

Prime Decomposition and Operator Responses in Spectral Natural-Number Arithmetic

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ABSTRACT: We construct decomposition and occupation operators on a generated correlation-spectral arithmetic system. Prime residue classes form the augmentation cotangent basis, and the corresponding one-word vectors span the degree-one kernel of a decomposition Hodge Laplacian. The complementary spectral gap is exactly one, giving sharp heat and resolvent approximations to the prime projection and summable Gaussian trace errors. Symmetric and exterior occupations retain logarithmic arithmetic energy and quadratic correlation energy on the same value space. A faithful extension of addition records relative inputs; its charge defect intertwines input multiplicative parity with the parity of the sum. Common-factor coordinates and exact square-divisor identities give scale transforms of the resulting signed responses. Combining additive and decomposition responses defines a Gaussian endpoint map. Under diagonal self-consistency, this map has a unique stable fixed point with an enclosure of the complete infinite response. For compact Laplace realizations of square energy and quadratic shell capacity, Weyl counting gives dimension three; further geometric and central-sign classifications hold in their stated realization classes. Counting applications recover Bertrand's theorem, the prime-number asymptotic and square-free sign densities through the occupation traces and their boundary responses.

KEYWORDS: Prime factorization; Hodge Laplacian; Prime projection; Möbius function; Riemann zeta function; Fock space

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1 Introduction

Unique factorization identifies a positive integer with a finite prime occupation profile. Logarithmic energy adds over the occupied prime directions and gives Euler products; quadratic correlation energy depends on the square of the evaluated integer and governs Gaussian response. Addition introduces a different identification: several ordered input pairs have the same sum. We construct operators that retain factorization and relative additive data on the correlation-spectral arithmetic system of [1].

The underlying data are an ordered Peano semiring of natural modes, its correlation Hilbert space and canonical transport across scales and orientations. Finite expression fibres retain ordered addition and multiplication before evaluation; a recursive positive source connects those histories to the modes by conditional Hilbert readout. Section 2 records these data and their source.

Prime modes admit three compatible descriptions. Cascade irreducibility defines the prime family in the arithmetic semiring. Their residue classes form the augmentation cotangent basis. The corresponding one-word vectors form an orthonormal basis of the degree-one harmonic subspace of the decomposition complex by Theorem 3.1. On the one-word space $\mathcal{K}_{b,1}$, let R_b be the decomposition Laplacian and let P_b^{pr} project onto its kernel. The composite spectral gap is one, attained at prime squares. Theorem 3.10 gives the sharp norm identities

$$\|e^{-\tau R_b} - P_b^{\text{pr}}\| = e^{-\tau}, \quad \|\lambda(R_b + \lambda I)^{-1} - P_b^{\text{pr}}\| = \frac{\lambda}{1 + \lambda} \quad (\tau \geq 0, \lambda > 0).$$

Gaussian weighting also gives summable trace errors. These estimates approximate the prime projector uniformly on the whole one-word space.

The additive construction retains the information discarded by scalar summation. Addition is a closable map whose fixed-sum fibres split into one longitudinal line and a transverse difference space. Its faithful extension records relative inputs, and the charge-defect identity intertwines the two input Liouville signs with the sign of the sum at the central

character. A separate unitary records the common factor and primitive coprime legs of an ordered pair, transporting the Gaussian scale by the square of the common factor.

Symmetric and exterior occupation give parallel realizations of the prime family. Symmetric occupation retains arbitrary exponents and Liouville parity on every positive mode. Exterior occupation selects square-free profiles on which the same parity gives the Möbius sign. Prime-direction transfer preserves the full signed response before an unsigned capacity estimate is taken. Square-divisor identities give mutually inverse dilation transforms between the Liouville and Möbius responses on the stated weighted scale spaces. Mellin transformation connects the quadratic Gaussian traces with the logarithmic Dirichlet traces.

The occupation formulas have direct precedents in Julia’s prime gas [2] and Spector’s interpretation of the Möbius sign as fermion parity [3]. Related constructions include the Bost–Connes arithmetic dynamical system [4] and the Bohr lift of Dirichlet series [5]. The bar and Koszul constructions have their standard homological sources [6, 7]. Here evaluation transports the generated square-energy law to prime occupations. The factor-word complex gives the gapped prime projector, while the faithful additive extension retains relative inputs and their multiplicative charge defect. Mellin and square-divisor identities relate these responses with their stated characters and scale normalizations. Appendix F compares these structures.

Normalized endpoint states combine the additive and decomposition responses. The permanent endpoint ratio gives an exact escort law. Under diagonal self-consistency, the returned endpoint retention equals the attenuation coordinate of the same response map. Theorem 6.5 proves that this map has a unique stable fixed point. Proposition 6.6 explains its magnitude through the first contributing shell; the infinite-tail and interval estimates enclose the complete response.

Representation fibres use the retained square coordinate in a further way. A compact Laplace realization of both square energy and quadratic shell capacity has the three-dimensional Weyl exponent. Affine energy–capacity compatibility within the round spherical class identifies the three-spherical parent and its antipodal projective compression. The nontrivial central character then selects the minimal half-spin module. The realization classes and response laws are specified in Appendix A, with calculations in in Appendices C–E.

The counting applications recover classical results on the same carrier. The central-binomial argument gives Bertrand’s theorem in Appendix G. Positive prime-power response and the Fock phase inequality supply the boundary data for the Wiener–Ikehara theorem in Mellin form. Appendix H uses these data to obtain the prime-number asymptotic and square-free sign densities, including the required boundary and summation arguments.

The Lean development cited in [8] provides formal counterparts of the underlying source arithmetic and history construction. The computational supplement contains the interval certificates and code for the endpoint enclosure. The accompanying correspondence and computational records identify the formal statements and numerical bounds.

Figure 1 shows how decomposition, occupation and additive input spaces are related through the common arithmetic data.

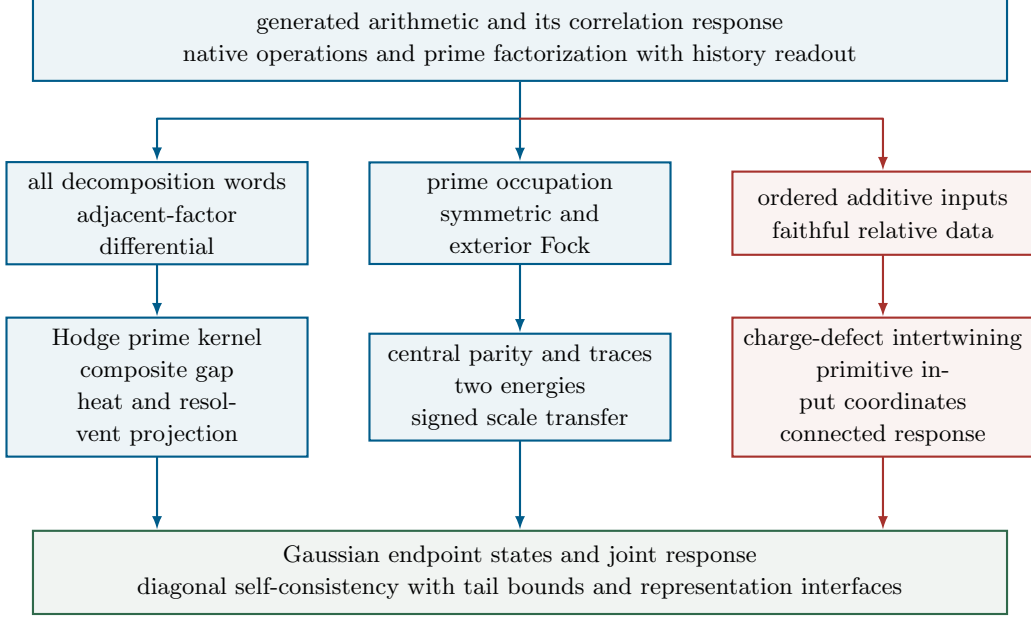


Figure 1. Decomposition and occupation with additive input data. The decomposition complex is formed from all nonunit-factor words and recovers the prime family through its degree-one kernel. Occupation uses factorization coordinates. The faithful additive map retains relative inputs through its defect environment including the order sign. Gaussian weighting combines their responses. Diagonal self-consistency supplies the endpoint fixed-point equation.

2 Arithmetic and history data

We use the natural modes, Gaussian weights and expression readouts of [1]. The normalizations below fix the Hilbert spaces on which decomposition and addition act.

2.1 Natural modes and correlation energy under transport

The Gaussian profile at a positive correlation scale b is $K_b(d) = \exp(-d^2/(2b^2))$. Its local information radius and total normalized correlation radius are

$$\ell_{UV}(b) = \sqrt{2}b, \quad L_{IR}(b) = \sqrt{2\pi}b, \quad \mathcal{W}_b = [\ell_{UV}(b), L_{IR}(b)], \quad \Delta_b = L_{IR}(b) - \ell_{UV}(b). \quad (2.1)$$

On $\mathcal{H}_b = L^2(\mathcal{W}_b)$ the Neumann generator A_b has orthonormal eigenvectors $\phi_{b,n}$ and energies

$$E_{b,n} = E_{b,1}n^2, \quad E_{b,1} = (\pi/\Delta_b)^2, \quad C_b(t) = e^{-tA_b/2}. \quad (2.2)$$

The response eigenvalue at the local endpoint is $\Lambda_{b,n} := e^{-b^2 E_{b,n}/2}$. The zero mode is constant. Response comparison orders the spectral projections before the coordinate n is introduced. Orientation gives the analytic lifts $u_{b,n}(z) = z^n$ and the family $\mathcal{U}_b$. Its native operations satisfy

$$u_{b,m} \boxplus_b u_{b,n} = u_{b,m+n}, \quad u_{b,m} \boxtimes_b u_{b,n} = u_{b,mn}. \quad (2.3)$$

Here $0_b = u_{b,0}$ is the constant homomorphism and $1_b = u_{b,1}$ is the identity circle map. The real Neumann modes and analytic modes are linked by the unitary lift $\mathcal{A}_b \phi_{b,n} = u_{b,n}$. The

winding map $\deg_b u_{b,n} = n$ certifies the ordered Peano semiring and its unique factorization [1, Sections 4–5 and Appendix B]. The prime family $\mathcal{P}_b$ consists of the positive nonunit cascade irreducibles. We write

$$u = \boxtimes_b \pi^{\boxtimes_b \nu_u(\pi)} \quad (u >_b 0_b) \quad (2.4)$$

for its finite exponent profile.

The integer and rational completions are denoted by $\mathbb{Z}_b$ and $\mathbb{Q}_b$. The real and complex completions are $\mathbb{R}_b$ and $\mathbb{C}_b$. An integer is a formal difference $[u, v]$ of natural modes. The equality $[u, v] = [u', v']$ holds when $u \boxplus_b v' = u' \boxplus_b v$. The difference has degree $\deg_b(u) - \deg_b(v)$. Fractions and metric completion give the remaining scalar fields [1, Section 12]. We use their canonical identifications with the standard scalar fields in the Hilbert constructions.

The composition algebra and augmentation are

$$\mathfrak{A}_b = \text{span}_{\mathbb{C}}(\mathcal{U}_b \setminus \{0_b\}), \quad \epsilon_b \left(\sum_u a_u u \right) = a_{1_b}, \quad I_b = \ker \epsilon_b. \quad (2.5)$$

Cascade is extended bilinearly in this algebra. Unique factorization identifies it with the polynomial algebra on the prime modes. The classes $[\pi]$ therefore form the basis of I_b/I_b^2 . Their finite symmetric products recover $\mathfrak{A}_b$ through

$$\text{Sym}_{\text{alg}}(I_b/I_b^2) \cong \mathfrak{A}_b, \quad \prod_{\pi} [\pi]^{\nu(\pi)} \longmapsto \boxtimes_b \pi^{\boxtimes_b \nu(\pi)}. \quad (2.6)$$

These are the prime-cotangent identifications of [1, Section 5]. The Hilbert constructions use the specified orthonormal bases for modes and occupations as well as for words. These bases are related by the displayed evaluation maps.

The endpoint times and dimensionless parameter are

$$t_{\text{UV}} = b^2, \quad t_{\text{IR}} = \pi b^2, \quad \beta = \frac{tE_{b,1}}{2}, \quad \frac{t_{\text{IR}}}{t_{\text{UV}}} = \pi. \quad (2.7)$$

The positive parameter t belongs to correlation response. Expression depth and assembly time are separate parameters. Changing b canonically identifies modes in the same response position and transports normalized energy and both arithmetic operations [1, Section 5]. This map acts entrywise on decomposition words and occupation profiles.

Gaussian time changes the root weight, whereas decomposition time acts on factorization words. The escort power compares normalized states at fixed spectral data. In the diagonal self-consistency equation, the decomposition attenuation equals the endpoint-retention output. Table 1 distinguishes these parameters.

2.2 Expressions and the two coherent readouts

The reduced expression object Expr_b retains the ordered children and their brackets together with the addition or cascade label at each internal vertex. Normalization imposes the stated zero and unit laws. Its evaluation val_b intertwines expression operations with

Table 1. Parameters of the occupation and endpoint responses.

Symbol	Normalization	Action
$b > 0$	covariance scale of the source Gaussian	fixes lengths and the dimensional energy unit
$t > 0$	$C_b(t) = e^{-tA_b/2}$	supplies Gaussian correlation response
$\theta > 0$	$\theta = t/b^2$	compares responses across correlation scales
$\beta > 0$	$\beta = tE_{b,1}/2$	gives the root weight $e^{-\beta n^2}$
$\tau \geq 0$	coefficient of Δ_b^{bar} or its transported response	runs decomposition flow toward the harmonic sector
$\eta > 0$	normalized power of the ultraviolet state	reaches the infrared state at $\eta = \pi$
$0 \leq \kappa \leq 1$	trial value in the τ coordinate	gives the diagonal endpoint-retention equation

(2.3). Each fibre $\text{Expr}_b(u) = \text{val}_b^{-1}(u)$ is finite and has size $h_b(u)$. The quotient by equal evaluation is $\mathcal{U}_b$ [1, Section 6 and Appendix D]. The expression Hilbert space and counting readout are

$$\mathcal{H}_b^{\text{expr}} = \ell^2(\text{Expr}_b), \quad Q_b^{\text{expr}}|e\rangle = h_b(u)^{-1/2}|u\rangle_b, \quad u = \text{val}_b(e). \quad (2.8)$$

This is a normalized Hilbert compression. Algebraic evaluation acts on expressions themselves and has no normalization coefficient.

A root channel for $u >_b 1_b$ is an ordered additive pair of positive child values or an ordered cascade pair of nonunit positive child values evaluating to u . Write $\mathcal{C}_b(u)$ for this finite set and $c_b(u)$ for its size. The recursive history law is

$$p_b(1_b) = 1, \quad p_b(\text{node}_c(e, f)) = \frac{p_b(e)p_b(f)}{c_b(u)}. \quad (2.9)$$

It has total mass one on each positive fibre. Writing $G_{b,\beta}(u)$ for the root Gaussian mass gives $\sum_{e \in \text{Expr}_b(u)} G_{b,\beta}(u)p_b(e) = G_{b,\beta}(u)$. Our normalization is $G_{b,\beta}(u_{b,n}) = e^{-\beta n^2}$. Its total positive mass is

$$\mathcal{Z}_b^+(\beta) := \sum_{u >_b 0_b} G_{b,\beta}(u) = \sum_{n \geq 1} e^{-\beta n^2} < \infty. \quad (2.10)$$

Adjoining the zero mode adds one to this mass. Thus the root marginal of the normalized source has the same normalization as the complete Gaussian trace. The construction and its phase extension are given in [1, Sections 6–7].

Given a probability π_u on each fibre define

$$Q_\pi|e\rangle = \sqrt{\pi_u(e)}|u\rangle_b, \quad \omega_{\pi,u} = \sum_{\text{val}_b e=u} \sqrt{\pi_u(e)}|e\rangle. \quad (2.11)$$

Orthonormality of the vectors $\omega_{\pi,u}$ gives $Q_\pi Q_\pi^* = I$. The expression space therefore splits into their closed span and $\ker Q_\pi$. Uniform weights give Q_b^{expr} . The resulting decomposition is

$$\mathcal{H}_b^{\text{expr}} = \overline{\text{span}}\{\omega_{\pi,u} : u \in \mathcal{U}_b\} \oplus \ker Q_\pi. \quad (2.12)$$

The choice $\pi_u = p_b$ on positive fibres gives

$$\mathcal{H}_b^{\text{hist}} = \bigoplus_{u >_b 0_b} \ell^2(\text{Expr}_b(u)), \quad Q_b^{\text{hist}}|e\rangle = \sqrt{p_b(e)}|u\rangle_b, \quad \omega_{b,u}^{\text{hist}} = (Q_b^{\text{hist}})^*|u\rangle_b. \quad (2.13)$$

These coisometry identities follow by summing the fibre probabilities. Probability pushforward sums the probabilities themselves, whereas the Hilbert readout uses their square roots. The history assembly operator gives the trace-class parent–child response. Its construction and the variance and entropy decompositions are established in [1, Sections 8–11].

3 Prime decomposition and occupation representations

The value readout gives the natural modes with their original Gaussian masses. Flattening the outer cascade skeleton of an expression retains an ordered word of evaluated nonunit factors with the same total mode. It forgets the child histories and brackets but preserves the adjacent factors that can be merged. Prime occupation instead records the multiplicities in the evaluated factorization.

The bar differential merges adjacent factors, so its image in degree one consists exactly of decomposable modes. Unique factorization identifies the composition algebra with the symmetric algebra on the prime cotangent space. The Koszul resolution [7] then computes the higher bar homology as its exterior powers. Symmetric occupation retains repeated prime factors; exterior occupation retains square-free profiles. The two Fock completions realize these occupation rules on orthonormal bases.

3.1 The reduced decomposition complex

Composition defines a canonical complex on the augmentation ideal by merging adjacent factors. For $r \geq 1$ set

$$\overline{B}_r(\mathfrak{A}_b) := I_b^{\otimes r}, \quad \overline{B}_0(\mathfrak{A}_b) := \mathbb{C}, \quad (3.1)$$

and write a basis tensor as $[u_1] \cdots [u_r]$ with every u_j a positive nonunit natural mode. Define $d_1 = 0$ and for $r \geq 2$ put

$$d_r[u_1] \cdots [u_r] := \sum_{j=1}^{r-1} (-1)^{j-1} [u_1] \cdots [u_j] \boxtimes_b [u_{j+1}] \cdots [u_r]. \quad (3.2)$$

Associativity of $\boxtimes_b$ gives $d_{r-1}d_r = 0$. The total mode

$$\text{Tot}_b[u_1] \cdots [u_r] := u_1 \boxtimes_b \cdots \boxtimes_b u_r \quad (3.3)$$

is preserved by d . Hence the complex is the algebraic direct sum of finite-dimensional total-mode fibres.

Theorem 3.1 (Decomposition homology). *The reduced decomposition complex satisfies*

$$H_r(\overline{B}_\bullet(\mathfrak{A}_b), d) \cong \bigwedge^r (I_b/I_b^2) \cong \bigwedge^r \text{span}_{\mathbb{C}} \mathcal{P}_b \quad (r \geq 0). \quad (3.4)$$

In degree one this gives

$$H_1(\overline{B}_\bullet(\mathfrak{A}_b), d) \cong I_b/I_b^2 \cong \text{span}_{\mathbb{C}} \mathcal{P}_b. \quad (3.5)$$

The total-mode fibre of a fixed positive mode u has one-dimensional homology in degree r precisely when u is square-free with exactly r distinct spectral-prime factors. Otherwise its homology is zero.

Proof. The differential into degree one has image I_b^2 and $d_1 = 0$. Thus the first homology is

$$H_1(\overline{B}_\bullet(\mathfrak{A}_b), d) = I_b/I_b^2. \quad (3.6)$$

For prime classes $\bar{\pi}, \bar{\varrho} \in I_b/I_b^2$ the antisymmetric word

$$[\pi \mid \varrho] - [\varrho \mid \pi] \quad (3.7)$$

is a cycle because the algebra is commutative. It represents $\bar{\pi} \wedge \bar{\varrho}$. Repetition of a prime direction gives zero after antisymmetrization.

The identity in [1, Theorem 5.3] is $\mathfrak{A}_b = \text{Sym}(I_b/I_b^2)$. Each finite set of prime generators is a regular sequence in the polynomial algebra it generates. The Koszul resolution [7, Corollary 4.5.5] is therefore a free resolution of its augmentation module. Tensoring with that module sets every prime generator to zero. The differential therefore vanishes and the resulting homology is the exterior algebra on the prime classes. The bar resolution computes the same homology. Every bar word uses finitely many generators. Exactness of filtered unions of vector spaces then gives (3.4) for the full algebra. The resolution preserves total mode. A wedge basis uses each prime at most once. This proves the fibre statement by unique factorization. $\square$

The homology calculation distinguishes two occupation rules. Arbitrary prime multiplicities reconstruct the positive composition algebra. In homology repeated prime factors disappear and a square-free mode contributes one exterior basis vector. Completing the corresponding orthonormal bases gives the symmetric and antisymmetric Fock spaces.

3.2 Symmetric and antisymmetric Fock functors

We use the standard symmetric and antisymmetric Fock constructions [9, 10]. Their one-component space is the Hilbert completion of the cotangent space in [1, Theorem 5.3] and is given by

$$\mathcal{H}_{b,\mathcal{P}} := \overline{\text{span}} \mathcal{P}_b \subset H^2(\mathbb{T}). \quad (3.8)$$

The family $\mathcal{P}_b$ is an orthonormal basis of this subspace. Let

$$\Gamma_s(\mathcal{H}_{b,\mathcal{P}}) := \bigoplus_{r \geq 0} \text{Sym}^r(\mathcal{H}_{b,\mathcal{P}}), \quad \Gamma_a(\mathcal{H}_{b,\mathcal{P}}) := \bigoplus_{r \geq 0} \bigwedge^r(\mathcal{H}_{b,\mathcal{P}}). \quad (3.9)$$

The degree-zero summand in each Fock space is the vacuum. Under evaluation it gives the empty cascade product 1_b . The vacuum therefore belongs to the positive arithmetic family even though its occupation number is zero. Theorem 3.1 identifies the algebraic span of the exterior occupation basis with the homology of the decomposition complex. The symmetric functor is the completion in the occupation norm of the free cotangent closure (2.6). For a finitely supported occupation profile $\nu : \mathcal{P}_b \rightarrow \mathbb{N}_0$ let $\Phi_{b,\nu}^s$ be the normalized symmetric occupation state. For $\varepsilon : \mathcal{P}_b \rightarrow \{0,1\}$ let $\Phi_{b,\varepsilon}^a$ be the corresponding antisymmetric occupation state whose wedge sign is fixed by the prime order.

Theorem 3.2 (Fock evaluation). *The basis maps*

$$\mathcal{E}_b^s \Phi_{b,\nu}^s := \boxtimes_{\pi \in \mathcal{P}_b} \pi^{\boxtimes_b \nu(\pi)}, \quad (3.10)$$

$$\mathcal{E}_b^a \Phi_{b,\varepsilon}^a := \boxtimes_{\pi \in \mathcal{P}_b} \pi^{\boxtimes_b \varepsilon(\pi)} \quad (3.11)$$

extend to unitaries from the symmetric Fock space onto $\overline{\text{span}}\{u_{b,n} : n \geq 1\}$ and from the antisymmetric Fock space onto the closed span of the square-free natural modes respectively.

Proof. The occupation states are orthonormal bases of the two Fock spaces. [1, Theorem 5.2] makes (3.10) a bijection onto all positive natural modes. Restricting each occupation number to 0 or 1 makes (3.11) a bijection onto precisely the square-free modes. A bijection between orthonormal bases extends uniquely to a unitary. $\square$

For a prime mode $\pi = u_{b,p}$ repeated symmetric occupation evaluates as

$$\mathcal{E}_b^s (\Phi_{b,r\delta_\pi}^s) = \pi^{\boxtimes_b r} = u_{b,p^r}. \quad (3.12)$$

Repeated symmetric occupation and iterated cascade thus give the same prime power under evaluation. Exterior occupation instead allows each prime direction at most once and evaluates onto the square-free sector.

The relation with the complete history source is an isometric embedding. Identify the positive mode Hilbert space with its value basis and define

$$\mathcal{V}_b^s = (Q_b^{\text{hist}})^* \mathcal{E}_b^s, \quad \mathcal{V}_b^a = (Q_b^{\text{hist}})^* \mathcal{E}_b^a. \quad (3.13)$$

The symmetric occupation of a profile evaluating to u maps to $\omega_{b,u}^{\text{hist}}$. The exterior construction does so on the square-free profiles. Fock evaluation is unitary and $(Q_b^{\text{hist}})^*$ is isometric. These facts prove the assertions. The range of $\mathcal{V}_b^s$ is the source value subspace of the history space. Its orthogonal complement consists of source-weighted history differences. Root-dependent energies and characters intertwine these maps on their diagonal domains.

