $\mathcal{X}$ is compact metric and μ is finite of full support. Let F be a fixed finite-dimensional Hilbert space. Every finite mode has a specified continuous representative. In the vector case we fix this common trivialization so differences at distinct points lie in the same space. Put $\mathcal{E}_0(K; x) = \sum_{\omega_j \le K} |\phi_j(x)|^2$. Condition C1 consists of the following estimates.

C1a. For some $d > 0$ and $C_0 < \infty$

$$\sup_x \mathcal{E}_0(K; x) \le C_0 (1 + K)^d. \tag{2.11}$$

C1b. The finite projections have continuous kernels and uniformly convergent spectral expansions have their continuous representatives. A Feller realization of the semigroup is one standard sufficient setting.

C1c. Each derivative used is a specified closed operator $D^\ell : \mathrm{Dom}\, D^\ell \subset \mathcal{H} \to L^2(\mathcal{X}, \mu; F_\ell)$ with $\mathcal{D}_{\mathrm{fin}} \subset \mathrm{Dom}\, D^\ell$ where F_ℓ is fixed and

$$\sup_x \sum_{\omega_j \le K} |D^\ell \phi_j(x)|^2 \le C_\ell (1 + K)^{d+2\ell}. \tag{2.12}$$

C1d. For every pair (ℓ, γ) used with $0 < \gamma \le 1$

$$\sup_{x \ne y} \frac{\sum_{\omega_j \le K} |D^\ell \phi_j(x) - D^\ell \phi_j(y)|^2}{d_{\mathcal{X}}(x, y)^{2\gamma}} \le C_{\ell,\gamma} (1 + K)^{d+2\ell+2\gamma}. \tag{2.13}$$

2.4 Local spectral counting

Heat-kernel bounds provide one way to verify C1a. For a scalar realization with diagonal expansion $p_t(x,x) = \sum_j e^{-t\lambda_j}|\phi_j(x)|^2$ suppose $p_t(x,x) \le Ct^{-d/2}$ for $0 < t \le 1$ uniformly in x. Taking $t = (1+K)^{-2}$ gives

$$\mathcal{E}_0(K;x) \le e\, p_{(1+K)^{-2}}(x,x) \le eC(1+K)^d. \tag{2.14}$$

Gaussian upper bounds for elliptic heat kernels are treated in [13, Chapter 3] and [14, Chapter 6]. The remaining parts of C1 require continuity together with derivative and difference estimates for the chosen realization.

The exponent here uses the frequency convention of Definition 2.1. Thus $\mathcal{E}_0(K;x) \lesssim (1+K)^d$ corresponds to exponent $d/2$ when the cutoff is expressed in eigenvalues $\Lambda = K^2$.

Condition C1a also gives the low-frequency Bernstein estimate

$$\|P_K f\|_{L^\infty(\mathcal{X})} \le C_0^{1/2}(1+K)^{d/2}\,\|f\|_2\,. \tag{2.15}$$

At each x Cauchy–Schwarz gives

$$|P_K f(x)| \le \left(\sum_{\omega_j \le K} |\phi_j(x)|^2\right)^{1/2} \left(\sum_{\omega_j \le K} |\,\langle f, \phi_j\rangle\,|^2\right)^{1/2}. \tag{2.16}$$

C1a and Parseval therefore yield (2.15).

Theorem 2.2 (Spectral regularity and tails)**.** Assume C1a and let $r > d/2$. Then

$$\|(I - P_K)f\|_{L^\infty(\mathcal{X})} \le C_{d,r}(1+K)^{d/2-r}\,\|f\|_{\mathcal{H}^r_A}\,. \tag{2.17}$$

If C1b also holds then every $f \in \mathcal{H}^r_A$ has a continuous representative and $\mathcal{H}^r_A \hookrightarrow C(\mathcal{X})$. If C1c is available through order m then for $r > d/2 + m$

$$\|D^m(I - P_K)f\|_{L^\infty(\mathcal{X})} \le C_{d,r,m}(1+K)^{d/2+m-r}\,\|f\|_{\mathcal{H}^r_A}\,. \tag{2.18}$$

Under C1c through order m and C1d at (m,γ) with $0 < \gamma < 1$ one also has

$$[D^m(I - P_K)f]_{C^{0,\gamma}(\mathcal{X})} \le C_{d,r,m,\gamma}(1+K)^{d/2+m+\gamma-r}\,\|f\|_{\mathcal{H}^r_A}\,, \tag{2.19}$$

whenever $r > d/2 + m + \gamma$. Hence $D^m f$ has a γ-Hölder representative. When the declared pointwise realization identifies $D^0, \dots, D^m$ with the standard derivative hierarchy this is the usual $C^{m,\gamma}$ conclusion.

Proof. Dyadic shells let us compare the growth in the local mode count with the decay supplied by the Sobolev weight. Put $R = 1 + K$ and decompose the tail into the shells $2^n R < 1 + B \le 2^{n+1}R$ for $n \ge 0$. These shells cover the support of $I - P_K$ including when $K = 0$. On the nth shell Cauchy–Schwarz and C1a together with the equivalence of $1 + \omega$ with $(1+\omega^2)^{1/2}$ give the bound

$$C_0^{1/2} 2^{(d+r)/2}(2^n R)^{d/2-r}\,\|f\|_{\mathcal{H}^r_A}\,. \tag{2.20}$$

The geometric series converges because $r > d/2$ and proves (2.17). One may take

$$C_{d,r} = \frac{C_0^{1/2} 2^{(d+r)/2}}{1 - 2^{d/2-r}}. \tag{2.21}$$

Under C1b every partial sum is continuous and the tail tends to zero uniformly. This proves the continuous embedding. Replacing C1a by the derivative estimate C1c gives (2.18) and uniform convergence of the derivative expansion. Its limit also exists in $L^2(\mathcal{X}, \mu; F_m)$ because $\mu(\mathcal{X}) < \infty$. Since $P_K f \to f$ in $\mathcal{H}$ closedness of D^m identifies this limit with $D^m f$ and places f in its domain. Applying (2.13) to the difference of two evaluations gives on the nth shell the additional factor $d_{\mathcal{X}}(x, y)^\gamma (2^n R)^\gamma$. Summability follows from $r > d/2 + m + \gamma$ and gives (2.19). The constants in the last two estimates are obtained from the displayed formula by replacing (d, C_0) with $(d + 2m, C_m)$ and $(d + 2m + 2\gamma, C_{m,\gamma})$ respectively. For the full Hölder representative apply the same argument to the entire expansion. Its finite low-frequency part satisfies C1d and its tails converge uniformly with the stated Hölder bound. □

These estimates turn weighted norm bounds into observation bounds. For products the remaining question is which weights survive the interaction of two input frequencies.

3 Interaction tensors and critical nonlinear reserve

We estimate a bilinear operation in an exponential norm by transferring the output weight to the two inputs. The support of the interaction tensor determines which frequency triples occur. An estimate on its absolute coefficients then controls the sum over those triples.

The weighted spaces themselves are standard Sobolev–Gevrey spaces. The question addressed here is which exponential costs are compatible with the nonzero coefficients of the given interaction and whether the size of those coefficients makes the resulting restriction necessary. Thus the cost condition is imposed on the actual interaction support rather than on all possible frequency triples. The admissible exponential cost is consequently tested against the interaction native to the model rather than selected by a preferred name for the regularity scale.

3.1 Polynomial interaction bounds

Let $\mathcal{M}$ denote either scalar multiplication m or a state bilinear map $\mathcal{B}$. In a corresponding orthonormal eigenbasis write

$$\widehat{\mathcal{M}(f, g)}_k = \sum_{i,j} \Gamma^{\mathcal{M}}_{ijk} \widehat{f}_i \widehat{g}_j. \tag{3.1}$$

We begin with inputs in the finite core. Since the exponential weight is compared at each nonzero interaction separately the summation estimate must remain valid after signs and phases have been removed.

For finite coefficient families define the absolute majorant interaction by

$$\widehat{\mathcal{M}^{\#}(f, g)}_k := \sum_{i,j} |\Gamma^{\mathcal{M}}_{ijk}| |\widehat{f}_i| \, |\widehat{g}_j|. \tag{3.2}$$

Definition 3.1 (Polynomially tame interaction with loss)**.** The interaction is polynomially tame from order s with loss $q \geq 0$ if there is $C_{s,q} < \infty$ such that

$$\left\| \mathcal{M}^{\#}(f,g) \right\|_{\mathcal{H}_A^{s-q}} \leq C_{s,q} \, \|f\|_{\mathcal{H}_A^s} \, \|g\|_{\mathcal{H}_A^s} \,, \tag{3.3}$$

for finite-core inputs. Asymmetric variants with different input orders are allowed when they are stated explicitly. Scalar multiplication usually has $q = 0$ above the algebra threshold. A first-order transport interaction naturally has $q = 1$. This is the form of condition C2 used below.

At other input orders C2 means the corresponding explicitly stated asymmetric version of (3.3). In Fourier and cosine realizations the absolute-majorant bound follows from convolution and the weighted ℓ^2 algebra estimate for $s > d/2$. Other geometries enter through estimates for this same absolute majorant.

3.2 Spectral costs compatible with the interaction

For the weights in (2.6) and (2.7) we seek conditions under which this transfer preserves the exponential reserve. For $\xi \in \Sigma_B$ write $E_\xi = \mathbf{1}_{\{\xi\}}(B)$ and define the basis-independent interaction support

$$\mathcal{I}_{\mathcal{M}} := \{(\xi,\eta,\zeta) \in \Sigma_B^3 : E_\zeta \mathcal{M}(E_\xi \mathcal{D}_{\mathrm{fin}}, E_\eta \mathcal{D}_{\mathrm{fin}}) \neq \{0\}\}. \tag{3.4}$$

In any eigenbasis this is exactly the set of frequency triples represented by at least one nonzero coefficient $\Gamma^{\mathcal{M}}_{ijk}$.

Definition 3.2 (Interaction spectral-cost cone)**.** The spectral-cost cone of $\mathcal{M}$ is

$$\mathfrak{W}_{\mathcal{M}} := \{\Psi \in \mathfrak{W}_0(\Sigma_B) : \quad \Psi(\zeta) \leq \Psi(\xi) + \Psi(\eta) \text{ for every } (\xi,\eta,\zeta) \in \mathcal{I}_{\mathcal{M}}\} \,. \tag{3.5}$$

For a finite composition $\mathcal{T}$ of operations its cone is the intersection of the cones of its bilinear operations with frequency labels identified in the corresponding intermediate spaces.

The inequalities are linear in the cost so the cone contains zero and is closed under nonnegative linear combinations and finite-valued pointwise limits. For a cost satisfying these interaction inequalities the initial data determine how much reserve α is available.

The square of a Neumann mode has a nonzero component at frequency $2n$. The linear frequency cost satisfies $2n = n + n$ whereas the quadratic cost gives $(2n)^2 > n^2 + n^2$ for $n > 0$. The analytic weight can therefore be transferred to the two inputs with the same reserve. The Gaussian weight requires a loss of reserve at the output. Summing these coefficientwise comparisons using the polynomial bound gives an estimate for the whole product.

Theorem 3.3 (No-loss reserve on the spectral-cost cone)**.** If $\mathcal{M}$ is polynomially tame from order s with loss q and $\Psi \in \mathfrak{W}_{\mathcal{M}}$ then for every $\alpha \geq 0$ the operator $\mathcal{M}$ has a unique continuous bilinear extension from finite-core inputs to the displayed weighted spaces satisfying

$$\|\mathcal{M}(f,g)\|_{\Psi,\alpha,s-q;A} \leq C_{s,q} \, \|f\|_{\Psi,\alpha,s;A} \, \|g\|_{\Psi,\alpha,s;A} \,. \tag{3.6}$$

The assertion applies separately to scalar and state interaction tensors. It also applies to a finite composition whose cost lies in the intersection cone provided that each remaining operation satisfies an estimate preserving the same cost and reserve and that the domain orders are compatible at successive operations.

Proof. We first transfer the exponential factors coefficientwise and then apply the assumed polynomial estimate. On a nonzero interaction the defining inequality gives

$$e^{\alpha\Psi(\omega_k)} \le e^{\alpha\Psi(\omega_i)} e^{\alpha\Psi(\omega_j)}. \tag{3.7}$$

Insert these factors in the absolute-majorant expansion (3.2) and apply (3.3) to the weighted input sequences. This proves the estimate on the finite core. If "in" and "out" denote the input and output norms in (3.6) bilinearity also gives

$$\begin{aligned}\big\|\mathcal{M}(f,g) - \mathcal{M}(f',g')\big\|_{\mathrm{out}} \le C_{s,q}\big(&\big\|f - f'\big\|_{\mathrm{in}} \|g\|_{\mathrm{in}} \\ &+ \|f'\|_{\mathrm{in}} \big\|g - g'\big\|_{\mathrm{in}}\big).\end{aligned} \tag{3.8}$$

The difference estimate gives the unique continuous extension from the dense finite core with both bounds unchanged. Successive application with the estimates for the other operations proves the last assertion. □

3.3 Power costs and the critical analytic layer

Suppose the interaction support satisfies

$$\Gamma^{\mathcal{M}}_{ijk} \neq 0 \quad \Longrightarrow \quad \omega_k \le \omega_i + \omega_j. \tag{3.9}$$

Put $\Phi_\beta(\omega) = \omega^\beta$ so that $\Phi_{2\theta} = \Psi_\theta$ in the notation of section 2. For $0 < \beta \le 1$

$$(x+y)^\beta \le x^\beta + y^\beta, \tag{3.10}$$

so $\Phi_\beta \in \mathfrak{W}_{\mathcal{M}}$. For $\beta > 1$

$$(x+y)^\beta \le 2^{\beta-1}(x^\beta + y^\beta). \tag{3.11}$$

Corollary 3.4 (Critical nonlinear reserve)**.** Let $\mathcal{M}$ be polynomially tame from order s with loss q and suppose (3.9). Then the following estimates hold.

(i) if $0 < \theta \le \frac{1}{2}$ and $\alpha \ge 0$ then

$$\|\mathcal{M}(f,g)\|_{\theta,\alpha,s-q;A} \le C_{s,q} \|f\|_{\theta,\alpha,s;A} \|g\|_{\theta,\alpha,s;A}\,; \tag{3.12}$$

(ii) if $\frac{1}{2} < \theta \le 1$ then the same estimate holds with output reserve α_{out} whenever

$$\alpha_{\mathrm{out}} \le 2^{1-2\theta}\alpha_{\mathrm{in}}, \tag{3.13}$$

and both inputs have reserve α_{in}.

Thus a binary product preserves the analytic reserve $e^{\sigma\sqrt{A}}$ whereas a Gaussian reserve $e^{\alpha A}$ is in general reduced to $e^{\alpha A/2}$.

Proof. Set $\beta = 2\theta$. Part (i) is Theorem 3.3. For $\beta > 1$ use (3.11) in the same proof. The weight transfers provided $2^{\beta-1}\alpha_{\text{out}} \le \alpha_{\text{in}}$. □

Sharpness depends on the size of the interaction coefficients as well as their support. Rapid coefficient decay could compensate for a superadditive cost even with a fixed polynomial loss q. We therefore test comparable input frequencies whose output nearly saturates the triangle law and whose coefficients decay too slowly to offset the exponential defect.

Fix $\beta > 1$. We call a sequence of nonzero interaction coefficients (i_n, j_n, k_n) and β-quantitatively non-degenerate when

$$\omega_{k_n} \longrightarrow \infty, \qquad \frac{\omega_{k_n}}{\omega_{i_n} + \omega_{j_n}} \longrightarrow 1, \tag{3.14}$$

and

$$0 < c \le \frac{\omega_{i_n}}{\omega_{j_n}} \le C < \infty, \qquad \frac{-\log\left|\Gamma^{\mathcal{M}}_{i_n j_n k_n}\right|}{\omega_{k_n}^{\beta}} \longrightarrow 0, \tag{3.15}$$

Under polynomial tameness a polynomial lower bound for these coefficients implies the last condition. The single-mode tame estimate also supplies a polynomial upper bound and all three frequencies are comparable along the sequence.

Theorem 3.5 (Single-mode obstruction and sharpness)**.** Fix $\alpha > 0$ and finite orders $s_{\text{in}}, s_{\text{out}}$. If

$$\|\mathcal{M}(f,g)\|_{\Psi,\alpha,s_{\text{out}};A} \le C\,\|f\|_{\Psi,\alpha,s_{\text{in}};A}\,\|g\|_{\Psi,\alpha,s_{\text{in}};A} \tag{3.16}$$

holds on the finite core then every nonzero interaction coefficient obeys

$$\alpha\big[\Psi(\omega_k) - \Psi(\omega_i) - \Psi(\omega_j)\big] \le \log\frac{C}{|\Gamma^{\mathcal{M}}_{ijk}|} + \frac{s_{\text{in}}}{2}\log(1+\omega_i^2) + \frac{s_{\text{in}}}{2}\log(1+\omega_j^2) - \frac{s_{\text{out}}}{2}\log(1+\omega_k^2). \tag{3.17}$$

Consequently a positive cost defect which dominates the logarithmic coefficient decay and every fixed polynomial weight rules out (3.16). In particular let $\frac{1}{2} < \theta \le 1$ and set $\beta = 2\theta$. Suppose that $\mathcal{M}$ has a β-quantitatively non-degenerate sequence. Then no constant C can satisfy

$$\|\mathcal{M}(f,g)\|_{\theta,\alpha,s_{\text{out}};A} \le C\,\|f\|_{\theta,\alpha,s_{\text{in}};A}\,\|g\|_{\theta,\alpha,s_{\text{in}};A}\,, \tag{3.18}$$

for all finite-core f, g and fixed finite polynomial orders.

Proof. For the necessary inequality take f and g to be the ith and jth modes scaled to have unit input norm. Retaining only the kth coefficient in the output norm gives (3.17) after taking logarithms.

Normalizing by the sum of the input frequencies lets us use strict convexity on a fixed compact interval. Write $a_n = \omega_{i_n}/(\omega_{i_n} + \omega_{j_n})$. Comparability places a_n in the compact interval $[c/(1+c), C/(1+C)] \subset (0,1)$. Strict convexity gives a uniform $\delta > 0$ such that $a_n^{\beta} + (1-a_n)^{\beta} \le 1 - \delta$. Frequency saturation then yields

$$\omega_{k_n}^{\beta} - \omega_{i_n}^{\beta} - \omega_{j_n}^{\beta} \ge c_{\beta,c,C}\,\omega_{k_n}^{\beta}, \tag{3.19}$$

eventually. Quantitative non-degeneracy makes $-\log|\Gamma^{\mathcal{M}}_{i_n j_n k_n}| = o(\omega^{\beta}_{k_n})$ whereas all polynomial weights contribute only logarithmically. This contradicts (3.17) with $\Psi(\omega) = \omega^{\beta}$. □

Together with the frequency support law and polynomial tameness quantitative non-degeneracy at each excluded exponent makes $\theta = 1/2$ the no-loss boundary within the power family. Here no loss means that the same positive reserve is used at both inputs and the output uniformly over finite-core inputs at fixed polynomial orders. For general costs the cone gives sufficient inequalities on the interaction support while (3.17) gives a necessary restriction involving the coefficient sizes. Under the hypotheses of Corollary 3.4 the reduced output reserve in (3.13) still gives a bilinear bound. For an evolution the available reserve also depends on semigroup smoothing and on the initial data and forcing.

The threshold concerns the stated bilinear interaction and its nonzero coefficients. Another native operation can have a different resource transformer. A linear parent–child response with Gaussian source weights is estimated from its weighted matrix entries. The bilinear frequency-triangle test concerns a different operation. Source bounds and multiplication estimates therefore remain distinct from the bounds for other admitted responses.

3.4 Transfer from Gaussian smoothing to analytic reserve

The Gaussian weight supplied by heat flow controls every fixed analytic weight. Completing the square gives the cost of this transfer and hence an analytic bound for successive bilinear estimates.

More precisely if $t > 0$ and $\sigma \geq 0$ and $g \in \mathcal{H}^{s}_{A}$ then

$$\left\| e^{-tA} g \right\|_{1/2,\sigma,s;A} \leq e^{\sigma^2/(4t)} \, \|g\|_{\mathcal{H}^{s}_{A}} \, . \tag{3.20}$$

Thus every positive heat time supplies an analytic certificate to which Corollary 3.4 applies. The multiplier on the left is $e^{\sigma\sqrt{\lambda} - t\lambda}$. Completing the square in its exponent gives the supremum over $\lambda \geq 0$ as $e^{\sigma^2/(4t)}$.

When the spectrum is unbounded the inverse heat operator is unbounded. An analytic certificate for a smoothed state therefore gives no uniform analytic bound for the unsmoothed state. On a fixed finite spectral window the inverse exists as a bounded operator with a bound depending on that window.

4 Certified constructions and limits

Product and heat-flow estimates together with differentiation bounds give norm bounds at each stage of a finite expression. Once an expression is used as input to another operation those bounds specify its available regularity and the loss the next step can afford. Retaining them with the construction avoids repeating the coefficient analysis. For an infinite continuation the approximants must carry a common stronger bound as well as a convergent sequence of values.

4.1 Histories and propagated bounds

Fix the source modes and a coefficient algebra together with a finite-description language for operator parameters. A history is a finite expression formed from these leaves by sums and scalar multiplication together with admitted operators and bilinear interactions. Bochner integrals are admitted with an integrable bound. Each operation must act on its declared domain.

Write $\mathsf{Val}(H)$ for the evaluated history. At each node a proved operation estimate maps available input bounds to output bounds. This is its certificate transformer. It is defined on the input resources for which the operation estimate holds and has a nonempty output there. The region $\mathsf{Res}(H)$ consists of the bounds obtained by this recursion together with the weakenings

$$(\Psi, \alpha, s, M) \in \mathsf{Res}(H) \implies (\Psi, \beta, r, M') \in \mathsf{Res}(H), \quad 0 \le \beta \le \alpha, \quad r \le s, \quad M' \ge M. \tag{4.1}$$

A presentation retains the construction and its value together with the region of bounds obtained in this way. Different expressions may have the same value and yield different bounds.

A presentation is therefore not another name for the value in a weighted space. The retained bounds determine which later operations are already justified and which finite cutoffs can be certified without proving a new estimate from the value alone. Equality of values does not erase this difference in their available analytic information.

There is a second use of histories when the state itself has coordinates indexed by constructions. In that setting a vector assigns amplitudes to histories and a value readout combines amplitudes with the same evaluated root. Distinct vectors can have the same readout while an operation on their history coordinates distinguishes them. Such an operation acts on the history state space. Whether it also defines an operation on the readout values is a separate compatibility question. This differs from two derivations of one function supplying different proved bounds. The function's full certificate region remains fixed in the latter case.

We call a history intrinsic when all of its leaves and operators and their parameters belong to this source language. A relative history may also use predeclared external inputs with their provenance retained. A spectral multiplier is intrinsic only when its construction is admitted by the source language. Its norm bound records a separate analytic property.

Model-native constructions. Relative to a specified realization a construction is model-native when its leaves and operators belong to the source system declared for that model together with its interactions and parameters and limiting procedures. Thus a model-native history is intrinsic in the terminology above. The term records provenance within the realization. The declared source language and closure rules determine the class of realizations in which uniqueness is stated.

For time integration the path and interval and the time dependence must have finite descriptions in the chosen language and a common integrable certificate bound. We refer to

this requirement as rule (E4). Appendix E specifies the source data and operator domains and finite-tree rules underlying these histories.

This recursion is sound. If every leaf has its declared bound and every node uses a verified transformer on admissible inputs then induction on the finite tree gives

$$\varnothing \neq \mathsf{Res}(H) \subseteq \mathsf{Cert}_{\mathfrak{S}}(\mathsf{Val}(H)). \tag{4.2}$$

Weakening decreases the spectral weight or increases the bound so it preserves the same inclusion.

For example the analytic heat and differentiation estimates are

$$\|e^{-tA} f\|_{1/2,\alpha+\delta,s;A} \leq e^{\delta^2/(4t)} \|f\|_{1/2,\alpha,s;A}, \quad t > 0,\ \delta \geq 0, \tag{4.3}$$

$$\|A^{q/2} f\|_{\theta,\beta,s;A} \leq C_{q,\theta} (\alpha - \beta)^{-q/(2\theta)} \|f\|_{\theta,\alpha,s;A}, \quad \alpha > \beta \geq 0. \tag{4.4}$$

The heat estimate follows from the scalar bound $\delta\omega - t\omega^2 \leq \delta^2/(4t)$ valid for $\omega \geq 0$.

The certificate loss in (4.4) is determined by a scalar maximum. If $a > 0$ and $q \geq 0$ and $0 < \theta \leq 1$ then

$$\sup_{x \geq 0} x^{q/2} e^{-ax^\theta} = \Big(\frac{q}{2e\theta a}\Big)^{q/(2\theta)}, \tag{4.5}$$

with the value one when $q = 0$. For $q > 0$ differentiation of $(q/2)\log x - ax^\theta$ places the maximum at $x = (q/(2a\theta))^{1/\theta}$. Substitution gives the displayed value. Consequently one may take $C_{q,\theta} = (q/(2e\theta))^{q/(2\theta)}$ in (4.4).

Minkowski's inequality gives

$$\Big\| \int_a^b F(t)\,\mathrm{d}t \Big\|_{\theta,\alpha,s;A} \leq \int_a^b \|F(t)\|_{\theta,\alpha,s;A}\,\mathrm{d}t. \tag{4.6}$$

Together with Theorem 3.3 these estimates propagate bounds through compositions of the admitted operations.

Example 4.1 (One value with two propagated bounds). Use unnormalized cosine coordinates on $[0, \pi]$ and retain the scalar conversion to the normalized source modes as part of the provenance. For $f = \sum_j a_j \cos(jx)$ set $M_4(f) = \sum_j 4^j |a_j|$. The cosine product identity gives $M_4(fg) \leq M_4(f) M_4(g)$. The expressions

$$(1 + \cos x)(1 - \cos x) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x)$$

give the propagated bounds 25 and 17/2. Since $2^j \sqrt{1 + j^2} \leq 4^j$ either bound M implies

$$\|f\|_{1/2,\log 2,1;A_N} \leq \sqrt{\pi} M, \qquad \|(I - P_K) f\|_{\mathcal{H}^1_{A_N}} \leq \sqrt{\pi} M 2^{-(K+1)}, \quad K \in \mathbb{N}_0. \tag{4.7}$$

Multiplication by a further input with bound $M_4(g)$ propagates $25 M_4(g)$ or $(17/2) M_4(g)$ for the same product. For an admitted differentiation either norm bound can be used in (4.4). The smaller constant can also certify a smaller sufficient cutoff in (4.7). Expanding all coefficients gives the exact support cutoff $K = 2$ from either expression.

4.2 Two resource levels and a common bound

We use two spectral norms because convergence and uniform tail control play different roles. Convergence in the weaker norm identifies the value while a common stronger bound controls the tails and passes to the limit by Fatou's lemma. A strict gap between the weights makes the stronger ball relatively compact in the weaker norm. Convergence in the weaker norm always means norm convergence in that space.

For fixed θ write $\boldsymbol{\rho} = (\theta, \alpha, s)$ and $[\boldsymbol{\rho}; M] = (\theta, \alpha, s, M)$. Define

$$w_{\boldsymbol{\rho}}(\lambda) = e^{\alpha\lambda^{\theta}}(1+\lambda)^{s/2}, \qquad \boldsymbol{\rho}_+ \succ \boldsymbol{\rho}_- \iff \alpha_+ \geq \alpha_-,\ s_+ \geq s_-,\ (\alpha_+, s_+) \neq (\alpha_-, s_-). \tag{4.8}$$

We use $\|\cdot\|_{\boldsymbol{\rho};A}$ for the corresponding weighted norm. An admissibility policy fixes in advance a set $\mathcal{R}_{\mathfrak{G}}$ of such strict pairs and a regularity floor $\varpi_{\mathfrak{G}}(\lambda) \geq 1$ tending to infinity with

$$w_{\boldsymbol{\rho}_-}(\lambda) \geq c_{\boldsymbol{\rho}}\varpi_{\mathfrak{G}}(\lambda), \qquad (\boldsymbol{\rho}_+, \boldsymbol{\rho}_-) \in \mathcal{R}_{\mathfrak{G}}. \tag{4.9}$$

We denote this policy by $\mathfrak{A}$. The floor specifies a minimum regularity for the admitted values. A positive Sobolev floor or a positive analytic radius gives a standard example.

Definition 4.2 (Certified presentation and generated class)**.** For $\boldsymbol{\rho}_+ \succ \boldsymbol{\rho}_-$ a sequence of histories (H_n) presents f with bound M when

$$[\boldsymbol{\rho}_+; M] \in \mathsf{Res}(H_n), \qquad \|\mathsf{Val}(H_n) - f\|_{\boldsymbol{\rho}_-;A} \longrightarrow 0. \tag{4.10}$$

Its presentation retains f and this sequence together with the resource pair and M. The intrinsic or relative presentations with this fixed resource pair and bound form a bounded presentation slice denoted $\mathscr{C}_{\mathfrak{G},\bullet}[\boldsymbol{\rho}_+ \to \boldsymbol{\rho}_-; M]$ where $\bullet$ = int or rel. The map U takes a presentation to its value. Taking the union of these slices over the policy's resource pairs and finite M gives $\mathscr{C}^{\mathrm{adm}}_{\mathfrak{G},\bullet}$. Its value image is the generatively regular class

$$\mathcal{N}^{\mathfrak{A}}_{\mathfrak{G},\bullet} = \mathsf{U}(\mathscr{C}^{\mathrm{adm}}_{\mathfrak{G},\bullet}). \tag{4.11}$$

Its elements are called generatively regular objects or functions when the spectral datum has a function realization.

For instance on the Neumann interval $[0, \pi]$ fix an admitted heat parameter $\tau > 0$. The histories which form $g_N = \sum_{n=0}^{N} 2^{-n}\phi_n$ and then apply $e^{-\tau A}$ have the common bound $[(1, \tau, 0); 2]$. Their values converge at resource $(1, \tau/2, 0)$ since the squared omitted norm is at most $4e^{-\tau(N+1)^2}$. The limiting state retains the stronger heat bound.

For the fixed source and policy membership requires at least one admissible presentation. The policy's regularity floor places its value in $\mathcal{H}$. Thus the class is both a subset of the ambient value space and the image of the presentation space under U which forgets the histories and their bounds.

Two presentations with the same image represent the same value. They therefore have the same full region $\mathsf{Cert}_{\mathfrak{G}}(f)$ while the chosen histories may supply different regions $\mathsf{Res}(H)$ of derived bounds as in Example 4.1. On their stated domains the admitted value

operations agree on equivalent values. The presentation determines which estimates are already available to justify their use.

An operation acting on history coordinates has a further requirement. A closed equation for the readout must give the same response whenever two history states have the same readout. In the bounded linear case this amounts to invariance of the invisible directions under the response operator.

Proposition 4.3 (Compatibility with a value readout)**.** Let $C : \mathcal{H}_{\mathrm{hist}} \to \mathcal{H}_{\mathrm{val}}$ be a coisometry and let L be bounded on $\mathcal{H}_{\mathrm{hist}}$. A bounded operator $\overline{\mathsf{L}}$ satisfying $C\mathsf{L} = \overline{\mathsf{L}}C$ exists if and only if the readout kernel is invariant

$$\mathsf{L}(\ker C) \subseteq \ker C. \tag{4.12}$$

When it exists $\overline{\mathsf{L}} = C\mathsf{L}C^*$ is unique and $Ce^{-\tau\mathsf{L}} = e^{-\tau\overline{\mathsf{L}}}C$ for $\tau \geq 0$. Conversely this semigroup identity for every $\tau \geq 0$ implies the operator identity. If L is self-adjoint then the kernel condition means that $\ker C$ and its orthogonal complement reduce L.

Proof. The operator identity gives $C\mathsf{L}z = 0$ for $z \in \ker C$. Conversely decompose $f = C^*Cf + (I - C^*C)f$. The second summand belongs to $\ker C$ so kernel invariance gives $C\mathsf{L}f = C\mathsf{L}C^*Cf$. Since $CC^* = I$ the induced operator is unique. Its intertwining with every power of L proves the exponential identity by the norm-convergent power series. Differentiating at zero proves the converse. Self-adjointness makes the orthogonal complement of an invariant closed subspace invariant. □

An invisible initial difference lies in $\ker C$. The criterion says that it remains invisible throughout the bounded linear evolution. If it fails then the readout omits a variable that can affect a later observation. The full state or the contribution of the eliminated variables must then be retained. This is a condition on the represented equation before any finite approximation is made. Operations on the full history space are certified by their own bounds while a reduced equation additionally requires this compatibility with readout.

Extending both the operation and its certificate to limits requires two estimates. The first controls differences in the weaker norms so that the output values converge. On strong input balls of radius M suppose

$$\|\mathcal{A}(F_1, \ldots, F_m) - \mathcal{A}(G_1, \ldots, G_m)\|_{Y_-} \leq \eta_{\mathcal{A},M}\left(\max_r \|F_r - G_r\|_{X_{r,-}}\right), \tag{4.13}$$

where $\eta_{\mathcal{A},M}(r) \to 0$ as $r \downarrow 0$. For a bilinear operation this follows by expanding the difference as in (3.8). The second estimate supplies a common stronger output bound. Together they give a limiting value with the strict resource gap required for its certified presentation.

Theorem 4.4 (Certified limits and operations)**.** Let $\mathfrak{G}$ satisfy C0 and suppose every primitive operation used in its syntax has been certified on the displayed input regions. The pointwise clause uses the spectral geometry C1 stated in section 2.3. The following statements hold.

(i) The resource region of every finite history is obtained by structural propagation.

(ii) Certified convergence is stable under a primitive operation when its transformer maps the selected common stronger input slices to a fixed strong output certificate $[\mathbf{p}_+; M_{\mathrm{out}}]$ for every history obtained by applying the operation and is locally difference-certified into the output space indexed by $\mathbf{p}_-$ where $\mathbf{p}_+ \succ \mathbf{p}_-$ and $M_{\mathrm{out}} < \infty$ is independent of the histories in the presenting sequences.

(iii) The value image $\mathsf{U}\mathscr{C}_{\mathfrak{G},\bullet}[\boldsymbol{\rho}_+ \to \boldsymbol{\rho}_-; M]$ is compact and hence complete in the $\boldsymbol{\rho}_-$ norm. Every value in that image has $\boldsymbol{\rho}_+$ norm at most M.

(iv) The value image of a slice embeds continuously into $\mathcal{H}$ whenever $\alpha_- > 0$ or $s_- \geq 0$. If C1a and C1b hold and $s_- > d/2$ then the inclusion is continuous into $C(\mathcal{X})$ for scalar data and into $C(\mathcal{X}; F)$ for a pointwise state realization.

(v) For every admissibility policy and either choice of $\bullet$ the value image of $\mathscr{C}^{\mathrm{adm}}_{\mathfrak{G},\bullet}$ is contained in the proper regularity class

$$\mathcal{H}_{\varpi_{\mathfrak{G}}} := \left\{ f : \sum_j \varpi_{\mathfrak{G}}(\lambda_j)^2 |f_j|^2 < \infty \right\} \subsetneq \mathcal{H}, \tag{4.14}$$

whenever $\mathcal{H}$ is infinite-dimensional.

The compactness and completeness in part (iii) concern the value image of a fixed bounded slice in its weaker norm. Retaining a discrete tree coordinate can destroy compactness. Passing to the union changes both the resource pair and the bound so completeness in a single norm must be considered separately. The continuous embeddings describe the regularity of admitted values without asserting that every continuous function has such a presentation.

Proof. Part (i) is (4.2). Operations. For part (ii) join each tuple of histories in the presenting sequences under the node for the operation. By hypothesis all output resource regions contain $[\mathbf{p}_+; M_{\mathrm{out}}]$. Equation (4.13) makes the output values Cauchy in the complete $\mathbf{p}_-$ space. Their limit together with the resulting histories and the strict output pair satisfies Definition 4.2. The same difference estimate shows that its value depends only on the input limits within the selected input slices. It agrees with the original operation wherever that operation is already defined. All provenance labels remain in the resulting histories. Admissibility additionally requires that $(\mathbf{p}_+, \mathbf{p}_-)$ belong to the policy for the resulting presentation.

Closure of a bounded slice. For part (iii) consider the inclusion from the $\boldsymbol{\rho}_+$ space into the $\boldsymbol{\rho}_-$ space. The inclusion is diagonal with entries equal to the weaker-to-stronger weight ratio. They tend to zero along an infinite spectrum by the strict resource gap so the inclusion is compact. In finite dimension it has finite rank. The strong ball of radius M is relatively compact in the weaker norm and coefficientwise Fatou preserves its strong bound in the weaker closure.

Relative compactness does not by itself ensure that the limit still has a presentation by histories. To prove this let f_m be values in the value image converging in the weaker norm. Choose from the mth presentation a history whose value lies within $1/m$ of f_m in that norm. The diagonal history sequence has the same tree-derived strong bound and converges to the limit. Thus the value image is compact and complete.

For part (iv) the weaker spectral weight has a positive lower bound when $\alpha_- > 0$ regardless of s_- and is at least one when $\alpha_- = 0$ and $s_- \geq 0$. This gives the continuous inclusion into $\mathcal{H}$. The continuous embedding for $s_- > d/2$ is Theorem 2.2.

For part (v) (4.9) bounds the $\mathcal{H}_{\varpi_{\mathfrak{G}}}$ norm of every represented value by a constant multiple of the $\boldsymbol{\rho}_-$ norm on each admissible slice. The inclusion in (4.14) follows. Since $\varpi_{\mathfrak{G}}(\lambda_j) \to \infty$ choose a subsequence j_n with $\varpi_{\mathfrak{G}}(\lambda_{j_n}) \geq n$ and set $f_{j_n} = 1/n$ with all other coefficients zero. Then $f \in \ell^2$ whereas its $\mathcal{H}_{\varpi_{\mathfrak{G}}}$ norm diverges. Hence the inclusion is proper in infinite dimension. □

For the same source and policy with the same declared external terminals

$$\mathcal{N}^{\mathfrak{A}}_{\mathfrak{G},\mathrm{int}} \subseteq \mathcal{N}^{\mathfrak{A}}_{\mathfrak{G},\mathrm{rel}} \subseteq \mathcal{H}_{\varpi_{\mathfrak{G}}}, \tag{4.15}$$

where $\mathcal{H}_{\varpi_{\mathfrak{G}}} \subsetneq \mathcal{H}$ in infinite dimension. The first inclusion follows because every intrinsic history is relative. The second is part (v) of Theorem 4.4.

Likewise let $\mathcal{A}$ be an operation admitted by the typed syntax on selected bounded presentation slices. Suppose its transformer produces a common strong output tree certificate $[\mathbf{p}_+; M_{\mathrm{out}}]$. Assume it is locally difference-certified into $\mathbf{p}_-$ and satisfies $\mathbf{p}_+ \succ \mathbf{p}_-$ with $(\mathbf{p}_+, \mathbf{p}_-) \in \mathcal{R}_{\mathfrak{G}}$ for the output type. Then its extension to the input limits takes values in $\mathcal{N}^{\mathfrak{A}}_{\mathfrak{G},\mathrm{rel}}$. If the presenting histories and the operation are intrinsic then the output belongs to $\mathcal{N}^{\mathfrak{A}}_{\mathfrak{G},\mathrm{int}}$. This is part (ii) of Theorem 4.4. A finite composition uses the composed certificate and difference transformers.

For a bilinear interaction the tame estimates at the stronger and weaker input orders supply the strong output bound and the weak difference estimate (3.8). The output pair must still retain an admissible margin.

For a function realization satisfying the relevant parts of C1 Theorem 2.2 converts the retained norm bound into continuity or derivative estimates at the stated order thresholds. This regularity depends on the weighted norm and therefore holds for every element of the corresponding spectral space.

Generated, effective, and trustworthy presentations. A generated presentation is a presentation in Definition 4.2. It retains an admissible history or an admissible certified limit of histories together with its source and bounds. An effective generated presentation additionally supplies procedures producing the finite histories and evaluating their values. It also supplies computable certificate constants and a computable convergence modulus. A trustworthy finite computation further specifies finite encodings of the data and accuracy request. A fixed procedure produces a finite record with a sound acceptance test whose proof is independent of the producer recurrence. Uniform success on the encoded input family and resource bounds in the chosen bit model are also required. These successive

requirements distinguish an analytic presentation from an effective approximation and from a finite computation with independently checkable evidence. The companion computation paper [12] realizes these requirements for explicit equation families.

The resource region of a history records upper bounds in normed spaces. A history may in addition carry a quadratic ledger for a declared positive response form. Such a ledger is not inserted into $\mathsf{Res}(H)$ as another monotone norm bound. It contains exact identities together with positive defects and signed increments. These have different logical roles. Its sound propagation requires compatibility of the transport with the response form and explicit control of every injection and approximation cross term. These requirements are stated independently of the admissibility policy so a generated presentation can be used with or without the additional quadratic record.

5 Quadratic observations and conservative ledgers

The certificates considered so far control a state in a weighted spectral norm. When an operation contracts or projects that state one may also need to know how a specified quadratic quantity has changed. For the squared Hilbert norm a projection separates retained and omitted components. A weighted or spatially resolved observation can couple those components and a new input interacts with the state already present. Recording these contributions makes it possible to follow the same quantity through several operations.

We use a quadratic ledger for this account. Its Gram matrix can be non-diagonal in the generator's eigenbasis so signs and phases in the input remain in the cross terms of the observation. The residual estimate and the ledger have distinct purposes. One bounds the distance to the solution and the other records the changes in a chosen quantity along the construction.

5.1 Response forms and compatible endpoint transport

Let $R : \mathcal{H} \to \mathcal{Y}$ be a bounded operator into a Hilbert space. The associated response form is

$$\mathcal{Q}_R(x) := \|Rx\|_{\mathcal{Y}}^2 = \langle x, \mathsf{Q}x\rangle\,, \qquad \mathsf{Q} = R^*R \geq 0. \tag{5.1}$$

An integral response

$$\mathcal{Q}_R(x) = \int_\Lambda \|R_\lambda x\|_{\mathcal{Y}_\lambda}^2 \, \mathrm{d}\nu(\lambda) \tag{5.2}$$

is included by taking R with values in the corresponding direct integral provided the displayed map is bounded on the state space under consideration. In spectral coordinates its Gram kernel is

$$G_{ij} = \int_\Lambda \langle R_\lambda \phi_i, R_\lambda \phi_j\rangle_{\mathcal{Y}_\lambda} \, \mathrm{d}\nu(\lambda), \qquad \mathcal{Q}_R(x) = \sum_{i,j} \overline{x_i} G_{ij} x_j. \tag{5.3}$$

The off-diagonal entries retain the cross terms of the observation.

The basic conservative cell is determined by a positive contraction. The following proposition states both the exact compatible case and the term that remains when the observation does not share the transport's spectral resolution.

Proposition 5.1 (Retained–defect decomposition)**.** Let M be a bounded self-adjoint operator with $0 \le M \le I$ and put

$$D = (I - M^2)^{1/2}. \tag{5.4}$$

Let $\mathsf{Q} \ge 0$ be bounded and self-adjoint. If $M\mathsf{Q} = \mathsf{Q}M$ then

$$\mathcal{Q}(x) = \mathcal{Q}(Mx) + \mathcal{Q}(Dx), \qquad \mathcal{Q}(x) = \langle x, \mathsf{Q}x\rangle\,. \tag{5.5}$$

For a general Q one has instead

$$\mathcal{Q}(x) = \mathcal{Q}(Mx) + \mathcal{Q}(Dx) + \mathcal{C}_{M,\mathsf{Q}}(x), \tag{5.6}$$

where

$$\mathcal{C}_{M,\mathsf{Q}}(x) = \frac{1}{2}\,\big\langle x, \big([M,[M,\mathsf{Q}]] + [D,[D,\mathsf{Q}]]\big)x\big\rangle\,. \tag{5.7}$$

The compatibility remainder need not have a fixed sign.

Proof. Functional calculus gives $MD = DM$ and $M^2 + D^2 = I$. If M commutes with Q then so does D and

$$M\mathsf{Q}M + D\mathsf{Q}D = \mathsf{Q}(M^2 + D^2) = \mathsf{Q}.$$

Taking the quadratic form proves (5.5). In general

$$\frac{1}{2}\big([M,[M,\mathsf{Q}]] + [D,[D,\mathsf{Q}]]\big) = \mathsf{Q} - M\mathsf{Q}M - D\mathsf{Q}D,$$

which proves (5.6). A double commutator with a positive operator is not generally positive so the last term is kept as a compatibility remainder rather than called a defect. □

The Gaussian endpoint cell is the principal example. If $A \ge 0$ is self-adjoint and $\tau > 0$ then

$$M_\tau = e^{-\tau A/4}, \qquad D_\tau = (I - e^{-\tau A/2})^{1/2} \tag{5.8}$$

satisfy the hypotheses. Every bounded positive spectral multiplier $\mathsf{Q} = q(A)$ therefore has the exact split

$$\mathcal{Q}(x) = \mathcal{Q}(M_\tau x) + \mathcal{Q}(D_\tau x). \tag{5.9}$$

The defect is the part of this quadratic account complementary to the contracted state. Thus the identity retains the full input quantity while the retained channel alone may decrease. In an evolution model the interpretation of this defect is supplied by the chosen transport and observation.

Signed sectors are most safely handled through positive sector forms. Let Γ be a bounded self-adjoint involution commuting with M and Q and put $P_\pm = (I \pm \Gamma)/2$. Then $\mathsf{Q}P_\pm \ge 0$ and Proposition 5.1 applies separately to

$$\mathcal{Q}_\pm(x) = \langle x, \mathsf{Q}P_\pm x\rangle\,. \tag{5.10}$$

The signed form $\langle x, \mathsf{Q}\Gamma x\rangle = \mathcal{Q}_+(x) - \mathcal{Q}_-(x)$ inherits the difference of these two exact ledgers. It is not itself a positive energy. Alternatively signs may be retained in the state and measured by the positive form (5.1). Then the Gram cross terms in (5.3) remain present.

5.2 One-step and finite-history ledgers

The retained–defect cell contains no new input. A generated history also has operations that inject a new direction and a finite computation may depart from the exact operation. These contributions must be expanded before an upper bound is taken.

Let $\mathcal{H}_r$ be Hilbert spaces with bounded positive operators Q_r and write

$$\mathcal{E}_r(x) = \langle x, \mathsf{Q}_r x\rangle\,. \tag{5.11}$$

For a bounded transport $T_r : \mathcal{H}_r \to \mathcal{H}_{r+1}$ define its defect operator by

$$\Delta_r = \mathsf{Q}_r - T_r^*\mathsf{Q}_{r+1}T_r. \tag{5.12}$$

The transport is ledger-compatible when $\Delta_r \geq 0$. This is a checkable relation between two declared quadratic observations. It is not implied by boundedness of T_r alone.

Theorem 5.2 (Finite conservative ledger)**.** Suppose $\Delta_r \geq 0$ for $0 \leq r < N$ and let

$$x_{r+1} = T_r x_r + j_r + e_r. \tag{5.13}$$

Here j_r is the new contribution admitted at the stage and e_r is the difference between the recorded and exact stage output. Set

$$\mathcal{D}_r := \langle x_r, \Delta_r x_r\rangle\,, \tag{5.14}$$

$$\mathcal{J}_r := 2\,\mathrm{Re}\,\langle T_r x_r, \mathsf{Q}_{r+1}j_r\rangle + \langle j_r, \mathsf{Q}_{r+1}j_r\rangle\,, \tag{5.15}$$

$$\mathcal{A}_r := 2\,\mathrm{Re}\,\langle T_r x_r + j_r, \mathsf{Q}_{r+1}e_r\rangle + \langle e_r, \mathsf{Q}_{r+1}e_r\rangle\,. \tag{5.16}$$

Then $\mathcal{D}_r \geq 0$ and the exact one-step identity is

$$\mathcal{E}_{r+1}(x_{r+1}) + \mathcal{D}_r = \mathcal{E}_r(x_r) + \mathcal{J}_r + \mathcal{A}_r. \tag{5.17}$$

Consequently

$$\mathcal{E}_N(x_N) + \sum_{r=0}^{N-1}\mathcal{D}_r = \mathcal{E}_0(x_0) + \sum_{r=0}^{N-1}\mathcal{J}_r + \sum_{r=0}^{N-1}\mathcal{A}_r. \tag{5.18}$$

If certified majorants $\mathcal{J}_r \leq \iota_r$ and $\mathcal{A}_r \leq \varepsilon_r$ are available then

$$\mathcal{E}_N(x_N) + \sum_{r=0}^{N-1}\mathcal{D}_r \leq \mathcal{E}_0(x_0) + \sum_{r=0}^{N-1}\iota_r + \sum_{r=0}^{N-1}\varepsilon_r. \tag{5.19}$$

Proof. Expansion of (5.13) gives

$$\begin{aligned}\mathcal{E}_{r+1}(x_{r+1}) &= \langle T_r x_r, \mathsf{Q}_{r+1}T_r x_r\rangle + \mathcal{J}_r + \mathcal{A}_r\\ &= \mathcal{E}_r(x_r) - \mathcal{D}_r + \mathcal{J}_r + \mathcal{A}_r.\end{aligned}$$

This proves the one-step identity. Summing cancels the intermediate energies and gives (5.18). The certified inequality follows by replacing the two signed increments by their proved upper bounds. □

The cross term in (5.15) measures how the new contribution changes the quantity already present. Its sign depends on their relative direction. Replacing $\mathcal{J}_r$ by $\mathcal{E}_{r+1}(j_r)$ would omit that interaction. The same issue occurs for e_r. The exact identity is therefore established before Cauchy–Schwarz or Young's inequality is used or a bound is obtained by interval arithmetic or another estimate to construct ι_r and ε_r. For propagation to an unknown output these majorants must be obtained from the incoming certificates or independently controlled increments. Their right-hand sides cannot contain the unknown terminal quantity being bounded.

Definition 5.3 (Ledger-certified history)**.** A finite history is ledger-certified for a declared sequence of quadratic forms when its selected stages have the form (5.13) and every transport has a verified positive defect operator (5.12) and its retained resource data include either the exact increments (5.15)–(5.16) or certified upper majorants for them. The resulting ledger accompanies the norm certificates of section 4.

Structural induction on a finite history and Theorem 5.2 show that grafting ledger-certified histories preserves (5.19) provided the output form of one stage is the input form of the next. Thus the local entries are reusable data. They record not merely that an output is bounded but how the certified quantity arrived there.

5.3 Certified limits and changes of representation

An exact finite ledger passes to a completed value only under a continuity condition on the observation. This requirement is separate from the stronger norm bound used to control spectral tails.

Passage to a limit. Let X_- be a normed space continuously embedded in $\mathcal{H}$ and let $x_n \to x$ in X_-. If $R : X_- \to \mathcal{Y}$ is bounded then $Rx_n \to Rx$ and hence $\mathcal{Q}_R(x_n) \to \mathcal{Q}_R(x)$. Thus an exact ledger passes to the limit when all its entries are given by such bounded responses and all terms converge. For a possibly unbounded closed positive quadratic form $\mathfrak{q}$ weak convergence $x_n \rightharpoonup x$ in $\mathcal{H}$ instead gives

$$\mathfrak{q}(x) \le \liminf_{n\to\infty} \mathfrak{q}(x_n), \tag{5.20}$$

provided the x_n lie in its form domain and the right-hand side is finite. Then x also belongs to that domain. This is weak lower semicontinuity of a closed positive form. Such an observation therefore preserves the corresponding limiting inequality. Equality requires further convergence. A common strong spectral bound alone does not make every unbounded observation continuous at the weaker level.

Evaluation and compression require a similar distinction. Let $C : \mathcal{H}_{\mathrm{exp}} \to \mathcal{H}_{\mathrm{val}}$ be a coisometry. Then C^*C is the projection onto the initial space of C and $P_{\mathrm{diff}} = I - C^*C$ projects onto $\ker C$. If $\mathsf{Q}_{\mathrm{val}} \ge 0$ and $\mathsf{Q}_{\mathrm{diff}} \ge 0$ is supported on $\ker C$ then the form

$$\mathsf{Q}_{\mathrm{exp}} = C^*\mathsf{Q}_{\mathrm{val}}C + \mathsf{Q}_{\mathrm{diff}} \tag{5.21}$$

has the exact decomposition

$$\langle x, \mathsf{Q}_{\mathrm{exp}}x\rangle = \langle Cx, \mathsf{Q}_{\mathrm{val}}Cx\rangle + \langle P_{\mathrm{diff}}x, \mathsf{Q}_{\mathrm{diff}}P_{\mathrm{diff}}x\rangle\,. \tag{5.22}$$

For identity forms this is $\|x\|^2 = \|Cx\|^2 + \|P_{\text{diff}}x\|^2$. The second term is the expression-difference contribution. It vanishes on the initial space of the coisometry.

The fibre direction can be chosen by a source law. Let a countable set of histories be partitioned into finite fibres $\mathscr{E}_n$ with $p_n(e) > 0$ and $\sum_{e\in\mathscr{E}_n} p_n(e) = 1$. The readout

$$C_p|e\rangle = \sqrt{p_n(e)}|n\rangle, \qquad V_p = C_p^*, \qquad P_{\text{diff}} = I - V_pC_p \tag{5.23}$$

is a coisometry since its adjoint sends $|n\rangle$ to orthonormal unit vectors on disjoint fibres. Uniform counting uses $p_n(e) = 1/|\mathscr{E}_n|$. A recursive source can give another law. For $x = V_pa + z$ where $a = C_px$ and $z = P_{\text{diff}}x$ the identity form always splits into $\|x\|^2 = \|a\|^2 + \|z\|^2$.

For a general bounded positive form the complete expression is

$$\langle x, \mathsf{Q}x\rangle = \langle a, \mathsf{Q}_{00}a\rangle + 2\,\mathrm{Re}\,\langle a, \mathsf{Q}_{01}z\rangle + \langle z, \mathsf{Q}_{11}z\rangle\,, \tag{5.24}$$

where $\mathsf{Q}_{00} = C_p\mathsf{Q}V_p$ and $\mathsf{Q}_{01} = C_p\mathsf{Q}|_{\ker C_p}$ with $\mathsf{Q}_{11} = P_{\text{diff}}\mathsf{Q}|_{\ker C_p}$. This follows by expanding $x = V_pa + z$ and using the adjoint identities. Equation (5.22) is its block-diagonal case. The mixed term measures the coupling between the readout direction and the difference component. Positivity of the whole form constrains its size but allows either sign. Retained and unresolved energies alone therefore determine the observation only in the block-diagonal case. The mixed term vanishes for all a, z exactly when the off-diagonal block is zero. For a positive self-adjoint response this is also the kernel-invariance condition of Proposition 4.3.