Arithmetic transport also acts on every factor in a bar word. Preservation of cascade makes this entrywise map commute with each adjacent merge. It therefore transports the differential and its homology together with the two Fock bases. Recursive transport preserves p_b and therefore also intertwines (3.13). These statements extend the canonical mode transport to the factorization and history realizations.

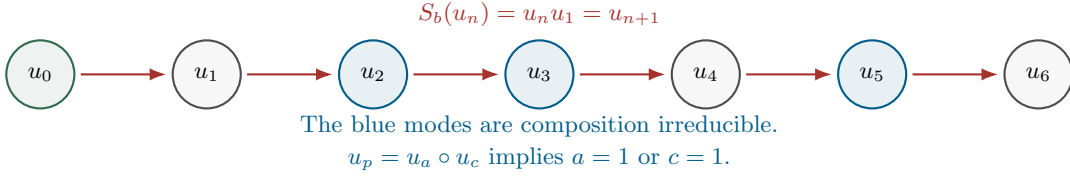
Example 3.3 (The first seven modes). The modes $u_{b,0}, \dots, u_{b,6}$ have normalized energies 0, 1, 4, 9, 16, 25, 36. Modulation by $u_{b,1}$ advances each mode to its successor. Cascade has identity $u_{b,1}$ and first irreducibles $u_{b,2}, u_{b,3}, u_{b,5}$. It also gives

$$u_{b,4} = u_{b,2} \boxtimes_b u_{b,2}, \quad u_{b,6} = u_{b,2} \boxtimes_b u_{b,3}.$$

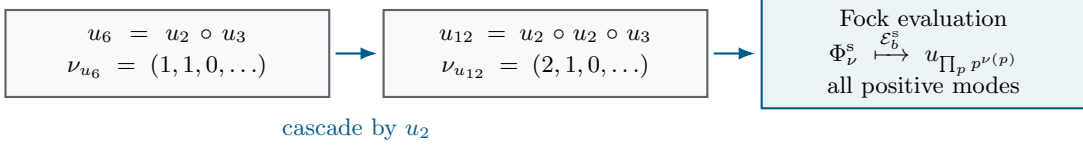
These seven modes exhibit the energy order and successor together with the first cascade factorizations.

Figure 2 places this calculation beside its occupation-space continuation. Panel (a) shows successor and composition irreducibility on the ordered modes. Panel (b) records the factorization data retained by symmetric Fock evaluation.

(a) Successor and irreducibility



(b) Factorization and occupation



(c) Two assemblies before evaluation

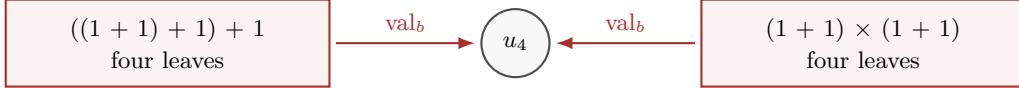


Figure 2. Low-order arithmetic on the generated mode family. Panel (a) shows successor as degree-one modulation and distinguishes the first composition-irreducible modes by blue borders. Panel (b) gives the factorizations of two modes and the change of occupation under cascade by u_2 . General Fock evaluation recovers every positive mode. Panel (c) gives distinct ordered addition and multiplication histories with the same evaluated mode. The additive history has source probability 1/8 and the multiplicative history has source probability 1/4.

The circle variable z records a natural mode by its frequency. To resolve its factorization we introduce a separate coordinate w_π for each prime direction. A pointwise power of a circle signal multiplies its frequency. A power of a prime coordinate increases that prime's occupation number. These operations generally give different natural modes.

The Bohr–Fock transform expresses factorization as multiplication of monomials. On the resulting occupation basis, correlation response and logarithmic arithmetic energy give different traces of the same modes. The following domination estimate controls the Gaussian sums and their parameter derivatives.

Lemma 3.4 (Uniform Gaussian domination). *Let $\beta_0 > 0$ and $C, M \geq 0$. Then*

$$\sum_{n \geq 1} \exp(C(\log(n+1))^2)(1+n)^M e^{-\beta n^2} \quad (3.14)$$

converges uniformly for $\beta \geq \beta_0$. Consequently a full-spectrum series whose fibre multiplicities and inserted spectral multipliers are bounded by the prefactor in (3.14) is absolutely and locally uniformly convergent. Finite parameter derivatives and limits may be passed through the series whenever the differentiated multipliers obey a bound of the same form.

Proof. Since $(\log(n+1))^2 = o(n^2)$ and $\log(1+n) = o(n^2)$ there exists n_0 such that

$$C(\log(n+1))^2 + M \log(1+n) \leq \frac{\beta_0}{2} n^2 \quad (n \geq n_0).$$

The tail of (3.14) is therefore bounded uniformly in $\beta \geq \beta_0$ by $\sum_{n \geq n_0} e^{-\beta_0 n^2/2}$. The finite initial segment is bounded. The same argument absorbs every fixed polynomial factor and every further $\exp(O((\log n)^2))$ factor. This proves the final statement by the Weierstrass test and dominated convergence. $\square$

3.3 Orientation and occupation gradings

The reflected circle carries the orientation space

$$\mathcal{H}_{\text{or}} := H_{+,0}^2(\mathbb{T}) \oplus H_{-,0}^2(\mathbb{T}), \quad \Gamma_{\text{or}}(f_+, f_-) := (f_+, -f_-). \quad (3.15)$$

Reflection exchanges the two summands and satisfies

$$\mathcal{J}\Gamma_{\text{or}} = -\Gamma_{\text{or}}\mathcal{J}. \quad (3.16)$$

This orientation grading distinguishes z^n from z^{-n} . The occupation grading $(-1)^N$ records the parity of the number of composition-irreducible factors. The two gradings therefore resolve additive orientation and multiplicative parity independently.

3.4 Divisibility and the occupation lattice

For a positive natural mode u let $\nu_u : \mathcal{P}_b \rightarrow \mathbb{N}_0$ be its unique exponent profile in (2.4). Then

$$u \mid_b v \iff \nu_u(\pi) \leq \nu_v(\pi) \quad (\pi \in \mathcal{P}_b), \quad (3.17)$$

$$\nu_{\gcd_b(u,v)}(\pi) := \min\{\nu_u(\pi), \nu_v(\pi)\}, \quad (3.18)$$

$$\nu_{\text{lcm}_b(u,v)}(\pi) := \max\{\nu_u(\pi), \nu_v(\pi)\}. \quad (3.19)$$

These definitions give

$$\gcd_b(u, v) \boxtimes_b \text{lcm}_b(u, v) = u \boxtimes_b v. \quad (3.20)$$

If $u \mid_b v$ the quotient is the mode with exponent profile $\nu_v - \nu_u$. The divisibility lattice is therefore the coordinate lattice $\mathbb{N}_0^{(\mathcal{P}_b)}$ supplied by the prime spectral basis.

For every integer $r \geq 2$ define

$$\mathcal{Q}_{b,r} := \left\{ u \in \mathcal{U}_b \setminus \{0_b\} : \nu_u \in r\mathbb{N}_0^{(\mathcal{P}_b)} \right\}. \quad (3.21)$$

Then

$$\mathcal{Q}_{b,r} = \{ v^{\boxtimes_b r} : v \in \mathcal{U}_b \setminus \{0_b\} \}, \quad \nu(\mathcal{Q}_{b,r}) = r\mathbb{N}_0^{(\mathcal{P}_b)}. \quad (3.22)$$

Unique spectral factorization gives $\nu_{v^{\boxtimes_b r}} = r\nu_v$. Conversely suppose $\nu_u = r\eta$ with $\eta \in \mathbb{N}_0^{(\mathcal{P}_b)}$. The finite profile η defines a unique positive natural mode v satisfying $u = v^{\boxtimes_b r}$. The r th powers are therefore exactly the modes whose occupation coordinates are all divisible by r .

The perfect-power sublattice is measured by intrinsic occupation operators. On the symmetric Fock basis define

$$\mathbb{N}_\pi \Phi_{b,\nu}^s := \nu(\pi) \Phi_{b,\nu}^s, \quad \pi \in \mathcal{P}_b. \quad (3.23)$$

3.5 The prime Hardy space and Bohr–Fock transform

For example $u_{b,12}$ has two occupations in the direction $u_{b,2}$ and one in $u_{b,3}$. Its prime-coordinate representative is $w_{u_{b,2}}^2 w_{u_{b,3}}$. In these coordinates adding occupation profiles becomes multiplication of monomials.

Let

$$\mathbb{T}^{\mathcal{P}_b} := \prod_{\pi \in \mathcal{P}_b} \mathbb{T} \quad (3.24)$$

with product Haar measure. For a finite occupation profile ν set

$$w^\nu := \prod_{\pi \in \mathcal{P}_b} w_\pi^{\nu(\pi)}. \quad (3.25)$$

The closed span of these nonnegative monomials is the Hardy space $H^2(\mathbb{T}^{\mathcal{P}_b})$. This product-torus realization is the Bohr model for Hilbert spaces of Dirichlet series [5]. Here the coordinate family is supplied by the prime cotangent space constructed in [1, Theorem 5.3].

The Bohr–Fock transform determined by the basis correspondence

$$\mathcal{B}_b \Phi_{b,\nu}^s := w^\nu \quad (3.26)$$

extends uniquely to a unitary

$$\mathcal{B}_b : \Gamma_s(\mathcal{H}_{b,\mathcal{P}}) \longrightarrow H^2(\mathbb{T}^{\mathcal{P}_b}). \quad (3.27)$$

On finite occupation sums the transform identifies symmetric occupation composition with pointwise polynomial multiplication according to

$$w^\nu w^\mu = w^{\nu+\mu}. \quad (3.28)$$

The evaluation map of Theorem 3.2 then gives

$$w^\nu \xrightarrow{(\mathcal{E}_b^s(\mathcal{B}_b))^{-1}} u_{b, \prod_\pi \deg_b(\pi)^{\nu(\pi)}}. \quad (3.29)$$

The Fock occupation states and the monomials w^ν are orthonormal bases, which proves unitarity. Multiplication adds exponent profiles, and (3.29) follows from unique spectral factorization.

The z and w variables have different roles. For a prime $\pi = u_{b,p}$ one has

$$w_\pi^r \longmapsto u_{b,p^r}(z) = z^{p^r}, \quad (3.30)$$

whereas the pointwise signal power is $u_{b,p}(z)^r = u_{b,rp}(z)$. Arithmetic exponentiation is the former. It is repeated cascade in the z domain and repeated occupation in the w domain.

Figure 3 distinguishes these operations. The symmetric branch intertwines occupation products with cascade. The exterior branch evaluates its ordered basis onto the square-free sector and retains its antisymmetric product in the occupation space.

Table 2 follows $u_{b,12}$. This is the least mode with two distinct prime directions and a repeated occupation. The Hodge entry uses orthonormal factorization words. Its degree-one bar Laplacian is $d_2 d_2^*$.

Example 3.5 (One mode seen by every representation). The factorization $12 = 2^2 \cdot 3$ gives occupation profile $(2, 1, 0, \dots)$ and excludes an exterior basis state. The four ordered words $(2, 6), (3, 4), (4, 3), (6, 2)$ give $d_2 d_2^*[u_{b,12}] = 4[u_{b,12}]$. Thus one factorization determines the occupation coordinates and square-free status together with the decomposition energy shown in the table.

3.6 Coefficient convolution

Finite linear combinations of occupation states are represented by their prime-coordinate polynomials. Multiplication of these polynomials collects all factorizations of the same output mode. This induces a convolution on coefficient families. Cascade remains the operation on individual modes.

Let a, c be finitely supported coefficient functions on $\mathcal{U}_b^+ := \mathcal{U}_b \setminus \{0_b\}$. Define

$$(a \star_b c)(u) := \sum_{v \boxtimes_b w = u} a(v)c(w). \quad (3.31)$$

The sum is finite by unique factorization.

For the Bohr polynomials

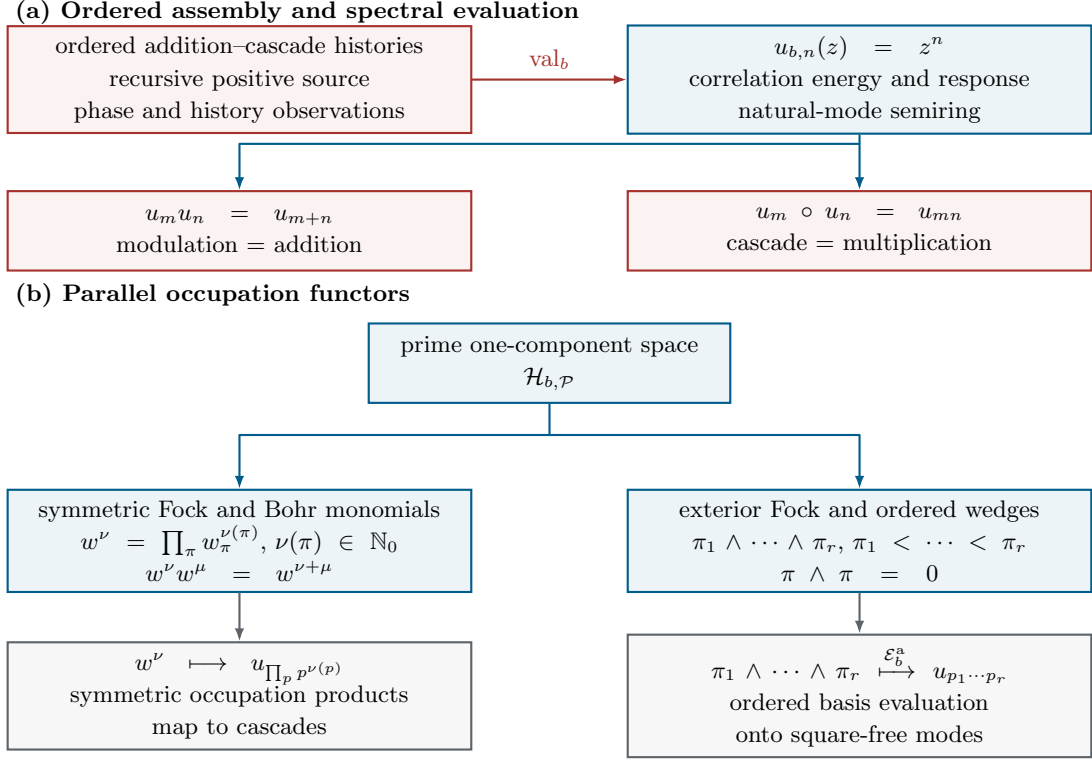
$$\mathcal{G}_b a := \sum_{u \in \mathcal{U}_b^+} a(u) w^{\nu_u}, \quad (3.32)$$

one has

$$\mathcal{G}_b(a \star_b c) = (\mathcal{G}_b a)(\mathcal{G}_b c). \quad (3.33)$$

The coefficient of w^{ν_u} in the product is the sum over $\nu_v + \nu_w = \nu_u$ or equivalently over $v \boxtimes_b w = u$.

Thus polynomial multiplication on occupation coordinates induces Dirichlet convolution on evaluated coefficients.



Both evaluations take values in the signal-mode space of panel (a).
 $w_\pi^r \mapsto u_{p^r}$ is arithmetic power and $u_p(z)^r = u_{rp}(z)$ is pointwise signal power.

Figure 3. Ordered assembly and its signal and occupation evaluations. The expression operations retain ordered inputs and brackets. Evaluation intertwines them with modulation and cascade on the generated mode family. This family retains its correlation response. The history source has the original Gaussian mode mass as its marginal. The same prime one-component space supports two different occupation functors. In the symmetric branch Bohr monomial multiplication evaluates as cascade. In the exterior branch the ordered wedge basis evaluates onto square-free modes. The occupation space retains its antisymmetric product. Here $\pi_i = u_{b,p_i}$ and the wedges follow the generated prime order. These basis correspondences distinguish arithmetic powers from pointwise signal powers.

Table 2. The mode $u_{b,12}$ across the representations generated in the paper. A repeated prime coordinate is retained by symmetric occupation and detected as a positive decomposition mode. Exterior occupation excludes it.

Representation	Data for degree 12
Correlation spectrum	$\phi_{b,12}$ with $E_{b,12}/E_{b,1} = 12^2 = 144$
Circle signal	$u_{b,12}(z) = z^{12}$
Cascade semiring	$u_{b,12} = u_{b,2} \boxtimes_b u_{b,2} \boxtimes_b u_{b,3}$
Symmetric Fock	$\Phi_{b, 2\delta_{u_{b,2}} + \delta_{u_{b,3}}}^s$ evaluated unitarily to $u_{b,12}$
Exterior Fock	no evaluating basis state because the $u_{b,2}$ direction is repeated
Degree-one	$\Delta_b^{\text{bar}}[u_{b,12}] = 4[u_{b,12}]$ from the four ordered nontrivial
bar-Hodge fibre	factorizations

3.7 Two energies and three Euler traces

The Gaussian response is governed by quadratic energy. For products of prime modes logarithmic energy is also useful because it adds over the factors. On the prime subspace define

$$\varepsilon_{\text{corr}}(u_{b,p}) = E_{b,p} = E_{b,1}p^2, \quad \varepsilon_{\text{ar}}(u_{b,p}) = \log p. \quad (3.34)$$

The same symbols denote the associated diagonal one-component operators on $\mathcal{H}_{b,p}$. They satisfy the functional-calculus identity

$$\varepsilon_{\text{ar}} = \frac{1}{2} \log \left(\frac{\varepsilon_{\text{corr}}}{E_{b,1}} \right). \quad (3.35)$$

For $t > 0$ and $\Re s > 1$ put

$$\Theta_{b,\mathcal{P}}(t) := \text{Tr}_{\mathcal{H}_{b,\mathcal{P}}} e^{-t\varepsilon_{\text{corr}}/E_{b,1}} = \sum_{u_{b,p} \in \mathcal{P}_b} e^{-tp^2}, \quad (3.36)$$

$$\mathcal{P}_b(s) := \text{Tr}_{\mathcal{H}_{b,\mathcal{P}}} \left(\frac{\varepsilon_{\text{corr}}}{E_{b,1}} \right)^{-s/2} = \sum_{u_{b,p} \in \mathcal{P}_b} p^{-s}. \quad (3.37)$$

Then

$$\mathcal{P}_b(s) = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1} \Theta_{b,\mathcal{P}}(t) dt. \quad (3.38)$$

This is the heat–Mellin bridge on the generated prime subspace. On the prime basis the normalized correlation energy has eigenvalue p^2 . For $a > 0$ and $\Re s > 0$ the scalar Gamma-integral gives

$$\int_0^\infty t^{s/2-1} e^{-at} dt = \Gamma(s/2) a^{-s/2}. \quad (3.39)$$

Taking $a = p^2$ and summing proves (3.38). The exchange is justified by $\sum_p p^{-\Re s} < \infty$ for $\Re s > 1$.

Second quantization sums either one-component energy over the occupied prime directions. For correlation energy the sum is $\sum_\pi \nu(\pi) E_b(\pi)$. Transporting A_b by Fock evaluation instead assigns the energy of the evaluated mode to the whole occupation state. Thus these

two extensions of prime correlation energy have different meanings. Logarithmic arithmetic energy is additive over factors. Its second quantization is therefore the logarithm of the evaluated degree and satisfies

$$d\Gamma(\varepsilon_{\text{ar}})\Phi_{b,\nu}^s = \log(\deg_b \mathcal{E}_b^s \Phi_{b,\nu}^s) \Phi_{b,\nu}^s. \quad (3.40)$$

For the complete occupation profile ν let $n(\nu) = \prod_{\pi} \deg_b(\pi)^{\nu(\pi)}$ and subtract the unit energy to obtain $\mathcal{E}_b(\nu) = n(\nu)^2 - 1$. Then

$$\mathcal{E}_b(\nu + \eta) = \mathcal{E}_b(\nu) + \mathcal{E}_b(\eta) + \mathcal{E}_b(\nu)\mathcal{E}_b(\eta). \quad (3.41)$$

The equality $n(\nu + \eta) = n(\nu)n(\eta)$ proves the identity by expansion. The vacuum is the unique zero of this energy. Its first nonvacuum value is three and is attained by the mode of degree two. The Gaussian source of the history system uses the whole root square and therefore retains the joint term in (3.41). Logarithmic energy instead resolves the additive occupation sum.

These occupation sums give three related generating functions. The interpretation of the Möbius sign as fermion parity appears in Spector [3]. Here the one-component space is the prime space of the generated semiring. Its occupation basis gives all three traces.

Theorem 3.6 (Symmetric and exterior trace identities with grading). *For $\Re s > 1$ one has*

$$Z_{\text{sym}}(s) := \text{Tr}_{\Gamma_s(\mathcal{H}_{b,\mathcal{P}})} e^{-s d\Gamma(\varepsilon_{\text{ar}})} = \prod_{u_{b,p} \in \mathcal{P}_b} (1 - p^{-s})^{-1} = \zeta(s), \quad (3.42)$$

$$Z_{\wedge}(s) := \text{Tr}_{\Gamma_a(\mathcal{H}_{b,\mathcal{P}})} e^{-s d\Gamma(\varepsilon_{\text{ar}})} = \prod_{u_{b,p} \in \mathcal{P}_b} (1 + p^{-s}) = \frac{\zeta(s)}{\zeta(2s)}, \quad (3.43)$$

$$\text{Str}_{\wedge}(s) := \text{Tr}_{\Gamma_a(\mathcal{H}_{b,\mathcal{P}})} (-1)^{\mathbf{N}} e^{-s d\Gamma(\varepsilon_{\text{ar}})} = \prod_{u_{b,p} \in \mathcal{P}_b} (1 - p^{-s}) = \frac{1}{\zeta(s)}. \quad (3.44)$$

Proof. Each symmetric prime coordinate contributes $\sum_{r \geq 0} p^{-rs} = (1 - p^{-s})^{-1}$. Each antisymmetric coordinate contributes $1 + p^{-s}$. Insertion of $(-1)^{\mathbf{N}}$ changes it to $1 - p^{-s}$. Absolute convergence follows from $\sum_p p^{-\Re s} \leq \sum_{n \geq 2} n^{-\Re s}$. Unique factorization identifies the symmetric expansion with all positive modes and the antisymmetric expansion with square-free modes. The elementary identity $1 + x = (1 - x^2)/(1 - x)$ proves the quotient in (3.43). $\square$

The three traces in Theorem 3.6 converge absolutely for $\Re s > 1$. Their meromorphic continuations are obtained from the completed Gaussian trace. Beyond the trace-class domain the formulas refer to those continued functions. Each zero of $\zeta(s)$ is a pole of the continued reciprocal with the same multiplicity.

Define the total occupation number and the spectral Liouville sign by

$$\Omega_b(u) := \sum_{\pi \in \mathcal{P}_b} \nu_u(\pi), \quad \lambda_b(u) := (-1)^{\Omega_b(u)}. \quad (3.45)$$

The sign λ_b is defined on every positive natural mode. The square-free selector is

$$\mathbf{q}_b(u) := \prod_{\pi \in \mathcal{P}_b} \mathbf{1}_{\{0,1\}}(\nu_u(\pi)). \quad (3.46)$$

Define the spectral Möbius function by

$$\mu_b(u) := \begin{cases} (-1)^{\sum_{\pi} \nu_u(\pi)}, & \nu_u(\pi) \in \{0,1\} \text{ for every } \pi, \\ 0, & \text{otherwise.} \end{cases} \quad (3.47)$$

Equivalently one has

$$\mu_b(u) = \mathbf{q}_b(u) \lambda_b(u). \quad (3.48)$$

The symmetric occupation carrier supplies the total occupation number and the Liouville sign on every positive mode. Its restriction to the antisymmetric sector agrees with exterior parity. Exterior nilpotence supplies the square-free selector. Their combination is the Möbius coefficient appearing in (3.44). It also satisfies

$$\mu_b \star_b \mathbf{1} = \delta_{1_b}. \quad (3.49)$$

The divisor sum factors into $\prod_{\pi|_b u} (1 - 1)$ unless $u = 1_b$.

Degree carries the occupation profile to the ordinary prime exponent profile and preserves square-free support and parity. Thus the ordinary Liouville and Möbius functions λ and μ on $\mathbb{N}_+$ satisfy

$$\lambda_b(u) = \lambda(\deg_b(u)), \quad \mu_b(u) = \mu(\deg_b(u)) \quad (u \in \mathcal{U}_b \setminus \{0_b\}). \quad (3.50)$$

Cutoff sums and cancellation estimates for μ_b consequently transfer exactly under the degree isomorphism.

The finite-cutoff counterpart is an ordinary finite-rank supertrace. Let

$$H_{\text{ar}} := d\Gamma(\varepsilon_{\text{ar}}), \quad Q_X := \mathbf{1}_{[0, \log X]}(H_{\text{ar}}), \quad X \geq 1. \quad (3.51)$$

An exterior occupation state has energy $\log \deg_b(u)$ and parity $\mu_b(u)$. Finiteness of the states of degree at most X implies that Q_X has finite rank and gives

$$\text{Tr}_{\Gamma_{\text{a}}(\mathcal{H}_{b,\mathcal{P}})} \left((-1)^N Q_X \right) = \sum_{\substack{u \in \mathcal{U}_b \setminus \{0_b\} \\ \deg_b(u) \leq X}} \mu_b(u). \quad (3.52)$$

Nonsquare-free modes contribute zero. The trace is the signed count of square-free modes whose sign is determined by the number of occupied prime directions.