This coherent amplitude readout is distinct from probability pushforward. The latter sums all probabilities in a fibre. For example if $\rho = \sum_e P(e)|e\rangle\langle e|$ is diagonal then the diagonal of $C_p\rho C_p^*$ is $\sum_{e\in\mathscr{E}_n} p_n(e)P(e)$ whereas the pushforward mass is $\sum_{e\in\mathscr{E}_n} P(e)$. A probability observation uses the latter map or its trace-preserving record-channel realization. An amplitude observation uses the specified coisometry. The two require different certificates even when their labels are the same root values.

The ledger therefore strengthens the meaning of a retained certificate in a composed calculation. A norm certificate controls admissibility and approximation. A quadratic ledger records the disposition of a selected observable. A finite checker can verify the algebraic entries of the ledger. Each certificate adds information about the same state.

6 Local generated evolution

A nonlinear expression may preserve the exponential weight while losing Sobolev regularity. The heat semigroup recovers this loss in the Duhamel integral at the expense of a singular time factor. Local existence therefore depends on the net derivative loss and on the time integrability of the growth and difference bounds for the complete nonlinearity. These bounds control the contraction and keep its Picard iterates in a common ball.

6.1 Generated nonlinearities

We use C4 for the following time assumptions. The forcing is strongly measurable in its stated weighted space. Every primitive nonlinearity is strongly measurable at fixed

input and locally Lipschitz on its declared domains and has measurable local bounds. The complete nonlinear expression must have growth and difference profiles in the time space required by the evolution theorem. This last check is made after the operations have been composed.

Let $\mathcal{N}_t$ be a finite expression built from the certified operations and operator terms of section 4 excluding time integration with compatible input and output types. Its leaves are copies of the state variable and independently certified coefficients. External coefficients retain their relative provenance. Constraint projections occur at the nodes where they act.

Proposition 6.1 (Local Lipschitz transformer)**.** Assume C2–C4 for every primitive node of $\mathcal{N}_t$ and suppose every interaction node satisfies the frequency support law (2.5). Require a compatible assignment of spectral orders to the tree. Each child output lies in the input domain certified for its parent and every node estimate is uniform in the analytic radius over the stated range. Fix $\sigma_0 \geq 0$ and $s \in \mathbb{R}$. Consider a certified ball

$$\mathbb{B}_{\sigma,s}(M) := \{u : \|u\|_{1/2,\sigma,s;A} \leq M\}, \qquad 0 \leq \sigma \leq \sigma_0. \tag{6.1}$$

Then the syntax tree determines a finite net derivative loss $q \geq 0$ and measurable functions $L_M, P_M \geq 0$ such that uniformly for $0 \leq \sigma \leq \sigma_0$

$$\|\mathcal{N}_t(u) - \mathcal{N}_t(v)\|_{1/2,\sigma,s-q;A} \leq L_M(t)\, \|u - v\|_{1/2,\sigma,s;A}\,, \tag{6.2}$$

and

$$\|\mathcal{N}_t(u)\|_{1/2,\sigma,s-q;A} \leq P_M(t), \tag{6.3}$$

for $u, v \in \mathbb{B}_{\sigma,s}(M)$. The profiles are finite sums and products of the primitive time bounds. Their time integrability must be certified after this recursion. Local integrability at individual nodes does not imply local integrability of the resulting profiles.

Proof. We track orders and time profiles together because a parent node may lower the regularity and multiply two time-dependent child bounds. At each node v record an output order s_v and a growth bound $P_v(t)$ and a difference bound $D_v(t)$ relative to the input state norm. A state leaf has $(P_v, D_v) = (M, 1)$. An independently bounded coefficient leaf has $(P_v, D_v) = (b_v(t), 0)$. A linear node with certified norm $a_v(t)$ multiplies both child bounds by $a_v(t)$ and assigns the output order supplied by its linear estimate. A sum adds the bounds at a common output order.

For a bilinear node with child orders s_1, s_2 put $r = \min(s_1, s_2)$. The compatible assignment requires an estimate at order r with the declared loss q_v and output order $s_v = r - q_v$. An asymmetric estimate may instead use the two child orders directly. Corollary 3.4 and the bilinear difference identity give with the corresponding certified constant $b_v(t)$

$$P_v = b_v P_1 P_2, \qquad D_v = b_v(D_1 P_2 + P_1 D_2). \tag{6.4}$$

Applying the rules from the leaves to the root gives measurable bounds $P_M = P_{\mathrm{root}}$ and $L_M = D_{\mathrm{root}}$. The root order determines the net loss $q = \max\{0, s - s_{\mathrm{root}}\}$. If $s_{\mathrm{root}} > s$ embed the output into the order-s space. This proves the two estimates. The time dependence remains in finite sums and products. Their integrability is checked at the root. □

For example $c(t) = t^{-1/2}$ belongs to $L^1(0,1)$ but the two-node expression $c(t)(c(t)u) = u/t$ has a nonintegrable Lipschitz profile. On a zero mode ϕ_0 of A every $u(t) = Ct\phi_0$ with $u(0) = 0$ solves its mild equation. Thus the composed expression can lose uniqueness even in one mode despite integrability of each coefficient separately. For a given expression Hölder's inequality can be applied to each product in (6.4).

6.2 Duhamel propagation

Consider

$$\partial_t u + Au = \mathcal{N}_t(u) + g(t), \qquad u(0) = u_0, \tag{6.5}$$

on the constrained state space. A mild solution satisfies

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}\big(\mathcal{N}_\tau(u(\tau)) + g(\tau)\big)\,\mathrm{d}\tau. \tag{6.6}$$

For the analytic scale functional calculus gives

$$\left\|e^{-hA}f\right\|_{1/2,\sigma,s;A} \le C_q(1 + h^{-q/2})\,\|f\|_{1/2,\sigma,s-q;A}\,, \qquad h > 0. \tag{6.7}$$

After the common exponential weight is removed the required multiplier is $(1+\lambda)^{q/2}e^{-h\lambda}$. One may take

$$C_q = 2^{\max\{q/2-1,0\}}\max\{1, (q/(2e))^{q/2}\}, \qquad q > 0, \qquad C_0 = 1. \tag{6.8}$$

Indeed $(1+\lambda)^{q/2}$ is at most $2^{\max\{q/2-1,0\}}(1+\lambda^{q/2})$. Maximizing the second term after multiplication by $e^{-h\lambda}$ gives the claim.

The singularity in (6.7) is locally integrable precisely when $q < 2$. The time profiles may also be unbounded so integrability of the kernel alone is insufficient. Hölder's inequality pairs the two factors and determines which time exponents control the Duhamel integral. Appendix A gives the resulting Volterra bound with its explicit time factor.

This is the smoothing estimate used in analytic-semigroup theory as in [4, Chapter 2]. The local construction follows the semilinear fixed-point argument of [5, Chapter 3] with the derivative loss and time profiles specified by the expression tree.

Theorem 6.2 (Local existence and reserve propagation)**.** Assume C0 and C2–C4. Suppose every interaction node satisfies (2.5) and let q be the net loss in Proposition 6.1. Suppose $0 \le q < 2$. Choose $p \in (1, \infty]$ with conjugate exponent p' such that

$$\frac{q}{2}p' < 1. \tag{6.9}$$

When $q = 0$ the endpoint $p = 1$ is also allowed. Let $u_0 \in \mathscr{G}^{(1/2)}_{\sigma_0,s}(A)$ with $\sigma_0 \ge 0$ and put

$$G(t) := \|g(t)\|_{1/2,\sigma_0,s-q;A}\,. \tag{6.10}$$

For some $T_0 > 0$ in the declared time domain assume $G \in L^p(0,T_0)$ and for every $M < \infty$ require the functions L_M and P_M of Proposition 6.1 to belong to $L^p(0,T_0)$.

Then there exist $0 < T_* \le T_0$ and a unique mild solution of (6.5) such that

$$u \in C\big([0,T_*);\mathscr{G}^{(1/2)}_{\sigma_0,s}(A)\big). \tag{6.11}$$

In particular a positive analytic reserve σ_0 is retained on this local interval. $\sigma_0 = 0$ gives the corresponding Sobolev endpoint. Writing $M(t) = \|u(t)\|_{1/2,\sigma_0,s;A}$ the proof gives the integral bound

$$M(t) \le \|u_0\|_{1/2,\sigma_0,s;A} + \int_0^t \mathcal{K}_q(t-\tau)\big(\mathcal{P}_\tau(M(\tau)) + G(\tau)\big)\,\mathrm{d}\tau, \tag{6.12}$$

where $P_M(t) = \mathcal{P}_t(M)$ may be chosen nondecreasing in M and

$$\mathcal{K}_q(h) = C_q(1 + h^{-q/2}). \tag{6.13}$$

Proof. We fix the ball before shortening the time interval so its growth and Lipschitz profiles remain fixed in the small-time estimates. For $0 < T \le T_0$ and $R > 0$ let $\mathfrak{X}_T(R)$ be the closed ball of radius R in $C([0,T];\mathscr{G}^{(1/2)}_{\sigma_0,s}(A))$ with the supremum norm. Define the Duhamel map by the right-hand side of (6.6). For a nonnegative profile a write

$$\mathcal{V}_T(a) := \sup_{0\le t\le T}\int_0^t \mathcal{K}_q(t-\tau)a(\tau)\,\mathrm{d}\tau \le \|\mathcal{K}_q\|_{L^{p'}(0,T)}\,\|a\|_{L^p(0,T)}\,. \tag{6.14}$$

Here $p' = \infty$ at the permitted endpoint $q = 0$ and $p = 1$. Equations (6.2) and (6.7) give the difference bound $\mathcal{V}_T(L_R)$. The growth estimate gives the following explicit conditions for contraction and invariance of the ball

$$\mathcal{V}_T(L_R) < 1, \qquad \mathcal{V}_T(P_R + G) \le R - \|u_0\|_{1/2,\sigma_0,s;A}\,. \tag{6.15}$$

Choose $R > \|u_0\|_{1/2,\sigma_0,s;A}$. For $p > 1$ condition (6.9) gives $\|\mathcal{K}_q\|_{L^{p'}(0,T)} \to 0$ as $T \downarrow 0$. For $q = 0$ and $p = 1$ the kernel is bounded and the integrals of the three fixed profiles tend to zero. Thus a sufficiently small T satisfies (6.15).

Strong measurability and the growth bound make the integral well defined. To check continuity split it into a short interval adjacent to the upper endpoint and its complement. The first part tends to zero by (6.14). On the complement strong continuity of the semigroup and dominated convergence apply. The Duhamel map therefore acts on the complete path space and Banach's theorem gives its fixed point. The difference estimate proves uniqueness first on a short interval in a common ball and then throughout the common interval by restarting. Applying the growth bound directly to the fixed-point identity yields (6.12). □

6.3 Positive-time generation of analytic radius

For Sobolev initial data we allow the analytic radius to vanish initially and increase with time. The decay of the heat multiplier compensates for the growing exponential weight. For periodic Navier–Stokes equations exponential spectral weights give Gevrey regularity under suitable forcing assumptions [6]. For fractional-Laplacian dissipation and analytic

nonlinearities built from Fourier multipliers Bae and Biswas establish positive-time Gevrey estimates from Sobolev data [7, section 2.2, Theorem 2.5]. Their time-dependent exponential weight has the square-root time scale for second-order dissipation. Here the uniform interaction bounds control the nonlinearity at each intermediate time and the forcing is measured at the same moving radius. Fix $c > 0$ and set

$$\sigma_c(t) := c\sqrt{t}. \tag{6.16}$$

The required semigroup estimate follows from the same scalar multiplier calculus as (6.7). For $0 \le \tau < t$ and $q \ge 0$ and $s \in \mathbb{R}$

$$\left\| e^{-(t-\tau)A} f \right\|_{1/2,\sigma_c(t),s;A} \le C_{q,c}\big(1 + (t-\tau)^{-q/2}\big)\, \|f\|_{1/2,\sigma_c(\tau),s-q;A}\,. \tag{6.17}$$

Moreover

$$\left\| e^{-tA} f \right\|_{1/2,\sigma_c(t),s;A} \le e^{c^2/4}\, \|f\|_{\mathcal{H}^s_A}\,, \tag{6.18}$$

and $e^{\sigma_c(t)B} e^{-tA} f \to f$ in $\mathcal{H}^s_A$ as $t \downarrow 0$. Put $h = t - \tau$ and $d = c(\sqrt{t} - \sqrt{\tau})$. Since $0 \le d/\sqrt{h} \le c$ the multiplier remaining after the two analytic weights are removed is

$$(1+\omega^2)^{q/2} e^{d\omega - h\omega^2} \le C_{q,c}(1 + h^{-q/2}), \tag{6.19}$$

by the change of variable $y = \sqrt{h}\,\omega$. This proves (6.17). Setting $\tau = 0$ and $q = 0$ gives (6.18) since $c\sqrt{t}\,\omega - t\omega^2 \le c^2/4$. The final assertion follows from pointwise multiplier convergence and dominated convergence in spectral coordinates.

Theorem 6.3 (Positive-time nonlinear analyticity)**.** Assume C0 and C2–C4. Suppose every interaction node satisfies (2.5). Let the finite nonlinear syntax have net loss $0 \le q < 2$ as in Proposition 6.1 and take $u_0 \in \mathcal{H}^s_A$. Fix $c > 0$ and choose p with $qp'/2 < 1$ (with the same $q = 0$ and $p = 1$ endpoint convention as in Theorem 6.2) and suppose the growth and Lipschitz bounds of that proposition hold uniformly on balls at radii $0 \le \sigma \le c\sqrt{T}$. Require

$$G_c(t) := \|g(t)\|_{1/2,\sigma_c(t),s-q;A} \in L^p(0,T), \tag{6.20}$$

and require the corresponding uniform L_M, P_M profiles to belong to $L^p(0,T)$. After decreasing $T > 0$ equation (6.5) has a unique mild solution such that

$$t \longmapsto e^{\sigma_c(t)B} u(t) \in C([0,T]; \mathcal{H}^s_A). \tag{6.21}$$

In particular $u(t) \in \mathscr{G}^{(1/2)}_{c\sqrt{t},s}(A)$ for every $t > 0$. For the homogeneous equation $g = 0$ the assumption on the initial datum is $u_0 \in \mathcal{H}^s_A$.

Proof. Let $\mathfrak{Y}_T$ be the space of paths for which the left-hand side of (6.21) is continuous with norm $\sup_{0\le t\le T} \|u(t)\|_{1/2,\sigma_c(t),s;A}$. Applying the moving weight identifies this space isometrically with the continuous paths in the Sobolev space so it is Banach. Choose a ball whose radius is strictly larger than the free-solution bound in (6.18). Estimate (6.17) replaces (6.7) in the fixed-point proof of Theorem 6.2. The Volterra kernel is still $C_{q,c}(1 + h^{-q/2})$ so Hölder's inequality and $qp'/2 < 1$ make the Duhamel map a contraction on a

sufficiently short interval. At the $q = 0$ and $p = 1$ endpoint use the bounded kernel and absolute continuity of the profile integrals instead. The same bound with $P_M + G_c$ makes the ball invariant. The free term is continuous at the origin by the multiplier convergence established above and the integral tends to zero there by the profile bound. At positive times split the integral near its upper endpoint as in the fixed-radius proof. The Volterra majorant controls the short part. Spectral multiplier continuity and dominated convergence handle the remainder after the moving weight is applied. □

6.4 Invariance of the admissible generated class

Theorem 6.2 constructs a mild solution in the weighted space. To obtain a presentation in the fixed source language we use histories for the approximating data and each Picard step with a common bound uniform in both the iteration and the data approximation.

For an analytic resource index $\boldsymbol{\rho}_\bullet = (1/2, \sigma_\bullet, s_\bullet)$ write

$$\boldsymbol{\rho}_\bullet^{-q} := (1/2, \sigma_\bullet, s_\bullet - q). \tag{6.22}$$

The strongest resource carries the common bound on every Picard history. A common time envelope for the nonlinearities together with uniform forcing bounds gives one interval and contraction bound for the family. At the intermediate resource the data and nonlinear approximation errors control convergence of the fixed points to the exact solution. Convergence at the weakest resource gives a presentation with a strict margin to the retained stronger bound. The semigroup and integral expressions at each step must also belong to the fixed language.

For a fixed quadratic nonlinearity with time-independent tame constants the same expression may be used at every approximation level. The nonlinear defect is then zero and each fixed resource ball has constant growth and difference envelopes. If the Picard constructions have finite mode support and continuous time coefficients then their integrals can be approximated by finite time sums. Admissibility of these sums requires a common strong certificate from the data and integral bounds including the derivative-recovery kernel when derivatives are lost.

Theorem 6.4 (Generated-flow invariance)**.** Let

$$\boldsymbol{\rho}_{++} \succ \boldsymbol{\rho}_{+} \succ \boldsymbol{\rho}_{-} \tag{6.23}$$

be analytic resource indices and suppose $(\boldsymbol{\rho}_{++}, \boldsymbol{\rho}_{-}) \in \mathcal{R}_{\mathfrak{G}}$. Assume that the hypotheses of Theorem 6.2 with the same $0 \le q < 2$ and p hold at both $\boldsymbol{\rho}_{++}$ and $\boldsymbol{\rho}_{+}$.

Assume the following three conditions.

(i) Data and their stronger bounds. There are relative generated histories H_n^0 together with time-dependent relative histories $H_n^g(t)$ admitted by rule (E4) and finite-syntax nonlinearities $\mathcal{N}_t^{(n)}$ whose coefficient paths are admitted time-dependent terminals. Put $u_{0,n} = \mathsf{Val}(H_n^0)$ and $g_n(t) = \mathsf{Val}(H_n^g(t))$ and assume that there are $M_0 < \infty$ and nonnegative measurable functions G_n such that

$$\begin{gathered} [\boldsymbol{\rho}_{++}; M_0] \in \mathsf{Res}(H_n^0) \quad \text{for every } n, \\ u_{0,n} \longrightarrow u_0 \quad \text{in } \boldsymbol{\rho}_{+}. \end{gathered} \tag{6.24}$$

$$[\boldsymbol{\rho}_{++}^{-q}; G_n(t)] \in \mathsf{Res}(H_n^g(t)) \quad \text{for a.e. } t,$$
$$\sup_n \|G_n\|_{L^p(0,T)} < \infty, \qquad g_n \longrightarrow g \quad \text{in } L^p(0,T;\boldsymbol{\rho}_+^{-q}). \tag{6.25}$$

Here a norm labelled by a resource index means the corresponding weighted spectral norm.

(ii) Nonlinear bounds and approximation. Let $L_M^{(n),++}$ and $P_M^{(n),++}$ be the strong profiles obtained by the node recursion of Proposition 6.1 so that the growth bound is a certificate transformer for the chosen nonlinear tree. Require these tree-derived profiles to admit for every M a common nonnegative envelope $h_M \in L^p(0,T)$ independent of n such that

$$L_M^{(n),++}(t) + P_M^{(n),++}(t) \le h_M(t) \quad \text{for every } n \text{ and a.e. } t. \tag{6.26}$$

Assume further that on every $\boldsymbol{\rho}_{++}$ ball of radius M

$$\eta_{n,M}(t) := \sup_{\|z\|_{\boldsymbol{\rho}_{++};A} \le M} \left\| \mathcal{N}_t^{(n)}(z) - \mathcal{N}_t(z) \right\|_{\boldsymbol{\rho}_+^{-q};A} \longrightarrow 0 \quad \text{in } L^p(0,T). \tag{6.27}$$

(iii) Admitted time constructions. Fix a dense set $D \subset [0,T]$ containing zero. At every Picard level and $t \in D$ the semigroup terms and integration interval together with the Duhamel kernel and nested integrand of the Picard iteration of (6.6) with data $u_{0,n}$ and g_n and nonlinearity $\mathcal{N}^{(n)}$ are admitted by the declared parameter and operator language and rule (E4) with the transformers in (i) and (ii). Whenever T is shortened use $D \cap [0,T]$.

After decreasing $T > 0$ using only the uniform displayed bounds every $u(t)$ admits a relative generated presentation $\widehat{u}_t$ satisfying

$$\mathsf{U}(\widehat{u}_t) = u(t), \qquad \widehat{u}_t \in \mathscr{C}_{\mathfrak{G},\mathrm{rel}}[\boldsymbol{\rho}_{++} \to \boldsymbol{\rho}_-; M] \subset \mathscr{C}^{\mathrm{adm}}_{\mathfrak{G},\mathrm{rel}}, \qquad 0 \le t \le T, \tag{6.28}$$

for one finite M independent of t. Moreover $u \in C([0,T]; \mathscr{G}^{(1/2)}_{\sigma_+,s_+}(A))$. The state is continuous in the intermediate norm and has a uniform stronger bound. If every leaf and operator term in these histories including the time-dependent semigroup and integral terms at $t \in D$ is intrinsic then rel may be replaced by int throughout.

Proof. Uniform stronger bounds. Soundness of the tree resources bounds $u_{0,n}$ by M_0 in the stronger space and $g_n(t)$ by $G_n(t)$ in the stronger forcing space. To pass the initial bound to u_0 use coefficientwise convergence in the intermediate space and Fatou's lemma for the stronger weighted sum. If $p < \infty$ first choose a subsequence realizing the lower limit of the stronger L^p norms and then pass to a further subsequence for which $g_n(t) \to g(t)$ in the intermediate space for almost every t. Pointwise Fatou followed by Fatou in time gives

$$\|g\|_{L^p(0,T;\boldsymbol{\rho}_{++}^{-q})} \le \liminf_{n\to\infty} \|g_n\|_{L^p(0,T;\boldsymbol{\rho}_{++}^{-q})}. \tag{6.29}$$

For $p = \infty$ convergence in the intermediate L^∞ norm yields pointwise intermediate convergence outside one null set and the same coefficientwise argument gives the corresponding

essential-supremum bound. Thus the limiting Duhamel map is defined at both stated resource levels.

For each n form the Picard sequence

$$u_n^{(0)}(t) = e^{-tA}u_{0,n},$$
$$u_n^{(m+1)}(t) = e^{-tA}u_{0,n} + \int_0^t e^{-(t-\tau)A}\big(\mathcal{N}_\tau^{(n)}(u_n^{(m)}(\tau)) + g_n(\tau)\big)\,\mathrm{d}\tau. \tag{6.30}$$

For every $t \in D$ each iterate has a relative generated history obtained by composing the admitted semigroup and nonlinear and Bochner operations with the data histories. The time-admissibility hypothesis supplies rule (E4) for the Bochner nodes. At the allowed endpoint $q = 0$ and $p = 1$ a uniform bound on the forcing profiles does not imply uniform small-time decay. We therefore absorb their full-interval contribution into the ball radius. Write T_0 for the initial endpoint in hypotheses (i)–(iii) and set

$$F_* := \|\mathcal{K}_q\|_{L^{p'}(0,T_0)} \sup_n \|G_n\|_{L^p(0,T_0)}, \qquad M := M_0 + F_* + 1.$$

The exponent condition makes F_* finite including the endpoint $q = 0$ and $p = 1$ where $p' = \infty$. With this M fixed choose $0 < T \le T_0$ so that $\mathcal{V}_T(h_M) < 1/2$. For $p > 1$ this follows from (6.14). For $p = 1$ use absolute continuity of the integral of h_M. Every approximating Duhamel map then has growth bound at most $M_0 + F_* + \mathcal{V}_T(h_M) < M$ and contraction constant below $1/2$ at $\boldsymbol{\rho}_{++}$. The limiting forcing has the same full-interval bound by (6.29). The limiting nonlinear profiles are also in L^p at this radius. A further shortening therefore makes the limiting Duhamel map invariant and contractive on the same strong ball. Consequently

$$\sup_{n,m}\sup_{0\le t\le T}\Big\|u_n^{(m)}(t)\Big\|_{\boldsymbol{\rho}_{++};A} \le M, \qquad [\boldsymbol{\rho}_{++}; M] \in \mathsf{Res}(H_n^{(m)}(t)) \quad (t \in D), \tag{6.31}$$

where $H_n^{(m)}(t)$ records these Picard operations. Indeed if the mth history has bound M then the next history has the tree-derived bound

$$M_0 + \int_0^t \mathcal{K}_q(t-\tau)\big(P_M^{(n),++}(\tau) + G_n(\tau)\big)\,\mathrm{d}\tau \le M. \tag{6.32}$$

The choice of ball and time made above proves this inequality uniformly in n. Induction and the resource-weakening rule (4.1) raise each upper bound to M and prove the resource assertion in (6.31).

Convergence at the intermediate resource. Let u_n be the fixed point for the nth data. Subtract its mild equation from that of u. The Volterra estimate at $\boldsymbol{\rho}_+$ together with (6.24) through (6.27) gives

$$\|u_n - u\|_{C([0,T];\boldsymbol{\rho}_+)} \longrightarrow 0. \tag{6.33}$$

Indeed split the nonlinear difference as

$$\mathcal{N}_t^{(n)}(u_n) - \mathcal{N}_t(u) = \big[\mathcal{N}_t^{(n)}(u_n) - \mathcal{N}_t(u_n)\big] + \big[\mathcal{N}_t(u_n) - \mathcal{N}_t(u)\big]. \tag{6.34}$$

The first bracket is bounded by $\eta_{n,M}$ because the approximate solutions stay in the common stronger ball. The second uses the intermediate Lipschitz profile of the limiting nonlinearity. Thus the common strong bound controls where the approximation defect is evaluated while only the limiting nonlinearity supplies the difference estimate at the intermediate level. After a further common decrease of T its Volterra contribution is absorbed. The remaining defects in the initial data and forcing and coefficients converge to zero.

Presentations at every time. Choose $m(n)$ so that

$$\left\| u_n^{(m(n))} - u_n \right\|_{C([0,T];\boldsymbol{\rho}_+)} \le 1/n.$$

At each $t \in D$ the diagonal sequence consists of relative histories and is uniformly bounded in $\boldsymbol{\rho}_{++}$ by (6.31) and converges to u in $\boldsymbol{\rho}_+$ and hence in $\boldsymbol{\rho}_-$. Definition 4.2 supplies the presentation and proves (6.28) on D. Continuity in the intermediate norm follows from Theorem 6.2 applied at $\boldsymbol{\rho}_+$. For an arbitrary $t \in [0,T]$ choose $t_j \in D$ with $t_j \to t$. Then $u(t_j) \to u(t)$ in $\boldsymbol{\rho}_-$ and every $u(t_j)$ belongs to the value image of the same slice with bound M. That image is closed by Theorem 4.4(iii). This gives the presentation at t. The identical argument applies to intrinsic histories. □

Applying the value map to the conclusion of Theorem 6.4 shows for the same policy $\mathfrak{A}$ that

$$u(t) \in \mathcal{N}^{\mathfrak{A}}_{\mathfrak{G},\mathrm{rel}}, \qquad 0 \le t \le T. \tag{6.35}$$

If every leaf and operator term in its approximating histories is intrinsic then rel may be replaced by int.

This conclusion is representational as well as analytic. The mild solution is an element of the weighted space by the local existence theorem. Under the additional history hypotheses it is also reached by the declared model operations while retaining a common stronger bound. That retained provenance is the sense in which the local dynamics remains inside the selected generated class.

The same diagonal histories can carry a quadratic account into the limiting flow. The finite transports and injections supply its entries. Continuity of the response at the convergence level identifies the terminal quantity with that of the solution. The accumulated positive defect passes through a lower limit so a certified inequality survives even when the individual stage accounts do not converge separately.

Quadratic ledgers along a generated flow. Assume the hypotheses of Theorem 6.4. Fix $t \in [0,T]$ and let $H_n(t)$ be a diagonal sequence of finite Picard histories presenting $u(t)$. Suppose these histories are ledger-certified in the sense of Definition 5.3 with terminal response $R : X_- \to \mathcal{Y}$ bounded at the weaker convergence level. Write their certified finite inequalities as

$$\mathcal{Q}_R(\mathsf{Val}(H_n(t))) + \mathcal{D}_n(t) \le \mathcal{E}_{0,n}(t) + \mathcal{I}_n(t) + \mathcal{A}_n(t), \qquad \mathcal{D}_n(t) \ge 0. \tag{6.36}$$

Then

$$\mathcal{Q}_R(u(t)) + \liminf_{n\to\infty} \mathcal{D}_n(t) \le \limsup_{n\to\infty} \big(\mathcal{E}_{0,n}(t) + \mathcal{I}_n(t) + \mathcal{A}_n(t)\big). \tag{6.37}$$

If every entry in the finite exact ledgers converges and the finite stage identities have a common limiting indexing then their equality passes to the limit. Indeed the presentation gives $\mathsf{Val}(H_n(t)) \to u(t)$ in X_-. Since $R : X_- \to \mathcal{Y}$ is bounded the terminal quadratic terms converge. Taking lower limits on the left and upper limits on the right of (6.36) proves (6.37). Under the last hypotheses every term in the exact finite identity converges so the equality is preserved.

The preceding presentation uses fixed resource indices. For the mild solution itself the local existence argument can be restarted at a fixed analytic radius when the solution norm stays bounded and the time profiles remain integrable. Retaining a generated presentation after a restart requires the approximation and time-construction hypotheses for the restarted data as well.

The usual continuation argument takes the following explicit form in the present weighted class. Let $T_{\max} < \infty$ be a candidate maximal existence time inside the admitted time domain of the coefficients. Suppose the same q and p remain admissible. Suppose that for some $\delta > 0$ and $M_* < \infty$ the functions L_{M_*+1} and P_{M_*+1} as well as G belong to $L^p(0, T_{\max} + \delta)$. If

$$M(t) \le M_*, \tag{6.38}$$

for $0 \le t < T_{\max}$ then the solution extends beyond $T_{\max}$. Thus failure of continuation in this fixed-reserve class entails divergence of the weighted norm or loss of the uniform time-integrability bounds or exit from the domain on which C4 is verified. To see this restart Theorem 6.2 at a time $t_0 < T_{\max}$ using the ball of radius $M_* + 1$. For $p > 1$ the Volterra bound tends to zero with the interval length uniformly in t_0. For $p = 1$ and $q = 0$ the same follows from absolute continuity of the integrals of the three fixed profiles on $[0, T_{\max} + \delta]$. The unit margin between the initial norm and the ball radius therefore gives a common positive existence time. Taking t_0 sufficiently close to $T_{\max}$ and using uniqueness extends the original solution past that time.

7 Nonlinear residuals and finite observations

Even an exact solution of a Galerkin equation can depart from the full evolution because the nonlinear term creates modes that the projection discards. The common stronger bound controls this omitted source as well as the initial tail. A stability estimate measures their accumulated effect on the state and an observation modulus translates the resulting error into the accuracy of the reported quantity.

The two norms serve different purposes. The stronger norm retains the spectral reserve needed to control unresolved modes whereas the weaker norm is the level at which the state error is propagated and observed. The ratio of their weights controls the omitted part of the initial state as well as frequencies regenerated by the nonlinear source. The cutoff is sufficient only when both contributions fit the same error budget.

7.1 Residual error propagation

Let v be an approximate path in the full state space. Its residual is the forcing that makes it satisfy the evolution equation exactly. For a strong path this is

$$r_v(t) := \dot{v}(t) + Av(t) - \mathcal{N}_t(v(t)) - g(t). \tag{7.1}$$

For a mild approximation a residual in the declared forcing space is specified by the perturbed Duhamel identity

$$v(t) = e^{-tA}v(0) + \int_0^t e^{-(t-\tau)A}\big(\mathcal{N}_\tau(v(\tau)) + g(\tau) + r_v(\tau)\big)\,\mathrm{d}\tau. \tag{7.2}$$

For the Lipschitz profile L_M at the weaker level write

$$H_M(T) := \sup_{0\le t\le T}\int_0^t \mathcal{K}_q(t-\tau)L_M(\tau)\,\mathrm{d}\tau. \tag{7.3}$$

This measures the amplification of a state difference through the nonlinear Duhamel term. The local error estimate absorbs this contribution when it is less than one.

Proposition 7.1 (Duhamel–Volterra residual bound)**.** Assume the hypotheses of Theorem 6.2 with $X = \mathscr{G}^{(1/2)}_{\sigma_0,s}(A)$ and $X_- = \mathscr{G}^{(1/2)}_{\sigma_0,s-q}(A)$ and suppose that the exact and approximate trajectories remain in the same certified ball of radius M. For $0 < T < T_*$ let $H_M(T)$ be given by (7.3). If $H_M(T) < 1$ and $r_v \in L^p(0,T;X_-)$ then

$$\sup_{0\le t\le T}\|u(t)-v(t)\|_X \le \frac{1}{1-H_M(T)}\Bigg[\|u(0)-v(0)\|_X + \sup_{0\le t\le T}\int_0^t \mathcal{K}_q(t-\tau)\,\|r_v(\tau)\|_{X_-}\,\mathrm{d}\tau\Bigg]. \tag{7.4}$$

The estimate can be iterated on a finite partition whose local stability numbers are less than one.

Proof. Subtract (7.2) from the exact mild identity. Equations (6.2) and (6.7) give with $E_T = \sup_{0\le t\le T}\|u(t)-v(t)\|_X$

$$E_T \le \|u(0)-v(0)\|_X + H_M(T)E_T + \sup_{0\le t\le T}\int_0^t \mathcal{K}_q(t-\tau)\,\|r_v(\tau)\|_{X_-}\,\mathrm{d}\tau. \tag{7.5}$$

Absorb the middle term. Integrability follows from $qp'/2 < 1$ or from the bounded kernel when $q = 0$ and $p = 1$ as in Theorem 6.2. Reapplying the argument after each partition time proves the last assertion. □

7.2 A common interval and a vanishing nonlinear residual

Spectral Galerkin equations are classical [15]. Selecting a cutoff from a prescribed tolerance requires the entire family on one time interval with an error constant independent of the cutoff. The stronger bound and common time envelopes provide these two forms of uniformity.

Theorem 7.2 (Galerkin residual and finite-window closure)**.** Let $0 \le q < 2$ and p satisfy (6.9) and also allow $q = 0$ and $p = 1$. Fix $\sigma \ge 0$ and $\delta > 0$. Choose $s \in \mathbb{R}$ and $\eta \ge 0$ and set

$$\begin{aligned} X &= \mathscr{G}^{(1/2)}_{\sigma,s}(A), & X_- &= \mathscr{G}^{(1/2)}_{\sigma,s-q}(A), \\ X^+ &= \mathscr{G}^{(1/2)}_{\sigma+\delta,s+\eta}(A), & X^+_- &= \mathscr{G}^{(1/2)}_{\sigma+\delta,s+\eta-q}(A). \end{aligned} \tag{7.6}$$

Assume C0 and C2–C4 at both levels with compatible full-tree estimates from X to X_- and from X^+ to X^+_- with the same net loss q and the frequency support law (2.5) at every interaction node. Fix $T_0 > 0$. For every $M < \infty$ require the composed growth and Lipschitz profiles at both levels to admit a common nonnegative envelope $h_M \in L^p(0, T_0)$ such that

$$L_M(t) + P_M(t) + L^+_M(t) + P^+_M(t) \le h_M(t) \quad \text{for a.e. } t. \tag{7.7}$$

These profiles and h_M are independent of K. Let $u_0 \in X^+$ and $g \in L^p(0, T_0; X^+_-)$. Consider the exact equation and the finite-dimensional equations

$$\partial_t v_K + A v_K = P_K\big(\mathcal{N}_t(v_K) + g(t)\big), \qquad v_K(0) = P_K u_0. \tag{7.8}$$

Then they have unique local mild solutions u and v_K on a common interval $[0, T]$ where $0 < T \le T_0$ and there is $M < \infty$ independent of K such that all these solutions remain in the X^+ ball of radius M. If

$$B^+_M(t) := P^+_M(t) + \|g(t)\|_{X^+_-}, \tag{7.9}$$

where P^+_M is the strong-level growth profile from Proposition 6.1 then for almost every t the full-space residual of v_K is

$$r_K(t) = -(I - P_K)\big(\mathcal{N}_t(v_K(t)) + g(t)\big) \tag{7.10}$$

and satisfies for almost every t

$$\|r_K(t)\|_{X_-} \le a^\star_K B^+_M(t) \le \overline{a}_K B^+_M(t), \qquad \begin{aligned} a^\star_K &:= \sup_{\substack{\omega \in \mathrm{Spec}(B) \\ \omega > K}} e^{-\delta\omega}(1+\omega^2)^{-\eta/2}, \\ \overline{a}_K &:= e^{-\delta K}(1+K^2)^{-\eta/2}, \end{aligned} \tag{7.11}$$

where the supremum over an empty omitted spectrum is defined to be zero.

After decreasing T so that $H_M(T) < 1$ one has the uniform state error

$$\sup_{0 \le t \le T} \|u(t) - v_K(t)\|_X \le a^\star_K C_T \le \overline{a}_K C_T, \tag{7.12}$$

where the finite constant

$$C_T := \frac{1}{1 - H_M(T)} \left[\|u_0\|_{X^+} + \sup_{0 \le t \le T} \int_0^t \mathcal{K}_q(t-\tau) B^+_M(\tau)\, \mathrm{d}\tau \right] \tag{7.13}$$

is independent of K. Consequently $v_K \to u$ in $C([0, T]; X)$ and for every $\varepsilon > 0$ the set

$$\{K \in \mathrm{Spec}(B) : a^\star_K C_T \le \varepsilon\} \tag{7.14}$$

is nonempty and has a least element.

Proof. We choose the ball and interval before fixing the cutoff. P_K is contractive in every space in (7.6). Thus the projected nonlinearities $P_K\mathcal{N}_t$ inherit the same profiles and the envelope (7.7). Choose $M > \|u_0\|_{X^+}$ and shorten the interval until the strong Duhamel growth fits this margin and the strong and weak difference maps have contraction constants below one. Hölder's inequality gives this simultaneously for every K when $p > 1$. For $q = 0$ and $p = 1$ use absolute continuity of the integrals of h_M and $\|g\|_{X^+_-}$. Theorem 6.2 therefore gives the common existence time and ball and also $B^+_M \in L^p$ and $H_M(T) < 1$. The inclusion $X^+ \hookrightarrow X$ is contractive so the same radius M bounds every trajectory in the weaker space used for stability. For each fixed K the right-hand side of the projected equation is integrable in the finite-dimensional retained space so v_K is absolutely continuous there. Since A commutes with P_K moving (7.8) to the full space gives (7.10).

The residual is supported on the omitted spectrum where the strict resource gap makes its weaker norm small. For every spectral frequency $\omega > K$ the ratio of the weaker forcing weight to the stronger one is

$$\frac{e^{\sigma\omega}(1+\omega^2)^{(s-q)/2}}{e^{(\sigma+\delta)\omega}(1+\omega^2)^{(s+\eta-q)/2}} = e^{-\delta\omega}(1+\omega^2)^{-\eta/2} \le a^\star_K \le \overline{a}_K. \tag{7.15}$$

The strong growth estimate gives $\|\mathcal{N}_t(v_K) + g(t)\|_{X^+_-} \le B^+_M(t)$ and proves (7.11). Both forcing orders have been lowered by the same loss q so their weight ratio is also the ratio of the state weights. Applying it to the initial data gives

$$\|(I - P_K)u_0\|_X \le a^\star_K \|u_0\|_{X^+}. \tag{7.16}$$

Proposition 7.1 now yields (7.12) and (7.13). The Volterra term is finite by $qp'/2 < 1$ and $B^+_M \in L^p$. At $q = 0$ and $p = 1$ use the bounded kernel.

The initial strong norm and common source profile are fixed before a cutoff is selected as is the weaker stability number. These determine C_T. The factor $a^\star_K$ is nonincreasing and tends to zero. In finite dimension it is exactly zero once the full spectrum is retained. Compact resolvent makes (7.14) a nonempty spectral upper tail with a least element. □

Generated presentations of the Galerkin family. Under the hypotheses of Theorem 7.2 suppose in addition that the exact data satisfy all hypotheses of Theorem 6.4 for the fixed source and policy. After a further common shortening of the interval every $u(t)$ admits a presentation in $\mathscr{C}^{\mathrm{adm}}_{\mathfrak{G},\mathrm{rel}}$. The same holds for every $v_K(t)$ whose projection term P_K is admitted by the fixed operator language or by the predeclared external terminals. The presentations are intrinsic when all leaves and operator terms are intrinsic. To verify the claim apply an admitted projection term to the approximating initial and forcing histories and to the nonlinear expression. Its contractivity preserves the common tree-derived profiles and the convergent data defects. Theorem 6.4 therefore applies on one common interval to the exact syntax and all admitted projected syntaxes. The provenance of each projection term is retained in the resulting histories.

The convergence and least-threshold conclusions of Theorem 7.2 use only its analytic hypotheses. The additional source and language assumptions in the preceding paragraph serve only to retain presentations of the exact and projected flows.

Residual and reconstruction budget. Under Proposition 7.1 suppose $\|u(0) - v(0)\|_X \le E_0$ and $\|r_v(t)\|_{X_-} \le D(t)$ where $D \ge 0$ belongs to $L^p(0,T)$. Let $w(t)$ be the state used for observation with $\|v(t) - w(t)\|_X \le E_{\mathrm{post}}(t)$. Define

$$E_{\mathrm{flow}}(T) := \frac{E_0 + \sup_{0\le t\le T} \int_0^t \mathcal{K}_q(t-\tau) D(\tau)\,\mathrm{d}\tau}{1 - H_M(T)}, \tag{7.17}$$

$$E_{\mathrm{total}}(t) := E_{\mathrm{flow}}(T) + E_{\mathrm{post}}(t). \tag{7.18}$$

Then

$$\|u(t) - w(t)\|_X \le E_{\mathrm{total}}(t), \qquad 0 \le t \le T. \tag{7.19}$$

This follows by inserting D into (7.4) and applying the triangle inequality to $u - v + v - w$. If the residual is a sum of terms with verified L^p majorants $D_1, \dots, D_m$ then the triangle inequality in X_- permits $D = \sum_{j=1}^m D_j$.

The two terms in the bound distinguish the evolution error from the error incurred when the state is reconstructed for observation. A smooth change of spectral selector or a truncated memory integral contributes to D when its defect is bounded in X_-. A hard change of projection gives a state jump instead. Its X-norm is added to the incoming error and the local estimate restarts with this initial bound. The L^p forcing budget accounts for the time-integrable defects between restarts. The projection and memory terms together with the state-transfer terms are derived in Appendix F.

For a positive quadratic observation the state-error estimate has a direct but distinct consequence. Let $R : X \to \mathcal{Y}$ be bounded and $\mathcal{Q}_R(z) = \|Rz\|_{\mathcal{Y}}^2$. Then

$$|\mathcal{Q}_R(u) - \mathcal{Q}_R(w)| \le \|R\|_{X\to\mathcal{Y}} \|u - w\|_X \left(\|Ru\|_{\mathcal{Y}} + \|Rw\|_{\mathcal{Y}}\right). \tag{7.20}$$

Indeed write the difference as the real part of the inner product of $R(u-w)$ and $R(u+w)$. If both states lie in the X-ball of radius M then the right-hand side is at most $2\,\|R\|_{X\to\mathcal{Y}}^2\, M\, \|u - w\|_X$. Thus (7.19) certifies the error in a quadratic observation. A conservative ledger supplies additional information. It records how much of the same quantity lies in a defect or unresolved channel. The state-error bound and ledger answer these different questions.

7.3 A finite approximation of the evolution operator

A finite calculation may replace the operator as well as the state. A small residual for the finite equation then controls only one part of the error. The finite operator can act differently from the complete one on the same computed path. For a bounded positive response this difference can be included explicitly in the forcing budget. Contractivity of the complete semigroup controls its accumulated effect over a finite interval.

Proposition 7.3 (Operator, residual, and readout errors)**.** Let $\mathsf{L} \ge 0$ be bounded and self-adjoint on a Hilbert space and let $f(\tau) = e^{-\tau\mathsf{L}} f_0$. Suppose L_N is bounded $\|\mathsf{L} - \mathsf{L}_N\| \le \delta_N$ and w is absolutely continuous on $[0, S]$. Write $r_N = w' + \mathsf{L}_N w$. Then

$$\|f(\tau) - w(\tau)\| \le \|f_0 - w(0)\| + \int_0^\tau \|r_N(r)\|\,\mathrm{d}r + \delta_N \int_0^\tau \|w(r)\|\,\mathrm{d}r =: E_N(\tau). \tag{7.21}$$

For a bounded readout C and an approximation $\widehat{C}$ with $\|C - \widehat{C}\| \le \epsilon_C$ on the range containing w

$$\|Cf(\tau) - \widehat{C}w(\tau)\| \le \|C\|E_N(\tau) + \epsilon_C\|w(\tau)\|. \tag{7.22}$$

If L_N is self-adjoint then the response observation obeys

$$|\langle f, \mathsf{L}f\rangle - \langle w, \mathsf{L}_N w\rangle| \le \|\mathsf{L}\|(\|f\| + \|w\|)\|f - w\| + \delta_N\|w\|^2. \tag{7.23}$$

Proof. The residual in the full equation is $w' + \mathsf{L}w = r_N + (\mathsf{L} - \mathsf{L}_N)w$. Variation of constants for $f - w$ and $\|e^{-\tau\mathsf{L}}\| \le 1$ give (7.21). Split the readout error as $C(f - w) + (C - \widehat{C})w$ to obtain (7.22). Finally add and subtract $\langle w, \mathsf{L}w\rangle$ and expand the difference of the two quadratic forms with the same operator L. Cauchy–Schwarz proves (7.23). □

The three terms in (7.21) distinguish the initial discrepancy and the residual of the finite equation together with the effect of replacing its operator. If the residual is already measured against the complete equation then the operator contribution is part of that residual and is counted there once. A coisometric readout has $\|C\| = 1$ and inherits the state bound without amplification. Approximating the readout adds its own error. The quadratic estimate similarly separates state error from error in the observation operator. These distinctions allow a finite response and its ledger to certify quantities for the complete equation. For the nonlinear evolutions above the corresponding full residual is propagated with the stated nonlinear stability factor.

7.4 Exponential tails and tolerance bandwidth

The same comparison of spectral weights bounds the L^2 tail of a single state. Here only the minimum weight on the omitted spectrum is needed.

If $f \in \mathscr{G}^{(\theta)}_{\alpha,s}(A)$ then the weighted coefficient sum gives the following tail estimate. Define

$$W_{\theta,\alpha,s}(K) := \inf_{\substack{\lambda\in\mathrm{Spec}(A)\\ \lambda>K^2}} e^{\alpha\lambda^\theta}(1+\lambda)^{s/2}, \tag{7.24}$$

with the convention $\inf \varnothing = +\infty$. This quantity is positive and

$$\|(I - P_K)f\|_2 \le W_{\theta,\alpha,s}(K)^{-1}\,\|f\|_{\theta,\alpha,s;A}\,. \tag{7.25}$$

If $s \ge 0$ this implies the simpler bound

$$\|(I - P_K)f\|_2 \le e^{-\alpha K^{2\theta}}(1+K^2)^{-s/2}\,\|f\|_{\theta,\alpha,s;A}\,. \tag{7.26}$$

For $\alpha > 0$ and arbitrary s the minimal weight tends to infinity as $K \to \infty$. For $\alpha = 0$ and $s < 0$ with A having unbounded spectrum no uniform L^2 tail bound follows from the negative-order norm alone. On the support of $I - P_K$ one pulls the infimum of the weight in (2.8) out of the omitted spectral sum. For $s \ge 0$ the weight is nondecreasing and its infimum is bounded below by its value at K^2. Exponential growth dominates every fixed negative power when $\alpha > 0$. If $\alpha = 0$ and $s < 0$ the weights instead tend to zero along an unbounded spectrum so they cannot control the omitted L^2 coefficients uniformly.

This estimate also separates individual approximation from a uniform window. Suppose $\mathcal{H}$ is infinite-dimensional and A has compact resolvent. For each fixed $f \in \mathcal{H}$ square summability of its spectral coefficients gives $\|(I - P_K)f\|_2 \to 0$. On the unit ball

$$\sup_{\|f\|_2 \le 1} \|(I - P_K)f\|_2 = 1 \qquad \text{for every finite } K. \tag{7.27}$$

The upper bound follows from orthogonality. A normalized eigenmode with $\lambda_j > K^2$ attains it. Thus a common L^2 bound gives individual approximation but no uniform small-error window on that ball. By (7.25) a family with common bound M in a fixed weighted norm has tail at most $M/W_{\theta,\alpha,s}(K)$. When $W_{\theta,\alpha,s}(K) \to \infty$ this supplies a uniform window for each tolerance with the error determined by the common certificate.

For a positive heat time the same tail estimate gives

$$\left\|(I - P_K)e^{-tA}g\right\|_2 \le e^{-tK^2} \|g\|_2 . \tag{7.28}$$

Since $e^{-t\lambda} > 0$ no nonzero spectral coefficient is annihilated. The decreasing tail nevertheless supplies a window for a prescribed accuracy.

7.5 The least certified spectral threshold

An observation family $\mathcal{O}$ is used on a specified bounded set $\mathcal{K} \subset X$. Condition C5 is the existence of a common increasing right-continuous modulus $\omega_{\mathcal{O},\mathcal{K}}$ with value zero at zero such that

$$|O(f) - O(g)| \le \omega_{\mathcal{O},\mathcal{K}}(\|f - g\|_X), \qquad O \in \mathcal{O}, \quad f, g \in \mathcal{K}. \tag{7.29}$$

For bounded linear observations one may use their common operator norm times the argument. Point evaluation is covered whenever the resource space embeds into continuous functions by Theorem 2.2. The general modulus also permits nonlinear observations on the bounded set actually visited by the trajectories.

Corollary 7.4 (Finite observation windows)**.** Under Theorem 7.2 assume that u and all v_K lie in a set with modulus (7.29). Then

$$\sup_{0 \le t \le T} \sup_{O \in \mathcal{O}} |O(u(t)) - O(v_K(t))| \le E_{\mathcal{O}}(K) := \omega_{\mathcal{O},\mathcal{K}}(a_K^\star C_T). \tag{7.30}$$

For each $\varepsilon > 0$ the set of spectral thresholds with $E_{\mathcal{O}}(K) \le \varepsilon$ has a least element

$$K_\varepsilon(\mathcal{O}) = \min\{K \in \mathrm{Spec}(B) : E_{\mathcal{O}}(K) \le \varepsilon\}. \tag{7.31}$$

This is the least threshold certified by the displayed error bound.

Proof. Apply the common modulus to (7.12). Since $a_K^\star \to 0$ right continuity at zero gives $E_{\mathcal{O}}(K) \to 0$. The spectral set is discrete and bounded below and has finite intersection with bounded intervals. A nonempty upper tail therefore has a least element. In finite dimension the full projection has zero error. □

Suppose also that a ledger supplies a nonincreasing bound $U(K) \downarrow 0$ for the unresolved part of a declared quadratic observation. Given separate tolerances $\varepsilon_{\mathrm{obs}}, \varepsilon_{\mathrm{led}} > 0$ the joint feasible set is

$$\{K \in \mathrm{Spec}(B) : E_{\mathcal{O}}(K) \le \varepsilon_{\mathrm{obs}}, \quad U(K) \le \varepsilon_{\mathrm{led}}\}. \tag{7.32}$$

It has a least element whenever it is nonempty. Under the stated decay it is nonempty in the same infinite-dimensional setting as Corollary 7.4. In finite dimension the full projection suffices when its unresolved ledger entry vanishes. This joint threshold certifies both the state-based observation error and the quadratic quantity left outside the retained channel.

A sharper state estimate or observation modulus may certify a smaller threshold in (7.31) within the same spectral set.

The certified threshold is where the infinite-dimensional analysis becomes a finite observational statement. It determines how many modes suffice for the stated observation bound. A computational realization also specifies how the coefficients are produced in finite arithmetic and how a submitted record is accepted. Its cost bounds concern both tasks.

For a Lipschitz observation modulus the sufficient cutoff is explicit. Suppose $\omega_{\mathcal{O},\mathcal{K}}(r) = L_{\mathcal{O}} r$ and $L_{\mathcal{O}} C_T > 0$. Every spectral threshold satisfying

$$K \ge \frac{1}{\delta} \max\left\{0, \log \frac{L_{\mathcal{O}} C_T}{\varepsilon}\right\} \tag{7.33}$$

has observation error at most ε. In finite dimension the full projection always suffices. If $L_{\mathcal{O}} C_T = 0$ every threshold suffices. Indeed since $\eta \ge 0$ (7.12) bounds the observation error by $L_{\mathcal{O}} C_T e^{-\delta K}$.

The cutoff depends on the stronger norm bounds and time profiles of the nonlinear and forcing terms over the whole interval. Additional approximation errors are included by applying the modulus in (7.29) to the total state bound (7.19). In that case the comparison set must also contain the reconstructed state. The general modulus applies to entropy of a spectral probability state. Appendix C of the companion computation paper [12] derives its common modulus from an exponential certificate and a trace bound with the normalization hypotheses stated explicitly.

Example 4.1 shows how a propagated constant enters a sufficient cutoff for one state. For an evolution the initial bound and strong source profile together with the stability factor determine C_T in (7.13). When histories supply these estimates over the common interval a smaller propagated bound can improve the observation budget while leaving the solution unchanged.

The common stronger bound in Figure 1 controls both the discarded source terms in the residual estimate and the approximating histories in the presentation theorem. The latter conclusion also requires the stated source and time-construction hypotheses.