3.8 Multiplicative compact-charge sector

Write the prime torus of (3.24) as

$$G_b^\times := \prod_{\pi \in \mathcal{P}_b} U(1)_\pi \quad (3.53)$$

with the product topology. For $g = (g_\pi)_\pi \in G_b^\times$ its action on the symmetric Fock basis is

$$U_b^\times(g)\Phi_{b,\nu}^s := \left(\prod_{\pi \in \mathcal{P}_b} g_\pi^{\nu(\pi)} \right) \Phi_{b,\nu}^s. \quad (3.54)$$

Each occupation vector has finite support and therefore a continuous character. Unitarity extends strong continuity from finite linear combinations to $\Gamma_s(\mathcal{H}_{b,\mathcal{P}})$. The coordinate generators are the commuting occupation operators $\mathbf{N}_\pi$ of (3.23). Cascade multiplication adds their joint integer charges according to

$$\nu_{u \boxtimes_b v}(\pi) = \nu_u(\pi) + \nu_v(\pi). \quad (3.55)$$

The arithmetic Hamiltonian from (3.40) is

$$H_b^\times := \sum_{\pi \in \mathcal{P}_b} \log \deg_b(\pi) \mathbf{N}_\pi = d\Gamma(\varepsilon_{\text{ar}}). \quad (3.56)$$

For $\Re s > 1$ the trace $\text{Tr } e^{-sH_b^\times}$ is the Euler partition function (3.42). The charge-addition law is the exponent law in unique spectral factorization.

The discrete group completion of the occupation cone is

$$\Lambda_b^\times := \text{Gp}(\mathbb{N}_0^{(\mathcal{P}_b)}) = \bigoplus_{\pi \in \mathcal{P}_b} \mathbb{Z}e_\pi. \quad (3.57)$$

Its positive cone is the symmetric-Fock occupation lattice. Pontryagin duality gives

$$\widehat{\Lambda_b^\times} \cong \prod_{\pi \in \mathcal{P}_b} U(1)_\pi = G_b^\times. \quad (3.58)$$

The compact prime torus and the signed occupation lattice are the two dual realizations of the same charge data.

For an integer $r \geq 2$ the coordinatewise power map

$$[r] : G_b^\times \longrightarrow G_b^\times, \quad [r](g)_\pi := g_\pi^r, \quad (3.59)$$

is surjective and has compact kernel

$$K_{b,r}^\times := \ker[r] = \prod_{\pi \in \mathcal{P}_b} \mu_r. \quad (3.60)$$

Here μ_r is the group of r th roots of unity. Each coordinate map is an r -fold covering of the circle. The full product map is a continuous surjective homomorphism with compact kernel. Each occupation character depends on only finitely many coordinates.

Theorem 3.7 (Finite-order occupation sectors). *For $[\alpha] \in \Lambda_b^\times / r\Lambda_b^\times$ define*

$$\mathcal{H}_{b,r}^{[\alpha]} := \overline{\text{span}} \{ \Phi_{b,\nu}^s : \nu \equiv \alpha \pmod{r\Lambda_b^\times} \}. \quad (3.61)$$

Then

$$\Gamma_s(\mathcal{H}_{b,\mathcal{P}}) = \bigoplus_{[\alpha] \in \Lambda_b^\times / r\Lambda_b^\times} \mathcal{H}_{b,r}^{[\alpha]} \quad (3.62)$$

orthogonally. The restriction of $U_b^\times$ to $K_{b,r}^\times$ acts on $\mathcal{H}_{b,r}^{[\alpha]}$ through the character $\kappa \mapsto \prod_\pi \kappa_\pi^{\alpha(\pi)}$. Its invariant sector is also called the untwisted sector and is given by

$$\mathcal{H}_{b,r}^{[0]} = \text{Ran } \mathcal{R}_{b,r}, \quad \mathcal{R}_{b,r} = \int_{K_{b,r}^\times} U_b^\times(\kappa) d\kappa, \quad (3.63)$$

and therefore evaluates exactly onto the arithmetic r th powers.

Proof. Two occupation basis states carry the same character of $K_{b,r}^\times$ exactly when their exponent profiles agree modulo r . Character orthogonality gives the orthogonal decomposition (3.62). Haar averaging retains the trivial character and annihilates every nontrivial sector. The surviving basis states are exactly those with $\nu \in r\mathbb{N}_0^{(\mathcal{P}_b)}$. Equation (3.22) identifies their evaluated range with the r th-power modes. $\square$

The involution

$$j_b^\times : \Lambda_b^\times \longrightarrow \Lambda_b^\times, \quad j_b^\times(\alpha) := -\alpha, \quad (3.64)$$

is the reflection of the signed charge completion. On the signed charge lattice define the logarithmic charge functional

$$\ell_b^\times(\alpha) := \sum_{\pi \in \mathcal{P}_b} \alpha(\pi) \log \deg_b(\pi). \quad (3.65)$$

The sum is finite because every lattice element has finite support. On $\ell^2(\Lambda_b^\times)$ let $J_b^\times |\alpha\rangle := |-\alpha\rangle$ and realize this functional by the diagonal logarithmic charge generator

$$L_b^\times |\alpha\rangle := \ell_b^\times(\alpha) |\alpha\rangle. \quad (3.66)$$

For $[\alpha] \in \Lambda_b^\times / r\Lambda_b^\times$ put

$$\tilde{\mathcal{H}}_{b,r}^{[\alpha]} := \overline{\text{span}} \{ |\gamma\rangle : \gamma \equiv \alpha \pmod{r\Lambda_b^\times} \} \subset \ell^2(\Lambda_b^\times). \quad (3.67)$$

Proposition 3.8 (Faithful signed charge and reflection). *The homomorphism $\ell_b^\times : \Lambda_b^\times \rightarrow \mathbb{R}$ has trivial kernel. Consequently one has*

$$\ker L_b^\times = \mathbb{C}|0\rangle \quad \text{on } \ell^2(\Lambda_b^\times). \quad (3.68)$$

The operator $J_b^\times$ is a unitary involution and

$$J_b^\times L_b^\times = -L_b^\times J_b^\times, \quad \left[J_b^\times, e^{-\sigma(L_b^\times)^2} \right] = 0 \quad (\sigma \geq 0). \quad (3.69)$$

The involution $J_b^\times$ also preserves the signed finite-order decomposition by sending

$$\tilde{\mathcal{H}}_{b,r}^{[\alpha]} \longleftrightarrow \tilde{\mathcal{H}}_{b,r}^{[-\alpha]}. \quad (3.70)$$

Inversion is odd on logarithmic charge and even on its Gaussian square. It exchanges conjugate signed torsion sectors. The positive Fock sectors in (3.62) are the intersections of these signed sectors with the nonnegative occupation cone. Inversion acts on their signed group completion. It sends each nonnegative-cone intersection to the nonpositive-cone intersection in the conjugate signed sector.

Proof. Write $\alpha = \alpha_+ - \alpha_-$ with nonnegative disjoint supports. If $\ell_b^\times(\alpha) = 0$ exponentiation gives

$$\prod_{\pi \in \mathcal{P}_b} \deg_b(\pi)^{\alpha_+(\pi)} = \prod_{\pi \in \mathcal{P}_b} \deg_b(\pi)^{\alpha_-(\pi)}. \quad (3.71)$$

Both products are finite. Unique spectral factorization gives $\alpha_+ = \alpha_-$ and hence $\alpha = 0$. This proves the kernel statement. Reflection permutes the orthonormal basis and changes the sign of $\ell_b^\times(\alpha)$. This proves unitarity and anticommutation on the diagonal operator domain. Even functional calculus gives the Gaussian commutator. Reduction modulo $r\Lambda_b^\times$ sends $[\alpha]$ to $[-\alpha]$ and proves the final statement. $\square$

Prime powers occupy a single prime coordinate. Their closed span is

$$\mathcal{A}_b^{\text{pp}} := \overline{\text{span}} \left\{ \Phi_{b,me_\pi}^s : \pi \in \mathcal{P}_b, m \geq 1 \right\}, \quad P_b^{\text{pp}} := \text{its orthogonal projector}. \quad (3.72)$$

Fock evaluation maps this axis exactly onto the prime-power natural modes.

The bar construction locates the prime subspace without first choosing prime coordinates: it starts with all words of nonunit factors and merges adjacent factors by cascade. On one-word vectors its Hodge operator vanishes precisely on irreducible modes. The positive spectral gap gives an explicit convergence rate to the prime projection. Since merging preserves total mode, Gaussian energy commutes with Hodge flow. The same estimate therefore gives a summable error in the correlation trace. The operator-theoretic language is that of Hilbert complexes [11]. In this direct-sum setting closedness and self-adjointness are verified on the finite-dimensional total-mode fibres.

3.9 Bar supercharge and decomposition closure

Declare the bar words in (3.1) orthonormal and let $\mathcal{K}_{b,r}$ be the Hilbert completion of $\overline{B}_r(\mathfrak{A}_b)$. For $n \geq 1$ let $\mathcal{K}_b(n)$ be the span of all bar words of total mode $u_{b,n}$ with the degree-zero vacuum assigned total mode 1_b . Every factor in a nonempty reduced bar word has degree at least two. A word in $\mathcal{K}_b(n)$ has length at most $\log_2 n$. Hence every $\mathcal{K}_b(n)$ is finite dimensional. The bar differential preserves these fibres. Let d_n be its finite-dimensional restriction and define

$$\mathcal{D}(d) := \left\{ x = (x_n) : \sum_{n \geq 1} \|d_n x_n\|^2 < \infty \right\}, \quad d := \bigoplus_{n \geq 1} d_n. \quad (3.73)$$

This maximal direct-sum realization is densely defined and closed. On

$$\mathcal{K}_b := \bigoplus_{r \geq 0} \mathcal{K}_{b,r} = \bigoplus_{n \geq 1} \mathcal{K}_b(n) \quad (3.74)$$

put $\mathcal{Q}_{b,n} := d_n + d_n^*$ and define

$$\begin{aligned}\mathcal{Q}_b &:= \bigoplus_{n \geq 1} \mathcal{Q}_{b,n}, & \mathcal{D}(\mathcal{Q}_b) &:= \left\{ x = (x_n) : \sum_{n \geq 1} \|\mathcal{Q}_{b,n} x_n\|^2 < \infty \right\}, \\ \Delta_b^{\text{bar}} &:= \mathcal{Q}_b^2, & \mathcal{D}(\Delta_b^{\text{bar}}) &:= \left\{ x = (x_n) : \sum_{n \geq 1} \|\mathcal{Q}_{b,n}^2 x_n\|^2 < \infty \right\},\end{aligned}\tag{3.75}$$

$$\Gamma_{\text{bar}}|_{\mathcal{K}_{b,r}} := (-1)^r I.$$

Each $\mathcal{Q}_{b,n}$ is a self-adjoint finite matrix. Their orthogonal direct sum is therefore self-adjoint on the displayed graph domain. On the algebraic bar core the identity $d^2 = 0$ gives

$$\|\mathcal{Q}_b x\|^2 = \|dx\|^2 + \|d^* x\|^2.\tag{3.76}$$

Applying this identity fibrewise and then summing over n shows that $\mathcal{D}(\mathcal{Q}_b) = \mathcal{D}(d) \cap \mathcal{D}(d^*)$. The nonnegative closed quadratic form

$$\mathfrak{q}_b^{\text{bar}}[x] := \|dx\|^2 + \|d^* x\|^2, \quad \mathcal{D}(\mathfrak{q}_b^{\text{bar}}) := \mathcal{D}(d) \cap \mathcal{D}(d^*),\tag{3.77}$$

has associated self-adjoint operator $\Delta_b^{\text{bar}} = \mathcal{Q}_b^2$. On the common operator domain this operator is $dd^* + d^*d$. The grading satisfies $\Gamma_{\text{bar}} \mathcal{Q}_b = -\mathcal{Q}_b \Gamma_{\text{bar}}$. This grading is the homological source of the exterior occupation grading. It remains distinct from the orientation grading (3.15).

The normalized correlation Hamiltonian is constant on each total-mode fibre and acts by

$$\mathbf{H}_b[u_1 | \cdots | u_r] := \deg_b(u_1 \boxtimes_b \cdots \boxtimes_b u_r)^2 [u_1 | \cdots | u_r],\tag{3.78}$$

with the vacuum in degree zero assigned total mode 1_b . It strongly commutes with d and with both $\mathcal{Q}_b$ and Δ_b^{bar} .

For a positive nonunit mode u define its ordered two-factor response by

$$r_b^{(2)}(u) := \text{card} \{(v, w) : v, w \in \mathcal{U}_b \setminus \{0_b, 1_b\}, \quad v \boxtimes_b w = u\}.\tag{3.79}$$

This finite quantity is defined entirely by the cascade law.

Proposition 3.9 (Exact local decomposition kernel). *For $n \geq 2$ the total-mode- $u_{b,n}$ fibre of $\mathcal{K}_{b,1}$ is one-dimensional and*

$$\Delta_b^{\text{bar}}[u_{b,n}] = r_b^{(2)}(u_{b,n})[u_{b,n}], \quad r_b^{(2)}(u_{b,n}) = d(n) - 2,\tag{3.80}$$

where $d(n)$ is the number of positive divisors of n . It follows that for $t > 0$ and $\tau \geq 0$ one has

$$\text{Tr}_{\mathcal{K}_{b,1}} \left(e^{-t\mathbf{H}_b} e^{-\tau \Delta_b^{\text{bar}}} \right) = \sum_{n=2}^{\infty} e^{-tn^2 - \tau(d(n)-2)}.\tag{3.81}$$

The zero eigenspace consists exactly of the spectral-prime modes and every composite mode has eigenvalue at least one.

Proof. The degree-one fibre of total mode $u_{b,n}$ has the single basis vector $[u_{b,n}]$. The degree-two fibre has one basis vector $[u_{b,a}|u_{b,c}]$ for each ordered factorization $ac = n$ with $a, c \geq 2$. Each satisfies $d_2[u_{b,a}|u_{b,c}] = [u_{b,n}]$. In the declared orthonormal bar basis the adjoint is therefore

$$d_2^*[u_{b,n}] = \sum_{\substack{ac=n \\ a,c \geq 2}} [u_{b,a}|u_{b,c}], \quad \|d_2^*[u_{b,n}]\|^2 = \#\{(a, c) : ac = n, a, c \geq 2\}. \quad (3.82)$$

Hence $d_2 d_2^*$ acts on $[u_{b,n}]$ by the number of these factorizations. Since $d_1 = 0$ this is the degree-one bar Laplacian. Choosing the first factor a identifies the ordered factorizations with the positive divisors other than 1 and n . This proves (3.80). Summation over the orthonormal degree-one basis proves (3.81). The condition $r_b^{(2)}(u) = 0$ is exactly composition irreducibility. A composite mode has at least one nontrivial factorization. $\square$

The same positive gap controls heat and resolvent localization and gives exact operator-norm errors together with a bound for the Gaussian trace error.

Theorem 3.10 (Heat and resolvent localization of the prime sector). *Let $R_b = \Delta_b^{\text{bar}}|_{\mathcal{K}_{b,1}}$ and put*

$$P_b^{\text{pr}} := \mathbf{1}_{\{0\}} \left(\Delta_b^{\text{bar}}|_{\mathcal{K}_{b,1}} \right). \quad (3.83)$$

Then P_b^{pr} is the orthogonal projection onto $\overline{\text{span}}\{[\pi] : \pi \in \mathcal{P}_b\}$.

(i) *For every $\tau \geq 0$ one has*

$$\left\| e^{-\tau \Delta_b^{\text{bar}}} |_{\mathcal{K}_{b,1}} - P_b^{\text{pr}} \right\| = e^{-\tau}. \quad (3.84)$$

Accordingly for $t > 0$ one has

$$0 \leq \text{Tr}_{\mathcal{K}_{b,1}} \left[e^{-tH_b} \left(e^{-\tau \Delta_b^{\text{bar}}} - P_b^{\text{pr}} \right) \right] \leq e^{-\tau} \sum_{\substack{n \geq 2 \\ n \text{ composite}}} e^{-tn^2}. \quad (3.85)$$

(ii) *For every $\lambda > 0$ one has*

$$\lambda(R_b + \lambda I)^{-1} - P_b^{\text{pr}} = (I - P_b^{\text{pr}})\lambda(R_b + \lambda I)^{-1}, \quad (3.86)$$

and

$$\left\| \lambda(R_b + \lambda I)^{-1} - P_b^{\text{pr}} \right\| = \frac{\lambda}{1 + \lambda}, \quad (3.87)$$

$$\left\| (I - P_b^{\text{pr}})(R_b + \lambda I)^{-1} \right\| = \frac{1}{1 + \lambda}. \quad (3.88)$$

Hence

$$P_b^{\text{pr}} = \lim_{\lambda \downarrow 0} \lambda(R_b + \lambda I)^{-1} \quad (3.89)$$

in operator norm. Equivalently the simple pole of the decomposition resolvent at zero has residue P_b^{pr} . Its compression to the composite sector is uniformly regular at the pole.

Proof. Proposition 3.9 diagonalizes R_b with eigenvalue zero on the prime modes and with positive integer eigenvalues on the composite modes. The least positive eigenvalue is one and is attained at every prime square. Functional calculus gives (3.84). Summing the composite contributions with their Gaussian weights proves (3.85). For the normalized resolvent it gives the identity on the prime block and the scalar multiplier $\lambda/(r + \lambda)$ on a composite eigenvector of response $r \geq 1$. The largest such multiplier is attained at $r = 1$. This proves (3.87). The regular resolvent multiplier is $(r + \lambda)^{-1}$ and has maximum $(1 + \lambda)^{-1}$. The norm limit and residue statement follow. $\square$

To place this projection in the history space identify a one-word vector $[u]$ for $u >_b 1_b$ with its value basis vector. Let V_1 be the restriction of $(Q_b^{\text{hist}})^*$ to this nonunit value space. There are then two prime projections on histories given by

$$\mathbf{P}_b^{\text{root}}|e\rangle = \mathbf{1}_{\mathcal{P}_b}(\text{val}_b(e))|e\rangle, \quad \mathbf{P}_b^{\text{src}} = V_1 P_b^{\text{pr}} V_1^*. \quad (3.90)$$

The first retains every history having a prime root. The second retains the source-weighted line over each prime root. Since these lines lie in the corresponding root fibres one has $\mathbf{P}_b^{\text{root}} \mathbf{P}_b^{\text{src}} = \mathbf{P}_b^{\text{src}}$. Their difference is the orthogonal projection onto the history differences inside prime-root fibres. Transporting the degree-one Hodge operator by V_1 gives an operator on $\text{Ran } V_1$. Its heat and resolvent estimates are exactly the preceding estimates under this isometry. Its kernel is the span of the prime source lines. The assembly operator acts on the full history space and has the parent–child relations proved in [1, Section 10].

Figure 4 shows the degree-one restriction. Each nontrivial two-factor decomposition contributes one unit to its eigenvalue. The prime modes have no such contribution and remain fixed under the heat flow. The composite modes decay.

There are two traces to distinguish. Restricting the Hodge flow to degree one and taking its large-time limit leaves the prime modes. Taking the alternating trace over all bar degrees instead retains the Euler characteristic of each total-mode fibre. The latter is the Möbius coefficient computed by the exterior homology.

Theorem 3.11 (Decomposition-flow closure). *For every $t > 0$ the Hodge flow has the following trace and localization properties.*

(i) *The degree-one Hodge flow converges to the prime heat trace.*

$$\lim_{\tau \rightarrow \infty} \text{Tr}_{\mathcal{K}_{b,1}} \left(e^{-t\mathbf{H}_b} e^{-\tau \Delta_b^{\text{bar}}} \right) = \sum_{\pi \in \mathcal{P}_b} e^{-t \deg_b(\pi)^2} = \Theta_{b,\mathcal{P}}(t). \quad (3.91)$$

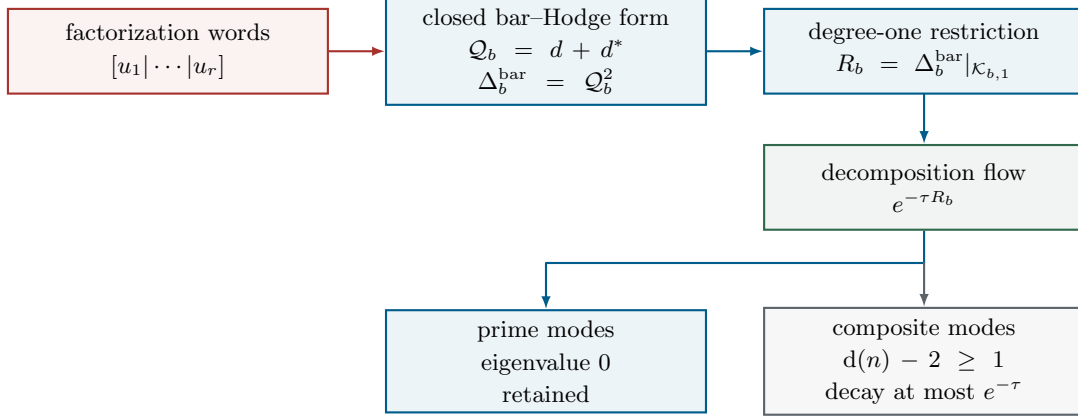
(ii) *The full graded heat trace is independent of $\tau \geq 0$ and is the Gaussian Möbius trace.*

$$\text{Str}_{\mathcal{K}_b} \left(e^{-t\mathbf{H}_b} e^{-\tau \Delta_b^{\text{bar}}} \right) = \sum_{u \in \mathcal{U}_b \setminus \{0_b\}} \mu_b(u) e^{-t \deg_b(u)^2}. \quad (3.92)$$

(iii) *At the local endpoint $\tau = 0$ the degree- r trace is the factorization-chain sum*

$$\text{Tr}_{\mathcal{K}_{b,r}} e^{-t\mathbf{H}_b} = \sum_{u_1, \dots, u_r \neq 0_b, 1_b} \exp \left[-t \deg_b(u_1 \boxtimes_b \dots \boxtimes_b u_r)^2 \right]. \quad (3.93)$$

At the closed endpoint $\tau = \infty$ the surviving degree- r space is $\bigwedge^r \mathcal{H}_{b,\mathcal{P}}$.



$$P_b^{\text{pr}} = \mathbf{1}_{\{0\}}(R_b), \quad \|e^{-\tau R_b} - P_b^{\text{pr}}\| = e^{-\tau}.$$

The least positive eigenvalue is attained by prime squares.

Figure 4. Hodge localization on the degree-one bar sector. The self-adjoint operator associated with the closed bar-Hodge form is first restricted to $\mathcal{K}_{b,1}$. This restriction tests the nontrivial two-factor decompositions of each nonunit mode. Its kernel consists of the prime directions. The heat multiplier of each composite mode is at most $e^{-\tau}$. Equality at prime squares gives the exact operator-norm convergence to the prime projector. This flow acts on multiplicative decomposition words. The parent-child assembly response acts on the separate ordered-history coefficient space.

Proof. On every total-mode fibre finite-dimensional Hodge decomposition gives

$$\ker \Delta_b^{\text{bar}} \cong H_\bullet(\overline{B}_\bullet(\mathfrak{A}_b), d). \quad (3.94)$$

Proposition 3.9 proves (i) directly. Theorem 3.1 identifies the degree- r kernel with the r th exterior power of the prime space. This proves the closed-endpoint statement in (iii). For each positive eigenvalue λ the map $\lambda^{-1/2} \mathcal{Q}_b$ exchanges the even and odd parts of its eigenspace and is an involution there. The two parity contributions to the supertrace therefore cancel. The harmonic contribution is $(-1)^r$ on a square-free mode with r prime factors. This proves (ii). Formula (3.93) is the definition of the bar-word basis and total-mode Hamiltonian. A word of total degree n has length at most $\log_2 n$. A crude bound for the number of such words is therefore $(1 + \log_2 n) n^{\log_2 n} = \exp(O((\log n)^2))$. This is dominated by $e^{tn^2/2}$ for all sufficiently large n . After multiplication by the heat weight e^{-tn^2} the bound in Lemma 3.4 therefore justifies absolute convergence and all displayed limiting interchanges. $\square$

For $r \geq 0$ and $n \geq 1$ let $c_{b,r}(n)$ be the number of ordered bar words of length r and total mode $u_{b,n}$. Thus $c_{b,0}(1) = 1$ and $c_{b,0}(n) = 0$ for $n > 1$. For $r \geq 1$ one has

$$c_{b,r}(n) := \#\{(n_1, \dots, n_r) : n_j \geq 2, n_1 \cdots n_r = n\}. \quad (3.95)$$

Theorem 3.12 (Bar-resolvent factorization). *For every $r \geq 0$ and $\Re s > 1$ one has*

$$\frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1} \text{Tr}_{\mathcal{K}_{b,r}}(e^{-tH_b}) dt = \sum_{n \geq 1} \frac{c_{b,r}(n)}{n^s} = (\zeta(s) - 1)^r. \quad (3.96)$$

For every fixed n only finitely many $c_{b,r}(n)$ are nonzero and

$$\mu_b(u_{b,n}) = \sum_{r \geq 0} (-1)^r c_{b,r}(n). \quad (3.97)$$

Consequently the complete alternating bar tower has the scalar resolvent

$$\sum_{n \geq 1} \frac{\mu_b(u_{b,n})}{n^s} = \frac{1}{\zeta(s)} \quad (\Re s > 1), \quad (3.98)$$

and in the nonempty region $|\zeta(s) - 1| < 1$ one has

$$\frac{1}{\zeta(s)} = \sum_{r \geq 0} (-1)^r (\zeta(s) - 1)^r. \quad (3.99)$$

The Dirichlet-series identity (3.98) continues this geometric expression holomorphically throughout $\Re s > 1$.