8 Neumann and Stokes–Fourier realizations

On the Neumann interval even periodic extension reduces the product estimates to Fourier convolution. The resulting bounds describe the local evolution and its Galerkin approximations and their observation errors in cosine coordinates. For the Stokes realization on the

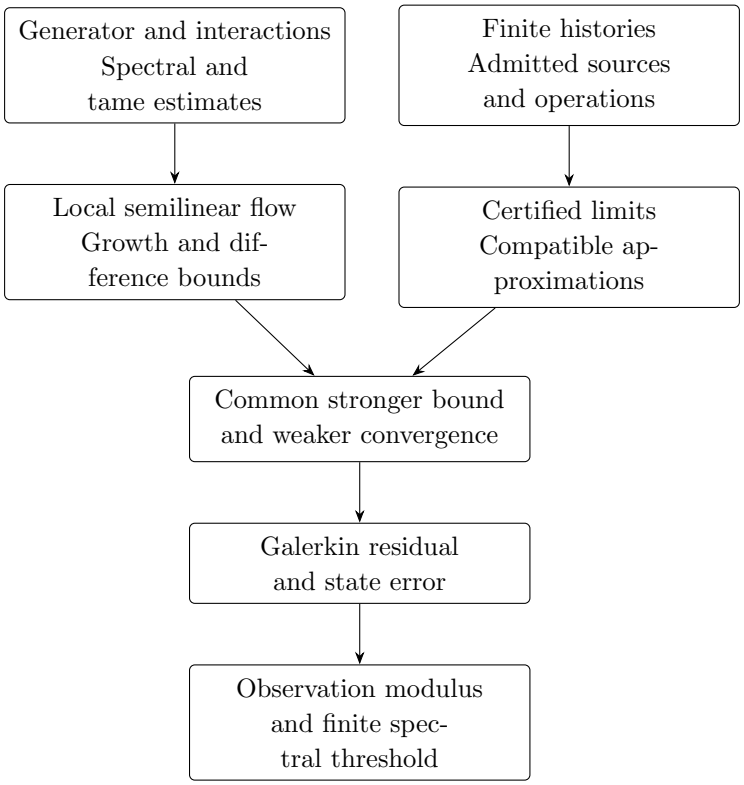

Figure 1. The left branch gives the local flow of Theorem 6.2 and the residual and observation bounds of Theorem 7.2 and Corollary 7.4. On the right Theorem 4.4 passes operations to certified limits and Theorem 6.4 retains presentations of the flow under its approximation and time-construction hypotheses. The two branches use a common stronger bound.

torus the Leray projection is contractive in the spectral norms while transport introduces one derivative of loss. Nonvanishing triads remain after projection and give the sharpness of the analytic exponent. The Neumann source also gives a conservative Gaussian endpoint cell and a coisometric value compression. These realize respectively the retained–defect and value–expression ledgers of section 5.

In both realizations the admitted operations are present in the model equations. The Neumann case uses reflection and scalar multiplication. The Stokes case uses the divergence-free constraint and Leray projection together with the transport interaction. Their role is to make the provenance of the construction explicit. Histories built from these declared operations are model-native relative to the declared source system. The reflected interval and constrained fluid equation provide the respective physical interpretations.

8.1 The Neumann correlation interval

Two correlation lengths fix the source interval and its Neumann frequency scale. For $b > 0$ take the endpoints

$$\ell_{\mathrm{UV}} = \sqrt{2}\, b, \qquad L_{\mathrm{IR}} = \sqrt{2\pi}\, b, \qquad L := \Delta_b = L_{\mathrm{IR}} - \ell_{\mathrm{UV}}. \tag{8.1}$$

Put $\mathcal{W}_b = [\ell_{\mathrm{UV}}, L_{\mathrm{IR}}]$. On $\mathcal{H}_b := L^2(\mathcal{W}_b)$ let A_b be the Neumann Laplacian with domain

$$\mathrm{Dom}\, A_b = \{f \in H^2(\mathcal{W}_b) : f'(\ell_{\mathrm{UV}}) = f'(L_{\mathrm{IR}}) = 0\}. \tag{8.2}$$

Translation by ℓ_{UV} gives a unitary $\mathcal{T}_b : \mathcal{H}_b \to L^2(0, L)$ and identifies A_b with

$$A_N := \mathcal{T}_b A_b \mathcal{T}_b^{-1} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad \mathrm{Dom}\, A_N = \{f \in H^2(0, L) : f'(0) = f'(L) = 0\}. \tag{8.3}$$

An orthonormal eigenbasis is

$$\phi_0 = L^{-1/2}, \qquad \phi_n = (2/L)^{1/2} \cos(n\pi x/L), \qquad \omega_n = n\pi/L. \tag{8.4}$$

On the source interval we write $\phi_{b,n} := \mathcal{T}_b^{-1}\phi_n$.

The semigroup convention on the translated interval is $T_\tau = e^{-\tau A_N}$. The minimal-change normalization uses

$$C_b(t) = e^{-tA_b/2}, \qquad \mathcal{T}_b C_b(t) \mathcal{T}_b^{-1} = T_{t/2}. \tag{8.5}$$

Thus change time t supplies Gaussian reserve $\alpha = t/2$ in the notation of section 2. The derivation of these source conventions including the length prescriptions and reflecting closure is given in Appendix C under its stated assumptions.

Translation preserves the spectrum and the norm estimates. The calculations use the translated modes whose source representatives are $\mathcal{E}_b^0 = \{\phi_{b,n} : n \geq 0\}$. The nonlinear operation is pointwise multiplication with coefficients given by the normalized cosine product law.

For the source language we take $\mathfrak{M}_{\mathfrak{G}} = \mathcal{E}_b^0$ and $\mathbb{K}_{\mathfrak{G}} = \mathbb{Q}$ and the standard inclusion $\mathbb{Q} \hookrightarrow \mathbb{R}$. Ambient analysis is performed over $\mathbb{F} = \mathbb{R}$. The normalized modes including their scale-dependent amplitudes are source terminals. Their rational linear combinations consequently have intrinsic finite histories. A product may have nonrational modal coefficients but these are determined by the declared multiplication tensor and retain that origin in the history. An independently supplied real coefficient is a labelled relative terminal unless an intrinsic term represents it. Thus the real finite cosine span is the algebra used in the estimates. Its rational source span need not itself be an algebra.

Proposition 8.1 (Neumann certification)**.** The datum (8.3) verifies C0 and C1 with spectral dimension $d = 1$. Its heat semigroup is conservative and symmetric and has the Markov property. Pointwise multiplication is polynomially tame on $\mathcal{H}^s_{A_N}$ for $s > 1/2$ and its interaction tensor obeys the frequency triangle law. The critical exponent in Corollary 3.4 is sharp.

Proof. The Neumann form on $H^1(0, L)$ defines the nonnegative self-adjoint operator (8.3) with compact resolvent. Even extension identifies its eigenbasis and heat flow with the even periodic ones. This gives a conservative symmetric Markov semigroup and strong continuity on continuous functions [13, 14]. Thus C0 and C1b hold. The normalized modes give the explicit C1a bound

$$\sum_{\omega_n \leq K} |\phi_n(x)|^2 \leq L^{-1} + 2L^{-1}\lfloor LK/\pi \rfloor \leq C_L(1 + K). \tag{8.6}$$

Each derivative contributes at most ω_n^ℓ to a mode so summing $\omega_n^{2\ell}$ below K gives C1c. The elementary bound $|\sin a - \sin b|, |\cos a - \cos b| \le \min\{2, |a-b|\}$ also gives

$$\left|D^\ell\phi_n(x) - D^\ell\phi_n(y)\right| \le C_{\ell,\gamma,L}\omega_n^{\ell+\gamma}\,|x-y|^\gamma\,, \qquad 0<\gamma\le 1, \tag{8.7}$$

and summation verifies C1d. For the interaction condition the product identity

$$2\cos(m\pi x/L)\cos(n\pi x/L) = \cos((m+n)\pi x/L) + \cos((m-n)\pi x/L) \tag{8.8}$$

places every output at frequency ω_{m+n} or $\omega_{|m-n|}$. To control the interaction without relying on cancellation extend the absolute cosine coefficients evenly to $\mathbb{Z}$. The normalized product coefficients are dominated by a fixed L-dependent multiple of convolution on this lattice. With $\langle n\rangle_L = (1+(\pi n/L)^2)^{1/2}$ and $s > 1/2$

$$\|a\|_{\ell^1} \le \Big(\sum_{n\in\mathbb{Z}}\langle n\rangle_L^{-2s}\Big)^{1/2}\|\langle n\rangle_L^s a_n\|_{\ell^2}. \tag{8.9}$$

The inequality $\langle m+n\rangle_L^s \le C_s(\langle m\rangle_L^s + \langle n\rangle_L^s)$ and Young's inequality therefore give

$$\|a*b\|_{\ell^2_s} \le C_s\big(\|a\|_{\ell^2_s}\|b\|_{\ell^1} + \|a\|_{\ell^1}\|b\|_{\ell^2_s}\big) \le C_{s,L}\|a\|_{\ell^2_s}\|b\|_{\ell^2_s}. \tag{8.10}$$

This proves the absolute-majorant estimate. To check sharpness take a sequence of squares. For $n \ge 1$

$$\phi_n^2 = L^{-1/2}\phi_0 + (2L)^{-1/2}\phi_{2n}. \tag{8.11}$$

The coefficient of ϕ_{2n} is independent of n and the frequency triangle is saturated exactly. This verifies quantitative non-degeneracy for every $\beta > 1$. □

Endpoint and expression ledgers. The source construction has two further structures that realize the abstract ledger. Between two Gaussian scale times $t_k < t_{k+1}$ put

$$M_k = \exp\left[-\frac{t_{k+1}-t_k}{4}A_b\right], \qquad D_k = (I - M_k^2)^{1/2}. \tag{8.12}$$

Functional calculus gives

$$\|M_k x\|_2^2 + \|D_k x\|_2^2 = \|x\|_2^2\,. \tag{8.13}$$

More generally every bounded positive spectral response $\mathsf{Q} = q(A_b)$ obeys

$$\langle x, \mathsf{Q}x\rangle = \langle M_k x, \mathsf{Q}M_k x\rangle + \langle D_k x, \mathsf{Q}D_k x\rangle\,. \tag{8.14}$$

The first term is retained at the next scale and the second is its positive complement. This is the Neumann instance of Proposition 5.1. The scale choice $t_k = \pi^k b^2$ and the corresponding bilateral conservative cascade are derived in [1]. Only the cell identity is needed here.

The source also distinguishes an arithmetic value from the reduced expressions evaluating to it [1]. A probability law on each finite expression fibre specifies a coherent value readout. Equation (5.22) then keeps the orthogonal difference component beside that

readout. The fibre law determines the readout direction. Uniform counting and recursive root-channel weights can select different initial directions.

Finiteness also permits comparison of certified histories without identifying them. If a recursively evaluated scalar budget $B(e)$ is attached to every expression e in the fibre of a fixed value n then

$$B_*(n) = \min_{\mathrm{Eval}(e)=n} B(e) \tag{8.15}$$

exists. For rational certified budgets it can be found by finite enumeration and exact comparison. Computable real budgets instead admit selection within any prescribed positive error by finite approximation. For a tuple of rational partially ordered resources the corresponding object is the finite Pareto-minimal frontier. This is an effectiveness statement for each fixed fibre. The cost of explicit enumeration depends on the fibre cardinality whose bound in [1] is exponential.

For a scalar polynomial repeated use of the product estimate gives the growth and difference bounds on a fixed ball. Each monomial is multiplied by a single time-dependent coefficient. The resulting profiles are therefore finite sums of the absolute coefficients with constant factors so their local integrability follows from that of the coefficients.

Scalar polynomial flows. Let $s > 1/2$ and $\sigma_0 \geq 0$ and

$$F_t(z) = \sum_{m=0}^{m_*} c_m(t) z^m, \qquad c_m \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{R}). \tag{8.16}$$

If $u_0 \in \mathscr{G}^{(1/2)}_{\sigma_0,s}(A_N)$ and $g \in L^1_{\mathrm{loc}}([0,\infty);\mathscr{G}^{(1/2)}_{\sigma_0,s}(A_N))$ then

$$\partial_t u + A_N u = F_t(u) + g(t), \qquad u(0) = u_0, \tag{8.17}$$

has a unique local mild solution in $C([0,T];\mathscr{G}^{(1/2)}_{\sigma_0,s}(A_N))$. Here the net loss is $q = 0$ so the endpoint $p = 1$ in Theorem 6.2 applies. If the data and their generated approximations satisfy all hypotheses of Theorem 6.4 for a fixed policy $\mathfrak{A}$ then each $u(t)$ admits a relative generated presentation and belongs to $\mathcal{N}^{\mathfrak{A}}_{\mathfrak{G},\mathrm{rel}}$. For the Galerkin conclusion suppose there are $\delta > 0$ and $\eta \geq 0$ such that

$$\begin{aligned} &u_0 \in \mathscr{G}^{(1/2)}_{\sigma_0+\delta,s+\eta}(A_N), \\ &g \in L^1_{\mathrm{loc}}\big([0,\infty);\mathscr{G}^{(1/2)}_{\sigma_0+\delta,s+\eta}(A_N)\big). \end{aligned} \tag{8.18}$$

Then the spectral Galerkin solutions satisfy Theorem 7.2 on a common local interval. To verify these claims Proposition 8.1 and Corollary 3.4 make the analytic class a Banach algebra. Repeated multiplication gives on every ball of radius M growth and Lipschitz profiles bounded by finite sums of $C_{m,s,L}|c_m(t)|M^m$ and $C_{m,s,L}m|c_m(t)|M^{m-1}$ respectively with the constant term treated as $|c_0(t)|\|\mathbf{1}\|_2 = \sqrt{L}|c_0(t)|$. These profiles are locally integrable and multiplication has no derivative loss. With (8.18) the estimates at both orders s and $s+\eta$ give a common L^1 envelope independent of the window. Theorem 6.2 gives the local solution. The additional presentation and stronger-data hypotheses allow us to apply Theorems 6.4 and 7.2 respectively.

Example 8.2 (Heat-prepared data and a generated nonlinear flow). Fix $s > 1/2$ together with a positive rational heat time τ and radii $0 \le \sigma_- < \sigma_+ < \sigma_{++}$. Take $\boldsymbol{\rho}_\bullet = (1/2, \sigma_\bullet, s)$ and a policy admitting $(\boldsymbol{\rho}_{++}, \boldsymbol{\rho}_-)$. Consider

$$\partial_t u + A_N u = u^2, \qquad u(0) = \sum_{j=0}^{\infty} 2^{-j} e^{-\tau\omega_j^2} \phi_j. \tag{8.19}$$

These data satisfy the approximation and time-construction hypotheses of Theorem 6.4. The resulting presentations are intrinsic when rational heat times and the Riemann sums used for time integration belong to the intrinsic language of the Neumann source.

Define the finite initial histories by

$$u_{0,n} = e^{-\tau A_N} \sum_{j=0}^{n} 2^{-j} \phi_j, \qquad M_0 = 2 \sup_{\omega \ge 0} e^{\sigma_{++}\omega - \tau\omega^2} (1+\omega^2)^{s/2}. \tag{8.20}$$

Gaussian decay dominates the analytic and polynomial weights so the supremum is finite. The finite sum has L^2 norm less than two and gives the common stronger certificate $[\boldsymbol{\rho}_{++}; M_0]$. An explicit upper bound follows by splitting the heat time into two equal parts and using $e^{\sigma_{++}\omega - \tau\omega^2/2} \le e^{\sigma_{++}^2/(2\tau)}$ and applying (6.8) with $q = s$ to the remaining factor. This larger explicit bound may replace M_0 throughout. The gap between the strongest and intermediate radii then gives the following estimate for the omitted coefficients

$$\|u_0 - u_{0,n}\|_{\boldsymbol{\rho}_+; A_N} \le M_0 e^{-(\sigma_{++} - \sigma_+)\omega_{n+1}}. \tag{8.21}$$

Take $g_n = g = 0$ and $\mathcal{N}^{(n)}(z) = \mathcal{N}(z) = z^2$. The forcing and nonlinearity approximation defects are zero. The cosine product estimate with the analytic triangle law gives at each of the three radii

$$P_M = C_{s,L} M^2, \qquad L_M = 2 C_{s,L} M. \tag{8.22}$$

Thus the same constant envelope bounds the complete growth and difference profiles in $L^\infty(0,T)$ with $q = 0$.

For rational $t \in [0,T]$ the time integrals admit approximations on positive rational subdivisions of $[0,t]$. At every fixed n and Picard depth the integrand has finitely many cosine modes and continuous time coefficients. The heat operation preserves its finite support and a product only adds the finitely many input frequencies. Its Bochner integral is consequently the limit in the stronger norm of Riemann sums at rational times. Each sum is a finite history of the admitted heat and product operations.

If the preceding Picard histories have bound M then the nonnegative quadrature weights give the same bound $tC_{s,L}M^2$ for the integral term. Choose $M > M_0$ and then T so that this term fits the margin $M - M_0$ and the local contraction test holds. The resulting estimate gives the common tree bound and verifies rule (E4) at every Picard level. Repeating the argument at each finite level treats nested integrals. All scalar weights and heat parameters in these histories are rational or source terms. The dense time set and certified completion then give the asserted presentations at every $t \in [0,T]$.

The strongest radius also supplies the gap required by Theorem 7.2. At target radius σ_+ set $\delta = \sigma_{++} - \sigma_+$ and $\eta = 0$. For $K = N\pi/L$ the first omitted frequency is $(N+1)\pi/L$ so

$$\sup_{0\le t\le T} \|u(t) - v_{N\pi/L}(t)\|_{\boldsymbol{\rho}_+;A_N} \le C_T e^{-\delta(N+1)\pi/L}. \tag{8.23}$$

Here C_T is (7.13) with the constant source profile $B_M^+ = C_{s,L} M^2$.

The same source modes serve other admitted polynomials. Their finite rational combinations have constant presenting sequences on permitted strict slices and hence are intrinsic generatively regular functions. Changing to unnormalized cosine coordinates retains the provenance of the scalar conversion factors. The operation clause of Theorem 4.4 preserves this status under admitted polynomial operations. The conclusion of Theorem 6.4 applies to the local flow when every leaf and operator term in the approximating trees is intrinsic.

The state estimate controls three familiar observations with explicit constants. The normalized mean $O_{\mathrm{av}}(f) = L^{-1}\int_0^L f(x)\,\mathrm{d}x$ has operator norm $L^{-1/2}$ on $L^2(0,L)$. Point evaluation $O_x(f) = f(x)$ has a common bound $C_{s,L}^{\mathrm{ev}}\|f\|_{\mathcal{H}^s_{A_N}}$ for all $x \in [0,L]$ when $s > 1/2$ by Theorem 2.2. These two families therefore have linear moduli in the analytic state norm at nonnegative radius with the displayed constants in Corollary 7.4.

The energy observation $O_{\mathrm{en}}(f) = \|f\|_2^2/2$ has Lipschitz constant M on the same radius-M ball since $|\|f\|_2^2 - \|h\|_2^2| \le 2M\|f-h\|_2$ there. A common constant for the mean and all point evaluations together with this energy is $L_{\mathcal{O}} = \max\{L^{-1/2}, C_{s,L}^{\mathrm{ev}}, M\}$. Example 8.2 then gives the explicit sufficient band

$$N_\varepsilon = \max\left\{0, \left\lceil \frac{L}{\pi\delta}\log\frac{L_{\mathcal{O}}C_T}{\varepsilon} - 1\right\rceil\right\}, \qquad L_{\mathcal{O}}C_T > 0. \tag{8.24}$$

This is the least nonnegative integer for which the state-error bound in (8.23) multiplied by the chosen common observation constant meets the tolerance.

For comparison with cosine coefficient estimates on $[0,\pi]$ the unnormalized expansion $f = \sum_{j\ge0} a_j\cos(jx)$ has normalized coefficients $\sqrt{\pi}\,a_0$ and $\sqrt{\pi/2}\,a_j$ for $j \ge 1$. Consequently

$$\|f\|_{1/2,\log 2,1;A_N} \le \sqrt{\pi}\sum_{j\ge0} 4^j|a_j|, \qquad \|f\|_{\mathcal{H}^1_{A_N}} \le \sqrt{\pi}\sum_{j\ge0}(1+j)|a_j|. \tag{8.25}$$

The first inequality is the convention used in Example 4.1. Both follow termwise from the orthonormal expansion and the triangle inequality. This fixes the norm conversion used by the companion computation paper [12].

For linear heat flow the largest omitted diagonal multiplier gives the exact operator norm of the truncation error and hence the worst error over the unit input ball. If $P_{b,N} = \sum_{n=0}^N |\phi_{b,n}\rangle\langle\phi_{b,n}|$ then

$$\|(I - P_{b,N})C_b(t)\|_{\mathcal{B}(\mathcal{H}_b)} = \exp\left[-\frac{\pi^2 t}{2L^2}(N+1)^2\right]. \tag{8.26}$$

Consequently for $t > 0$ and an operator tolerance $0 < \varepsilon < 1$ the least nonnegative band index meeting this uniform tolerance is

$$N_{\mathrm{eff}}(t,\varepsilon) = \max\left\{0, \left\lceil\sqrt{\frac{2L^2\log(1/\varepsilon)}{\pi^2 t}} - 1\right\rceil\right\}. \tag{8.27}$$

The threshold corresponding to this band is $K_N = N\pi/L$. We take the source parameter language to contain this formula for integer $N \geq 0$ so these projections have finite descriptions as required for their generated presentations. The norm identity also holds independently of that syntax choice. For the nonlinear flow the residual and stability estimates in sections 6 and 7 supply the additional terms in the observation budget.

8.2 History sources and their certified response

The Neumann source also admits a description in which distinct expressions with the same value remain separate coordinates [1]. Reading only the value combines those coordinates. To determine whether this reduction preserves an evolution one must know how the response acts on the differences it removes. This realization makes that issue explicit. It also shows how a normalized source controls spectral bounds on a space with many histories at each root.

Let $\mathscr{E}_n$ be the reduced ordered positive expressions of value n subject to the source unit and absorbing-zero reductions. Equivalently $\mathscr{E}_1 = \{1\}$ and $\mathscr{E}_n$ is the disjoint union of the child products $\mathscr{E}_a \times \mathscr{E}_b$ over ordered additive channels $a + b = n$ with $a, b \geq 1$ and multiplicative channels $ab = n$ with $a, b \geq 2$. The operation label is retained. Every child root is smaller than its parent so this recursion gives finite fibres. There are $c_n = n - 1 + \mathrm{d}(n) - 2$ channels at $n \geq 2$ where $\mathrm{d}(n)$ is the number of positive divisors. The source law is

$$p(1) = 1, \qquad p(\mathrm{node}_c(e_L, e_R)) = c_n^{-1} p(e_L) p(e_R), \qquad n = n(e). \tag{8.28}$$

The factor c_n^{-1} selects a root channel and the two child laws then distribute its mass over the histories within that channel. Equal root channel weights therefore need not give equal probabilities to complete expressions. At root four the four unbalanced additive trees have probability $1/8$ and the balanced additive and multiplicative trees have probability $1/4$ each.

On $\mathcal{H}_{\mathrm{hist}} = \ell^2(\bigcup_{n\geq 1} \mathscr{E}_n)$ define $\widetilde{A}|e\rangle = n(e)^2|e\rangle$ and let P_N^{hist} retain roots at most N. The value space is $\ell^2(\mathbb{N}_{>0})$. These are dimensionless root coordinates. The physical Neumann eigenvalue is $E_{b,1} n^2$ with $E_{b,1} = \pi^2/L^2$. Thus the source mass parameter is $\beta = tE_{b,1}/2$ under (8.5). The zero mode can be adjoined as a separate coordinate. No pointwise realization of the history coefficients is needed.

Proposition 8.3 (Source normalization and spectral certificates)**.** For $\beta > 0$ put

$$I_\beta(e) = e^{-\beta n(e)^2} p(e), \qquad \Psi_\beta = \sum_e \sqrt{I_\beta(e)}\, |e\rangle, \qquad C_p|e\rangle = \sqrt{p(e)}\, |n(e)\rangle. \tag{8.29}$$

Then $\sum_{e\in\mathscr{E}_n} p(e) = 1$ and $C_p C_p^* = I$. Moreover

$$\|\Psi_\beta\|^2 = \sum_{n\geq 1} e^{-\beta n^2}, \qquad C_p \Psi_\beta = \sum_{n\geq 1} e^{-\beta n^2/2} |n\rangle, \tag{8.30}$$

$$\|\Psi_\beta\|^2_{\boldsymbol{\rho};\widetilde{A}} = \sum_{n\geq 1} e^{-\beta n^2 + 2\alpha n^{2\theta}} (1 + n^2)^s. \tag{8.31}$$

The last sum is finite when $0 < \theta < 1$ and $\alpha \geq 0$ and $s \geq 0$. For $\theta = 1$ it is finite when $0 \leq 2\alpha < \beta$ and $s \geq 0$. Its tail gives the exact squared norm of $(I - P_N^{\mathrm{hist}})\Psi_\beta$ in that resource.

Proof. Induct on the root. Each child product has total conditional mass one and the c_n channel weights sum to one. This proves normalization. The vectors $C_p^*|n\rangle = \sum_{e \in \mathscr{E}_n} \sqrt{p(e)}|e\rangle$ are orthonormal and hence prove coisometry. Sum the squared source coefficients on each fibre to obtain both norm formulas and apply C_p to obtain the readout. For $\theta < 1$ the negative quadratic exponent dominates $2\alpha n^{2\theta} + s\log(1+n^2)$. For $\theta = 1$ the remaining quadratic coefficient is $-(\beta - 2\alpha) < 0$. This proves convergence. □

At root n summing over the histories recovers exactly the mass $e^{-\beta n^2}$. The spectral norm consequently depends on this mass and the root weight rather than on the number of expressions sharing the root. All history coordinates remain present. Normalization controls their total contribution to the certificate. Choose an admissible strict resource pair satisfying the displayed conditions at the stronger level. Finite source sums have a common strong certificate and converge in the weaker norm. Square roots of the rational probabilities can be approached from below by rational numbers. Applying the native root Gaussian multiplier to these finite sums preserves the same certificate. This gives an intrinsic presentation of Ψ_β in the lifted source language. When β is effectively given with a known positive lower bound then Gaussian tail estimates and rational square-root approximation also supply an effective modulus.

The parent–child response. For a nonleaf e put $b_e = |e\rangle - |e_L\rangle - |e_R\rangle$ and set

$$\mathsf{L}_{\beta,N} = \sum_{\substack{e \text{ nonleaf} \\ n(e) \leq N}} I_\beta(e)|b_e\rangle\langle b_e|. \tag{8.32}$$

The quadratic form of each summand measures the squared difference between the parent amplitude and the sum of its child amplitudes weighted by the source mass. These operators use the root law and Gaussian factors together with coordinate contractions. These are included among the native finite operations on the lifted source. The parent differs from its children. Repeated children give coefficient -2 so $\|b_e\|^2$ is three or five. The cutoff in (8.32) selects parent relations. An omitted relation may still act on retained child coordinates. This explains why this cutoff is generally different from the coordinate compression $P_N^{\mathrm{hist}}\mathsf{L}_\beta P_N^{\mathrm{hist}}$.

Proposition 8.4 (Weighted response and generated evolution)**.** For $\beta > 0$ the operators (8.32) converge in trace norm to a nonnegative self-adjoint operator L_β. For $N \geq 1$

$$\delta_N := \|\mathsf{L}_\beta - \mathsf{L}_{\beta,N}\| \leq 5 \sum_{n>N} e^{-\beta n^2} \leq \frac{5e^{-\beta N^2}}{2\beta N}. \tag{8.33}$$

Write $w_{\boldsymbol{\rho}}(n^2) = e^{\alpha n^{2\theta}}(1+n^2)^{s/2}$ with $s \geq 0$. If

$$\kappa_{\boldsymbol{\rho},\beta} := 5\sum_{n \geq 2} e^{-\beta n^2} w_{\boldsymbol{\rho}}(n^2) < \infty, \qquad \delta_{\boldsymbol{\rho},N} := 5\sum_{n>N} e^{-\beta n^2} w_{\boldsymbol{\rho}}(n^2), \tag{8.34}$$

then on $X_{\boldsymbol{\rho}} = \mathscr{G}^{(\theta)}_{\alpha,s}(\widetilde{A})$

$$\|\mathsf{L}_{\beta,N}\|,\ \|\mathsf{L}_{\beta}\| \le \kappa_{\boldsymbol{\rho},\beta}, \qquad \|\mathsf{L}_{\beta} - \mathsf{L}_{\beta,N}\| \le \delta_{\boldsymbol{\rho},N} \longrightarrow 0. \tag{8.35}$$

These series converge for $\theta < 1$ and every fixed $\alpha \ge 0$ or for $\theta = 1$ and $\alpha < \beta$. On a bounded interval $0 \le \tau \le S$

$$\|e^{-\tau \mathsf{L}_{\beta,N}}\|_{X_{\boldsymbol{\rho}} \to X_{\boldsymbol{\rho}}} \le e^{S\kappa_{\boldsymbol{\rho},\beta}}, \tag{8.36}$$

$$\sup_{\tau \le S} \|e^{-\tau \mathsf{L}_{\beta,N}} - e^{-\tau \mathsf{L}_{\beta}}\|_{X_{\boldsymbol{\rho}} \to X_{\boldsymbol{\rho}}} \le S e^{S\kappa_{\boldsymbol{\rho},\beta}} \delta_{\boldsymbol{\rho},N}. \tag{8.37}$$

Consequently this evolution preserves generated presentations on strict resource pairs for which both levels satisfy these bounds provided the native time parameters and output pair are admitted by the policy.

Proof. A positive rank-one summand has trace $I_\beta(e)\|b_e\|^2$. Summing on a root fibre bounds it by $5e^{-\beta n^2}$. This proves trace-norm convergence and the first tail estimate. Since $e^{-\beta x^2}$ decreases the tail is bounded by its integral from N to infinity. This is at most $e^{-\beta N^2}/(2\beta N)$.

Let $W = w_{\boldsymbol{\rho}}(\widetilde{A})$. On finite coordinates the weighted rank-one norm is $\|Wb_e\|\|W^{-1}b_e\|$. Every child root is smaller than $n(e)$ and $1 \le w_{\boldsymbol{\rho}}(n^2)$ is nondecreasing. Thus $\|Wb_e\| \le \sqrt{5}\, w_{\boldsymbol{\rho}}(n(e)^2)$ and $\|W^{-1}b_e\| \le \sqrt{5}$. Summing the resulting bound with the normalized root masses proves (8.35). The limits agree with the Hilbert-space operator by density and continuous inclusion of $X_{\boldsymbol{\rho}}$ in $\mathcal{H}_{\mathrm{hist}}$. The exponential series gives (8.36). Duhamel's identity for the two bounded generators gives (8.37).

To retain a presentation take finite-root presenting inputs with common strong bound M. Admitted root projections allow this choice. Apply the finite Taylor sum of $e^{-\tau \mathsf{L}_{\beta,N}}$ to each input. Every partial sum has strong bound $e^{S\kappa_+}M$ by its absolute series. At degree d its operator remainder at either resource level is at most $e^{S\kappa}(S\kappa)^{d+1}/(d+1)!$. Choose N and d tending to infinity. This remainder together with (8.37) and the weaker input error all tend to zero. The finite native constructions and their common strong bound therefore present the evolved value by Definition 4.2. Computable source parameters together with input moduli and bounds on these series make the same construction effective. □

In the unweighted Hilbert norm both semigroups are contractions and give the sharper finite-time response error $S\delta_N$. The weighted bound controls a different quantity. It keeps the strong certificate available for subsequent operations and completion. The response can redistribute amplitude between roots with different weights so unweighted contractivity alone would not provide this estimate. Both bounds use the normalized mass at each root. Enumeration and storage still involve the history coordinates themselves and their cost depends on the fibre cardinalities.

Here β specifies the Gaussian distribution of source mass while τ measures evolution under the parent–child response. Holding the source fixed the estimates apply on every finite assembly interval. A uniform family with $\beta \ge \beta_0 > 0$ uses β_0 in the tail majorants with the same strict resource restrictions at that lower endpoint. Varying the source scale and evolving its response are thus distinct operations with their own bounds.

The response also has exact quadratic accounts. For $f(\tau) = e^{-\tau \mathsf{L}_\beta} f_0$ differentiation gives

$$\|f(\tau)\|^2 + 2\int_0^\tau \langle f(r), \mathsf{L}_\beta f(r)\rangle \, \mathrm{d}r = \|f_0\|^2, \tag{8.38}$$

$$\langle f(\tau), \mathsf{L}_\beta f(\tau)\rangle + 2\int_0^\tau \|\mathsf{L}_\beta f(r)\|^2 \, \mathrm{d}r = \langle f_0, \mathsf{L}_\beta f_0\rangle \, . \tag{8.39}$$

Boundedness justifies differentiation and integration. The same identities hold at finite parent cutoff and pass to the limit by operator-norm and uniform finite-time convergence. They retain the cumulative response beside the current state quantity as in the finite ledgers.

What the value readout retains. The root multiplier intertwines C_p with the corresponding value multiplier because its weight is constant on each fibre. The assembly response generally fails the kernel condition in Proposition 4.3. To see this let $t_1, \ldots, t_4$ be the unbalanced additive histories of four and let t_5, t_6 be the balanced additive and multiplicative histories. Set

$$z = \frac{1}{2\sqrt{2}} \sum_{j=1}^{4} |t_j\rangle - \frac{1}{2}(|t_5\rangle + |t_6\rangle). \tag{8.40}$$

The probabilities above give $\|z\| = 1$ and $C_p z = 0$. Nevertheless

$$\langle |1\rangle, C_p \mathsf{L}_\beta z\rangle = -\frac{e^{-16\beta}}{4\sqrt{2}} - \frac{1 - 1/\sqrt{2}}{8} e^{-25\beta} < 0. \tag{8.41}$$

Indeed the four unbalanced parent relations at root four each have one unit child. Their combined contribution gives the first term. A higher relation contributing to this matrix element must have a unit history and a root-four history as children. Only the two ordered additive channels at root five do so. Their total is $(e^{-25\beta}/2) \sum_{j=1}^{6} p(t_j) z(t_j)$ which is the second term displayed. All other relations miss one of the two supports. This also verifies the complete rather than only the finite-cutoff response computed in [1].

The two initial states 0 and z have the same value readout but the derivatives of their readouts differ at $\tau = 0$ by $-C_p \mathsf{L}_\beta z$. Thus a value alone does not determine its later response in this realization. The full history state retains the missing variable. Eliminating that variable produces a memory contribution to the value equation and a general quadratic observation retains its mixed term with the readout component. This differs from Example 4.1 where two derivations establish different bounds for one fixed cosine function. Here the distinction is between states whose initially identical readouts conceal different responses.

8.3 The periodic Stokes state datum

In the Stokes realization the Leray projection is contractive in the spectral norms and imposes the divergence-free constraint. The interaction is projected transport.

Let $\mathbb{T}^3_L = (\mathbb{R}/L\mathbb{Z})^3$ and set

$$\mathcal{H}_{\mathrm{amb}} := \left\{ u \in L^2(\mathbb{T}^3_L;\mathbb{R}^3) : \int_{\mathbb{T}^3_L} u \, \mathrm{d}x = 0 \right\}, \qquad \mathcal{V} := \{ u \in \mathcal{H}_{\mathrm{amb}} : \nabla \cdot u = 0 \}. \tag{8.42}$$

Let $\Pi : \mathcal{H}_{\mathrm{amb}} \to \mathcal{V}$ be the orthogonal Leray projector and

$$A_S = -\Pi\Delta = -\Delta\big|_{\mathcal{V}}, \qquad \mathcal{B}(u,v) = -\Pi(u \cdot \nabla v). \tag{8.43}$$

The pointwise realization is the inclusion $\mathcal{J}$ of $\mathcal{V}$ into $L^2(\mathbb{T}^3_L;\mathbb{R}^3)$ with the common fiber $F = \mathbb{R}^3$. Fourier polynomials supply the specified continuous representatives. Here the divergence-free Fourier polarizations form $\mathfrak{M}_{\mathfrak{G}}$. The real coefficient field or equivalently the complex field together with conjugate symmetry belongs to the PDE input datum. We use the normalization

$$\begin{gathered} u(x) = L^{-3/2} \sum_{k \in (2\pi/L)\mathbb{Z}^3 \setminus \{0\}} \widehat{u}(k) e^{ik\cdot x}, \\ k \cdot \widehat{u}(k) = 0, \qquad \widehat{u}(-k) = \overline{\widehat{u}(k)}. \end{gathered} \tag{8.44}$$

Parseval's identity then has no volume factor. $\|u\|_2^2 = \sum_k |\widehat{u}(k)|^2$. An orthonormal basis of each two-dimensional plane $k^\perp$ resolves the vector coefficient into modal coefficients.

Proposition 8.5 (Stokes–Fourier certification)**.** The state datum $(\mathcal{H}_{\mathrm{amb}}, \Pi, \mathcal{V}, A_S, \mathcal{B})$ verifies C0 through C3 with $d = 3$. The state interaction tensor is supported on Fourier triads $k = p + q$. It is polynomially tame with one derivative loss for every $s > 3/2$ and the frequency support satisfies

$$|k| \le |p| + |q|. \tag{8.45}$$

For every $\beta > 1$ there are nonvanishing and β-quantitatively non-degenerate triads at arbitrarily large comparable frequencies. Consequently the analytic exponent $\theta = \frac{1}{2}$ is sharp for the state interaction.

Proof. The Fourier eigenvalues are positive and have finite multiplicity. They tend to infinity as required by C0. Each nonzero k has two polarizations of pointwise norm $L^{-3/2}$. The lattice count $C_L(1+K)^3$ gives C1a with $d = 3$. Smooth finite kernels and componentwise heat continuity give C1b and the derivative multipliers give C1c. For $0 < \gamma \le 1$

$$\left| e^{ik\cdot x} - e^{ik\cdot y} \right| \le C_{\gamma,L} |k|^\gamma d_{\mathbb{T}^3_L}(x,y)^\gamma. \tag{8.46}$$

Summing (8.46) with derivative factor $|k|^\ell$ gives C1d. The multiplier $\Pi_k = I - k \otimes k/|k|^2$ commutes with the Laplacian and preserves Fourier polynomials. It is contractive at every spectral weight as required by C3.

For the nonlinear estimate Fourier convolution gives the following expression when $k \neq 0$.

$$\widehat{\mathcal{B}(u,v)}(k) = -iL^{-3/2} \Pi_k \sum_{p+q=k} (\widehat{u}(p) \cdot q) \widehat{v}(q). \tag{8.47}$$

The zero-mode coefficient vanishes because the integral of $u \cdot \nabla v$ is zero for periodic divergence-free u. Hence the state interaction preserves the mean-zero constraint as well.

At each triad incompressibility gives $\widehat{u}(p)\cdot q = \widehat{u}(p)\cdot k$. Thus the absolute tensor is bounded by $C_L|k|$ times convolution of the absolute polarization coefficients. This identity places the derivative on the output frequency at each individual triad so it remains available after taking absolute values. Writing $\langle k\rangle = (1+|k|^2)^{1/2}$ we have

$$\begin{aligned} \langle k\rangle^{s-1}|k| &\le \langle k\rangle^s, \\ \|a*b\|_{\ell^2_s} &\le C_s\big(\|a\|_{\ell^2_s}\|b\|_{\ell^1} + \|a\|_{\ell^1}\|b\|_{\ell^2_s}\big) \\ &\le C_{s,L}\|a\|_{\ell^2_s}\|b\|_{\ell^2_s}. \end{aligned} \tag{8.48}$$

These follow from Young's inequality and $\sum_k \langle k\rangle^{-2s} < \infty$ for $s > 3/2$. Inserting $e^{\sigma|k|} \le e^{\sigma|p|}e^{\sigma|q|}$ proves

$$\Big\|\mathcal{B}^{\#}(u,v)\Big\|_{1/2,\sigma,s-1;A_S} \le C_{s,L}\,\|u\|_{1/2,\sigma,s;A_S}\,\|v\|_{1/2,\sigma,s;A_S}\,, \tag{8.49}$$

with $q = 1$ and a constant independent of $\sigma \ge 0$. The polynomial threshold is the usual Sobolev product threshold as in the product estimates in [16, Chapter 2].

Sharpness also requires a triad that survives the Leray projection. Work first in the complexified Fourier space and omit the common factor $2\pi/L$. Take

$$p_n = (n,0,0), \qquad q_n = (n,1,0), \qquad k_n = p_n + q_n = (2n,1,0), \tag{8.50}$$

with divergence-free polarizations $a = e_2$ at p_n and $b = e_3$ at q_n. Then $a\cdot q_n = 1$ and $b \perp k_n$. Restoring $\kappa_L = 2\pi/L$ and the normalized modes in (8.44) the output coefficient along e_3 at $\kappa_L k_n$ is $-i\kappa_L L^{-3/2}$. Moreover

$$\frac{|k_n|}{|p_n|+|q_n|} \longrightarrow 1, \qquad \frac{|p_n|}{|q_n|} \longrightarrow 1. \tag{8.51}$$

The coefficient is bounded below independently of n so (3.15) holds for every $\beta > 1$. The same coefficient test can be made in the real state space. Take

$$U_n(x) = \sqrt{2}L^{-3/2}e_2\cos(\kappa_L p_n\cdot x), \qquad V_n(x) = \sqrt{2}L^{-3/2}e_3\cos(\kappa_L q_n\cdot x).$$

These are real unit modes. Direct differentiation gives

$$\mathcal{B}(U_n,V_n) = \kappa_L L^{-3}e_3\big[\sin(\kappa_L k_n\cdot x) + \sin(\kappa_L(q_n-p_n)\cdot x)\big].$$

Both output polarizations survive the Leray projection because e_3 is orthogonal to both output frequencies. The coefficient at the normalized sine mode of frequency $\kappa_L k_n$ is $\kappa_L/(\sqrt{2}L^{3/2})$. This is again independent of n. Moreover $\mathcal{B}(V_n,U_n) = \mathcal{B}(U_n,U_n) = \mathcal{B}(V_n,V_n) = 0$ so this output is also present in $\mathcal{B}(U_n+V_n,U_n+V_n)$. □

Appendix B computes the exponential cost excess for these mode tests in both realizations.

An explicit tame constant follows from a bound on the lattice sum in the coefficient ℓ^1 estimate including its omitted tail. All sums below run over $k \in (2\pi/L)\mathbb{Z}^3$. Bounding each integer coordinate by $LR/(2\pi)$ gives

$$N_L(R) := \#\{k : |k| \le R\} \le D_L(1+R)^3, \qquad D_L := \max\{1, L/\pi\}^3. \tag{8.52}$$

For polynomial weights dyadic shells give a geometric series with the required exponent. Let $s > 3/2$. For $K \geq 1$ split the omitted frequencies into $2^j K < |k| \leq 2^{j+1}K$ for $j \geq 0$. The number of points in the jth shell is at most $D_L(1+2^{j+1}K)^3 \leq 27 D_L 2^{3j} K^3$. Hence

$$\sum_{|k|>K} (1+|k|^2)^{-s} \leq 27 D_L K^{3-2s} \sum_{j\geq 0} 2^{j(3-2s)} = T_{L,s} K^{3-2s}, \tag{8.53}$$

where $T_{L,s} := 27 D_L/(1-2^{3-2s})$. The full weighted sum is at most $N_L(1)+T_{L,s} \leq 8D_L + T_{L,s}$.

Effective tame constants. To keep the frequency and volume normalizations explicit we set

$$S_{s,L} := \sum_{k\in(2\pi/L)\mathbb{Z}^3} \langle k\rangle^{-2s}, \qquad c_s := 2^{\max\{s-1,0\}}, \qquad s > 3/2.$$

The inequalities $\langle p+q\rangle^s \leq c_s(\langle p\rangle^s + \langle q\rangle^s)$ and $\|a\|_{\ell^1} \leq S_{s,L}^{1/2}\|a\|_{\ell^2_s}$ give the convolution constant $2c_s S_{s,L}^{1/2}$ in (8.48). The passage from scalar convolution to the state tensor also requires a polarization factor. Write $u_{p,a}$ and $v_{q,b}$ for the coefficients in orthonormal polarizations $a, b \in \{1,2\}$ and put $U_p = (\sum_a |u_{p,a}|^2)^{1/2}$ and $V_q = (\sum_b |v_{q,b}|^2)^{1/2}$. Each tensor coefficient is bounded by $L^{-3/2}|k|$ at $p+q=k$. Summing the two input polarizations and taking the norm of the two output coefficients therefore bounds the absolute majorant at k by $2\sqrt{2}L^{-3/2}|k|(U*V)_k$. Since $|k|\langle k\rangle^{s-1} \leq \langle k\rangle^s$ one may take $4\sqrt{2}c_s L^{-3/2} S_{s,L}^{1/2}$ in (8.49) uniformly in $\sigma \geq 0$.

Suppose that $L > 0$ and $s > 3/2$ are effectively given. Finite sums over integer-index cubes can then be combined with the tail estimate (8.53) at the inscribed physical radius. This gives certified upper bounds for $S_{s,L}$ to any prescribed accuracy. The same argument applies to the cosine convolution with its one-dimensional lattice and the normalization in (8.4).

The Leray multiplier Π_k in (8.47) accounts for the pressure after its elimination from the equation. The Stokes semigroup is contractive in the analytic norms on $\mathcal{V}$. The Stokes estimates in Table 1 use unit viscosity. For the physical equation with viscosity $\nu > 0$ the linear generator in (6.5) is

$$A_\nu := \nu A_S. \tag{8.54}$$

If K and σ denote physical wavenumber and physical analyticity radius relative to A_S respectively then the corresponding abstract parameters for A_ν are

$$K_\nu = \sqrt{\nu}\, K, \qquad \sigma_\nu = \frac{\sigma}{\sqrt{\nu}}, \qquad e^{\sigma_\nu \sqrt{A_\nu}} = e^{\sigma\sqrt{A_S}}. \tag{8.55}$$

These identities allow us to apply the evolution theorem to A_ν while expressing the tail in the physical A_S frequency. For the following generated-class assertion $\mathfrak{G}$ and its fixed policy $\mathfrak{A}$ refer to A_ν. Any policy stated in physical A_S coordinates is transported by (8.55) with equivalent polynomial weights absorbed into the certificate constants.

Local Stokes–Fourier flow. Let $s > 3/2$ and $\sigma_0 \geq 0$ with $\nu > 0$. If

$$u_0 \in \mathscr{G}^{(1/2)}_{\sigma_0,s}(A_S), \qquad g \in L^\infty_{\mathrm{loc}}\big([0,\infty); \mathscr{G}^{(1/2)}_{\sigma_0,s-1}(A_S)\big), \tag{8.56}$$

Estimate	Explicit choice or bound	Conditions and normalization	Source
Differentiation reserve	$C_{q,\theta} = \big(q/(2e\theta)\big)^{q/(2\theta)}$	$q > 0$ and $0 < \theta \le 1$ with factor $(\alpha-\beta)^{-q/(2\theta)}$ for $\alpha > \beta \ge 0$. The constant for $q = 0$ is 1.	(4.4) and (4.5)
Semigroup recovery	$C_q = 2^{\max\{q/2-1,0\}} \cdot \max\{1, (q/(2e))^{q/2}\}$	$q > 0$ with kernel $C_q(1 + h^{-q/2})$ for $h > 0$. At $q = 0$ the constant is $C_0 = 1$.	(6.7) and (6.8)
Scalar Fourier convolution	$2c_s S_{s,L}^{1/2}$	$s > 3/2$ on $(2\pi/L)\mathbb{Z}^3$ with coefficient ℓ_s^2 norms.	(8.48)
Stokes absolute interaction	$4\sqrt{2}c_s L^{-3/2} S_{s,L}^{1/2}$	$s > 3/2$ and $\sigma \ge 0$ with orthonormal polarizations. One derivative of loss at unit viscosity.	(8.49)
Galerkin residual factor	$\overline{a}_K = e^{-\delta K}(1 + K^2)^{-\eta/2}$	$\delta > 0$ and $\eta \ge 0$ as a factor of the common strong source bound $B_M^+(t)$.	(7.11)
Polynomial lattice tail	$T_{L,s} K^{3-2s}$ $T_{L,s} = 27 D_L/(1 - 2^{3-2s})$	$s > 3/2$ and $K \ge 1$. This bounds $\sum_{\lvert k\rvert > K} \langle k\rangle^{-2s}$ on the same lattice.	(8.53)

Table 1. Explicit constants and certified bounds. Here $c_s = 2^{\max\{s-1,0\}}$ and $S_{s,L} = \sum_{k \in (2\pi/L)\mathbb{Z}^3} \langle k\rangle^{-2s}$ with $D_L = \max\{1, L/\pi\}^3$. The lattice tail certifies finite-sum upper bounds for $S_{s,L}$. The scalar convolution row concerns Fourier coefficients. The Stokes row includes the volume and polarization factors.

then the projected equation

$$\partial_t u + \nu A_S u = \mathcal{B}(u, u) + g(t), \qquad u(0) = u_0, \tag{8.57}$$

has a unique local mild solution in $C([0,T]; \mathscr{G}_{\sigma_0,s}^{(1/2)}(A_S))$. The certified loss is $q = 1$ and $p = \infty$ so $qp'/2 = 1/2 < 1$. If the data and their generated approximations satisfy all hypotheses of Theorem 6.4 for the fixed policy then each $u(t)$ admits a presentation in $\mathscr{C}_{\mathfrak{G},\mathrm{rel}}^{\mathrm{adm}}$ and belongs to $\mathcal{N}_{\mathfrak{G},\mathrm{rel}}^{\mathfrak{A}}$. The source of the PDE inputs is retained in these relative presentations. To obtain the generated presentation we apply (8.49) with the generator $A_\nu = \nu A_S$ with analytic parameter $\sigma_0/\sqrt{\nu}$ and physical frequency convention (8.55). The fixed viscosity changes only equivalent polynomial weights and the semigroup constants. Theorem 6.2 gives the local solution and the additional approximation hypotheses permit the use of Theorem 6.4.

Example 8.6 (Finite Stokes data). Fix $\nu > 0$ and a divergence-free and mean-zero finite Fourier polynomial u_0 with its coefficients declared as relative input terminals. Take $g = 0$

and three analytic resource levels of common order $s > 3/2$ with strictly decreasing radii in the A_ν convention (8.55). On any policy admitting the strongest-to-weakest pair these data satisfy Theorem 6.4.

The constant initial sequence $u_{0,n} = u_0$ and the unchanged nonlinearity $\mathcal{N}^{(n)}(z) = \mathcal{B}(z, z)$ have zero approximation defects. A finite Fourier polynomial has a finite stronger norm. Transporting (8.49) to A_ν gives the common profiles CM^2 and $2CM$ with $q = 1$ and $p = \infty$. Their Duhamel majorant is integrable because the recovery kernel is bounded by a constant times $1 + h^{-1/2}$.

The relative language includes the fixed Fourier modes and Leray projection. It also admits the viscosity and heat operations together with the corresponding time integrals. At each finite Picard depth the time integrand has finite Fourier support and continuous coefficients. Its Bochner integral is therefore the stronger-norm limit of finite time sums. The integral transformer supplies the common certificate from the preceding majorant. Left-endpoint sums in the integration variable bound the increasing kernel $1 + (t - s)^{-1/2}$ by its integral so the approximating histories have that same common bound and avoid the singular endpoint. This verifies the time-construction requirement on a dense set of times. The fixed-point argument then gives relative generative regularity on one common local interval. Since the initial polynomial lies at every larger radius it also supplies the gap needed for the two-level Galerkin estimate and its finite-observation consequences.

In both realizations in Table 2 the time bounds are independent of K because spectral projection is contractive at every selected resource level. Generated presentations also require the admitted data and operators together with the time constructions stated in the table.

8.4 Time-integrated spectral tails

For a moving frequency threshold the time integrability of the tail depends on how the norm and analytic radius vary along the chosen threshold path. A positive analytic radius gives an exponentially small tail at a fixed time but the size of that tail along a trajectory also depends on the norm and on how the radius changes. The following estimates keep these quantities separate so that a time-dependent window can be tested against the available certificate. For fixed $s \geq 0$ and radius parameter $\sigma > 0$ define

$$M_{\sigma,s}(t) := \left\| e^{\sigma\sqrt{A_S}} (I + A_S)^{s/2} u(t) \right\|_2 , \tag{8.58}$$

and

$$\Theta_{3,s}(\sigma, K) := \left(\sum_{\substack{k \in (2\pi/L)\mathbb{Z}^3 \\ |k| > K}} e^{-2\sigma|k|} (1 + |k|^2)^{-s} \right)^{1/2} . \tag{8.59}$$

Proposition 8.7 (Three-dimensional analytic tail)**.** For every state with finite certificate (8.58) and every $K \geq 0$ we have

$$\|P_{>K} u(t)\|_{L^\infty} \leq L^{-3/2} \Theta_{3,s}(\sigma, K) M_{\sigma,s}(t). \tag{8.60}$$

Requirement and use	Neumann polynomial flow	Stokes–Fourier flow
Generator with modes and constraints (C0 through C3)	Proposition 8.1 with $d = 1$. Cosine product and scalar Markov semigroup.	Proposition 8.5 with $d = 3$. Fourier triads with commuting Leray projection and viscosity rescaling (8.55).
Nonlinear growth and difference bounds	(8.10) for $s > 1/2$ with loss $q = 0$. Repeated multiplication gives the polynomial bounds.	(8.49) for $s > 3/2$ with loss $q = 1$. On a radius-M ball the profiles are CM^2 and $2CM$.
Time-integrability bounds (C4)	Finite sums of $C_m\lvert c_m(t)\rvert M^m$ and $C_m m\lvert c_m(t)\rvert M^{m-1}$ are in L^1 by (8.16) with the forcing measured in the same no-loss analytic space.	Time-independent interaction bounds and the L^∞ forcing in (8.56) give $p = \infty$ and $qp'/2 = 1/2$.
Stronger data for Galerkin residual decay	(8.18) supplies $\delta > 0$ and the stronger forcing bound.	The assumptions are u_0 at $(\sigma_0 + \delta, s + \eta)$ and g in L^∞ at $(\sigma_0 + \delta, s + \eta - 1)$ with $\delta > 0$ and $\eta \geq 0$ using (8.49) at both levels.
Presentation of the local solution	Example 8.2 checks all three groups of hypotheses for heat-prepared data and a quadratic flow. Other inputs retain their declared provenance.	Example 8.6 verifies the theorem for finite Fourier input data with relative provenance for the PDE structure.
Observation error (C5)	The mean is bounded on L^2. Point evaluation uses $s > 1/2$ and Theorem 2.2.	Bounded linear observations use their operator norm. Point observations use C1 and the embedding in Theorem 2.2.

Table 2. Hypotheses for the two local realizations. The stronger initial and forcing data in the fourth row give the common two-level bounds required by Theorem 7.2.