In degree one the decomposition response itself satisfies

$$\sum_{n \geq 2} \frac{r_b^{(2)}(u_{b,n})}{n^s} = (\zeta(s) - 1)^2, \quad \Re s > 1, \quad (3.100)$$

Proof. Mellin integration of the r th trace in (3.93) replaces the Gaussian factor of a word by $(n_1 \cdots n_r)^{-s}$. Absolute convergence for $\Re s > 1$ allows the factors to separate and gives $(\sum_{n \geq 2} n^{-s})^r$ and hence (3.96).

Let $f_b(1) = 0$ and $f_b(n) = 1$ for $n \geq 2$. Then $c_{b,r} = f_b^{\star_b r}$. Since every factor in a word is at least two $c_{b,r}(n) = 0$ for $r > \log_2 n$. Therefore the pointwise sum $c_b := \sum_{r \geq 0} (-1)^r c_{b,r}$ is finite and

$$(\delta_1 + f_b) \star_b c_b = \delta_1.$$

The function $\delta_1 + f_b$ is the constant-one divisor function. Its convolution inverse is the internally defined Möbius function. This proves (3.97). Dirichlet transformation proves (3.98). The identity $r_b^{(2)}(u_{b,n}) = c_{b,2}(n)$ holds for $n \geq 2$. The case $r = 2$ proves (3.100). $\square$

At $\tau = 0$ every finite factorization chain contributes. The limit $\tau \rightarrow \infty$ retains the harmonic part, represented by exterior products of prime directions. Degree one gives the prime heat trace; the complete graded trace retains the Möbius coefficient of each root.

4 Additive roots and relative input data

An additive history retains both child expressions. Evaluating the children gives an ordered pair of natural modes, whereas evaluating the root retains only their sum. The intervening fixed-sum fibre records the relative inputs. Uniform and source-weighted compressions connect this fibre to the history space; a faithful dilation retains the input pair together with its multiplicative charge defect.

4.1 Addition as a closable two-input map

Let $\mathcal{L}_b^+$ be the Hilbert space whose orthonormal basis $|n\rangle_b$ for $n \geq 1$ is identified with the positive natural modes. Put

$$N_b|n\rangle_b := n|n\rangle_b. \quad (4.1)$$

On the dense domain

$$\mathcal{D}_0 := \text{span}\{|m\rangle_b \otimes |n\rangle_b : m, n \geq 1\} \subset \mathcal{L}_b^+ \otimes \mathcal{L}_b^+$$

of finite support vectors define the addition map

$$A_{+,b}(|m\rangle_b \otimes |n\rangle_b) := |m+n\rangle_b. \quad (4.2)$$

At sum 5 the two inputs $|1, 4\rangle_b$ and $|2, 3\rangle_b$ together with their reversals $|3, 2\rangle_b$ and $|4, 1\rangle_b$ all map to $|5\rangle_b$. Here $|m, n\rangle_b := |m\rangle_b \otimes |n\rangle_b$. On their span addition sums the four coefficients. It therefore retains one linear combination and annihilates the three-dimensional subspace whose coefficients sum to zero. The longitudinal line and its transverse complement distinguish the common output from these relative input directions.

Proposition 4.1 (Closure and adjoint of the additive merge). *All input pairs with the same sum have the same image. Under the orthogonal decomposition*

$$\mathcal{L}_b^+ \otimes \mathcal{L}_b^+ = \bigoplus_{k \geq 2} \mathcal{F}_{b,k}, \quad \mathcal{F}_{b,k} := \text{span}\{|m, k-m\rangle_b : 1 \leq m \leq k-1\}, \quad (4.3)$$

the map $A_{+,b}$ is closable. Its closure is the maximal orthogonal direct sum of the row functionals on the fibres and has domain

$$\mathcal{D}(\bar{A}_{+,b}) = \left\{ x \in \mathcal{L}_b^+ \otimes \mathcal{L}_b^+ : \sum_{k \geq 2} \left| \sum_{m=1}^{k-1} x_{m, k-m} \right|^2 < \infty \right\}. \quad (4.4)$$

The adjoint has domain

$$\mathcal{D}(A_{+,b}^*) = \left\{ y = \sum_{k \geq 1} y_k |k\rangle_b : \sum_{k \geq 2} (k-1) |y_k|^2 < \infty \right\} \quad (4.5)$$

and acts by

$$A_{+,b}^* y = \sum_{k \geq 2} y_k \sum_{m=1}^{k-1} |m, k-m\rangle_b. \quad (4.6)$$

In particular one has $A_{+,b}^* |k\rangle_b = \sum_{m=1}^{k-1} |m, k-m\rangle_b$ for $k \geq 2$.

Proof. On $\mathcal{F}_{b,k} \cong \mathbb{C}^{k-1}$ the restriction of $A_{+,b}$ is the bounded row functional

$$A_k(x_1, \dots, x_{k-1}) = \sum_{m=1}^{k-1} x_m, \quad \|A_k\| = \sqrt{k-1}.$$

Thus $A_{+,b}$ on finite-support vectors is the algebraic direct sum of the operators A_k . For $y = \sum_{k \geq 1} y_k |k\rangle_b$ and every finite-support x one has

$$\langle A_{+,b} x, y \rangle = \sum_{k \geq 2} \sum_{m=1}^{k-1} \overline{x_{m,k-m}} y_k.$$

The representing vector in the input space has squared norm $\sum_{k \geq 2} (k-1) |y_k|^2$. This proves (4.5) and (4.6). Finite-support output vectors lie in the adjoint domain. Thus $A_{+,b}^*$ is densely defined and $A_{+,b}$ is closable. Taking the double adjoint gives the maximal direct sum of the A_k . Its output is square summable precisely under the condition in (4.4). $\square$

The same fibre calculation gives the additive covariance identities. On $\mathcal{D}_0$ put

$$N_1 := N_b \otimes I, \quad N_2 := I \otimes N_b, \quad N_+ := N_1 + N_2, \quad (4.7)$$

Then one has

$$N_b A_{+,b} = A_{+,b} N_+. \quad (4.8)$$

For every integer $a \geq 1$ the multiplicative dilation $D_a |n\rangle_b = |an\rangle_b$ satisfies

$$D_a A_{+,b} = A_{+,b} (D_a \otimes D_a). \quad (4.9)$$

Both sides of (4.8) send a basis pair to $(m+n)|m+n\rangle_b$. Both sides of (4.9) send it to $|a(m+n)\rangle_b$. Thus addition assigns coefficient one to every ordered basis pair and commutes with common integer dilation of the inputs.

4.2 Expression-root compression

The complete additive-root space retains both child histories at a fixed sum. For $k \geq 2$ define

$$\mathcal{R}_{b,k}^+ := \bigoplus_{\substack{m+n=k \\ m,n \geq 1}} \mathcal{H}_{b,u_b,m}^{\text{expr}} \otimes \mathcal{H}_{b,u_b,n}^{\text{expr}}. \quad (4.10)$$

Assembly at the root gives the isometry

$$M_{+,b,k}(|e\rangle \otimes |f\rangle) := |e \boxplus_{\text{expr}} f\rangle \quad \text{from } \mathcal{R}_{b,k}^+ \text{ into } \mathcal{H}_{b,u_b,k}^{\text{expr}}. \quad (4.11)$$

The restrictions $m, n \geq 1$ ensure that the displayed additive node is not removed by normalization. Put

$$Q_{b,k}^{\text{root}}(|e\rangle \otimes |f\rangle) := (h_b(u_{b,m}) h_b(u_{b,n}))^{-1/2} |m, n\rangle_b, \quad (4.12)$$

$$D_{b,k}^+ |m, n\rangle_b := \left(\frac{h_b(u_{b,m}) h_b(u_{b,n})}{h_b(u_{b,k})} \right)^{1/2} |m, n\rangle_b. \quad (4.13)$$

Proposition 4.2 (Counting compression and additive-root quotient). *The map Q_b^{expr} is a coisometry with initial space $\mathcal{H}_{b,\text{av}}^{\text{expr}}$. Equation (2.12) is its orthogonal initial-kernel decomposition. For every $k \geq 2$ the map $Q_{b,k}^{\text{root}}$ is a coisometry onto the fixed-sum fibre $\mathcal{F}_{b,k}$ of (4.3) and satisfies*

$$Q_{b,k}^{\text{expr}} M_{+,b,k} = A_{+,b} D_{b,k}^+ Q_{b,k}^{\text{root}}, \quad (4.14)$$

where $Q_{b,k}^{\text{expr}}$ is the restriction of Q_b^{expr} to $\mathcal{H}_{b,u_{b,k}}^{\text{expr}}$. Thus $\mathcal{F}_{b,k}$ is obtained by compressing the full expression space to its root-input values. It is distinct from a literal depth-one subspace.

Proof. Equation (2.12) with uniform weights proves the first assertion. The tensor product of the two fibre coisometries proves that $Q_{b,k}^{\text{root}}$ is a coisometry. Root assembly is injective on the displayed basis because ordered children and bracketing remain part of the normalized expression.

For $e \in \text{Expr}_b(u_{b,m})$ and $f \in \text{Expr}_b(u_{b,n})$ both sides of (4.14) send $|e\rangle \otimes |f\rangle$ to $h_b(u_{b,k})^{-1/2}|k\rangle_b$. This proves the intertwining identity and the stated interpretation of the fixed-sum fibre. $\square$

Cascade roots admit the same construction with $m, n \geq 2$ and value mn . The restrictions exclude the zero and unit multiplications removed by normalization. The resulting root space retains the two child expressions. Flattening its outer cascade skeleton instead gives the bar-word compression described after [1, Theorem 6.1].

The recursive source gives a second root-input compression. On the summand with child values m, n define

$$Q_{b,k}^{\text{root,hist}}(|e\rangle \otimes |f\rangle) = \sqrt{p_b(e)p_b(f)}|m, n\rangle_b. \quad (4.15)$$

It is a coisometry by the tensor-product form of (2.11). The source recursion gives

$$Q_{b,k}^{\text{hist}} M_{+,b,k} = c_b(u_{b,k})^{-1/2} A_{+,b} Q_{b,k}^{\text{root,hist}}. \quad (4.16)$$

Both sides send a child-history pair to $\sqrt{p_b(e)p_b(f)/c_b(u_{b,k})}|k\rangle_b$. This identity records the probability of selecting an additive channel among all available root channels. The fixed-sum input geometry itself uses its specified orthonormal value-pair basis. Both readouts therefore act through explicit maps on the same input geometry.

Readout at an additive root has two stages. The maps $Q_{b,k}^{\text{root}}$ and $Q_{b,k}^{\text{root,hist}}$ replace each child history by its value and retain the ordered pair $|m, n\rangle_b$. Application of $A_{+,b}$ then merges that pair to its sum. The faithful additive realization retains the relative-input information lost at this second stage.

4.3 Common-factor and primitive input coordinates

The prime-occupation coordinates also separate the scale shared by two inputs from the part on which they are coprime. Let

$$\mathfrak{P}_2 := \{(r, s) \in \mathbb{N}_+^2 : \gcd(r, s) = 1\}. \quad (4.17)$$

For $m, n \geq 1$ define d, r, s by

$$\nu_{u_{b,d}}(\pi) = \min\{\nu_{u_{b,m}}(\pi), \nu_{u_{b,n}}(\pi)\} \quad (\pi \in \mathcal{P}_b), \quad m = dr, \quad n = ds, \quad (r, s) \in \mathfrak{P}_2, \quad (4.18)$$

where the minimum is taken coordinatewise. This is the occupation-space form of the internal gcd constructed in [1, Appendix B].

Proposition 4.3 (Two-leg common-factor decomposition). *The assignment*

$$U_{\text{gcd},b}|m, n\rangle_b := |d; r, s\rangle_b \quad (4.19)$$

extends to a unitary

$$U_{\text{gcd},b} : \mathcal{L}_b^+ \otimes \mathcal{L}_b^+ \longrightarrow \ell^2(\mathbb{N}_+ \times \mathfrak{P}_2). \quad (4.20)$$

If $\mathbf{q}_b(u_{b,m})\mathbf{q}_b(u_{b,n}) = 1$ then

$$\mu_b(u_{b,m})\mu_b(u_{b,n}) = \mu_b(u_{b,d})^2\mu_b(u_{b,r})\mu_b(u_{b,s}) = \mathbf{q}_b(u_{b,d})\mu_b(u_{b,r})\mu_b(u_{b,s}). \quad (4.21)$$

The quadratic Gaussian weight transforms by

$$e^{-\beta(m^2+n^2)/2} = e^{-d^2\beta(r^2+s^2)/2}. \quad (4.22)$$

Thus the common factor is an exact dilation coordinate and (r, s) retains the primitive ordered pair.

Proof. Unique factorization gives a bijection $(m, n) \leftrightarrow (d, r, s)$ with $(r, s) = 1$. Its linearization is the unitary in (4.20). If both inputs are square-free then d, r, s are pairwise coprime and square-free. Multiplicativity of μ_b on coprime modes gives the first equality in (4.21). The identity $\mu_b(u)^2 = \mathbf{q}_b(u)$ gives the second. Substitution of $m = dr$ and $n = ds$ proves (4.22). $\square$

4.4 Additive charge defect and faithful environment

Common dilation preserves the additive merge. The merge itself mixes prime-charge coordinates. The additive charge defect records the information lost in the one-output mode. Write

$$\boldsymbol{\nu}_b(n) := \sum_{\pi \in \mathcal{P}_b} \nu_{u_{b,n}}(\pi) e_\pi \in \mathbb{N}_0^{(\mathcal{P}_b)} \subset \Lambda_b^\times \quad (4.23)$$

and define the additive charge defect

$$\delta_{+,b}(m, n) := \boldsymbol{\nu}_b(m) + \boldsymbol{\nu}_b(n) - \boldsymbol{\nu}_b(m+n) \in \Lambda_b^\times. \quad (4.24)$$

Negative coordinates are allowed because the defect belongs to the signed group completion containing the Fock positive cone.

Theorem 4.4 (Additive charge-defect identities). *For all $m, n, r, a \geq 1$ the defect is symmetric and satisfies*

$$\boldsymbol{\nu}_b(m) + \boldsymbol{\nu}_b(n) = \boldsymbol{\nu}_b(m+n) + \delta_{+,b}(m, n), \quad (4.25)$$

$$\delta_{+,b}(m, n) + \delta_{+,b}(m+n, r) = \delta_{+,b}(n, r) + \delta_{+,b}(m, n+r), \quad (4.26)$$

$$\delta_{+,b}(am, an) = \boldsymbol{\nu}_b(a) + \delta_{+,b}(m, n). \quad (4.27)$$

Equation (4.24) is the exact additive two-coboundary of the prime-occupation map. It also satisfies

$$\exp(\ell_b^\times(\delta_{+,b}(m, n))) = \frac{mn}{m+n}. \quad (4.28)$$

The augmented data

$$(m+n, \delta_{+,b}(m,n), \epsilon(m,n)), \quad \epsilon(m,n) := \text{sgn}(m-n), \quad (4.29)$$

determine the ordered pair (m,n) uniquely.

Proof. The conservation identity is the definition and symmetry is immediate. Expanding both sides of (4.26) gives $\nu_b(m) + \nu_b(n) + \nu_b(r) - \nu_b(m+n+r)$. Unique occupation factorization gives $\nu_b(an) = \nu_b(a) + \nu_b(n)$. This proves (4.27). Equation (3.65) gives

$$\ell_b^\times(\delta_{+,b}(m,n)) = \log m + \log n - \log(m+n),$$

and exponentiation proves (4.28). If $k = m+n$ and $\varrho = \exp(\ell_b^\times(\delta_{+,b}(m,n)))$ then $mn = k\varrho$ and the unordered pair $\{m,n\}$ is the root multiset of $x^2 - kx + k\varrho$. The sign in (4.29) fixes its order including the diagonal case $m = n$. $\square$

For $k \geq 2$ let $\mathcal{D}_{b,k}$ be the finite set of admissible pairs $(\delta_{+,b}(m, k-m), \epsilon(m, k-m))$ with $1 \leq m \leq k-1$. Put

$$\mathcal{E}_{+,b} := \bigoplus_{k \geq 2} \ell^2(\mathcal{D}_{b,k}). \quad (4.30)$$

The reconstruction part of Theorem 4.4 gives $|\mathcal{D}_{b,k}| = k-1$. Thus the defect labels give a unitary relabelling

$$W_{+,b} : \mathcal{L}_b^+ \otimes \mathcal{L}_b^+ \longrightarrow \bigoplus_{k \geq 2} \mathbb{C}|k\rangle_b \otimes \ell^2(\mathcal{D}_{b,k}), \quad W_{+,b}|m,n\rangle_b := |m+n\rangle_b \otimes |\delta_{+,b}(m,n), \epsilon(m,n)\rangle. \quad (4.31)$$

Transport the compact charge action to $\mathcal{L}_b^+$ by

$$\widehat{U}_b^\times(g)|n\rangle_b := \chi_{\nu_b(n)}(g)|n\rangle_b, \quad \chi_\alpha(g) := \prod_{\pi \in \mathcal{P}_b} g_\pi^{\alpha(\pi)}, \quad (4.32)$$

and let the defect environment carry $U_{\text{def}}(g)|\delta, \epsilon\rangle := \chi_\delta(g)|\delta, \epsilon\rangle$.

For $k \geq 2$ consider the fixed-sum fibre $\mathcal{F}_{b,k}$ from (4.3). Its longitudinal unit vector is

$$v_{b,k} := \frac{1}{\sqrt{k-1}} \sum_{m=1}^{k-1} |m\rangle_b \otimes |k-m\rangle_b. \quad (4.33)$$

The normalized merge isolates the common output direction. The associated fibre projectors are

$$\widetilde{A}_{+,b}(|m\rangle_b \otimes |n\rangle_b) := \frac{1}{\sqrt{m+n-1}} |m+n\rangle_b, \quad (4.34)$$

$$P_L^{(k)} := |v_{b,k}\rangle\langle v_{b,k}|, \quad P_T^{(k)} := I_{\mathcal{F}_{b,k}} - P_L^{(k)}. \quad (4.35)$$

Theorem 4.5 (Canonical addition with faithful dilation and central character). (i) Fix $k \geq 2$. Among coisometries $B_k : \mathcal{F}_{b,k} \rightarrow \mathbb{C}|k\rangle_b$ that are invariant under every permutation of the ordered-pair basis of $\mathcal{F}_{b,k}$ the restriction $\widetilde{A}_{+,b}|_{\mathcal{F}_{b,k}}$ is unique up to a phase on the one-dimensional target. Requiring positive basis coefficients fixes that phase and gives (4.34).

Suppose an isometric dilation of the deterministic merge has the form

$$V_k|m, k-m\rangle_b = |k\rangle_b \otimes e_{k,m} \quad (1 \leq m \leq k-1) \quad (4.36)$$

and retains enough environment information to distinguish the ordered inputs orthogonally. Then its environment has dimension at least $k-1$. At equality it is unitarily equivalent to $\ell^2(\mathcal{D}_{b,k})$. The dilation is then unitarily equivalent to $W_{+,b}|_{\mathcal{F}_{b,k}}$. The defect environment is therefore the minimal faithful orthogonal dilation of addition and is uniquely determined up to unitary equivalence.

The completed addition map satisfies compact-charge covariance in the form

$$W_{+,b}(\widehat{U}_b^\times(g) \otimes \widehat{U}_b^\times(g)) = (\widehat{U}_b^\times(g) \otimes U_{\text{def}}(g))W_{+,b}. \quad (4.37)$$

For

$$|\omega_{b,k}\rangle := \frac{1}{\sqrt{k-1}} \sum_{d \in \mathcal{D}_{b,k}} |d\rangle, \quad (4.38)$$

the normalized merge is the uniform-environment compression

$$\widetilde{A}_{+,b}|_{\mathcal{F}_{b,k}} = (I_{\mathbb{C}|k\rangle_b} \otimes \langle \omega_{b,k}|)W_{+,b}|_{\mathcal{F}_{b,k}}. \quad (4.39)$$

The projector $P_L^{(k)}$ retains the common defect superposition and $P_T^{(k)}$ retains its orthogonal relative information.

(ii) For a finitely supported signed charge $\alpha \in \Lambda_b^\times$ set

$$\chi_{\text{cen}}(\alpha) := (-1)^{\sum_{\pi \in \mathcal{P}_b} \alpha(\pi)}. \quad (4.40)$$

Define the Liouville involution on $\mathcal{L}_b^+$ and the defect involution on $\mathcal{E}_{+,b}$ by

$$\Gamma_b^{\text{Lio}}|n\rangle_b := \lambda_b(u_{b,n})|n\rangle_b, \quad (4.41)$$

$$\Gamma_{\text{def}}|\delta, \epsilon\rangle := \chi_{\text{cen}}(\delta)|\delta, \epsilon\rangle. \quad (4.42)$$

Then the charge-completed addition unitary satisfies

$$W_{+,b}(\Gamma_b^{\text{Lio}} \otimes \Gamma_b^{\text{Lio}})W_{+,b}^* = \Gamma_b^{\text{Lio}} \otimes \Gamma_{\text{def}}. \quad (4.43)$$

Equivalently for every $m, n \geq 1$ one has

$$\lambda_b(u_{b,m})\lambda_b(u_{b,n}) = \lambda_b(u_{b,m+n})\chi_{\text{cen}}(\delta_{+,b}(m, n)). \quad (4.44)$$

Combining this identity with the square-free selectors gives

$$\mu_b(u_{b,m})\mu_b(u_{b,n}) = \mathbf{q}_b(u_{b,m})\mathbf{q}_b(u_{b,n})\lambda_b(u_{b,m+n})\chi_{\text{cen}}(\delta_{+,b}(m, n)). \quad (4.45)$$

Proof. Permutation invariance gives $B_k|m, k-m\rangle_b = c|k\rangle_b$ with $(k-1)|c|^2 = 1$. Positive basis coefficients therefore select $c = (k-1)^{-1/2}$. Isometry of (4.36) makes the $e_{k,m}$ orthonormal. The environment therefore has dimension at least $k-1$. At equality sending this basis to the $k-1$ distinct defect labels of Theorem 4.4 is the required unitary equivalence.

On a basis pair the character on the input side of (4.37) is $\chi_{\nu_b(m)+\nu_b(n)}$. Equation (4.25) factors it as $\chi_{\nu_b(m+n)}\chi_{\delta_{+,b}(m,n)}$. This is the output character. Equation (4.39) sends every basis pair in the k th fibre to $(k-1)^{-1/2}|k\rangle_b$ as in (4.34). Pulling back the rank-one projector $|\omega_{b,k}\rangle\langle\omega_{b,k}|$ gives $P_L^{(k)}$. Its orthogonal complement gives $P_T^{(k)}$.

In (4.37) take the central compact charge $g_\pi = -1$ in every prime direction. Since every occupation and defect charge has finite support its character is respectively λ_b or χ_{cen} . The resulting intertwining relation is

$$W_{+,b}(\Gamma_b^{\text{Lio}} \otimes \Gamma_b^{\text{Lio}}) = (\Gamma_b^{\text{Lio}} \otimes \Gamma_{\text{def}})W_{+,b}.$$

Unitarity of $W_{+,b}$ allows right multiplication by $W_{+,b}^*$ and gives (4.43). Evaluation on a basis pair gives (4.44). The identity $\mu_b = \mathbf{q}_b\lambda_b$ gives the final identity. $\square$

The Liouville identity acts on every positive occupation profile. Multiplication by the two square-free selectors gives the exterior Möbius identity. The defect environment retains the ordered inputs through either readout.

4.5 Connected Gaussian response and center–difference coordinates

The square of the total degree differs from the sum of the two input energies by a bilinear cross-term. This term varies among pairs with the same sum, so it measures relative input data even when the total Gaussian weight is fixed.

Theorem 4.6 (Connected additive Gaussian response). *The closed quadratic form*

$$K_{+,b} := N_+^2 - N_1^2 - N_2^2 = 2N_1N_2 \succeq 0 \quad (4.46)$$

is the connected two-input response of the additive channel. It is the unique symmetric bilinear cross-term obtained by polarizing the generated quadratic energy $Q(n) = n^2$. For every $\beta > 0$ one has

$$e^{-\beta N_+^2} = e^{-\beta N_1^2} e^{-\beta N_2^2} e^{-\beta K_{+,b}}. \quad (4.47)$$

On the basis pair (m, n) the connected response is $2mn$.