If instead $u(t) \in H^r$ with $r > 3/2$ and $K > 0$ then

$$\|P_{>K}u(t)\|_{L^\infty} \leq C_{L,r}K^{3/2-r}\|u(t)\|_{H^r}. \tag{8.61}$$

Proof. With the normalization (8.44) we obtain

$$\begin{aligned}\|P_{>K}u(t)\|_\infty &\leq L^{-3/2}\sum_{|k|>K}|\widehat{u}(t,k)| \\ &\leq L^{-3/2}\Theta_{3,s}(\sigma,K)\Big(\sum_k e^{2\sigma|k|}\langle k\rangle^{2s}|\widehat{u}(t,k)|^2\Big)^{1/2}.\end{aligned} \tag{8.62}$$

The second factor is the stated analytic norm. This proves the first bound. Replacing the exponential weight by a polynomial weight and applying the lattice estimate (8.53) gives the Sobolev bound for $K \geq 1$. For $0 < K < 1$ use the full weighted sum bounded above and increase the constant. One may also apply Theorem 2.2 with $d = 3$ and use $(1 + K)^{3/2-r} \leq K^{3/2-r}$ for $K > 0$. □

Let u be a strongly measurable state trajectory. Let $K(t) \geq k_{\mathrm{IR}} > 0$ be a measurable spectral threshold and define

$$R(t) := \frac{\left\|P_{>K(t)}u(t)\right\|_{L^\infty}}{\nu K(t)}. \tag{8.63}$$

Theorem 8.8 (Time-integrated tail criterion)**.** On a finite interval $[0,T]$ let $\sigma(t) > 0$ be measurable when an analytic certificate is used. Either of the conditions

$$t \longmapsto \frac{C_{L,r}}{\nu} K(t)^{1/2-r} \left\|u(t)\right\|_{H^r} \in L^4(0,T), \qquad r > \frac{3}{2}, \tag{8.64}$$

$$t \longmapsto \frac{L^{-3/2}\Theta_{3,s}(\sigma(t),K(t))}{\nu K(t)} M_{\sigma(t),s}(t) \in L^4(0,T) \tag{8.65}$$

implies $R \in L^4(0,T)$.

Proof. Divide (8.61) or (8.60) by $\nu K(t)$ and apply the stated time integrability. □

Explicit estimates for the lattice tails in (8.65) are given in Appendix D. The criterion measures their product with the evolving spectral norm. It therefore accounts for a shrinking analytic radius or a growing norm through the chosen frequency path while keeping the required time integrability explicit.

To combine the tail with the retained low frequencies suppose in addition that $u \in L^\infty(0,T;L^2)$ and $K \in L^4(0,T)$. Bernstein's inequality and the definition of R give

$$\|u(t)\|_\infty \leq C_L(1+K(t))^{3/2}\|u(t)\|_2 + \nu K(t)R(t). \tag{8.66}$$

The first term belongs to $L^2(0,T)$ since the interval is finite. The second does so by Hölder's inequality when $R \in L^4(0,T)$. Together these estimates give $u \in L^2(0,T;L^\infty)$ under the stated energy and frequency-path bounds.

9 Conclusion

The competition between diffusion and interaction determines what a finite spectral window must retain. Diffusion suppresses high frequencies but the nonlinearity can create them again. The stronger bound and the gap between spectral weights control both the omitted initial modes and the nonlinear source outside the window. Stability converts these two contributions into one error estimate that is uniform over the Galerkin family on a common local interval. An observation modulus then determines the least cutoff certified by that estimate.

The interaction support determines which exponential bounds remain available at the next nonlinear step. In the Neumann and Stokes examples the frequency triangle law preserves analytic reserve while non-degenerate high-frequency interactions force a loss of Gaussian reserve. Thus the distinction between the two weights comes from the operation being performed. A stronger estimate supplied by linear smoothing must be tested against that operation before it is reused.

For compatible constructions the common stronger bound also survives completion. Convergence in the weaker norm identifies the limit and the retained histories specify how it is reached from the declared source. Different derivations of the same state may establish different bounds and therefore certify different sufficient windows. Improving such a bound changes the available approximation without changing the function. This is the operational value of retaining a presentation.

A quadratic ledger follows the same construction at the level of a specified observation. Compatible transport separates retained quantity from positive defect. Injection and approximation change the account through signed increments that include their interaction with the current state. Establishing the identity before bounding these terms preserves their signs under composition. A general observation also retains its mixed terms and compatibility remainder. Continuity determines which equalities pass to a limit. For closed unbounded forms the corresponding conclusion is a lower-semicontinuous inequality.

The minimal-change construction supplies the Neumann datum under the stated composition and scaling laws together with the endpoint and reflecting conditions. Stokes provides an independent constrained realization. On the Neumann history source conditional normalization gives weighted state and response bounds without a fibre-cardinality factor. The Gaussian operator tail controls finite-time approximation although the parent–child response has no uniform positive decay rate. This response can also turn an initially invisible history difference into a later observation. Kernel invariance decides whether the readout has a closed bounded linear evolution. Otherwise the eliminated variables must be accounted for together with their memory contribution and quadratic coupling. A finite memory horizon then requires a bound for the actual eliminated kernel.

These results select a model-relative range of states within the ambient function space. Its members are obtained from declared sources and operations while retaining the estimates required for continued use. Along a longer trajectory each restart must preserve the needed spectral bound and coefficient domain together with time-integrable profiles. Window transfer and memory approximation enter the new residual and initial error budgets. Their quadratic increments enter the ledger when that observation is requested.

For trustworthy computation these estimates have complementary tasks. The full-equation residual controls the accuracy of a finite path. The ledger records changes in the chosen quadratic quantity. The readout bound transfers state accuracy to the reported result. Effective parameters and convergence moduli make the tests executable. The work needed to represent the finite data and to construct and check them determines their computational cost. This preserves the connection between a state's source and subsequent evolution and what a finite calculation can certify about it.

A Functional-calculus estimates

We record the standard form identities for heat increments to fix the normalization of the generator and its fractional powers. The Volterra estimate gives the explicit time factor used in Theorem 6.2.

A.1 Infinitesimal and fractional variation

Let A be a nonnegative self-adjoint operator on $\mathcal{H}$ and put $T_t = e^{-tA}$. The short-time change in the heat evolution defines the extended quadratic form

$$\mathcal{Q}_1(f) := \lim_{t\downarrow 0} t^{-1} \langle f, (I - T_t)f \rangle . \tag{A.1}$$

By the spectral theorem the limit in (A.1) exists in $[0, \infty]$ and satisfies

$$\mathcal{Q}_1(f) = \|A^{1/2} f\|_2^2, \qquad \operatorname{Dom} \mathcal{Q}_1 = \operatorname{Dom} A^{1/2}. \tag{A.2}$$

This form is closed and determines A uniquely. If T_t is a symmetric Markov semigroup then it is a Dirichlet form. Indeed with the spectral resolution E_A and measure $\mu_f(S) = \langle f, E_A(S)f \rangle$ the difference quotient is $\int (1 - e^{-t\lambda})/t \, \mathrm{d}\mu_f(\lambda)$. Its integrand increases to λ as $t \downarrow 0$ so monotone convergence proves the identity including the value $+\infty$. The form norm is the graph norm of $A^{1/2}$ so the form is closed. The representation theorem then recovers A uniquely [17, Chapter VI]. The Markov assertion follows from the equivalence of symmetric Markov semigroups and Dirichlet forms [18, Theorem 1.4.1].

The same increments describe fractional variation when integrated over all positive times with a power-law weight.

For $0 < r < 1$ and $f \in \mathcal{H}$ the corresponding fractional identity is

$$\|A^{r/2} f\|_2^2 = \frac{r}{\Gamma(1 - r)} \int_0^\infty t^{-1-r} \langle f, (I - T_t)f \rangle \, \mathrm{d}t, \tag{A.3}$$

where both sides may be infinite. For $\lambda > 0$ substitution of $z = t\lambda$ followed by integration by parts gives

$$\int_0^\infty t^{-1-r}(1 - e^{-t\lambda}) \, \mathrm{d}t = \lambda^r \frac{\Gamma(1 - r)}{r}.$$

Both sides vanish at $\lambda = 0$. The integrands are nonnegative so Tonelli's theorem permits integration against μ_f even when the resulting integral is infinite.

Self-adjointness and positivity suffice for both spectral identities. In the scalar realization the additional Markov hypothesis gives contraction under 1-Lipschitz maps fixing zero.

A.2 The Volterra time estimate

The contraction argument in Theorem 6.2 requires the singular smoothing bound to be integrable in time. The relevant Volterra estimate makes the dependence on the time interval explicit. Let $1 < p \le \infty$ and let p' be conjugate to p and suppose $0 \le \gamma < 1/p'$. Then

$$\sup_{0 \le t \le T} \int_0^t (t - \tau)^{-\gamma} |h(\tau)| \, \mathrm{d}\tau \le \frac{T^{1/p' - \gamma}}{(1 - \gamma p')^{1/p'}} \, \|h\|_{L^p(0,T)} . \tag{A.4}$$

For $p = \infty$ the displayed constant is interpreted with $p' = 1$. For $\gamma = 0$ the endpoint $p = 1$ is also valid with the right-hand side $\|h\|_{L^1(0,T)}$. The estimate follows from Hölder's inequality with the norm of the kernel $r^{-\gamma}$ on $(0, t)$ bounded by its norm on $(0, T)$. Taking $\gamma = q/2$ gives the integrability condition (6.9). At $\gamma p' = 1$ the kernel's $L^{p'}$ norm diverges. This identifies the endpoint at which an additional time estimate is needed.

B Sharp interaction sequences

Neumann and Stokes interactions admit single-mode tests that rule out a bilinear estimate with the same positive exponential reserve at both inputs and the output when the frequency cost has power $\beta > 1$. In each test a nonvanishing output coefficient accompanies a cost excess that dominates every fixed polynomial weight.

The normalized Neumann interaction in (8.11) has $\Gamma_{nn,2n} = (2L)^{-1/2}$. For $\beta > 1$ the exponential cost excess along these triples is exactly

$$\omega_{2n}^{\beta} - 2\omega_n^{\beta} = (2^{\beta} - 2)(\pi n/L)^{\beta}. \tag{B.1}$$

The coefficient is constant and the inputs have equal frequency so Theorem 3.5 applies to this positive cost excess for every fixed choice of polynomial orders.

For the Stokes interaction use the real unit modes in the proof of Proposition 8.5 with frequencies p_n, q_n, k_n from (8.50) and $\kappa_L = 2\pi/L$. That calculation gives the nonzero coefficient $\kappa_L/(\sqrt{2}L^{3/2})$ at the normalized sine mode of frequency $\kappa_L k_n$ after the Leray projection. Its independence of n leaves the growth of the exponential weight uncompensated. The cost excess is

$$\begin{aligned} |\kappa_L k_n|^{\beta} - |\kappa_L p_n|^{\beta} - |\kappa_L q_n|^{\beta} &= \kappa_L^{\beta}\big[(4n^2+1)^{\beta/2} - n^{\beta} - (n^2+1)^{\beta/2}\big] \\ &= (2^{\beta} - 2)(\kappa_L n)^{\beta} + o(n^{\beta}). \end{aligned} \tag{B.2}$$

The inputs are comparable and asymptotically saturate the frequency triangle law. Together with the displayed nonzero coefficient (B.2) verifies all hypotheses of Theorem 3.5. Thus the real state space already contains the required sharpness sequence.

Restricted interaction support or exponentially decaying coefficients can change the admissible costs.

C The Neumann spectrum from minimal change

The Neumann datum in section 8 has a source construction that fixes its interval length and heat-time convention together with its mode ordering. The derivation follows [1, sections 2, 3 and 4]. Four groups of assumptions have distinct roles. Minimality specifies the class of orthogonal representations. Unbiased composition on the resulting Lie algebra is specified by convolution and self-similarity together with invariance and covariance. Two length functionals specify how the correlation profile is read. Finally conservation with zero endpoint flux specifies the reflecting closure. These are structural inputs. The carrier dimension and Gaussian law follow from them. So do the endpoint values and Neumann boundary condition with the resulting square spectrum and cosine product. They provide one source realization of the abstract spectral datum. The abstract analysis begins once this datum and its interaction estimates have been fixed. The construction therefore supplies provenance for the spectral realization. A physical interpretation additionally identifies the datum with a concrete state space and specifies its evolution law and observations.

C.1 Minimal reversible change

We minimize the dimension of the real carrier within the class of orthogonal actions defined below. The resulting representation metric supplies the unit for the covariance normalization.

An orthogonal change datum is a triple (V, G, ρ) in which V is a nonzero finite-dimensional real inner-product space. The group G is a nontrivial compact connected Lie group and $\rho : G \to O(V)$ is a faithful continuous representation irreducible over $\mathbb{R}$. Its Lie algebra carries the representation metric

$$\langle X, Y \rangle_\rho := -\frac{1}{\dim_{\mathbb{R}} V} \operatorname{Tr}_V(d\rho(X) d\rho(Y)). \tag{C.1}$$

The datum is minimal when $\dim_{\mathbb{R}} V$ is least among all such data. Faithfulness makes the displayed metric positive definite. A skew-adjoint $d\rho(X)$ has zero squared Hilbert–Schmidt norm only if it vanishes and then $X = 0$.

Every minimal orthogonal change datum has $\dim_{\mathbb{R}} V = 2$ with $G \simeq SO(2)$ and the standard rotation representation up to orthogonal equivalence. If X_0 is either unit Lie generator and $J = d\rho(X_0)$ then

$$J^{\mathsf{T}} = -J, \qquad J^2 = -I, \qquad J^{\mathsf{T}} J = I. \tag{C.2}$$

Thus a choice of orientation makes V a one-dimensional complex Hilbert space through $(a + \mathrm{i}c)v = av + cJv$. The conclusion is short but fixes the normalization used below. In dimension one $O(1) = \{\pm 1\}$ has no nontrivial connected subgroup so faithfulness excludes that dimension. The standard action of $SO(2)$ on $\mathbb{R}^2$ is faithful and real irreducible. Hence the minimum is two. For a minimal datum connectedness places the image in $SO(2)$. Compactness makes the image closed and a nontrivial connected closed subgroup of the circle is the whole circle. Faithfulness therefore identifies G with $SO(2)$. The unit generators of $\mathfrak{so}(2)$ in the metric (C.1) are the two standard skew-symmetric quarter turns. This proves (C.2) and the complex scalar rule.

C.2 Unbiased composition and the Gaussian law

Unbiased composition specifies the law of the Lie-algebra increments. Convolution describes addition of independent increments. Square-root scaling fixes their spread as a function of the accumulation parameter and orthogonal invariance treats the two directions equally. The covariance is normalized in the representation metric.

Assume that the Lie algebra of a minimal datum carries Borel probability measures $(\mu_t)_{t>0}$ with

(A1) $\mu_s * \mu_t = \mu_{s+t}$ and $\mu_t \Rightarrow \delta_0$ as $t \downarrow 0$.

(A2) $\mu_t = (D_{\sqrt{t}})_{\#}\mu_1$ where $D_r X = rX$.

(A3) μ_t is invariant under the full orthogonal group of the representation metric.

(A4) μ_t has finite nonzero second moment and covariance $tI_{\mathfrak{g}}$.

Either unit coordinate identifies $\mathfrak{g}$ with $\mathbb{R}$. The four assumptions then determine the compositional law uniquely.

$$d\mu_t(a) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{a^2}{2t}\right) \mathrm{d}a. \tag{C.3}$$

Let $\varphi_t(k)$ be the characteristic function. Convolution and weak continuity make $t \mapsto \varphi_t(k)$ a continuous multiplicative semigroup with limit one. Orthogonal invariance makes it real. The identity $\varphi_t(k) = \varphi_{t/(2m)}(k)^{2m}$ makes it nonnegative and it cannot vanish because $\varphi_{t/n}(k) \to 1$. Hence $\varphi_t(k) = e^{-t\psi(k)}$ for a continuous even $\psi \geq 0$. Square-root self-similarity gives $\psi(rk) = r^2\psi(k)$ so $\psi(k) = ck^2$. The covariance normalization fixes $c = 1/2$. Uniqueness of characteristic functions then gives (C.3).

Two canonical length functionals assign an interval to this law. They measure the first displacement carrying one unit of logarithmic cost and the profile's mass per unit peak. Both prescriptions scale with the standard deviation. For $b > 0$ the density at $t = b^2$ has the peak-normalized profile

$$K_b(d) := \exp\left(-\frac{d^2}{2b^2}\right). \tag{C.4}$$

Let $F : (0, 1] \to [0, \infty)$ be continuous and satisfy $F(rs) = F(r) + F(s)$ with $F(1) = 0$. Normalize it by $F(e^{-1}) = 1$. Then $H(x) = F(e^{-x})$ is a continuous additive function on $[0, \infty)$. It follows that $F(r) = -\log r$. The local information is therefore

$$\mathcal{D}_b(d) := F(K_b(d)/K_b(0)) = \frac{d^2}{2b^2}. \tag{C.5}$$

Let m be Haar measure on $(\mathbb{R}, +)$ normalized by $m([0, 1]) = 1$. For a nonzero nonnegative profile $K \in C(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we define

$$\mathcal{L}(K) := \frac{\int_{\mathbb{R}} K(d)\, \mathrm{d}d}{\|K\|_\infty}. \tag{C.6}$$

The denominator is finite and positive. The normalization of Haar measure fixes the translation-invariant mass in the numerator. Thus $\mathcal{L}$ measures mass per unit peak and is invariant under positive amplitude rescaling and obeys $\mathcal{L}(K(\cdot/a)) = a\mathcal{L}(K)$ for $a > 0$.

The two prescriptions now give the endpoints directly. The first unit of local information and the peak-equivalent global width are

$$\ell_{\mathrm{UV}}(b) := \inf\{d > 0 : \mathcal{D}_b(d) \geq 1\} = \sqrt{2}\, b, \qquad L_{\mathrm{IR}}(b) := \mathcal{L}(K_b) = \sqrt{2\pi}\, b. \tag{C.7}$$

Consequently

$$\mathcal{W}_b = [\ell_{\mathrm{UV}}(b), L_{\mathrm{IR}}(b)], \qquad \Delta_b = |\mathcal{W}_b| = b(\sqrt{2\pi} - \sqrt{2}). \tag{C.8}$$

The first formula is the unique positive solution of $d^2/(2b^2) = 1$. The second follows from $K_b(0) = 1$ and the Gaussian integral $\int_{\mathbb{R}} K_b(d)\, \mathrm{d}d = \sqrt{2\pi}\, b$.

C.3 Reflecting closure and the spectral carrier

We impose reflecting boundary conditions on the prescribed interval. The covariance normalization in (C.3) fixes the local diffusion coefficient. On the full line convolution by μ_t has Fourier multiplier $e^{-t\xi^2/2}$ and infinitesimal generator $\frac{1}{2}\partial_d^2$. The conservative reflecting realization of this differential operator on the generated interval has Markov generator $-A_b/2$ when A_b denotes the nonnegative Laplacian. This fixes the factor $1/2$ in $C_b(t) = e^{-tA_b/2}$.

For the local Laplacian on $\mathcal{W}_b$ we assume separated homogeneous self-adjoint boundary conditions that conserve the constant state and give zero flux at each endpoint. A separated condition has the form $\alpha_* f + \beta_* f' = 0$ with $(\alpha_*, \beta_*) \in \mathbb{R}^2 \setminus \{(0,0)\}$. The constant function belongs to the operator domain precisely when $\alpha_* = 0$. Then $\beta_* \neq 0$ and the condition is $f' = 0$. This also gives zero flux. The resulting operator on the real space $L^2(\mathcal{W}_b; \mathbb{R})$ is

$$A_b f = -f'', \qquad \operatorname{Dom} A_b = \{f \in H^2(\mathcal{W}_b) : f'(\ell_{\mathrm{UV}}) = f'(L_{\mathrm{IR}}) = 0\}. \tag{C.9}$$

Theorem C.1 (Minimal change to the Neumann spectrum)**.** Under the assumptions of this appendix A_b is nonnegative self-adjoint with compact resolvent. Its kernel consists exactly of constants and $C_b(t) = e^{-tA_b/2}$ is a conservative symmetric Markov semigroup. After the change of variable $x = d - \ell_{\mathrm{UV}}$ its complete orthonormal eigenfamily is

$$\phi_{b,0} = \Delta_b^{-1/2}, \qquad \lambda_{b,0} = 0, \qquad \phi_{b,n} = \sqrt{\frac{2}{\Delta_b}} \cos \frac{n\pi x}{\Delta_b}, \qquad \lambda_{b,n} = \left(\frac{n\pi}{\Delta_b}\right)^2 \quad (n \geq 1). \tag{C.10}$$

The family of nonzero spectral projections

$$\mathfrak{E}_b := \{\mathbf{1}_{\{\lambda\}}(A_b) : \lambda \in \operatorname{Spec}(A_b)\} \tag{C.11}$$

consists of rank-one projections including the projection onto the zero-eigenvalue space. For any $\tau > 0$ the relation

$$P \prec_b Q \quad \Longleftrightarrow \quad \operatorname{Tr}(PC_b(\tau)) > \operatorname{Tr}(QC_b(\tau)) \tag{C.12}$$

is independent of τ and well-orders the carrier with order type ω. Its ordered enumeration supplies the coordinate n. Choosing the real unit representative positive at $x = 0$ gives the family in (C.10). Its finite real span is a pointwise algebra and is uniformly dense in $C(\mathcal{W}_b; \mathbb{R})$. Write $E_{b,n} = \mathbf{1}_{\{\lambda_{b,n}\}}(A_b)$ for the single-mode projection and $P_{b,N} = \sum_{n=0}^{N} E_{b,n}$ for the cumulative projection. For $t \geq 0$ we have

$$\|(I - P_{b,N})C_b(t)\|_{\mathcal{B}(L^2(\mathcal{W}_b))} = \exp\left[-\frac{\pi^2 t}{2\Delta_b^2}(N+1)^2\right]. \tag{C.13}$$

Proof. The densely defined closed form $\int_{\mathcal{W}_b} f'g' \, \mathrm{d}d$ on $H^1(\mathcal{W}_b)$ determines a nonnegative self-adjoint operator. Integration by parts identifies its domain with (C.9). The resolvent maps into $H^2(\mathcal{W}_b)$. Its inclusion in L^2 is compact. If the form energy vanishes then f is constant. This proves the kernel assertion.

For a positive eigenvalue the equation $-f'' = \lambda f$ has solutions $c\cos(\sqrt{\lambda}x) + d\sin(\sqrt{\lambda}x)$. The condition at 0 gives $d = 0$ and the condition at Δ_b gives $\sqrt{\lambda}\Delta_b = n\pi$. Their squared

integrals are $\Delta_b/2$ for $n \geq 1$ whereas the constant has squared integral Δ_b. This yields (C.10). Even extension to the circle of length $2\Delta_b$ identifies this family with its even Fourier basis. This proves completeness.

Reflection also gives the kernel needed to verify positivity and conservation. With $g_t(z) = (2\pi t)^{-1/2} e^{-z^2/(2t)}$ the kernel for $t > 0$ and $0 \leq x, y \leq \Delta_b$ it is

$$p_b(t; x, y) = \sum_{m \in \mathbb{Z}} \big[g_t(x - y + 2m\Delta_b) + g_t(x + y + 2m\Delta_b) \big]. \tag{C.14}$$

The series is locally uniformly convergent for positive time. It is symmetric and nonnegative. Integrating in y unfolds the reflected intervals into the real line. This gives mass one. Its cosine multipliers are $e^{-t\lambda_{b,n}/2}$ so it represents $C_b(t)$ and proves the conservative symmetric Markov properties.

For the ordering assertion functional calculus gives on a single mode

$$\operatorname{Tr}(E_{b,n} C_b(\tau)) = e^{-\tau\lambda_{b,n}/2}. \tag{C.15}$$

The eigenvalues are simple and strictly increasing. Hence (C.12) orders the carrier independently of τ with finitely many predecessors for every element and infinitely many elements. Its order type is ω. Each real eigenline has exactly one unit vector positive at 0.

The cosine product identity gives closure of the finite real span under multiplication. It contains constants and separates points because $x \mapsto \cos(\pi x/\Delta_b)$ is strictly decreasing on the interval. The real Stone–Weierstrass theorem gives uniform density. The omitted diagonal multiplier has its largest value at $n = N + 1$. This proves (C.13). □

D Lattice tails for Stokes spectral estimates

The Stokes observation estimate (8.65) requires bounds for a weighted sum over omitted frequencies. We use the lattice $k \in (2\pi/L)\mathbb{Z}^3$ and the counting constant in (8.52). The polynomial weight is summable when $s > 3/2$. At $s = 0$ a positive exponential weight supplies the required summability.

For $s > 3/2$ the full polynomial sum bounds the tail when $K < 1$. Equation (8.53) gives its decay when $K \geq 1$. Bounding the exponential factor by its value at the threshold gives the following bound for $\sigma \geq 0$ and $K \geq 0$.

$$\sum_{|k| > K} e^{-2\sigma|k|} (1 + |k|^2)^{-s} \leq C_{L,s} e^{-2\sigma K} (1 + K^2)^{3/2 - s}. \tag{D.1}$$

Here one may take $C_{L,s} = 2^{s-3/2}(8D_L + T_{L,s})$.

For $s = 0$ integration of the counting function bounds the full exponential sum.

$$\sum_k e^{-\sigma|k|} = \sigma \int_0^\infty e^{-\sigma r} N_L(r)\, \mathrm{d}r \leq 16 D_L (1 + \sigma^{-3}), \qquad \sigma > 0. \tag{D.2}$$

The identity follows by summing the nonnegative integrals $e^{-\sigma|k|} = \int_{|k|}^\infty \sigma e^{-\sigma r}\, \mathrm{d}r$. It includes the zero mode through $N_L(0) = 1$. The integral bound from (8.52) is $D_L(1 +$

$3\sigma^{-1} + 6\sigma^{-2} + 6\sigma^{-3}$). This is at most the displayed expression. On the omitted region $e^{-2\sigma|k|} \le e^{-\sigma K} e^{-\sigma|k|}$. Taking square roots therefore yields the following bound for $\sigma > 0$ and $K \ge 0$.

$$\Theta_{3,0}(\sigma, K) \le 4D_L^{1/2}\big(1 + \sigma^{-3/2}\big)e^{-\sigma K/2}. \tag{D.3}$$

Both bounds are uniform in K. In (8.65) they give explicit sufficient time-integrability conditions in terms of the paths $\sigma(t)$ and $K(t)$ and $M_{\sigma(t),s}(t)$.

E Finite histories and operator domains

The finite histories of section 4 record input sources and the successive operations applied to them. Their types specify the operator domains and ensure that each composition is well defined.

Generated terminals. A generated terminal datum specifies a modal alphabet $\mathfrak{M}_{\mathfrak{G}} \subset \mathcal{D}_{\mathrm{fin}}$ and a unital source coefficient algebra $\mathbb{K}_{\mathfrak{G}}$ with a faithful representation $\iota_{\mathfrak{G}} : \mathbb{K}_{\mathfrak{G}} \hookrightarrow \mathbb{F}$. It also specifies a finite-description parameter language $\mathfrak{P}_{\mathfrak{G}}$ and a typed list $\mathfrak{T}_{\mathrm{ext}}$ of admitted external terminals. The intrinsic finite source core is the set

$$\mathcal{D}_{\mathrm{fin}}^{\mathfrak{G}} := \mathrm{span}^{\mathrm{fin}}_{\iota_{\mathfrak{G}}(\mathbb{K}_{\mathfrak{G}})} \mathfrak{M}_{\mathfrak{G}}, \tag{E.1}$$

and each element used in a history is constructed from specified modal and coefficient leaves. Its presentation retains this finite linear combination. Each external leaf is supplied with a provenance label and an independently verified certificate.

Here "finite-description" means a finite term in the declared language for a parameter or kernel. Such a term records how the object is specified. Effective evaluation requires computational data as well. A parameter term evaluates to a finite tuple with scalar or index or interval entries. A multiplier primitive requires a finite formula from the source operations. Naming its values $(h(\lambda_j))_{j\ge 0}$ is insufficient. For the Neumann and Stokes realizations we may take $\mathbb{K}_{\mathfrak{G}} = \mathbb{Q}$ for intrinsic modal histories and $\mathbb{F} = \mathbb{R}$ (or $\mathbb{C}$ with a reality condition) for ambient analysis. A real initial state or coefficient path not represented by an intrinsic term remains an external terminal with its own certificate.

A finite operator formula must be accompanied by the domain conditions needed for its action. The following rules retain both the formula and these conditions under composition.

Intrinsic operator terms. The typed class $\mathsf{Op}_{\mathrm{int}}(\mathfrak{G})$ is the smallest class of finite rooted operator terms containing the following operators on their stated domains.

$$I, \qquad e^{-tA}, \qquad P_K, \qquad (I + \tau A)^{-1}, \qquad A^{r/2}, \qquad \Pi, \tag{E.2}$$

These operators are included whenever the relevant parameters t, K, τ, r are terms of $\mathfrak{P}_{\mathfrak{G}}$ and the displayed operator has the required type. It is closed under finite composition and finite direct sums as well as tensor contractions. A subordination or Bochner integral is admitted only when its kernel and measure as well as its interval and integrand are given by finite intrinsic terms and the resulting operator has a verified transformer. Any further multiplier $h(A)$ or smooth cutoff $\chi(B/K)$ is an external operator terminal unless it has

been reduced to these rules or explicitly included as a finite-description primitive in the source datum. The same condition applies to differential operators and integral kernels.

In the completion of section 4 a limiting presentation retains its approximating histories. For a multiplier $h(A)$ the finite description specifies its source. The transformer supplies the bound used when it acts on a history. These are separate requirements on the same operator term.

An operator obtained from native finite matrices can also act through certified completion. To make this statement precise let $T_N : X_+ \to Y_+$ be finite native operator terms with proved uniform bounds $\|T_N\|_{X_+\to Y_+} \le C_+$. Suppose that their weaker extensions obey $\|T_N - T\|_{X_-\to Y_-} \le \delta_N \to 0$. Suppose also that T is bounded between these weaker spaces and that $Y_+ \to Y_-$ is an admissible strict output pair. If histories H_N present f with common stronger bound M then $T_N H_N$ have the common output certificate $C_+ M$ and

$$\|T_N \mathsf{Val}(H_N) - Tf\|_{Y_-} \le \delta_N \|\mathsf{Val}(H_N)\|_{X_-} + \|T\|_{X_-\to Y_-} \|\mathsf{Val}(H_N) - f\|_{X_-} \longrightarrow 0.$$

The finite terms and their uniform transformers therefore present Tf under Definition 4.2. An effective version retains procedures for those terms together with computable bounds and convergence moduli. Boundedness of an independently specified operator supplies only an estimate. Its native approximation must still be exhibited.

Intrinsic and relative histories. An intrinsic history $H \in \mathsf{Hist}_{\mathrm{int}}(\mathfrak{G})$ is a finite typed rooted tree formed by the following rules.

(E1) A modal atom in $\mathfrak{M}_{\mathfrak{G}}$ and a coefficient atom in $\mathbb{K}_{\mathfrak{G}}$ are intrinsic leaves. Finite sums and scalar products are explicit algebraic nodes.

(E2) An operator term in $\mathsf{Op}_{\mathrm{int}}(\mathfrak{G})$ may act on a history lying in its declared domain.

(E3) Scalar multiplication m or a state interaction $\mathcal{B}$ or a constraint or a finite tensor contraction may be applied when its type is compatible and its C2 and C3 certificates have been verified.

(E4) A Bochner time integral is permitted when the path and interval as well as the time dependence have finite descriptions in the relevant history language and a common integrable certificate bound.

The relative class $\mathsf{Hist}_{\mathrm{rel}}(\mathfrak{G}; \mathfrak{T}_{\mathrm{ext}})$ obeys the same rules but also permits the declared external value and operator terminals as labelled leaves. Hence

$$\mathsf{Hist}_{\mathrm{int}}(\mathfrak{G}) \subset \mathsf{Hist}_{\mathrm{rel}}(\mathfrak{G}; \mathfrak{T}_{\mathrm{ext}}). \tag{E.3}$$

As in section 7 a residual is constructed from the equation and the admitted projection. Its history therefore records those operations rather than an additional input leaf.

F Moving windows, memory, and restarted budgets

A moving spectral window and a truncated memory contribute different error terms. Transferring a reconstructed state at a restart adds another contribution. Continuous defects with the required forcing bounds enter (7.17) while jumps enter the next initial-error bound. The projections and multipliers retain the source declarations of section 4.

F.1 Retained and unresolved states

Write $\mathcal{F}_t(u) = -Au + \mathcal{N}_t(u) + g(t)$ and let $P = P_K$ and $Q = I - P$. Applying these projections to the evolution gives

$$\dot{u}_P = P\mathcal{F}_t(u_P + u_Q), \qquad \dot{u}_Q = Q\mathcal{F}_t(u_P + u_Q). \tag{F.1}$$

For mild paths this notation denotes the corresponding projected Duhamel identities. The difference between the retained equation and the equation evaluated on the retained state alone is

$$\mathcal{C}_P(t;u) = P\mathcal{F}_t(u) - P\mathcal{F}_t(Pu) = P\big(\mathcal{N}_t(u) - \mathcal{N}_t(Pu)\big). \tag{F.2}$$

The linear and forcing terms cancel because $PA = AP$ and $P^2 = P$. The spectral projections are contractions in the resource spaces X, X_- of Proposition 7.1 so u and Pu belong to the same norm ball. The local difference estimate gives

$$\|\mathcal{C}_P(t;u)\|_{X_-} \le L_M(t)\,\|Qu\|_X\,, \qquad \|u\|_X \le M. \tag{F.3}$$

This closure residual measures how the unresolved state changes the retained equation. The full-space residual (7.10) measures the source omitted when the Galerkin solution is substituted into the original equation.

For a fixed window suppose that u is an absolutely continuous $\mathcal{H}$-valued solution of $\dot{u} = \mathcal{F}_t(u)$ and P is a fixed orthogonal projection. Then

$$\|u\|_2^2 = \|Pu\|_2^2 + \|Qu\|_2^2\,, \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Pu\|_2^2 = \operatorname{Re}\langle Pu, P\mathcal{F}_t(u)\rangle \tag{F.4}$$

almost everywhere. Orthogonality gives the first identity. Since P is fixed and bounded Pu is absolutely continuous with derivative $P\dot{u}$. The Hilbert-space chain rule gives the second identity.

For a bounded positive observation operator G put $\mathcal{Q}_{\mathsf{G}}(z) = \langle z, \mathsf{G}z\rangle$. The corresponding fixed window identity is

$$\mathcal{Q}_{\mathsf{G}}(u) = \mathcal{Q}_{\mathsf{G}}(Pu) + \mathcal{Q}_{\mathsf{G}}(Qu) + 2\operatorname{Re}\langle Pu, \mathsf{G}Qu\rangle\,. \tag{F.5}$$

The last term is a compatibility term. It vanishes when P reduces G. This holds in particular when G is a spectral multiplier of A. In that case the retained and unresolved entries are positive and form an exact ledger. For a general Gram observation deleting the cross term would change the observed quantity.

If P is fixed the retained form also satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}_{\mathsf{G}}(Pu) = \operatorname{Re}\langle P\mathcal{F}_t(u), \mathsf{G}Pu\rangle \tag{F.6}$$

almost everywhere. This is a balance identity. The sign of its right-hand side is determined by the equation and the observation. Positivity of G alone does not determine it.

For a finite non-diagonal Gram matrix the compatible split can be written without discarding its off-diagonal block. Suppose

$$\mathsf{G} = \begin{pmatrix} G_{11} & G_{12} \\ G_{12}^* & G_{22} \end{pmatrix} \geq 0, \qquad G_{11} > 0, \tag{F.7}$$

relative to retained and unresolved coordinates. Completing the square gives

$$\langle (x,y), \mathsf{G}(x,y) \rangle = \left\langle x + G_{11}^{-1} G_{12} y, G_{11}(x + G_{11}^{-1} G_{12} y) \right\rangle + \left\langle y, (G_{22} - G_{12}^* G_{11}^{-1} G_{12}) y \right\rangle . \tag{F.8}$$

The Schur complement is positive. Equation (F.8) is the Gram-orthogonal retained–unresolved ledger. It also shows why the coordinate split with the cross term simply omitted is not an equivalent quadratic observation.

F.2 Hard and smooth window motion

For a hard window the projection is constant between switches. When the state is absolutely continuous and there are finitely many switches the derivative of the retained path is a measure containing both the jumps and the continuous evolution between them.

Proposition F.1 (Hard-window crossing measure)**.** Let $P(t)$ be a right-continuous and piecewise constant path of finite-rank spectral cutoff projections with finitely many switches on $[0,T]$. Let $u : [0,T] \to \mathcal{H}$ be absolutely continuous. With $\Delta P(\tau) = P(\tau) - P(\tau-)$ and switch set $\mathcal{X}_P \subset (0,T]$ the path Pu has bounded variation and

$$\mathrm{d}(Pu) = P(t)\dot{u}(t)\,\mathrm{d}t + \sum_{\tau \in \mathcal{X}_P} \Delta P(\tau) u(\tau) \delta_\tau . \tag{F.9}$$

In particular

$$P(t)u(t) - P(0)u(0) = \int_0^t P(s)\dot{u}(s)\,\mathrm{d}s + \sum_{\substack{\tau \in \mathcal{X}_P \\ \tau \leq t}} \Delta P(\tau) u(\tau). \tag{F.10}$$

Proof. Integrate on each interval of constant projection. At a switch the continuity of u leaves precisely the increment $\Delta P(\tau)u(\tau)$. The integral of $\|\dot{u}\|$ and the finite sum of these jump norms bound the variation. Summing the increments proves both identities. □

The integrated identity includes a terminal switch and carries no atom at the initial time. The projection path rather than the frequency parameter alone determines the jumps. In particular an isolated contact with a threshold contributes no switch when the two one-sided projections agree and the right-continuous representative is used. The energy transferred at a switch is exactly

$$\frac{1}{2}\left(\|P(\tau)u(\tau)\|_2^2 - \|P(\tau-)u(\tau)\|_2^2 \right) = \frac{1}{2} \operatorname{Re} \langle u(\tau), \Delta P(\tau) u(\tau) \rangle . \tag{F.11}$$

The unresolved energy has the opposite jump. For the general quadratic observation the jump is

$$\mathcal{Q}_{\mathsf{G}}(P(\tau)u(\tau)) - \mathcal{Q}_{\mathsf{G}}(P(\tau-)u(\tau)). \tag{F.12}$$

When both one-sided projections reduce G the resolved and unresolved ledgers acquire opposite jumps. Otherwise the change in the compatibility term of (F.5) must also be retained.

For a smooth selector put $B = A^{1/2}$ and $S_\kappa = \chi(B/\kappa)$ where $\chi \in C^1([0,\infty);[0,1])$ and $C_\chi = \sup_{r\geq 0} r|\chi'(r)| < \infty$.

If κ is positive and absolutely continuous on $[0,T]$ with a positive lower bound and u is an absolutely continuous solution of $\dot{u} = \mathcal{F}_t(u)$ then

$$\partial_t(S_\kappa u) = S_\kappa \mathcal{F}_t(u) + \dot{\kappa}(\partial_\kappa S_\kappa)u, \qquad \partial_\kappa S_\kappa = -\frac{B}{\kappa^2}\chi'(B/\kappa) \tag{F.13}$$

almost everywhere. Moreover

$$\|u\|_2^2 = \|S_\kappa u\|_2^2 + \|(I - S_\kappa)u\|_2^2 + E_{\mathrm{ov}}(\kappa; u), \tag{F.14}$$

where

$$E_{\mathrm{ov}}(\kappa; u) = 2\sum_j \chi(\omega_j/\kappa)(1 - \chi(\omega_j/\kappa))|\widehat{u}_j|^2, \qquad 0 \leq E_{\mathrm{ov}} \leq \tfrac{1}{2}\|u\|_2^2 . \tag{F.15}$$

The derivative multiplier has norm at most C_χ/κ. Spectral dominated convergence applies because κ is absolutely continuous with a positive lower bound. These properties justify the product rule in (F.13). Expanding $u = S_\kappa u + (I - S_\kappa)u$ gives the energy identity. The scalar bound $0 \leq 2r(1-r) \leq 1/2$ controls its cross term.

The retained energy consequently satisfies

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|S_\kappa u\|_2^2 = {} & \operatorname{Re}\langle S_\kappa u, S_\kappa \mathcal{F}_t(u)\rangle \\ & + \dot{\kappa}\operatorname{Re}\langle S_\kappa u, (\partial_\kappa S_\kappa)u\rangle . \end{aligned} \tag{F.16}$$

If selectors converge pointwise on the spectrum to a $\{0,1\}$-valued cutoff then their overlap tends to zero by dominated convergence. This requires the stated pointwise convergence also at any eigenfrequency on the threshold. Using the overlap to bound a state or observation error requires an estimate for the chosen reconstruction. An exact quadratic partition can instead be obtained by choosing $S^*S + T^*T = I$. The field identity $Su + Tu = u$ requires a separate condition. For a positive form G the analogous exact partition is $S^*\mathsf{G}S + T^*\mathsf{G}T = \mathsf{G}$. If this identity is not available then the difference is a compatibility remainder of the kind in Proposition 5.1 rather than a positive unresolved energy.

F.3 Elimination and the memory budget

An unresolved component can affect the retained state even when it is absent from the reported value. Solving its equation separates this effect into the response of the unresolved initial state and the response to its later excitation by the retained variables. Substitution

gives a history integral in the retained equation. Consider a linear block system on $P\mathcal{H}\oplus Q\mathcal{H}$ with blocks $L_{PP}, L_{PQ}, L_{QP}, L_{QQ}$.

Suppose L_{PP}, L_{PQ}, L_{QP} are bounded between their respective spaces and L_{QQ} generates a strongly continuous semigroup on $Q\mathcal{H}$. Then every mild solution satisfies

$$\begin{aligned}\dot{u}_P(t) &= L_{PP}u_P(t) + L_{PQ}e^{tL_{QQ}}u_Q(0)\\ &\quad + \int_0^t L_{PQ}e^{(t-s)L_{QQ}}L_{QP}u_P(s)\,\mathrm{d}s.\end{aligned} \tag{F.17}$$

The retained equation has continuous right-hand side. This is the exact state-space memory identity. Variation of constants gives $u_Q(t) = e^{tL_{QQ}}u_Q(0) + \int_0^t e^{(t-s)L_{QQ}}L_{QP}u_P(s)\,\mathrm{d}s$. Substitute this expression in $\dot{u}_P = L_{PP}u_P + L_{PQ}u_Q$. Strong continuity of the semigroup and boundedness of the coupling operators give a continuous right-hand side on compact time intervals.

The history response of section 8.2 provides a case in which the eliminated component is invisible to the initial readout but affects its later evolution. Put $V_p = C_p^*$ and $P_{\mathrm{diff}} = I - V_pC_p$. With $f = V_pa + z$ its positive response has blocks

$$\mathsf{L}_{00} = C_p\mathsf{L}_\beta V_p, \qquad \mathsf{L}_{01} = C_p\mathsf{L}_\beta|_{\ker C_p}, \qquad \mathsf{L}_{11} = P_{\mathrm{diff}}\mathsf{L}_\beta|_{\ker C_p}.$$

The complete equation is therefore

$$a' = -\mathsf{L}_{00}a - \mathsf{L}_{01}z, \qquad z' = -\mathsf{L}_{01}^*a - \mathsf{L}_{11}z. \tag{F.18}$$

Both diagonal blocks are nonnegative self-adjoint. Eliminating z gives

$$a'(\tau) = -\mathsf{L}_{00}a(\tau) - \mathsf{L}_{01}e^{-\tau\mathsf{L}_{11}}z(0) + \int_0^\tau \mathsf{L}_{01}e^{-(\tau-r)\mathsf{L}_{11}}\mathsf{L}_{01}^*a(r)\,\mathrm{d}r. \tag{F.19}$$

These identities follow by applying C_p and P_{diff} and then using variation of constants in the second equation. The nonzero matrix element (8.41) proves that the coupling block is present in this source. The initial term records a difference already present at $\tau = 0$. The integral records differences excited by the subsequent readout path. Even when $z(0) = 0$ the state can enter the difference space and later affect the readout. Its quadratic coupling is the mixed term in (5.24).

The Gaussian estimate (8.33) controls the effect of high-root relations on a finite interval. Decay with elapsed assembly time is a different question. In fact the complete response has no uniform positive spectral gap. A unit history vector v above root N obeys $\mathsf{L}_{\beta,N}v = 0$. The positive tail $T_N = \mathsf{L}_\beta - \mathsf{L}_{\beta,N}$ consequently gives

$$\langle v, \mathsf{L}_\beta v\rangle \le \delta_N, \qquad \|\mathsf{L}_\beta v\|^2 \le \delta_N\,\langle v, \mathsf{L}_\beta v\rangle\,, \qquad \delta_N \longrightarrow 0. \tag{F.20}$$

Use $T_N^2 \le \delta_N T_N$ for the second inequality. A parent relation for v gives strictly positive response but the ratio in this estimate can be arbitrarily small. In particular differentiating a proposed uniform exponential decay estimate for the response form at zero would contradict (F.20).

A finite memory horizon can instead be assessed from the actual coupling kernel. For a horizon ℓ the omitted part of (F.19) has the bound

$$\left\| \int_0^{(\tau-\ell)_+} \mathsf{L}_{01} e^{-(\tau-r)\mathsf{L}_{11}} \mathsf{L}_{01}^* a(r)\, \mathrm{d}r \right\| \le \sup_{r\le\tau} \|a(r)\| \int_\ell^{\max\{\tau,\ell\}} \|\mathsf{L}_{01} e^{-v\mathsf{L}_{11}} \mathsf{L}_{01}^*\|\, \mathrm{d}v. \tag{F.21}$$

Contractivity bounds the integral by $\|\mathsf{L}_{01}\|^2(\tau-\ell)_+$. This certifies the omission when that bound fits the allocated residual tolerance. A shorter memory at the same tolerance requires a sharper kernel estimate. Alternatively one can retain the finite history state and use the Gaussian parent tail to bound its operator error. The two approximations remove different information and are certified by different estimates.

For a memory horizon $\ell \ge 0$ suppose in addition that $\left\|e^{rL_{QQ}}\right\| \le M_Q e^{-\beta r}$ with $\beta > 0$. The omitted history of a continuous retained path w has the bound

$$\left\| \int_0^{(t-\ell)_+} L_{PQ} e^{(t-s)L_{QQ}} L_{QP} w(s)\, \mathrm{d}s \right\| \le \frac{M_Q \|L_{PQ}\| \|L_{QP}\|}{\beta} e^{-\beta\ell} \sup_{s\le t} \|w(s)\|. \tag{F.22}$$

For $t > \ell$ the time differences in the omitted integral are at least ℓ. Integration of the exponential bound gives the displayed factor. For $t \le \ell$ the integral is empty. The unresolved initial state contributes separately with bound $M_Q \|L_{PQ}\| e^{-\beta t} \|u_Q(0)\|$. These contributions enter the residual budget in the norm used for the block and semigroup estimates. For a nonlinear unresolved equation the analogous elimination requires estimates for the dependence of its solution map on the retained history. When a quadratic ledger is also kept the exact eliminated contribution is part of the state before a memory approximation is made. Truncating the history produces an approximation increment including its cross term with the retained memory contribution as in (5.16). Bounding only the norm of the omitted memory controls the state error but does not by itself give its quadratic increment.

F.4 Auxiliary resources and scale

Let $\rho \in \mathfrak{R}$ record the auxiliary choices of an approximation. These may specify the transition width and memory horizon together with the time step. Write $E(K, \rho, t)$ for its certified total state-error bound as obtained from the residual estimate and any direct postprocessing error as in (7.18). Let $\Sigma = \mathrm{Spec}(B) \cup \{0\}$ with repetitions removed. For fixed (ρ, t) set

$$K_\varepsilon(t;\rho) = \min\{K \in \Sigma : E(K, \rho, t) \le \varepsilon\} \tag{F.23}$$

when the feasible set is nonempty. This is the least threshold certified by the chosen upper bound with the auxiliary resources fixed and the threshold restricted to the declared candidate set.

Suppose $E(K, \rho, t)$ is nonincreasing in K and tends to zero along the unbounded discrete spectrum. In finite dimension replace this limit by vanishing at the largest threshold.

Then (F.23) exists. If a dilation by $a > 0$ transports the spectrum to $a^{-1}\Sigma$ and the budget obeys $E^{(a)}(K/a, \rho^{(a)}, t^{(a)}) = E(K, \rho, t)$ then

$$K_\varepsilon^{(a)}(t^{(a)}; \rho^{(a)}) = a^{-1} K_\varepsilon(t; \rho). \tag{F.24}$$

The limiting or terminal condition gives feasibility. Discreteness and nonnegativity of the spectrum give a least feasible level. Monotonicity makes the feasible set an upper tail. Dilation maps this set bijectively to the transported feasible set and preserves its ordering.

For the Neumann interval this transport is explicit. The unitary map $(U_a f)(x) = a^{-1/2} f(x/a)$ from $L^2(0, L)$ to $L^2(0, aL)$ satisfies

$$A_{aL} U_a = a^{-2} U_a A_L, \quad P_{K/a}^{aL} U_a = U_a P_K^L, \quad e^{-a^2 t A_{aL}} U_a = U_a e^{-t A_L}. \tag{F.25}$$

Each identity follows on the cosine basis and extends on its natural domain. If the nonlinear term and forcing are transported as $\mathcal{N}_{a^2 t}^{(a)}(U_a u) = a^{-2} U_a \mathcal{N}_t(u)$ and $g^{(a)}(a^2 t) = a^{-2} U_a g(t)$ the entire mild equation is transported. Using $\|U_a v\|_{X^{(a)}} = \|v\|_X$ and the corresponding transported forcing norm preserves the error budget. Conversion back to an unscaled Sobolev convention carries the explicit norm comparison constants. The scaling statement therefore transports the equation with its norms and error budget.

Joint resource selection minimizes a prescribed cost over $E(K, \rho, t) \le \varepsilon$. This allows both the threshold and the auxiliary resources to vary. Here $\mathfrak{R}$ has a metrizable topology and Σ has its discrete topology.

If this feasible set is closed and contains a point of finite cost and the cost is lower semicontinuous with compact finite sublevel sets then a minimum is attained. Indeed choose a minimizing sequence in the feasible set. A finite-cost feasible point allows its tail to be placed in one compact cost sublevel. A convergent subsequence then has a feasible limit by closedness and lower semicontinuity shows that the limit attains the infimum. When several minimizers exist a specified tie-breaking rule selects a representative. Uniqueness of the minimizer itself requires an additional hypothesis. Computing the least certified threshold also requires deciding boundary equality. Certified upper bounds and quantitative decay give a strictly feasible sufficient window.

F.5 Temporal selection and transfer

A change in the least certified threshold can occur only when an error bound meets the tolerance. A finiteness condition on these contacts therefore gives a projection path to which the hard-window balance applies.

Fix ρ. Suppose $E(K, \rho, t)$ is nonincreasing in K and continuous in t. Assume that the least feasible threshold is locally bounded. For each compact time interval J and each finite K_* assume that

$$\bigcup_{K \in \Sigma \cap [0, K_*]} \{t \in J : E(K, \rho, t) = \varepsilon\} \tag{F.26}$$

is finite. Replacing the minimal threshold at each contact time by its right limit gives a right-continuous and piecewise constant feasible selector with left limits and finitely many switches on compact intervals. At a terminal endpoint the selector takes the pointwise

minimal value. The construction is elementary but records the point at which temporal regularity enters. Local boundedness restricts the possible thresholds to finitely many levels on a compact interval. Away from the finite contact set continuity fixes the sign of $E(K,\rho,t)-\varepsilon$ at every such level so the least feasible level is locally constant. Its one-sided limits exist. A right-limit level is feasible at the contact time by continuity and is at least the pointwise minimum. The stated replacement therefore preserves feasibility and gives the asserted path including a possible final jump.

The resulting projection path meets the projection hypotheses of Proposition F.1. The balance applies to an absolutely continuous state path. Given $0<\varepsilon_{\mathrm{in}}<\varepsilon_{\mathrm{out}}$ and a pair (K,ρ) whose current continuous budget starts below $\varepsilon_{\mathrm{in}}$ the next event is the first time that budget reaches $\varepsilon_{\mathrm{out}}$ with value $+\infty$ if it never does. Continuity gives a positive interval before that event.

At an event time τ let v^- be the reconstructed state before transfer and v^+ the initial state after transfer. If $\|v^+-v^-\|_X\le\Delta_{\mathrm{tr}}$ and the old state error is bounded by e^- then the new initial budget is

$$e^+=e^-+\Delta_{\mathrm{tr}}. \tag{F.27}$$

When transfer starts from a postprocessed state e^- includes the postprocessing error. The estimate on the new interval assumes this initial bound together with a verified common solution ball. It also requires a residual envelope and a local stability number. For a finite partition write $H_j<1$ for the local stability number. Write D_j for the integrated residual bound and Δ_j for transfer at its right endpoint. Repeated use of Proposition 7.1 gives

$$e_{j+1}\le\frac{e_j+D_j}{1-H_j}+\Delta_j. \tag{F.28}$$

The recurrence carries the accumulated error into the next interval. Changes in memory initialization contribute to that interval's residual bound while window jumps contribute their X-norm through Δ_j.

If the same intervals carry ledger-certified transports their quadratic budgets compose separately as

$$\mathcal{E}_{j+1}+\mathcal{D}_j\le\mathcal{E}_j+\iota_j+\varepsilon_j,\qquad \mathcal{D}_j\ge 0, \tag{F.29}$$

where ι_j contains the complete increment from newly introduced data and ε_j contains the complete transfer or approximation increment. Summation gives

$$\mathcal{E}_N+\sum_{j<N}\mathcal{D}_j\le\mathcal{E}_0+\sum_{j<N}\iota_j+\sum_{j<N}\varepsilon_j. \tag{F.30}$$

This account accompanies the state-error recurrence (F.28). A restart is accepted only after the transferred state lies in the next analytic domain and both sets of incoming bounds have been re-established.

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Declarations

Conflict of interest. The author declares no competing interests.

Data availability. This work uses no empirical dataset. All proofs and model estimates are contained in the article and its appendices.

Lean Formalization of Arithmetic and Basic Physical Interfaces [19] provides independent formal checks of the common source construction and its finite interfaces. The spectral estimates and completion and evolution arguments used here are proved in the article and its appendices. The library is archived at https://doi.org/10.5281/zenodo.23254521.

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