Proof. The operators N_1 and N_2 are nonnegative and strongly commute. Expansion of $(N_1 + N_2)^2$ proves (4.46). Joint functional calculus proves (4.47). If a symmetric bilinear form B satisfies $Q(x+y) = Q(x) + Q(y) + B(x, y)$ for all positive generated degrees then direct substitution gives $B(x, y) = 2xy$. Its joint functional-calculus realization is therefore $2N_1N_2$. This proves uniqueness. $\square$

The reflecting double retains a second two-input coordinate system. It is lost when the two input legs are evaluated at the same circle point. Let

$$\Lambda_{\text{cd}} := \{(N, D) \in \mathbb{Z}^2 : N \equiv D \pmod{2}\}. \quad (4.48)$$

The map

$$C : \mathbb{Z}^2 \longrightarrow \Lambda_{\text{cd}}, \quad C(m, n) := (m+n, m-n) \quad (4.49)$$

is a group isomorphism with inverse $m = (N+D)/2$ and $n = (N-D)/2$. Denote the character of $\widehat{\Lambda_{\text{cd}}}$ indexed by (N, D) by $e_{N,D}$.

Theorem 4.7 (Center-difference Gaussian prime-pair field). *For $\beta > 0$ the full oriented prime-pair field*

$$\mathcal{K}_{b,\beta}^{\text{pr},2} := \sum_{\pi,\rho \in \mathcal{P}_b} \exp \left[-\frac{\beta}{2} (\deg_b(\pi)^2 + \deg_b(\rho)^2) \right] e_{\deg_b(\pi) + \deg_b(\rho), \deg_b(\pi) - \deg_b(\rho)} \quad (4.50)$$

belongs to $L^2(\widehat{\Lambda_{\text{cd}}})$. Its Fourier coefficient at $(N, D) \in \Lambda_{\text{cd}}$ is

$$\widehat{\mathcal{K}}_{b,\beta}^{\text{pr},2}(N, D) = e^{-\beta(N^2 + D^2)/4} \mathbf{1}_{\mathcal{P}_b}(u_{b,(N+D)/2}) \mathbf{1}_{\mathcal{P}_b}(u_{b,(N-D)/2}), \quad (4.51)$$

where the coefficient is zero unless $(N \pm D)/2$ are positive integers. The Gaussian energy and the multiplicative product are recovered simultaneously from the two uncompressed coordinates by

$$p^2 + q^2 = \frac{N^2 + D^2}{2}, \quad pq = \frac{N^2 - D^2}{4}. \quad (4.52)$$

Let P_N^Σ and P_D^Δ be the Fourier projections onto a fixed first and second coordinate respectively. Then

$$\left\| P_N^\Sigma \mathcal{K}_{b,\beta}^{\text{pr},2} \right\|_2^2 = e^{-\beta N^2/2} \sum_{\substack{p+q=N \\ p,q \text{ prime}}} e^{-\beta(p-q)^2/2}, \quad (4.53)$$

$$\left\| P_D^\Delta \mathcal{K}_{b,\beta}^{\text{pr},2} \right\|_2^2 = e^{-\beta D^2/2} \sum_{\substack{p-q=D \\ p,q \text{ prime}}} e^{-\beta(p+q)^2/2}. \quad (4.54)$$

The fixed-sum norm is positive exactly when N has an ordered prime decomposition. The fixed-difference norm is positive exactly when D occurs as an ordered prime gap. Goldbach fibres and prime gap fibres are therefore the two coordinate projections of one Gaussian-weighted two-input field.

Proof. The parity condition in (4.48) makes C bijective. Its Pontryagin dual therefore identifies the two oriented input circles with $\widehat{\Lambda_{\text{cd}}}$. Gaussian summability gives

$$\sum_{p,q \text{ prime}} e^{-\beta(p^2 + q^2)} < \infty,$$

so (4.50) is an L^2 Fourier series. For each (N, D) there is at most one ordered pair $p = (N + D)/2$ and $q = (N - D)/2$. This proves (4.51). The identities in (4.52) follow by expanding $(p+q)^2$ and $(p-q)^2$. Parseval's identity on each coordinate fibre gives (4.53) and (4.54). All their summands are strictly positive. This proves the two positivity criteria. $\square$

On the reflection-even diagonal the same two-input information is compressed to the familiar product-to-sum formula

$$\cos(p\theta) \cos(q\theta) = \frac{1}{2} [\cos((p+q)\theta) + \cos((p-q)\theta)]. \quad (4.55)$$

This identity describes pointwise spectral mixing. Arithmetic cascade is a different operation given by $u_{b,p} \circ u_{b,q} = u_{b,pq}$. The two-variable field in (4.50) keeps sum and difference coordinates separate. The one-variable cosine compression identifies them.

The connected response (4.46) uses N_+ directly. On each fixed-sum fibre N_+ acts as the scalar k . The operator $A_{+,b}^* N_b^2 A_{+,b}$ instead couples all pairs through the longitudinal rank-one direction. It vanishes on the transverse subspace where $K_{+,b}$ is positive.

4.6 Longitudinal and transverse quotient

For $k \geq 2$ the fibre $m + n = k$ has $k - 1$ positive input pairs. The normalized merge of (4.34) combines these fibre maps into a global coisometry.

Theorem 4.8 (Transverse quotient and fixed-sum geometry). (i) *The normalized merge is a coisometry onto $\overline{\text{span}}\{|k\rangle_b : k \geq 2\}$. In particular one has*

$$\tilde{A}_{+,b} \tilde{A}_{+,b}^* = I. \quad (4.56)$$

It follows that

$$P_L := \tilde{A}_{+,b}^* \tilde{A}_{+,b}, \quad P_T := I - P_L \quad (4.57)$$

are orthogonal projectors. The transverse two-input space is

$$\mathcal{T}_{+,b} := P_T(\mathcal{L}_b^+ \otimes \mathcal{L}_b^+) \cong \frac{\mathcal{L}_b^+ \otimes \mathcal{L}_b^+}{\text{Ran } \tilde{A}_{+,b}^*}. \quad (4.58)$$

For every $k \geq 2$ let $K_{+,b}^{T,(k)}$ be the finite-dimensional positive compression $P_T^{(k)} K_{+,b}|_{\mathcal{F}_{b,k}} P_T^{(k)}$. Define

$$K_{+,b}^T := \bigoplus_{k \geq 2} K_{+,b}^{T,(k)}, \quad \mathcal{D}(K_{+,b}^T) := \left\{ x = (x_k) : \sum_{k \geq 2} \|K_{+,b}^{T,(k)} x_k\|^2 < \infty \right\}. \quad (4.59)$$

Then $K_{+,b}^T$ is a nonnegative self-adjoint operator on $\mathcal{T}_{+,b}$. The algebraic direct sum of the transverse fibres is a core.

(ii) On $\mathcal{F}_{b,k}$ one has

$$P_L^{(k)} \mathcal{F}_{b,k} = \mathbb{C} v_{b,k}, \quad P_T^{(k)} \mathcal{F}_{b,k} = v_{b,k}^\perp \cap \mathcal{F}_{b,k}, \quad (4.60)$$

and hence

$$\dim \mathcal{F}_{b,k} = k - 1, \quad \dim P_T^{(k)} \mathcal{F}_{b,k} = k - 2. \quad (4.61)$$

Define the complete fibre capacity and the normalized longitudinal weight by

$$C_{+,b}(k) := \dim \mathcal{F}_{b,k} = k - 1, \quad \xi_{+,b}(k) := \frac{\text{rank } P_L^{(k)}}{\dim \mathcal{F}_{b,k}} = \frac{1}{k - 1}. \quad (4.62)$$

They obey the exact reciprocal normalization

$$C_{+,b}(k) \xi_{+,b}(k) = 1 \quad (k \geq 2). \quad (4.63)$$

In the ordered basis indexed by $1 \leq m \leq k - 1$ one has

$$\langle m, k - m | P_T^{(k)} | m', k - m' \rangle = \delta_{m,m'} - \frac{1}{k - 1}. \quad (4.64)$$

The connected response has the exact transverse trace

$$\mathrm{Tr}_{P_T^{(k)} \mathcal{F}_{b,k}} K_{+,b}^T = \frac{k(k+1)(k-2)}{3}. \quad (4.65)$$

For every $\beta > 0$ the untruncated additive transverse response is the convergent positive spectral sum

$$\Pi_{+,b}^T(\beta) := \mathrm{Tr}_{\mathcal{T}_{+,b}} \left(K_{+,b}^T e^{-\beta N_+^2} \right) = \sum_{k=3}^{\infty} \frac{k(k+1)(k-2)}{3} e^{-\beta k^2}. \quad (4.66)$$

Proof. The adjoint is

$$\tilde{A}_{+,b}^* |k\rangle_b = \frac{1}{\sqrt{k-1}} \sum_{m=1}^{k-1} |m\rangle_b \otimes |k-m\rangle_b = v_{b,k}. \quad (4.67)$$

Applying $\tilde{A}_{+,b}$ returns $|k\rangle_b$. This proves the coisometry identity and the projector and quotient assertions. Each $K_{+,b}^{T,(k)}$ is a positive self-adjoint matrix. Their maximal orthogonal direct sum is positive and self-adjoint on (4.59) with finite-support vectors as a graph core. The range of P_L in the k th fibre is $\mathbb{C}v_{b,k}$. This proves (4.60)–(4.64). On the basis vector indexed by m the connected response has eigenvalue $2m(k-m)$. Therefore

$$\mathrm{Tr}_{\mathcal{F}_{b,k}} K_{+,b} = 2 \sum_{m=1}^{k-1} m(k-m) = \frac{k(k^2-1)}{3}, \quad (4.68)$$

$$\langle v_{b,k}, K_{+,b} v_{b,k} \rangle = \frac{1}{k-1} \mathrm{Tr}_{\mathcal{F}_{b,k}} K_{+,b} = \frac{k(k+1)}{3}. \quad (4.69)$$

Subtracting the longitudinal trace gives (4.65). Since $N_+ = kI$ on $\mathcal{F}_{b,k}$ it commutes with both fibre projectors. Summing the finite-fibre traces proves (4.66). Gaussian decay gives absolute and locally uniform convergence by Lemma 3.4. $\square$

The quotient removes the uniform merge direction in each fixed-sum fibre and retains the relative two-input response. This information is lost under scalar merging but remains available to the transverse operator.

Example 4.9 (The fixed-sum fibre at $k=5$). Put $e_j := e_{u_{b,j}}$ for $j=2,3,5$. The fibre has four ordered basis pairs

$$|1,4\rangle_b, \quad |2,3\rangle_b, \quad |3,2\rangle_b, \quad |4,1\rangle_b.$$

Their normalized longitudinal vector is their half-sum. The transverse quotient therefore has rank three. The connected Gaussian response has diagonal values (8, 12, 12, 8). The faithful defect labels are

$$(2e_2 - e_5, -), \quad (e_2 + e_3 - e_5, -), \quad (e_2 + e_3 - e_5, +), \quad (2e_2 - e_5, +).$$

Scalar addition retains the common output 5. The transverse projector retains three relative directions. The signed defect environment still distinguishes all four ordered inputs.

4.7 Fixed-sum response and Schur reduction

The longitudinal expectation is the response of the uniform input vector. A second scalar response is obtained by fixing its longitudinal component and allowing the transverse component to vary so as to minimize the quadratic form. This elimination of the transverse variables is expressed by a Schur complement. In the present fibre it can be evaluated exactly. Let

$$H_j := \sum_{m=1}^j \frac{1}{m}, \quad H_0 := 0, \quad (4.70)$$

and write the connected response on $\mathcal{F}_{b,k} = \mathbb{C}v_{b,k} \oplus P_T^{(k)} \mathcal{F}_{b,k}$ as

$$K_{+,b}|\mathcal{F}_{b,k} = \begin{pmatrix} A_{b,k} & B_{b,k}^* \\ B_{b,k} & C_{b,k} \end{pmatrix}. \quad (4.71)$$

Theorem 4.10 (Exact transverse Schur response). *For every $k \geq 2$ the transverse block $C_{b,k}$ is positive and invertible on $P_T^{(k)} \mathcal{F}_{b,k}$ with the empty block understood when $k = 2$. Exact elimination of the transverse environment gives the scalar shorted response*

$$\mathcal{S}_{+,b}(k) := A_{b,k} - B_{b,k}^* C_{b,k}^{-1} B_{b,k} = \frac{k(k-1)}{H_{k-1}} > 0. \quad (4.72)$$

Equivalently one has

$$\mathcal{S}_{+,b}(k) = \inf_{w \in P_T^{(k)} \mathcal{F}_{b,k}} \langle v_{b,k} + w, K_{+,b}(v_{b,k} + w) \rangle. \quad (4.73)$$

The longitudinal expectation and the transverse self-energy are therefore

$$A_{b,k} = \frac{k(k+1)}{3}, \quad \Sigma_{+,b}^T(k) := A_{b,k} - \mathcal{S}_{+,b}(k) \geq 0, \quad (4.74)$$

with strict inequality for $k \geq 4$. Eliminating the transverse component therefore changes the longitudinal expectation by the nonnegative quantity $\Sigma_{+,b}^T(k)$.

Proof. In the fixed-sum basis the connected response is the positive diagonal matrix

$$D_k = \text{diag}\{2m(k-m) : 1 \leq m \leq k-1\}.$$

Every compression of D_k is positive definite. This proves the claim about $C_{b,k}$. The Schur-complement variational principle gives (4.73). The constraint $w \perp v_{b,k}$ is equivalent to $\langle v_{b,k}, v_{b,k} + w \rangle = 1$. Minimizing the D_k -energy under this constraint gives

$$v_{b,k} + w_{\min} = \frac{D_k^{-1} v_{b,k}}{\langle v_{b,k}, D_k^{-1} v_{b,k} \rangle}, \quad \mathcal{S}_{+,b}(k) = \frac{1}{\langle v_{b,k}, D_k^{-1} v_{b,k} \rangle}.$$

The remaining matrix element is explicit and equals

$$\begin{aligned} \langle v_{b,k}, D_k^{-1} v_{b,k} \rangle &= \frac{1}{k-1} \sum_{m=1}^{k-1} \frac{1}{2m(k-m)} \\ &= \frac{1}{2k(k-1)} \sum_{m=1}^{k-1} \left(\frac{1}{m} + \frac{1}{k-m} \right) = \frac{H_{k-1}}{k(k-1)}. \end{aligned} \quad (4.75)$$

This proves (4.72). The value of $A_{b,k}$ was computed in the proof of Theorem 4.8(ii). Positivity of the Schur correction follows from the block decomposition. Equality holds exactly when $v_{b,k}$ is an eigenvector of D_k . This occurs for $k = 2, 3$. For $k \geq 4$ the diagonal entries $2m(k - m)$ are not all equal. $\square$

The fixed-sum fibre retains the ordered pair, its longitudinal line gives the common additive output, and its transverse quotient retains relative input directions. The Schur response measures the change in longitudinal energy when those directions are eliminated.

5 Occupation sign and scale transport

Total occupation parity gives the Liouville sign on every positive mode; restriction to square-free profiles gives the Möbius sign. Both pull back to histories through the root value. Square-divisor identities relate their complete scale responses, while prime-direction transfers retain the sign and Gaussian weight of each occupation.

5.1 Square-dilation equivalence of complete and exterior parity responses

The complete occupation grading and its exterior reduction can be compared before an asymptotic estimate is made. Let $\Phi : (0, \infty) \rightarrow \mathbb{C}$ be locally bounded and suppose that $\Phi(x) = O(x^{-A})$ as $x \rightarrow \infty$ for some $A > 1$. In particular one has

$$\sum_{d,m \geq 1} |\Phi(d^2 m/L)| < \infty \quad (L > 0). \quad (5.1)$$

The Gaussian profiles used in the spectral response satisfy this condition. Define the complete responses

$$\mathcal{B}_{\lambda,\Phi}^+(L) := \sum_{n \geq 1} \lambda_b(u_{b,n}) \Phi(n/L), \quad \mathcal{B}_{\mu,\Phi}^+(L) := \sum_{n \geq 1} \mu_b(u_{b,n}) \Phi(n/L). \quad (5.2)$$

Proposition 5.1 (Complete-occupation square-dilation equivalence). *The responses in (5.2) satisfy*

$$\mathcal{B}_{\mu,\Phi}^+(L) = \sum_{d \geq 1} \mu(d) \mathcal{B}_{\lambda,\Phi}^+(L/d^2), \quad (5.3)$$

$$\mathcal{B}_{\lambda,\Phi}^+(L) = \sum_{d \geq 1} \mathcal{B}_{\mu,\Phi}^+(L/d^2). \quad (5.4)$$

For $\sigma > \frac{1}{2}$ let

$$\mathcal{H}_\sigma^{\text{sc}} := L^2\left((0, \infty), L^{-2\sigma} \frac{dL}{L}\right), \quad (\mathcal{S}_{d^2} F)(L) := F(L/d^2). \quad (5.5)$$

Then the operator series

$$\mathcal{D}_\mu := \sum_{d \geq 1} \mu(d) \mathcal{S}_{d^2}, \quad \mathcal{D}_+ := \sum_{d \geq 1} \mathcal{S}_{d^2} \quad (5.6)$$

converge in operator norm on $\mathcal{H}_\sigma^{\text{sc}}$. They are mutually inverse and obey

$$\|\mathbf{D}_\mu\| \leq \frac{\zeta(2\sigma)}{\zeta(4\sigma)}, \quad \|\mathbf{D}_+\| \leq \zeta(2\sigma). \quad (5.7)$$

Consequently one has

$$\mathcal{B}_{\mu,\Phi}^+ \in \mathcal{H}_\sigma^{\text{sc}} \iff \mathcal{B}_{\lambda,\Phi}^+ \in \mathcal{H}_\sigma^{\text{sc}}. \quad (5.8)$$

The same comparison respects the exact finite cutoff. If

$$\mathcal{B}_{\vartheta,\Phi,N}^+(L) := \sum_{1 \leq n \leq N} \vartheta_b(u_{b,n}) \Phi(n/L), \quad \vartheta \in \{\lambda, \mu\}, \quad (5.9)$$

then

$$\mathcal{B}_{\mu,\Phi,N}^+(L) = \sum_{d \leq \sqrt{N}} \mu(d) \mathcal{B}_{\lambda,\Phi,\lfloor N/d^2 \rfloor}^+(L/d^2), \quad (5.10)$$

$$\mathcal{B}_{\lambda,\Phi,N}^+(L) = \sum_{d \leq \sqrt{N}} \mathcal{B}_{\mu,\Phi,\lfloor N/d^2 \rfloor}^+(L/d^2). \quad (5.11)$$

Put $\phi_\Phi(L) := \Phi(1/L)$ and $\lfloor_{*,\Phi} := \mathcal{B}_{*,\Phi}^+ - \phi_\Phi$. The responses with the unit mode removed satisfy

$$\lfloor_{\mu,\Phi} = \mathbf{D}_\mu \lfloor_{\lambda,\Phi} + (\mathbf{D}_\mu - I)\phi_\Phi, \quad (5.12)$$

$$\lfloor_{\lambda,\Phi} = \mathbf{D}_+ \lfloor_{\mu,\Phi} + (\mathbf{D}_+ - I)\phi_\Phi. \quad (5.13)$$

Whenever $\phi_\Phi \in \mathcal{H}_\sigma^{\text{sc}}$ membership of either centered response in $\mathcal{H}_\sigma^{\text{sc}}$ is therefore equivalent to membership of the other. The Gaussian profiles above satisfy this condition.

Proof. The coefficient identities

$$\mu(n) = \lambda(n) \sum_{d^2|n} \mu(d), \quad \lambda(n) = \sum_{d^2|n} \mu(n/d^2) \quad (5.14)$$

follow from $\mu^2(n) = \sum_{d^2|n} \mu(d)$ and $\lambda(d^2m) = \lambda(m)$. Reindexing $n = d^2m$ is justified by (5.1) and gives (5.3) and (5.4). The same reindexing with $d^2m \leq N$ gives (5.10) and (5.11).

A change of scale gives

$$\|\mathbf{S}_{d^2} F\|_{\mathcal{H}_\sigma^{\text{sc}}} = d^{-2\sigma} \|F\|_{\mathcal{H}_\sigma^{\text{sc}}}. \quad (5.15)$$

Hence both series in (5.6) converge in operator norm when $\sigma > \frac{1}{2}$ and satisfy

$$\sum_{d \geq 1} |\mu(d)| d^{-2\sigma} = \sum_{d \geq 1} \mu(d)^2 d^{-2\sigma} = \frac{\zeta(2\sigma)}{\zeta(4\sigma)}, \quad \sum_{d \geq 1} d^{-2\sigma} = \zeta(2\sigma).$$

Absolute operator convergence permits the Cauchy product. The coefficient of $\mathbf{S}_{k^2}$ in either $\mathbf{D}_\mu \mathbf{D}_+$ or $\mathbf{D}_+ \mathbf{D}_\mu$ is $\sum_{d|k} \mu(d) = \mathbf{1}_{\{k=1\}}$. Thus the two operators are mutually inverse. Subtracting the common unit-mode term gives (5.12) and (5.13). $\square$

This equivalence holds in each fixed weighted space. Its constants grow as $\sigma \downarrow \frac{1}{2}$. Estimates transfer between the complete Liouville response and the exterior Möbius response through the displayed square-dilation operators, including the unit-mode terms for centered responses.

5.2 Character-twisted traces and marked prime-power response

The occupation representation also carries the characters of the free multiplicative monoid. Let

$$\chi : \mathcal{U}_b^+ \longrightarrow \{z \in \mathbb{C} : |z| \leq 1\}, \quad \chi(1_b) = 1, \quad \chi(u \boxtimes_b v) = \chi(u)\chi(v), \quad (5.16)$$

and define the diagonal twist on either occupation space by

$$U_\chi \Phi_{b,\nu} := \prod_{\pi \in \mathcal{P}_b} \chi(\pi)^{\nu_\pi} \Phi_{b,\nu}. \quad (5.17)$$

It is a contraction and is unitary when $|\chi(\pi)| = 1$ on every prime direction.

The marked prime-power space retains individual prime directions and their positive repetition numbers. Define

$$\mathcal{J}_b := \ell^2(\{(\pi, m) : \pi \in \mathcal{P}_b, m \in \mathbb{N}_+\}) \quad (5.18)$$

with orthonormal basis $|\pi, m\rangle$ and put

$$\mathbf{N}_b^{\text{pp}}|\pi, m\rangle := \deg_b(\pi)^m |\pi, m\rangle, \quad \mathbf{L}_b^{\text{pp}}|\pi, m\rangle := \log \deg_b(\pi) |\pi, m\rangle, \quad (5.19)$$

$$U_\chi^{\text{pp}}|\pi, m\rangle := \chi(\pi)^m |\pi, m\rangle. \quad (5.20)$$

Theorem 5.2 (Twisted occupation traces and marked prime-power response). (i) For $\Re s > 1$ one has

$$Z_b^\times(s, \chi) := \text{Tr}_{\Gamma_s(\mathcal{H}_{b,\mathcal{P}})} \left(U_\chi e^{-s d\Gamma(\varepsilon_{\text{ar}})} \right) = \sum_{u \in \mathcal{U}_b^+} \frac{\chi(u)}{\deg_b(u)^s} = \prod_{\pi \in \mathcal{P}_b} \left(1 - \frac{\chi(\pi)}{\deg_b(\pi)^s} \right)^{-1}, \quad (5.21)$$

$$\text{Str}_{\wedge,b}(s, \chi) := \text{Tr}_{\Gamma_a(\mathcal{H}_{b,\mathcal{P}})} \left((-1)^{\mathbf{N}} U_\chi e^{-s d\Gamma(\varepsilon_{\text{ar}})} \right) = \prod_{\pi \in \mathcal{P}_b} \left(1 - \frac{\chi(\pi)}{\deg_b(\pi)^s} \right) = \frac{1}{Z_b^\times(s, \chi)}. \quad (5.22)$$

(ii) For $\Re s > 1$ one has

$$\begin{aligned} \mathcal{T}_b^{\text{pp}}(s, \chi) &:= \text{Tr}_{\mathcal{J}_b} (\mathbf{L}_b^{\text{pp}} U_\chi^{\text{pp}} (\mathbf{N}_b^{\text{pp}})^{-s}) \\ &= \sum_{\pi \in \mathcal{P}_b} \sum_{m \geq 1} \frac{\log \deg_b(\pi) \chi(\pi)^m}{\deg_b(\pi)^{ms}} = -\frac{\partial}{\partial s} \log Z_b^\times(s, \chi). \end{aligned} \quad (5.23)$$

For the continuous phase character

$$\chi_\tau(u) := e^{-i\tau \log \deg_b(u)}, \quad \tau \in \mathbb{R}, \quad (5.24)$$

one has

$$Z_b^\times(s, \chi_\tau) = \zeta(s + i\tau), \quad \mathcal{T}_b^{\text{pp}}(s, \chi_\tau) = -\frac{\zeta'}{\zeta}(s + i\tau). \quad (5.25)$$

Proof. On the π coordinate the symmetric trace is the absolutely convergent geometric series $\sum_{m \geq 0} \chi(\pi)^m \deg_b(\pi)^{-ms}$. The graded exterior trace is $1 - \chi(\pi) \deg_b(\pi)^{-s}$. Multiplication over the prime coordinates is justified by absolute convergence in $\Re s > 1$. Unique occupation factorization identifies the coefficient of each symmetric state with $\chi(u) \deg_b(u)^{-s}$.

The first equality follows by taking the diagonal trace on $\mathcal{J}_b$. Termwise differentiation of the absolutely convergent Euler product in part (i) gives the logarithmic derivative. The identity $\chi_\tau(u) \deg_b(u)^{-s} = \deg_b(u)^{-(s+i\tau)}$ proves the shift identities. $\square$

The marked basis retains the prime direction and its repetition number separately. The logarithmic derivative weights each such pair by $\log \deg_b(\pi)$ and gives the von Mangoldt weight on prime powers. The subspace with repetition number one identifies with the prime Hodge kernel. Keeping all repetition numbers gives the logarithmic response as a trace that resolves the individual phase factors.

5.3 Completed Gaussian trace

Let

$$\Theta_+(t) := \sum_{n \geq 1} e^{-\pi n^2 t}. \quad (5.26)$$

For $\Re s > 1$ termwise Mellin transformation gives

$$\pi^{-s/2} \Gamma(s/2) Z_b^\times(s, 1) = \int_0^\infty \Theta_+(t) t^{s/2} \frac{dt}{t}. \quad (5.27)$$

The Gaussian Poisson identity implies $\Theta_+(t) = t^{-1/2} \Theta_+(t^{-1}) + (t^{-1/2} - 1)/2$. Splitting (5.27) at $t = 1$ therefore yields

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \Theta_+(t) \left(t^{s/2} + t^{(1-s)/2} \right) \frac{dt}{t}. \quad (5.28)$$

Proposition 5.3 (Completed arithmetic trace). *The function*

$$\Xi_b(s) := \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) Z_b^\times(s, 1) \quad (5.29)$$

extends to an entire function and satisfies

$$\Xi_b(s) = \Xi_b(1-s). \quad (5.30)$$

Its logarithmic response is

$$-\frac{\Xi'_b(s)}{\Xi_b(s)} = -\frac{1}{s} - \frac{1}{s-1} + \frac{1}{2} \log \pi - \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} + \mathcal{T}_b^{\text{pp}}(s, 1). \quad (5.31)$$

Proof. Equation (5.28) is invariant under $s \mapsto 1-s$. On every compact subset of the s -plane the integrand and each of its s -derivatives are bounded by an integrable multiple of $e^{-\pi t}$ times a fixed power of t and $\log t$ for $t \geq 1$. The integral is therefore normally convergent and entire. Multiplication by $s(s-1)/2$ removes the two displayed poles. Logarithmic differentiation together with Theorem 5.2 proves (5.31). $\square$

The reflection formula extends the Gaussian Mellin trace to the whole plane. Its logarithmic derivative separates the completion terms from the marked prime-power response.

5.4 Full-charge Gaussian–Fock readout

For $w = (w_\pi)_\pi \in \mathbb{T}^{\mathcal{P}_b}$ let

$$\chi_w(u) := \prod_{\pi \in \mathcal{P}_b} w_\pi^{\nu_u(\pi)}, \quad u \in \mathcal{U}_b^+. \quad (5.32)$$

Write $U_w := U_{\chi_w}$ for the diagonal unitary (5.17). Define the phase-resolved Gaussian trace

$$\Theta_b(t; w) := \sum_{u \in \mathcal{U}_b^+} \chi_w(u) e^{-\pi t \deg_b(u)^2}, \quad t > 0. \quad (5.33)$$

The series converges absolutely and uniformly in w for every $t \geq t_0 > 0$.

Theorem 5.4 (Full-charge Gaussian–Fock readout). *For $w \in \mathbb{T}^{\mathcal{P}_b}$ and $\Re s > 1$ one has*

$$\begin{aligned} \int_0^\infty \Theta_b(t; w) t^{s/2} \frac{dt}{t} &= \pi^{-s/2} \Gamma(s/2) \sum_{u \in \mathcal{U}_b^+} \frac{\chi_w(u)}{\deg_b(u)^s} \\ &= \pi^{-s/2} \Gamma(s/2) \prod_{\pi \in \mathcal{P}_b} \left(1 - \frac{w_\pi}{\deg_b(\pi)^s} \right)^{-1}. \end{aligned} \quad (5.34)$$

Equivalently the Mellin transform of the phase-resolved Gaussian scale trace is the phase-resolved symmetric-Fock trace of Theorem 5.2.

Let D be a trace-class operator diagonal in the symmetric occupation basis with diagonal coefficients d_ν . Then the compact-charge transform

$$\mathcal{F}_b^\times D(w) := \text{Tr}(U_w D) = \sum_{\nu \in \mathbb{N}_0^{(\mathcal{P}_b)}} d_\nu w^\nu \quad (5.35)$$

determines every coefficient d_ν and is therefore injective. The coefficients are recovered by

$$d_\nu = \int_{\mathbb{T}^{\mathcal{P}_b}} \mathcal{F}_b^\times D(w) \overline{w}^\nu dw. \quad (5.36)$$

Evaluation at the identity $w = 1$ is the scalar trace compression and retains the sum of the diagonal coefficients.

Proof. Apply (3.39) with $a = \pi \deg_b(u)^2$. Since $|\chi_w(u)| = 1$ the sum of absolute Mellin integrals is $\pi^{-\sigma/2} \Gamma(\sigma/2) \zeta(\sigma) < \infty$ for $\sigma = \Re s > 1$. Fubini gives the Dirichlet series and Theorem 5.2 gives the Euler product.

The trace-class condition gives $(d_\nu) \in \ell^1$. Hence (5.35) converges uniformly. Character orthogonality on every finite coordinate torus followed by the directed limit over finite coordinate sets proves (5.36). Setting $w = 1$ in (5.35) gives the coefficient sum. $\square$

The Mellin integral decomposes into scaled copies of the permanent dimensionless window. Partition $(0, \infty)$ into the intervals $[\pi^k, \pi^{k+1}]$ indexed by $k \in \mathbb{Z}$. The substitution $t = \pi^k u$ gives the exact scale-tower identity

$$\int_0^\infty \Theta_b(t; w) t^{s/2} \frac{dt}{t} = \int_1^\pi \sum_{k \in \mathbb{Z}} \Theta_b(\pi^k u; w) \pi^{ks/2} u^{s/2} \frac{du}{u} \quad (5.37)$$

in the same half-plane. For $w = 1$ the Gaussian–Poisson reflection used in (5.28) folds the negative scale half of this tower onto the positive half and supplies the displayed unit and zero-mode completion terms. For a general charge w the unfolded tower (5.37) is the defining scale representation.

For the Gaussian state this recovery applies at every positive scale. The charge variable retains the occupation data through the Mellin transform. Evaluation at $w = 1$ gives the scalar partition trace.

Exterior occupation gives the Möbius sign by combining occupation parity with square-free support. The relation of this two-element group action to the tangent Gaussian depends on the operator class. Among affine first-order operators annihilation of the Gaussian and commutator normalization select the annihilation operator up to phase and determine its adjoint. Symmetric quadratic ordering gives a half-step spectrum and its nontrivial central character. A nontrivial lift that commutes with finite prime permutations acts uniformly on the prime directions. Exterior products then carry this character to the occupation grading.

Signed and unsigned occupation traces measure this central action along the Gaussian flow. Their ratio changes according to the mean-energy difference between the two sign sectors. The Gaussian construction and the projective half-spin representation of Appendix E meet at their common central character.

5.5 Gaussian half-step from minimal quadratic closure

Normalize the tangent Gaussian by

$$g_b(y) := (\pi b^2)^{-1/4} e^{-y^2/(2b^2)}. \quad (5.38)$$

The logarithmic derivative of this state is linear in y . This singles out affine first-order differential operators as the minimal class in which to seek a normalized annihilator. Fixing its commutator leaves only the phase of the annihilator. Its first-order ladder directions act on the Schwartz space in $L^2(\mathbb{R})$ as

$$a_b := \frac{1}{\sqrt{2}} \left(b \frac{d}{dy} + \frac{y}{b} \right), \quad a_b^* := \frac{1}{\sqrt{2}} \left(-b \frac{d}{dy} + \frac{y}{b} \right). \quad (5.39)$$

The normalized ordered quadratic closure of these directions is the affine class

$$\mathfrak{Q}_b := \{ \mathbf{H}_{b,\alpha} := \alpha a_b^* a_b + (1 - \alpha) a_b a_b^* : 0 \leq \alpha \leq 1 \}. \quad (5.40)$$

Its coefficients are nonnegative and have unit sum. The principal quadratic part is therefore fixed while only the relative coefficients of the two orderings vary. Ordering neutrality gives neither ordering priority. Let $\mathcal{O}_b$ exchange the two ordered quadratic expressions $a_b^* a_b$ and $a_b a_b^*$ within $\mathfrak{Q}_b$.

The identity $a_b a_b^* = a_b^* a_b + I$ shows that the ordering involution sends $\mathbf{H}_{b,\alpha}$ to $\mathbf{H}_{b,1-\alpha}$. Its unique fixed coefficient is $\alpha = 1/2$. Thus the order-neutral generator is

$$\mathbf{H}_b^G := \frac{1}{2} (a_b^* a_b + a_b a_b^*) = a_b^* a_b + \frac{1}{2} I = \frac{1}{2} \left(-b^2 \frac{d^2}{dy^2} + \frac{y^2}{b^2} \right). \quad (5.41)$$

This full-line generator belongs to the tangent Gaussian datum. The finite-window Neumann generator A_b arises only after correlation closure. The two operators thus belong to successive stages of the construction and act on different Hilbert spaces. The Hermite basis and oscillator closure are standard [12, Chapter 1]. Their role here is fixed by the Gaussian annihilator and its commutator normalization.

Theorem 5.5 (Gaussian ladder normalization and oscillator central sign). (i) *Let*

$$A = c_1 b \frac{d}{dy} + c_2 \frac{y}{b} + c_0 \quad (5.42)$$

be a first-order differential operator with complex affine coefficients on the Schwartz core. If $Ag_b = 0$ and $[A, A^] = I$ then a phase $e^{i\gamma} \in U(1)$ exists such that*

$$A = e^{i\gamma} a_b, \quad A^* = e^{-i\gamma} a_b^*. \quad (5.43)$$

*The ordered quadratic products A^*A and AA^* are therefore independent of this phase. Within the stated affine first-order class the normalized Gaussian and commutator normalization determine the pair up to its $U(1)$ gauge.*

(ii) *The closure of H_b^G is positive and self-adjoint and has an orthonormal Hermite basis $(h_{b,n})_{n \geq 0}$ satisfying*

$$H_b^G h_{b,n} = \left(n + \frac{1}{2}\right) h_{b,n}. \quad (5.44)$$

Its ground state is $h_{b,0} = g_b$ with unnormalized profile K_b . The unitary phase flow

$$U_b^G(\theta) := e^{-i\theta H_b^G} \quad (5.45)$$

obeys

$$U_b^G(2\pi) = -I, \quad U_b^G(4\pi) = I. \quad (5.46)$$

The half-unit in (5.44) gives the nontrivial central character of the double-valued phase lift.

Proof. The logarithmic derivative of g_b is $g'_b/g_b = -y/b^2$. Consequently $Ag_b = 0$ forces $c_2 = c_1$ and $c_0 = 0$. The formal adjoint is then

$$A^* = \overline{c_1} \left(-b \frac{d}{dy} + \frac{y}{b} \right). \quad (5.47)$$

The identity $[b d/dy, y/b] = I$ gives $[A, A^*] = 2|c_1|^2 I$. The commutator normalization then gives $|c_1| = 1/\sqrt{2}$ and proves (5.43). The phase cancels in both quadratic products.

The unitary dilation $U_b f(y) = b^{-1/2} f(y/b)$ intertwines H_1^G with H_b^G . The standard Hermite expansion [12, Chapter 1] therefore gives $h_{b,n} = U_b h_{1,n}$ and the self-adjoint closure with domain

$$\mathcal{D}(H_b^G) = \left\{ \sum_{n \geq 0} c_n h_{b,n} : \sum_{n \geq 0} \left(n + \frac{1}{2}\right)^2 |c_n|^2 < \infty \right\}, \quad (5.48)$$

The ground state scales to g_b . On every basis vector the phase at 2π is $e^{-2\pi i(n+1/2)} = -1$. Completeness and squaring give (5.46). □

5.6 Möbius grading from the Gaussian central character

Let

$$Z_2^G := \{I, \sigma_b\}, \quad \sigma_b := U_b^G(2\pi) = -I, \quad (5.49)$$

and let ε be its nontrivial one-dimensional character satisfying $\varepsilon(\sigma_b) = -1$.

For a finite permutation η of the prime basis let U_η denote the unitary determined by $U_\eta \pi = \eta(\pi)$. A central-sign lift is called prime-natural when its one-particle action commutes with every U_η . The condition assigns the central character independently of the prime label; the prime directions retain their distinct spectral energies. The nontrivial lift preserves the Gaussian central character. This character relates the tangent oscillator to the prime one-particle space and the projective half-spin module while keeping their representations distinct.

Proposition 5.6 (Prime-natural central lift and Möbius grading). (i) *Let V be the unitary involution representing σ_b on $\mathcal{H}_{b,\mathcal{P}}$. If*

$$VU_\eta = U_\eta V \quad \text{for every finite permutation } \eta \text{ of } \mathcal{P}_b, \quad (5.50)$$

then $V = I$ or $V = -I$. The nontrivial central character uniquely selects $V = -I$. Its unique exterior lift compatible with decomposable products is

$$\Gamma_V := \bigoplus_{r \geq 0} \bigwedge^r V, \quad \Gamma_V|_{\wedge^r \mathcal{H}_{b,\mathcal{P}}} = (-1)^r I. \quad (5.51)$$

Thus the Gaussian central sign attaches uniformly to every generated prime direction. Exterior degree records its accumulated character. (ii) Denote the selected one-particle representation and its exterior lift by

$$\mathcal{H}_{b,\mathcal{P}}^{\text{cen}} := \mathcal{H}_{b,\mathcal{P}} \otimes \varepsilon, \quad V_b := I_{\mathcal{H}_{b,\mathcal{P}}} \otimes \varepsilon(\sigma_b) = -I_{\mathcal{H}_{b,\mathcal{P}}}, \quad \Gamma_{\text{cen}} := \Gamma_a(V_b) \quad \text{on } \Gamma_a(\mathcal{H}_{b,\mathcal{P}}^{\text{cen}}), \quad (5.52)$$

where Γ_a denotes antisymmetric second quantization.

On the r -particle exterior sector one has

$$\Gamma_{\text{cen}}|_{\wedge^r \mathcal{H}_{b,\mathcal{P}}^{\text{cen}}} = (-1)^r I. \quad (5.53)$$

For $\Re s > 1$ this gives

$$\text{Tr}_{\Gamma_a(\mathcal{H}_{b,\mathcal{P}}^{\text{cen}})} \left(\Gamma_{\text{cen}} e^{-s d\Gamma(\varepsilon_{\text{ar}})} \right) = \prod_{\pi \in \mathcal{P}_b} (1 - \deg_b(\pi)^{-s}) = \frac{1}{\zeta(s)}. \quad (5.54)$$

The sign in this supertrace is the second-quantized central character of the Gaussian half-step under the stated prime-natural lift. Exterior nilpotence supplies the independent square-free selector.

Proof. In the prime basis commutation with every finite permutation makes the diagonal entries of V equal and the off-diagonal entries equal. A nonzero common off-diagonal entry would give $V\pi$ infinitely many coefficients of the same nonzero modulus. This would

contradict $V\pi \in \ell^2(\mathcal{P}_b)$. Thus $V = cI$ and the unitary involution condition gives $c = \pm 1$. The nontrivial character selects $c = -1$. Compatibility with decomposable exterior products requires $\Gamma_V(v_1 \wedge \cdots \wedge v_r) = (Vv_1) \wedge \cdots \wedge (Vv_r)$. This determines the lift and its sign $(-1)^r$. Each prime coordinate of the exterior Fock space has occupation zero or one and therefore contributes $1 - \deg_b(\pi)^{-s}$ to the trace. Absolute convergence on $\Re s > 1$ and the Euler identity already established in Theorem 3.6 prove (5.54). $\square$

5.7 Central overlap and energy–entropy flow

The central character determines the occupation sign. Its cancellation along the arithmetic scale is measured by an overlap. For a square-free positive mode u denote the corresponding exterior occupation vector by $\Phi_{b,u}^a$ including the vacuum vector for $u = 1_b$.

Theorem 5.7 (Central overlap and its energy–entropy evolution). (i) For $X \geq 1$ define

$$\Psi_{b,X}^{\text{sf}} := \sum_{\substack{u \in \mathcal{U}_b \setminus \{0_b\} \\ \deg_b(u) \leq X, \mathbf{q}_b(u)=1}} \Phi_{b,u}^a, \quad \mathbf{Q}_b(X) := \|\Psi_{b,X}^{\text{sf}}\|^2. \quad (5.55)$$

Then

$$\sum_{\substack{u \in \mathcal{U}_b \setminus \{0_b\} \\ \deg_b(u) \leq X}} \mu_b(u) = \left\langle \Psi_{b,X}^{\text{sf}}, \Gamma_{\text{cen}} \Psi_{b,X}^{\text{sf}} \right\rangle. \quad (5.56)$$

Define the normalized central angle by

$$\cos \vartheta_b(X) := \frac{\langle \Psi_{b,X}^{\text{sf}}, \Gamma_{\text{cen}} \Psi_{b,X}^{\text{sf}} \rangle}{\mathbf{Q}_b(X)}. \quad (5.57)$$

The finite Möbius sum then equals $\mathbf{Q}_b(X) \cos \vartheta_b(X)$.

For $\beta > 0$ the Gaussian-smoothed vector

$$\Psi_{b,\beta}^{\text{sf}} := \sum_{\substack{u \in \mathcal{U}_b \setminus \{0_b\} \\ \mathbf{q}_b(u)=1}} e^{-\beta \deg_b(u)^2 / 2} \Phi_{b,u}^a \quad (5.58)$$

belongs to the exterior Fock space and satisfies

$$\left\langle \Psi_{b,\beta}^{\text{sf}}, \Gamma_{\text{cen}} \Psi_{b,\beta}^{\text{sf}} \right\rangle = \sum_{u \in \mathcal{U}_b \setminus \{0_b\}} \mu_b(u) e^{-\beta \deg_b(u)^2}, \quad (5.59)$$

$$\|\Psi_{b,\beta}^{\text{sf}}\|^2 = \sum_{u \in \mathcal{U}_b \setminus \{0_b\}} \mu_b(u)^2 e^{-\beta \deg_b(u)^2}. \quad (5.60)$$

(ii) The Gaussian overlap admits an exact energy–entropy evolution. Split the square-free modes according to their central sign and set

$$Z_{b,\pm}^{\text{sf}}(\beta) := \sum_{\substack{u \in \mathcal{U}_b \setminus \{0_b\} \\ \mathbf{q}_b(u)=1, \mu_b(u)=\pm 1}} e^{-\beta \deg_b(u)^2}, \quad (5.61)$$

$$\mathcal{E}_{b,\pm}^{\text{sf}}(\beta) := \frac{1}{Z_{b,\pm}^{\text{sf}}(\beta)} \sum_{\substack{u \in \mathcal{U}_b \setminus \{0_b\} \\ \mathbf{q}_b(u)=1, \mu_b(u)=\pm 1}} \deg_b(u)^2 e^{-\beta \deg_b(u)^2}. \quad (5.62)$$

Both partition functions are strictly positive for $\beta > 0$.

Define

$$c_b(\beta) := \frac{Z_{b,+}^{\text{sf}}(\beta) - Z_{b,-}^{\text{sf}}(\beta)}{Z_{b,+}^{\text{sf}}(\beta) + Z_{b,-}^{\text{sf}}(\beta)}, \quad \mathfrak{r}_b(\beta) := \text{artanh } c_b(\beta). \quad (5.63)$$

Then $c_b(\beta)$ is precisely the normalized Gaussian Möbius overlap in (5.59)–(5.60) and satisfies

$$\mathfrak{r}_b(\beta) = \frac{1}{2} \log \frac{Z_{b,+}^{\text{sf}}(\beta)}{Z_{b,-}^{\text{sf}}(\beta)}, \quad (5.64)$$

$$\mathfrak{r}'_b(\beta) = \frac{1}{2} \left(\mathcal{E}_{b,-}^{\text{sf}}(\beta) - \mathcal{E}_{b,+}^{\text{sf}}(\beta) \right), \quad (5.65)$$

$$c'_b(\beta) = \frac{1 - c_b(\beta)^2}{2} \left(\mathcal{E}_{b,-}^{\text{sf}}(\beta) - \mathcal{E}_{b,+}^{\text{sf}}(\beta) \right). \quad (5.66)$$

In particular the permanent endpoint ratio $\beta_{\text{IR}}/\beta_{\text{UV}} = \pi$ gives the exact residual-energy law

$$\mathfrak{r}_b(\pi\beta) - \mathfrak{r}_b(\beta) = \frac{1}{2} \int_{\beta}^{\pi\beta} \left(\mathcal{E}_{b,-}^{\text{sf}}(t) - \mathcal{E}_{b,+}^{\text{sf}}(t) \right) dt. \quad (5.67)$$

Let $S_{b,\pm}^{\text{sf}}(\beta)$ be the Shannon entropy of the normalized Gibbs weights in the two sign sectors. Then

$$2\mathfrak{r}_b(\beta) = S_{b,+}^{\text{sf}}(\beta) - S_{b,-}^{\text{sf}}(\beta) - \beta \left(\mathcal{E}_{b,+}^{\text{sf}}(\beta) - \mathcal{E}_{b,-}^{\text{sf}}(\beta) \right). \quad (5.68)$$

Proof. The exterior occupation vectors are orthonormal. On the vector indexed by u the sign given by Proposition 5.6(ii) is $(-1)^{\Omega_b(u)} = \mu_b(u)$ because the summation is restricted to square-free modes. Taking the indicated inner products proves (5.56) and (5.59). Omitting the central sign gives the two norm identities. Gaussian summability proves convergence of the smoothed vector.

Equations (5.59) and (5.60) are respectively the difference and sum of the two partition functions. This proves the formula for c_b . The identity for $\mathfrak{r}_b$ follows from $\text{artanh}((a-b)/(a+b)) = \frac{1}{2} \log(a/b)$. Gaussian decay makes all required moments locally summable in each sign sector. Applying (6.34) to the two partition functions proves (5.65). The chain rule $c'_b = (1 - c_b^2)\mathfrak{r}'_b$ proves (5.66). Integration over $[\beta, \pi\beta]$ gives the endpoint identity. Subtracting the two entropy formulas in (6.35) gives (5.68). $\square$

5.8 Prime-direction dilation transfer

Exterior occupation also gives an exact dilation pairing in every prime direction. Fix $\pi = u_{b,p} \in \mathcal{P}_b$ and let $\mathcal{A}_{b,p}^0$ be the closed exterior-Fock span of square-free modes whose π -occupation is zero. Define the prime-creation isometry

$$W_{b,p} : \mathcal{A}_{b,p}^0 \longrightarrow \mathcal{A}_{b,p}^1, \quad W_{b,p} \Phi_{b,u}^a := \Phi_{b,u \boxtimes_b \pi}^a. \quad (5.69)$$

Here $\mathcal{A}_{b,p}^1$ is the corresponding occupied sector.

Theorem 5.8 (Signed prime-direction transfer and unsigned capacity bound). *Let $D_b \Phi_{b,u}^a := \deg_b(u) \Phi_{b,u}^a$. On $\mathcal{A}_{b,p}^0$ one has*

$$\Gamma_{\text{cen}} W_{b,p} = -W_{b,p} \Gamma_{\text{cen}}, \quad D_b^2 W_{b,p} = p^2 W_{b,p} D_b^2. \quad (5.70)$$

Define the unoccupied-sector vector and its signed and unsigned Gaussian responses by

$$\Psi_{b,\beta}^{(p,0)} := \sum_{\substack{u \in \mathcal{U}_b \setminus \{0_b\} \\ \mathbf{q}_b(u)=1, \nu_u(\pi)=0}} e^{-\beta \deg_b(u)^2/2} \Phi_{b,u}^a, \quad (5.71)$$

$$A_{b,p}^0(\beta) := \left\langle \Psi_{b,\beta}^{(p,0)}, \Gamma_{\text{cen}} \Psi_{b,\beta}^{(p,0)} \right\rangle, \quad Z_{b,p}^0(\beta) := \|\Psi_{b,\beta}^{(p,0)}\|^2. \quad (5.72)$$

Then

$$\Psi_{b,\beta}^{\text{sf}} = \Psi_{b,\beta}^{(p,0)} + W_{b,p} \Psi_{b,p^2\beta}^{(p,0)}, \quad (5.73)$$

$$A_b^{\text{sf}}(\beta) := \left\langle \Psi_{b,\beta}^{\text{sf}}, \Gamma_{\text{cen}} \Psi_{b,\beta}^{\text{sf}} \right\rangle = A_{b,p}^0(\beta) - A_{b,p}^0(p^2\beta), \quad (5.74)$$

$$Z_b^{\text{sf}}(\beta) := \|\Psi_{b,\beta}^{\text{sf}}\|^2 = Z_{b,p}^0(\beta) + Z_{b,p}^0(p^2\beta). \quad (5.75)$$

The normalized central overlap obeys the strict bound

$$|c_b(\beta)| \leq \frac{Z_{b,p}^0(\beta) - Z_{b,p}^0(p^2\beta)}{Z_{b,p}^0(\beta) + Z_{b,p}^0(p^2\beta)} < 1. \quad (5.76)$$

Proof. Prime creation changes the exterior degree by one. This proves the first identity in (5.70). Evaluation multiplies the natural degree by p and proves the second. Every square-free occupation vector lies in exactly one of the orthogonal sectors $\mathcal{A}_{b,p}^0$ and $\mathcal{A}_{b,p}^1$. The occupied vector has the unique form $W_{b,p} \Phi_{b,u}^a$ with $\nu_u(\pi) = 0$. The energy covariance therefore proves (5.73). Orthogonality and the central anticommutation give the signed difference and unsigned sum.

For the signed difference one has

$$A_{b,p}^0(\beta) - A_{b,p}^0(p^2\beta) = \sum_{\substack{\mathbf{q}_b(u)=1 \\ \nu_u(\pi)=0}} \mu_b(u) \left(e^{-\beta \deg_b(u)^2} - e^{-p^2\beta \deg_b(u)^2} \right).$$

The factor in parentheses is strictly positive. The triangle inequality therefore bounds the absolute value by $Z_{b,p}^0(\beta) - Z_{b,p}^0(p^2\beta)$. The denominator is (5.75). Strictness follows because both positive partition values are nonzero. $\square$

The recursions retain signed and unsigned responses separately. The capacity bound follows only after the triangle inequality. For several prime directions, dilation and sign exchange combine into a positive transfer on the pair of central sectors. To express its normalized one-prime response, define

$$c_{b,p}^0(\beta) := \frac{A_{b,p}^0(\beta)}{Z_{b,p}^0(\beta)}, \quad r_{b,p}(\beta) := \frac{Z_{b,p}^0(p^2\beta)}{Z_{b,p}^0(\beta)}. \quad (5.77)$$

Theorem 5.9 (Finite-prime transfer and normalized central response). (i) Let $S \subset \mathcal{P}_b$ be finite. Write $p_E := \prod_{u_{b,p} \in E} p$ for $E \subset S$ with $p_\emptyset = 1$ and let $Z_{b,S,\pm}^0(\beta)$ denote the Gaussian partition functions of the square-free modes which occupy no direction in S and have central sign ± 1 . On scalar functions of β put

$$(T_a F)(\beta) := F(a\beta), \quad \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (5.78)$$

Then the full central-sector vector is obtained from the unoccupied vector by the commuting positivity-preserving transfers

$$\begin{pmatrix} Z_{b,+}^{\text{sf}} \\ Z_{b,-}^{\text{sf}} \end{pmatrix} = \prod_{u_{b,p} \in S} (I + \sigma_x T_{p^2}) \begin{pmatrix} Z_{b,S,+}^0 \\ Z_{b,S,-}^0 \end{pmatrix}. \quad (5.79)$$

Equivalently the basis of the total and signed responses gives

$$Z_b^{\text{sf}}(\beta) = \sum_{E \subset S} Z_{b,S}^0(p_E^2 \beta) = \left[\prod_{u_{b,p} \in S} (I + T_{p^2}) \right] Z_{b,S}^0(\beta), \quad (5.80)$$

$$A_b^{\text{sf}}(\beta) = \sum_{E \subset S} (-1)^{|E|} A_{b,S}^0(p_E^2 \beta) = \left[\prod_{u_{b,p} \in S} (I - T_{p^2}) \right] A_{b,S}^0(\beta). \quad (5.81)$$

The unsigned capacity and the central residual are respectively the even and odd eigenchannels of one positive occupation transfer.

(ii) For every prime direction $u_{b,p}$ and every $\beta > 0$ one has

$$c_b(\beta) = \frac{c_{b,p}^0(\beta) - r_{b,p}(\beta) c_{b,p}^0(p^2 \beta)}{1 + r_{b,p}(\beta)} \quad (5.82)$$

$$= \frac{1 - r_{b,p}(\beta)}{1 + r_{b,p}(\beta)} c_{b,p}^0(\beta) - \frac{r_{b,p}(\beta)}{1 + r_{b,p}(\beta)} (c_{b,p}^0(p^2 \beta) - c_{b,p}^0(\beta)). \quad (5.83)$$

If $\mathcal{E}_{b,p,\pm}^0$ are the two conditional mean energies in the p -unoccupied sector then

$$c_{b,p}^0(p^2 \beta) - c_{b,p}^0(\beta) = \frac{1}{2} \int_{\beta}^{p^2 \beta} (1 - c_{b,p}^0(t)^2) (\mathcal{E}_{b,p,-}^0(t) - \mathcal{E}_{b,p,+}^0(t)) dt. \quad (5.84)$$

The unsigned capacity satisfies

$$Z_{b,p}^0(\beta) = \frac{p}{p+1} \frac{1}{\zeta(2)} \frac{\sqrt{\pi}}{2\sqrt{\beta}} + O_p(\beta^{-1/4}), \quad \beta \downarrow 0, \quad (5.85)$$

and consequently

$$r_{b,p}(\beta) \longrightarrow \frac{1}{p}, \quad \frac{1 - r_{b,p}(\beta)}{1 + r_{b,p}(\beta)} \longrightarrow \frac{p-1}{p+1}. \quad (5.86)$$

The scale-transport term accompanying the explicit one-prime coefficient is the integrated central-sector energy imbalance in (5.84).

Proof. Every square-free occupation pattern has a unique decomposition into a pattern avoiding S and a subset $E \subset S$. Occupying E multiplies the natural degree by p_E . It therefore replaces β by $p_E^2\beta$ and changes the central sign by $(-1)^{|E|}$. Adding one prime direction therefore leaves the sign sector unchanged for the unoccupied contribution and exchanges the two sectors for the occupied contribution. This is the positive operator $I + \sigma_x T_{p^2}$. Distinct occupation directions and their dilations commute. Expanding their product proves (5.79). The vectors $(1, 1)$ and $(1, -1)$ diagonalize σ_x with eigenvalues 1 and -1 . This gives (5.80) and (5.81).

Divide the signed recursion (5.74) by the unsigned recursion (5.75). This gives (5.82). Elementary rearrangement gives (5.83). Applying Theorem 5.7 inside the closed p -unoccupied sector proves (5.84).

Let $Q_{b,p}(X)$ count internally constructed square-free modes of degree at most X whose $u_{b,p}$ -occupation is zero. The exterior-Fock Möbius selector gives the square-divisor identity

$$Q_{b,p}(X) = \sum_{\substack{d \leq \sqrt{X} \\ p \nmid d}} \mu_b(u_{b,d}) \left(\left\lfloor \frac{X}{d^2} \right\rfloor - \left\lfloor \frac{X}{pd^2} \right\rfloor \right) = \frac{p}{p+1} \frac{X}{\zeta(2)} + O_p(\sqrt{X}).$$

Stieltjes summation against $e^{-\beta x^2}$ yields (5.85). Replacing β by $p^2\beta$ and taking the ratio proves (5.86). □

This occupation grading acts on a different space from the orientation grading of the two circle channels. Additive index parity is different again because it acts on the natural-mode degree. Their values lie in the same two-element group. The underlying actions and the data they distinguish are specified separately.

6 Endpoint states and joint response

Normalizing the Gaussian root masses gives trace-one endpoint states. Their spectral probabilities admit exact escort and entropy identities. On additive input fibres, the Gaussian energy and decomposition response also define a joint positive trace. These two parameters remain independent until diagonal self-consistency equates the decomposition attenuation with the returned endpoint ratio.

6.1 Normalized correlation states

Adjoin the singleton zero expression with intensity one. Normalize the diagonal intensities on the counting history space by $1 + \mathcal{Z}_b^+(\beta)$. The probability of a root mode is the sum of its diagonal history probabilities. It is therefore $G_{b,\beta}(u)/(1 + \mathcal{Z}_b^+(\beta))$ for a positive root and $1/(1 + \mathcal{Z}_b^+(\beta))$ at zero. This is probability pushforward under val_b . The source-compatible Hilbert map of Section 2 instead acts on amplitude vectors. The two constructions recover the Gaussian weights in their respective probability and Hilbert formulations. The finite phase extension remains confined to positive roots.

For $t > 0$ define the heat trace and normalized correlation operator by

$$Z_b(t) := \text{Tr } e^{-tA_b/2}, \quad \rho_b(t) := \frac{e^{-tA_b/2}}{Z_b(t)}. \quad (6.1)$$

The spectral decomposition in Section 2.1 gives

$$Z_b(t) = \sum_{n=0}^{\infty} e^{-tE_{b,n}/2}, \quad \rho_b(t) = \sum_{n=0}^{\infty} \varpi_{b,n}(t) |\phi_{b,n}\rangle \langle \phi_{b,n}|, \quad (6.2)$$

where

$$\varpi_{b,n}(t) := \frac{e^{-tE_{b,n}/2}}{Z_b(t)}. \quad (6.3)$$

For every $t > 0$ the operator $\rho_b(t)$ is positive and trace class with trace one. The rank-one projections

$$P_{b,n} := |\phi_{b,n}\rangle \langle \phi_{b,n}|, \quad n \geq 0, \quad (6.4)$$

form its spectral projective measurement and satisfy

$$\varpi_{b,n}(t) > 0, \quad \sum_{n=0}^{\infty} \varpi_{b,n}(t) = 1, \quad \text{Pr}_t(n) = \text{Tr}(\rho_b(t)P_{b,n}) = \varpi_{b,n}(t). \quad (6.5)$$

Positivity and trace class follow from functional calculus and [1, Proposition 3.5]. Division by the finite positive trace gives (6.5). Thus the spectral projective measurement of $\rho_b(t)$ has weights $\varpi_{b,n}(t)$.

The zero mode and the positive natural-mode sector may be sampled separately. Put

$$Z_b^+(t) := Z_b(t) - 1, \quad \varpi_{b,n}^+(t) := \frac{e^{-tE_{b,n}/2}}{Z_b^+(t)}, \quad n \geq 1. \quad (6.6)$$

This is the conditional distribution given a positive-energy outcome. Conditioning further on the spectral-prime sector gives

$$Z_b^{\mathcal{P}}(t) := \sum_{u_{b,p} \in \mathcal{P}_b} e^{-tE_{b,p}/2}, \quad \varpi_{b,p}^{\mathcal{P}}(t) := \frac{e^{-tE_{b,p}/2}}{Z_b^{\mathcal{P}}(t)}. \quad (6.7)$$

Both sums converge for every $t > 0$. The first law ranges over every positive natural mode. The second ranges only over the modes selected by composition irreducibility.

Conditioning on the prime subspace at the ultraviolet endpoint gives the normalized law

$$\mathbb{P}_b^{\mathcal{P}}(u_{b,p} \mid \mathcal{P}_b) := \varpi_{b,p}^{\mathcal{P}}(b^2) = \frac{\Lambda_{b,p}}{\sum_{u_{b,q} \in \mathcal{P}_b} \Lambda_{b,q}}. \quad (6.8)$$

This is the probability of the prime-mode outcome $u_{b,p}$ conditional on observing a prime mode. Uniform counting on the finite set of primes below a degree cutoff gives another sampling rule. The two laws use the same prime modes with different weights and normalizations.

For $x \geq 2$ uniform sampling in the prime sector below degree x has capacity entropy $\log \pi_b(x)$. Here $\pi_b(x)$ counts the generated prime modes up to degree x . The full positive-degree sector has capacity entropy $\log[x]$. A window in logarithmic degree describes the same counting cutoff after the change of coordinate $x = e^r$. The quadratic Neumann energy instead determines the Gaussian sampling law in (6.7).

6.2 Endpoint escorting and observable transport

Let

$$\rho_b^{\text{UV}} := \rho_b(b^2), \quad \rho_b^{\text{IR}} := \rho_b(\pi b^2). \quad (6.9)$$

For $\eta > 0$ define the normalized power path by

$$\rho_{b,\eta} := \frac{(\rho_b^{\text{UV}})^\eta}{\text{Tr}(\rho_b^{\text{UV}})^\eta}, \quad \langle O \rangle_\eta := \text{Tr}(O \rho_{b,\eta}). \quad (6.10)$$

The path joins the two endpoint states at $\eta = 1$ and $\eta = \pi$.

Theorem 6.1 (Endpoint escorting and response). *The two finite correlation endpoints satisfy*

$$\rho_b^{\text{IR}} = \frac{(\rho_b^{\text{UV}})^\pi}{\text{Tr}(\rho_b^{\text{UV}})^\pi}. \quad (6.11)$$

The equivalent probability identities are

$$\varpi_{b,n}^{\text{IR}} = \frac{(\varpi_{b,n}^{\text{UV}})^\pi}{\sum_j (\varpi_{b,j}^{\text{UV}})^\pi}, \quad \frac{\varpi_{b,n}^{\text{IR}}}{\varpi_{b,m}^{\text{IR}}} = \left(\frac{\varpi_{b,n}^{\text{UV}}}{\varpi_{b,m}^{\text{UV}}} \right)^\pi. \quad (6.12)$$

The same formulas hold after conditioning either on the positive-energy sector or on the spectral-prime sector.

Every bounded self-adjoint spectral observable $O = f(A_b)$ satisfies

$$\langle O \rangle_{\text{IR}} = \frac{\text{Tr}(O(\rho_b^{\text{UV}})^\pi)}{\text{Tr}(\rho_b^{\text{UV}})^\pi}. \quad (6.13)$$

Differentiation along the path (6.10) gives

$$\frac{d}{d\eta} \langle O \rangle_\eta = \text{Cov}_\eta(O, \log \rho_b^{\text{UV}}), \quad (6.14)$$

and hence

$$|\langle O \rangle_{\text{IR}} - \langle O \rangle_{\text{UV}}| \leq \int_1^\pi \sqrt{\text{Var}_\eta(O) \text{Var}_\eta(\log \rho_b^{\text{UV}})} d\eta. \quad (6.15)$$

All moments in these formulas are finite.

Proof. Functional calculus gives $(\rho_b^{\text{UV}})^\eta = e^{-\eta b^2 A_b/2} / Z_b(b^2)^\eta$. Normalization at $\eta = \pi$ proves the operator and probability identities. Restriction to either specified spectral sector gives the conditional laws. Evaluation against O yields (6.13). Differentiating its normalized trace along the path gives (6.14). Gaussian decay controls every energy moment locally uniformly for $\eta > 0$ and justifies differentiation. Cauchy–Schwarz and integration from 1 to π give (6.15). $\square$

For a probability vector p satisfying $\sum_j p_j^\eta < \infty$ the map

$$\mathcal{E}_\eta(p)_j := \frac{p_j^\eta}{\sum_m p_m^\eta}, \quad \eta > 0, \quad (6.16)$$

is the escort transformation of generalized statistical mechanics [13]. Root escorting followed by the history lift leaves the conditional probabilities $p_b(e)$ unchanged. On the positive sector this follows from

$$P_{b,\eta\beta}(e) = \mathcal{E}_\eta(q_{b,\beta})(u)p_b(e), \quad u = \text{val}_b(e).$$

Raising the full history probability to the power η would instead replace the conditional law on each fibre by the normalization of $p_b(e)^\eta$. The root deformation and the history lift are therefore performed in the stated order. Their entropy is given by the conditional decomposition in [1, equation (8.21)]. The root law satisfies

$$\varpi_b^{\text{IR}} = \mathcal{E}_\pi(\varpi_b^{\text{UV}}), \quad (6.17)$$

so its exponent is the correlation-time ratio $t_{\text{IR}}/t_{\text{UV}} = \pi$. The covariance bound describes how a commuting observable changes along this endpoint path.

6.3 Scale-independent mode law and prime conditioning

Set

$$q_* := \sqrt{2\pi} - \sqrt{2}, \quad c_G := \frac{\pi^2}{2q_*^2} = 4.135182382\dots \quad (6.18)$$

The identity $\Delta_b = bq_*$ gives the mode law at a dimensionless scale $t = \theta b^2$.

$$\frac{tE_{b,n}}{2} = \theta c_G n^2, \quad \varpi_{b,n}(\theta b^2) = \frac{e^{-\theta c_G n^2}}{\sum_{j \geq 0} e^{-\theta c_G j^2}}. \quad (6.19)$$

The distribution depends on b only through $\theta = t/b^2$. The ultraviolet positive-mode ratios are therefore

$$\frac{\varpi_{b,n}^+(b^2)}{\varpi_{b,1}^+(b^2)} = e^{-c_G(n^2-1)}. \quad (6.20)$$

The first values are

$$\frac{\varpi_{b,2}^+}{\varpi_{b,1}^+} \approx 4.096 \times 10^{-6}, \quad \frac{\varpi_{b,3}^+}{\varpi_{b,1}^+} \approx 4.294 \times 10^{-15}, \quad \frac{\varpi_{b,4}^+}{\varpi_{b,1}^+} \approx 1.153 \times 10^{-27}. \quad (6.21)$$

Every component retains positive weight. The Gaussian factor orders the relative likelihoods along the complete spectrum.

Spectral primes satisfy $E_{b,p} = p^2 E_{b,1}$ and hence

$$\frac{\varpi_{b,p}^{\mathcal{P}}(\theta b^2)}{\varpi_{b,q}^{\mathcal{P}}(\theta b^2)} = e^{-\theta c_G(p^2-q^2)} \quad (u_{b,p}, u_{b,q} \in \mathcal{P}_b). \quad (6.22)$$

Relative probabilities within the prime sector are therefore determined by the same quadratic energy law as on the full mode family.

The escort deformation orders the full mode family by correlation energy and hence by degree magnitude. The comparison of cascade products with their factors is expressed for positive integers $m, n \geq 2$ by

$$D_{b,\theta}(m, n) := \frac{\varpi_{b,mn}(\theta b^2)}{\varpi_{b,m}(\theta b^2)\varpi_{b,n}(\theta b^2)}. \quad (6.23)$$

Then

$$D_{b,\theta}(m, n) = Z_b(\theta b^2) e^{-\theta c_G(m^2 n^2 - m^2 - n^2)}, \quad (6.24)$$

$$\frac{D_{b,\pi}(m, n)}{D_{b,1}(m, n)} = \frac{Z_b(\pi b^2)}{Z_b(b^2)} e^{-(\pi-1)c_G(m^2 n^2 - m^2 - n^2)}. \quad (6.25)$$

These identities compare the Gaussian weight of a product degree with the weights of its factors. Composition irreducibility supplies the prime selector used in (6.7). The energy law then supplies the relative weights within that selected family.

6.4 Spectral entropy and information recovery

The generated operator and its scale determine a normalized spectral state. Let A be a nonnegative operator with compact resolvent satisfying $\text{Tr } e^{-\gamma A} < \infty$ for every $\gamma > 0$. Put

$$Z_A(\gamma) = \text{Tr } e^{-\gamma A}, \quad \rho_{A,\gamma} = Z_A(\gamma)^{-1} e^{-\gamma A}, \quad S_A(\gamma) = -\text{Tr } \rho_{A,\gamma} \log \rho_{A,\gamma}. \quad (6.26)$$

Here γA is dimensionless. The dimensional Neumann Laplacian uses $\gamma = t/2$ and its normalized square spectrum uses $\beta = \gamma E_{b,1}$.

Let $\langle E \rangle_t := \sum_n \varpi_{b,n}(t) E_{b,n}$ and define the spectral entropy

$$S_{\text{sp}}(t) := - \sum_{n=0}^{\infty} \varpi_{b,n}(t) \log \varpi_{b,n}(t). \quad (6.27)$$

Differentiating the Gaussian partition function at $t > 0$ gives

$$\frac{d}{dt} \log Z_b(t) = -\frac{1}{2} \langle E \rangle_t, \quad (6.28)$$

$$\frac{d^2}{dt^2} \log Z_b(t) = \frac{1}{4} \text{Var}_t(E), \quad (6.29)$$

$$\frac{d}{dt} \langle E \rangle_t = -\frac{1}{2} \text{Var}_t(E) \leq 0, \quad (6.30)$$

$$S_{\text{sp}}(t) = \log Z_b(t) + \frac{t}{2} \langle E \rangle_t, \quad (6.31)$$

$$\frac{d}{dt} S_{\text{sp}}(t) = -\frac{t}{4} \text{Var}_t(E) \leq 0. \quad (6.32)$$

The Fisher information of the family $\varpi_b(t)$ with respect to t is

$$\mathcal{I}_t := \sum_n \varpi_{b,n}(t) (\partial_t \log \varpi_{b,n}(t))^2 = \frac{1}{4} \text{Var}_t(E), \quad (6.33)$$

and hence $-S'_{\text{sp}}(t) = t \mathcal{I}_t$.

The Gaussian spectrum permits termwise differentiation for every $t > 0$. The logarithmic derivative of Z_b gives (6.28). Differentiating the Gibbs average gives (6.30) and hence (6.29). Substitution of $\log \varpi_{b,n} = -t E_{b,n}/2 - \log Z_b$ gives the entropy formula and its derivative. The identity $\partial_t \log \varpi_{b,n} = -(E_{b,n} - \langle E \rangle_t)/2$ proves (6.33). The identity $\text{Tr } \rho_b(t) = 1$ expresses conservation of total probability. Equation (6.30) gives the accompanying monotone evolution of mean spectral energy.

Under the hypotheses of (6.26) let λ_n be the distinct eigenvalues with multiplicities m_n and $p_n = m_n e^{-\gamma \lambda_n} / Z_A$. Applying the same calculation with γ as the parameter gives

$$\partial_\gamma \log Z_A = -\langle A \rangle_\gamma, \quad \partial_\gamma \langle A \rangle_\gamma = -\text{Var}_{\rho_{A,\gamma}}(A). \quad (6.34)$$

Including the multiplicities in the entropy gives

$$S_A = H(p) + \sum_n p_n \log m_n = \log Z_A + \gamma \text{Tr}(A \rho_{A,\gamma}), \quad S'_A(\gamma) = -\gamma \text{Var}_{\rho_{A,\gamma}}(A). \quad (6.35)$$

The entropy tends to $\log m_0$ as $\gamma \rightarrow \infty$. As $\gamma \downarrow 0$ it tends to $\log D$ on a D -dimensional carrier and to $+\infty$ on an infinite-dimensional carrier. Adding a scalar multiple of the identity to A leaves the state and its entropy unchanged.

Each eigenvalue of the density matrix in shell n is p_n/m_n . Summing its entropy gives the first identity. Trace class at $\gamma/2$ controls every polynomial spectral moment at γ and justifies differentiation and the variance identity. Shift the ground energy to zero. For fixed $\gamma_0 > 0$ dominated convergence with the heat trace at γ_0 gives $Z_A(\gamma) \rightarrow m_0$ and $\gamma \text{Tr}(A e^{-\gamma A}) \rightarrow 0$ as $\gamma \rightarrow \infty$. In finite dimension the state tends to I/D at the other endpoint. In infinite dimension $Z_A(\gamma) \rightarrow \infty$ and $S_A \geq \log Z_A$ for the shifted nonnegative operator. Finally the scalar factor introduced by an energy shift cancels in the normalization.

Uniform weights within a shell are fixed by invariance under all unitary changes of basis in that shell. They also maximize entropy for prescribed shell probabilities with finite lifted entropy. A conditional state σ_n in shell n satisfies $S(\sigma_n) \leq \log m_n$ with equality only for I/m_n . Thus multiplicity contributes to state entropy through its probability weight. The separate quantity $S_{\text{cap}}(K) = \log \text{rank } \mathbf{1}_{[0,K^2]}(A)$ is the maximum entropy on the retained carrier whenever that carrier is nonempty. It measures the carrier's capacity.

The spectral entropy and the spatial entropy in the de Bruijn identity belong to different probability laws. Gaussian convolution spreads a spatial density and increases its entropy. Normalized spectral weights concentrate toward the zero mode as high energies are suppressed. Their Shannon entropy therefore decreases.

Combining [1, Theorem 3.6] with the Gibbs identities (6.28)–(6.33) gives the two entropy derivatives

$$\frac{d}{dt} H_{\text{pos},b}(t) = \frac{1}{2} \mathcal{I}_{\text{pos},b}(t) \geq 0, \quad \frac{d}{dt} S_{\text{sp}}(t) = -\frac{t}{4} \text{Var}_t(E) \leq 0. \quad (6.36)$$

Both evolutions use the same heat semigroup. It acts on spatial densities in the first identity and defines normalized mode weights in the second. The spatial Fisher information and the spectral energy variance give the respective rates.

Endpoint probabilities also recover the generator gaps. For $0 < t_1 < t_2$ the recovery formula is

$$E_{b,n} - E_{b,m} = -\frac{2}{t_2 - t_1} \log \left[\frac{\varpi_{b,n}(t_2)/\varpi_{b,m}(t_2)}{\varpi_{b,n}(t_1)/\varpi_{b,m}(t_1)} \right]. \quad (6.37)$$

Two normalized correlation states therefore determine all spectral energy gaps.

6.5 Theta representation and controlled tails

With

$$\alpha_b(t) := \frac{\pi^2 t}{2\Delta_b^2}, \quad (6.38)$$

the heat trace is

$$Z_b(t) = \sum_{n=0}^{\infty} e^{-\alpha_b(t)n^2} = \frac{1 + \vartheta_3(0, e^{-\alpha_b(t)})}{2}. \quad (6.39)$$

Jacobi's identity [14]

$$\vartheta_3(0, e^{-\alpha}) = \sqrt{\frac{\pi}{\alpha}} \vartheta_3(0, e^{-\pi^2/\alpha}) \quad (6.40)$$

relates the short- and long-scale heat traces. For every $N \geq 1$ one has

$$\sum_{n>N} e^{-\alpha n^2} \leq \int_N^{\infty} e^{-\alpha x^2} dx \leq \frac{e^{-\alpha N^2}}{2\alpha N}. \quad (6.41)$$

This bounds the truncation error of the complete heat trace at each N .

6.6 Equivalence of Gaussian scales

For $b, b' > 0$ define

$$(U_{b,b'} f)(d') := \sqrt{\frac{b}{b'}} f\left(\frac{b}{b'} d'\right), \quad d' \in \mathcal{W}_{b'}. \quad (6.42)$$

Theorem 6.2 (Scale equivalence). *The map $U_{b,b'} : \mathcal{H}_b \rightarrow \mathcal{H}_{b'}$ is unitary and satisfies*

$$A_{b'} U_{b,b'} = \left(\frac{b}{b'}\right)^2 U_{b,b'} A_b, \quad U_{b,b'} \phi_{b,n} = \phi_{b',n}. \quad (6.43)$$

For every $\theta > 0$ one has

$$\rho_{b'}(\theta b'^2) U_{b,b'} = U_{b,b'} \rho_b(\theta b^2). \quad (6.44)$$

The analytic lifts identify $u_{b,n}$ with $u_{b',n}$ and induce a Peano-semiring isomorphism $\mathcal{U}_b \cong \mathcal{U}_{b'}$. Their restrictions to the composition-irreducible modes also identify the corresponding prime subspaces and their two Fock functors.

Proof. A change of variables proves unitarity. Differentiation gives the generator relation. Functional calculus then intertwines the heat operators at equal dimensionless scale. Equality of their traces proves (6.44). The explicit cosine formulas give the component identity. The induced map on analytic modes sends $u_{b,n}$ to $u_{b',n}$ and preserves degree together with modulation and cascade. It therefore preserves irreducibility and extends to the symmetric and antisymmetric Fock spaces. $\square$

6.7 Normalized decomposition flow and endpoint compatibility

On the degree-one space put

$$R_b := \Delta_b^{\text{bar}}|_{\mathcal{K}_{b,1}}, \quad Z_b^{\text{dec}}(\beta, \tau) := \text{Tr}_{\mathcal{K}_{b,1}} e^{-\beta H_b - \tau R_b}, \quad (6.45)$$

and define the normalized state for $\beta > 0$ and $\tau \geq 0$ by

$$\rho_b^{\text{dec}}(\beta, \tau) := \frac{e^{-\beta H_b - \tau R_b}}{Z_b^{\text{dec}}(\beta, \tau)}. \quad (6.46)$$

The two generators strongly commute by (3.78).

Theorem 6.3 (Normalized localization and joint response). *The normalized state satisfies the following properties for every $\beta > 0$ and $\tau \geq 0$.*

(i) *The mean decomposition response decreases monotonically.*

$$\partial_\tau \log Z_b^{\text{dec}}(\beta, \tau) = -\langle R_b \rangle_{\beta, \tau}, \quad (6.47)$$

$$\partial_\tau \langle R_b \rangle_{\beta, \tau} = -\text{Var}_{\beta, \tau}(R_b) \leq 0. \quad (6.48)$$

(ii) *Let*

$$\rho_{b, \mathcal{P}}(\beta) := \frac{P_b^{\text{pr}} e^{-\beta H_b}}{\text{Tr}(P_b^{\text{pr}} e^{-\beta H_b})}. \quad (6.49)$$

Then $\rho_b^{\text{dec}}(\beta, \tau)$ converges in trace norm to $\rho_{b, \mathcal{P}}(\beta)$ as $\tau \rightarrow \infty$. With

$$Z_{\mathcal{P}}(\beta) := \sum_{\pi \in \mathcal{P}_b} e^{-\beta \deg_b(\pi)^2}, \quad Z_C(\beta) := \sum_{\substack{n \geq 2 \\ n \text{ composite}}} e^{-\beta n^2}, \quad (6.50)$$

the quantitative bound is

$$\left\| \rho_b^{\text{dec}}(\beta, \tau) - \rho_{b, \mathcal{P}}(\beta) \right\|_1 \leq 2e^{-\tau} \frac{Z_C(\beta)}{Z_{\mathcal{P}}(\beta)}. \quad (6.51)$$

(iii) *The joint response of the same state satisfies*

$$\nabla_{(\beta, \tau)}^2 \log Z_b^{\text{dec}}(\beta, \tau) = \begin{pmatrix} \text{Var}_{\beta, \tau}(H_b) & \text{Cov}_{\beta, \tau}(H_b, R_b) \\ \text{Cov}_{\beta, \tau}(H_b, R_b) & \text{Var}_{\beta, \tau}(R_b) \end{pmatrix} \succeq 0. \quad (6.52)$$

For $a > 0$ and $c \geq 0$ put $G_{a, c} := aH_b + cR_b$ and

$$\rho_{a, c}(s) := \frac{e^{-sG_{a, c}}}{\text{Tr} e^{-sG_{a, c}}}, \quad s > 0. \quad (6.53)$$

Then

$$\frac{d}{ds} \langle G_{a, c} \rangle_s = -\text{Var}_s(G_{a, c}) \leq 0, \quad (6.54)$$

$$\frac{d}{ds} S(\rho_{a, c}(s)) = -s \text{Var}_s(G_{a, c}) \leq 0, \quad (6.55)$$

where $S(\rho) := -\text{Tr}(\rho \log \rho)$.

Let $m_C(\beta, \tau)$ be the total probability of the composite modes in $\rho_b^{\text{dec}}(\beta, \tau)$. Its odds satisfy the decay bound

$$\frac{m_C(\beta, \tau)}{1 - m_C(\beta, \tau)} \leq e^{-\tau} \frac{m_C(\beta, 0)}{1 - m_C(\beta, 0)}. \quad (6.56)$$

Correlation energy and decomposition response form a joint directed flow. The positive integer decomposition gap removes composite mass at least exponentially in τ .

Proof. The operators H_b and R_b strongly commute and are diagonal in the natural-mode basis. Thus $Z_b^{\text{dec}} = \sum_{n \geq 2} e^{-\beta n^2 - \tau(d(n)-2)}$. Every required moment is locally uniformly summable for $\beta > 0$ and $\tau \geq 0$ because $d(n) - 2 \leq n$ and Gaussian decay controls polynomial factors. Termwise differentiation gives (i). The first derivatives of $\log Z_b^{\text{dec}}$ are $-\langle H_b \rangle$ and $-\langle R_b \rangle$. Differentiating again gives the covariance matrix in (6.52). Its quadratic form at (x, y) is $\text{Var}(xH_b + yR_b)$ and is therefore nonnegative.

Along $(\beta, \tau) = s(a, c)$ apply (6.34) and (6.35) to $G_{a,c}$. They give the energy and entropy derivatives in (iii).

Write

$$B_\tau(\beta) := \sum_{\substack{n \geq 2 \\ n \text{ composite}}} e^{-\beta n^2 - \tau(d(n)-2)}. \quad (6.57)$$

The positive integer gap $d(n) - 2 \geq 1$ gives $B_\tau(\beta) \leq e^{-\tau} B_0(\beta) = e^{-\tau} Z_C(\beta)$. Thus $Z_b^{\text{dec}} = Z_{\mathcal{P}} + B_\tau$. The prime block loses mass $B_\tau/(Z_{\mathcal{P}} + B_\tau)$ relative to the conditioned prime state and the composite block has exactly that mass. Orthogonality of the blocks makes their two contributions add to the trace-norm bound in (ii). Finally $m_C/(1 - m_C) = B_\tau/Z_{\mathcal{P}}$ proves (6.56) from the same gap estimate. $\square$

For $\eta > 0$ and a positive trace-one operator ρ satisfying $0 < \text{Tr} \rho^\eta < \infty$ define $\text{Esc}_\eta(\rho) := \rho^\eta / \text{Tr} \rho^\eta$. On spectral states commuting with P_b^{pr} define Hodge conditioning by

$$\text{Hdg}_b(\rho) := \frac{P_b^{\text{pr}} \rho P_b^{\text{pr}}}{\text{Tr}(P_b^{\text{pr}} \rho)}. \quad (6.58)$$

Functional calculus gives the following identity whenever the denominator is nonzero.

$$\text{Hdg}_b(\text{Esc}_\eta(\rho)) = \text{Esc}_\eta(\text{Hdg}_b(\rho)). \quad (6.59)$$

With c_G from (6.18) the diagram

$$\begin{array}{ccc} \rho_b^{\text{dec}}(c_G, 0) & \xrightarrow{\text{Esc}_\pi} & \rho_b^{\text{dec}}(\pi c_G, 0) \\ \text{Hdg}_b \downarrow & & \downarrow \text{Hdg}_b \\ \rho_{b,\mathcal{P}}(c_G) & \xrightarrow{\text{Esc}_\pi} & \rho_{b,\mathcal{P}}(\pi c_G) \end{array} \quad (6.60)$$

is a commutative square. One obtains the same prime-conditioned infrared state by conditioning before or after escort transport. Hodge-flow convergence gives these conditional states as trace-norm limits.

The scale equivalence of Theorem 6.2 induces the analytic basis unitary $V_{b,b'}u_{b,n} := u_{b',n}$. It lifts to the bar Hilbert spaces by

$$\overline{U}_{b,b'}[u_1|\cdots|u_r] := [V_{b,b'}u_1|\cdots|V_{b,b'}u_r]. \quad (6.61)$$

Preservation of cascade and degree implies

$$\overline{U}_{b,b'}d_b = d_{b'}\overline{U}_{b,b'}, \quad \overline{U}_{b,b'}\Delta_b^{\text{bar}} = \Delta_{b'}^{\text{bar}}\overline{U}_{b,b'}, \quad \overline{U}_{b,b'}\mathbf{H}_b = \mathbf{H}_{b'}\overline{U}_{b,b'}. \quad (6.62)$$

The Hodge projector and normalized decomposition flow are consequently invariant under changes of b . So are the spectral gap and convergence rate.

6.8 Joint additive and decomposition response

The joint trace weights each additive pair by its total Gaussian energy, its transverse response and the decomposition attenuation of both inputs. At fixed attenuation, the infrared-to-ultraviolet ratio measures the fraction retained between the permanent end-points. Setting the attenuation equal to this ratio gives a scalar fixed-point equation. The first contributing fibre bounds its denominator from below; Gaussian decay controls the higher fibres and the derivative of the ratio.

The fixed-sum input. Let $\mathcal{L}_b^+$ be the positive natural-mode Hilbert space. Multiplicative occupation vectors define a compact character action. Addition acts on two-input fibres

$$\mathcal{F}_{b,k} = \text{span}\{|m\rangle_b \otimes |k-m\rangle_b : 1 \leq m \leq k-1\}.$$

The normalized merge has a one-dimensional longitudinal range in every fibre. Its orthogonal complement retains the relative two-input data erased by scalar addition. The fibre dimension and the squared coefficient of each input pair in the normalized longitudinal vector satisfy

$$C_{+,b}(k) \xi_{+,b}(k) = (k-1) \frac{1}{k-1} = 1.$$

The multiplicative character and additive transverse quotient remain available together with the Gaussian UV/IR response before scalar compression.

Equation (4.16) obtains this input space from the complete additive-root histories. Occupation and decomposition operators act on the child values, while the transverse compression retains their relative directions.

6.9 Diagonal full-spectrum closure

The endpoint closure is defined directly on these carriers. Let

$$\widehat{r}_b(n) := \begin{cases} 0, & n = 1 \text{ or } n \text{ is prime,} \\ d(n) - 2, & n \text{ is composite,} \end{cases} \quad n \geq 1, \quad (6.63)$$

be the transported degree-one decomposition response. On the fixed-sum fibre indexed by $k \geq 3$ put

$$a_{k,m} := 2m(k-m) \left(1 - \frac{2}{k-1}\right) + \frac{k(k^2-1)}{3(k-1)^2}, \quad 1 \leq m \leq k-1. \quad (6.64)$$

These coefficients are the diagonal weights of the positive transverse compression on each fixed-sum fibre. Equation (4.64) gives the compression matrix and equation (B.62) computes its diagonal entries. Summing the fibres gives the complete response

$$\Pi_{+,b}(\beta, \tau) := \sum_{k=3}^{\infty} e^{-\beta k^2} \sum_{m=1}^{k-1} a_{k,m} e^{-\tau[\widehat{r}_b(m) + \widehat{r}_b(k-m)]}. \quad (6.65)$$

The two-leg source embedding $(Q_b^{\text{hist}})^* \otimes (Q_b^{\text{hist}})^*$ is an isometry. Conjugating a positive value-input operator by this embedding preserves its nonzero spectrum and trace. Thus the response in (6.65) is also its trace on the source-weighted two-history subspace. It retains the same weights $a_{k,m}$ without a history multiplicity factor.

Here τ controls decomposition attenuation and β controls the Gaussian observation scale. The permanent correlation endpoints give the dimensionless parameters

$$\beta_{\text{UV}} := \frac{\pi^2}{2(\sqrt{2\pi} - \sqrt{2})^2}, \quad \beta_{\text{IR}} := \pi\beta_{\text{UV}}. \quad (6.66)$$

Their numerical values are

$$\beta_{\text{UV}} = 4.1351823821\dots, \quad \beta_{\text{IR}} = 12.9910585929\dots$$

Their full-spectrum retention map is

$$\mathfrak{R}_b(\kappa) := \frac{\Pi_{+,b}(\beta_{\text{IR}}, \kappa)}{\Pi_{+,b}(\beta_{\text{UV}}, \kappa)}, \quad 0 \leq \kappa \leq 1. \quad (6.67)$$

Definition 6.4 (Diagonal self-consistency). For an input attenuation τ , evaluate the joint response at the two permanent endpoints and use their retention ratio as the attenuation in the next evaluation. The diagonal continuation law $\mathbf{C}_b^{\text{diag}}$ identifies a stationary input with this returned value. A state satisfying this condition has $\tau = \kappa$ and $\kappa = \mathfrak{R}_b(\kappa)$. The diagonal continuation law acts on the response constructed over the generated arithmetic carrier and uses the spectral coefficients of that response.

Theorem 6.5 (Full-spectrum endpoint closure under diagonal self-consistency). *The series (6.65) is finite and strictly positive. It converges locally uniformly for $\beta > 0$ and $\tau \geq 0$. The map (6.67) sends $[0, 1]$ into $(0, 1)$ and satisfies*

$$\sup_{0 \leq \kappa \leq 1} |\mathfrak{R}'_b(\kappa)| < 8.70 \times 10^{-47}. \quad (6.68)$$

The self-consistency equation

$$\kappa = \mathfrak{R}_b(\kappa) \quad (6.69)$$

has a unique solution $\kappa_b^* \in (0, 1)$. It is invariant under every scale equivalence $b \mapsto b'$ and lies in the certified interval

$$2.429273782017158 \times 10^{-35} < \kappa_b^* < 2.429273782017160 \times 10^{-35}. \quad (6.70)$$

For this endpoint response Definition 6.4 therefore determines a single stable retention capacity from the complete additive-decomposition spectrum.

Proof. The operator representation and the local uniform convergence of the full series are established in Theorem B.8. The derivative identity and the $k = 3$ lower contribution together with the positive-series majorants appear in (B.64)–(B.66) and are proved in Appendix B. They give the displayed contraction estimate. Banach’s fixed-point theorem then gives a unique stable solution. All quantities in the defining response are dimensionless. The solution is therefore unchanged by $b \mapsto b'$.

The interval in (6.70) follows from the finite evaluation together with the iteration estimate (B.69) and the Gaussian tail estimate (B.77). These errors are combined in the outward-rounded enclosure (B.79). $\square$

Proposition 6.6 (Leading-shell form of the diagonal closure). *Set*

$$\mathfrak{r}_0 := e^{-9(\beta_{\text{IR}} - \beta_{\text{UV}})}, \quad \mathfrak{a}(\beta) := \frac{1}{4} \sum_{k=4}^{\infty} \frac{k(k+1)(k-2)}{3} e^{-\beta(k^2-9)}. \quad (6.71)$$

The following bounds hold uniformly for $0 \leq \kappa \leq 1$.

$$\frac{\mathfrak{r}_0}{1 + \mathfrak{a}(\beta_{\text{UV}})} \leq \mathfrak{R}_b(\kappa) \leq \mathfrak{r}_0(1 + \mathfrak{a}(\beta_{\text{IR}})). \quad (6.72)$$

They imply

$$\sup_{0 \leq \kappa \leq 1} \left| \frac{\mathfrak{R}_b(\kappa)}{\mathfrak{r}_0} - 1 \right| < 8.948 \times 10^{-13}, \quad |\kappa_b^* - \mathfrak{R}_b(0)| < 2.12 \times 10^{-81}. \quad (6.73)$$

Thus the $k = 3$ shell fixes the scale of the endpoint retention. The higher shells give the displayed uniform relative correction. The remaining dependence on the feedback coordinate is smaller still at the fixed point.

Proof. The $k = 3$ fibre has zero decomposition response and contributes $4e^{-9\beta}$. The exact transverse trace (4.65) and positivity together with $e^{-\kappa h_{k,m}} \leq 1$ give

$$\Pi_{+,b}(\beta, \kappa) = 4e^{-9\beta}(1 + \delta_\beta(\kappa)), \quad 0 \leq \delta_\beta(\kappa) \leq \mathfrak{a}(\beta).$$

Taking the ratio at β_{IR} and β_{UV} proves (6.72). The tail starts at $k \geq 4$ so (B.65) gives

$$\mathfrak{a}(\beta) \leq \frac{1}{4}B(\beta).$$

The certified bounds in (B.67) therefore give $\mathfrak{a}(\beta_{\text{UV}}) < 8.9475 \times 10^{-13}$ and $\mathfrak{a}(\beta_{\text{IR}}) < 1.06975 \times 10^{-39}$. This proves the first estimate in (6.73). The second is (B.80). $\square$

The small fixed-point value comes from the endpoint separation and the lowest contributing energy, $k^2 = 9$. Higher fibres change this leading scale by less than 8.948×10^{-13} in relative value; the contraction estimate separately controls feedback. Taking this scalar trace leaves the occupation and additive carriers intact: compact-charge readout recovers their diagonal occupation coefficients, and each fixed-sum fibre retains its longitudinal line and transverse quotient.

Appendix B also decomposes the endpoint state into the multiplicative vacuum, prime-power axes and their orthogonal complement. Theorem B.9 gives the resulting trace budget, and Theorem B.9 gives its entropy decomposition.

7 Representation-fibre interfaces

The constructions retain root modes and their weighted expression fibres. A representation lift assigns coefficient capacity to that root. We compare compact Laplace realizations of the quadratic root energy and square coefficient capacity. Reproducing both sequences fixes the Weyl exponent; the exact spherical and central-sign statements use the realization classes specified below.

7.1 Value data and shell capacity

For a positive mode $u = u_{b,n}$ the normalized correlation eigenvalue is n^2 . The source readout Q_b^{hist} recovers $|u\rangle_b$ from its source-weighted history line. A representation-fibre lift assigns the coefficient space

$$\mathcal{F}_n \cong \text{End}(\mathbb{C}_b^n), \quad \dim \mathcal{F}_n = n^2,$$

with the response-compatibility law of Appendix C. Thus the same square coordinate supplies the root energy and the capacity of this specified lift. In a basis of $\mathbb{C}_b^n$ the n^2 matrix units count the independent coefficient directions. The response acts on this entire coefficient fibre according to the stated compatibility law.

These capacities count different objects. The Neumann eigenspace has rank one, the root phase group has order n , the expression and expression-phase fibres have cardinalities h_n and nh_n , and the representation fibre has dimension n^2 . Recursive probability normalizes each history fibre to its root mass. Representation capacity counts coefficient directions over that root.

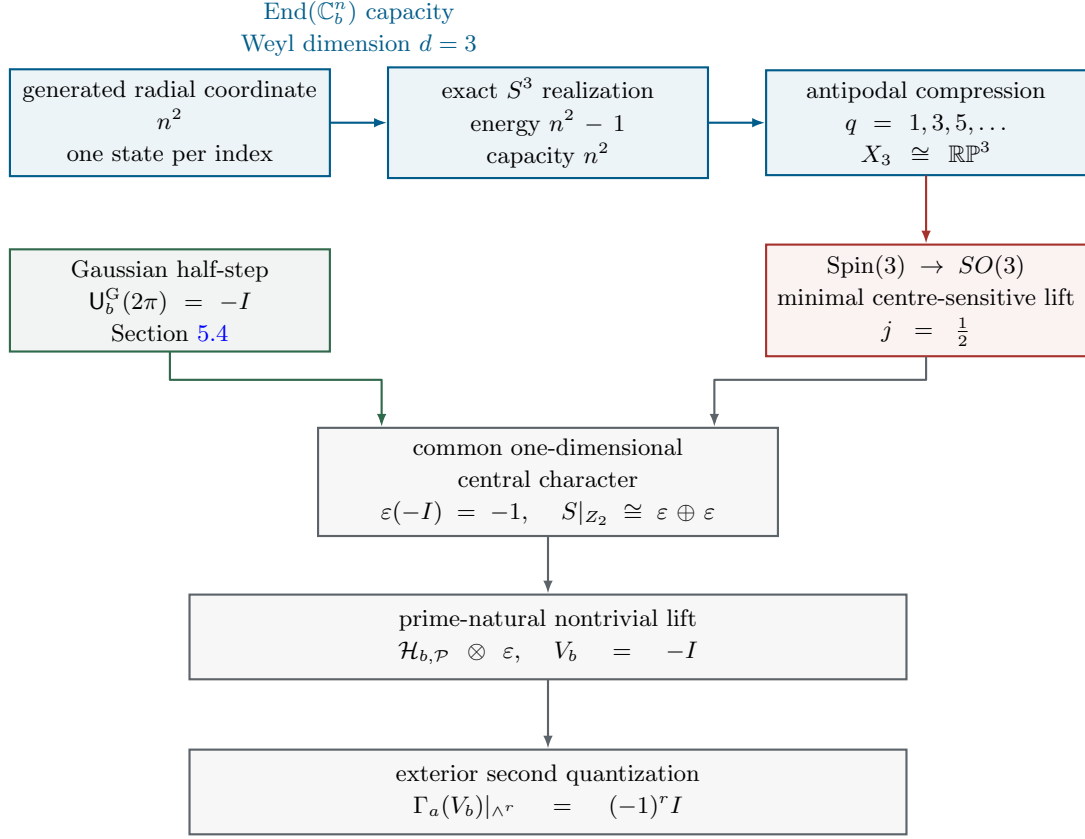
Suppose a complete compact Laplace realization reproduces the square-energy coordinate and square capacity, with the finite initial exceptions allowed in Definition C.2. Summing the shell capacities gives an eigenvalue count of order $\Lambda^{3/2}$. Weyl's law then gives dimension three (Proposition C.3).

7.2 Projective and central-sign realizations

Within the round spherical affine energy-capacity class the exact shell identity selects the three-spherical parent and its antipodal projective compression. The proofs in Appendices C and D retain the normalization and multiplicities of the Laplace spectrum together with the acting groups. The scalar/traceless splitting of a coefficient fibre is invariant under diagonal conjugation. Each summand remains in that fibre's Laplace eigenspace. This includes its scalar matrix direction.

The resulting three-dimensional rotation group has the double cover $\text{Spin}(3) \cong SU(2)$. Requiring its nontrivial central element to act by a negative sign selects the two-dimensional half-spin module of smallest positive dimension (Theorem E.1). Its restriction to the centre has the same nontrivial character as the Gaussian half-step of Section 5.4. The prime-natural occupation lift carries that character to the exterior grading. These are intertwining statements on their specified spaces.

Figure 5 compares the two central actions.



The Gaussian and projective half-spin routes agree on the central character.
Their full metaplectic and $SU(2)$ representations remain distinct.

Figure 5. The Gaussian central sign and its projective geometric continuation. The radial square spectrum has one state per index. A response-compatible square-capacity lift has three-dimensional Weyl growth. Exact affine classification in the round spherical class identifies the three-spherical parent. Antipodal compression and the minimal centre-sensitive rotational lift provide the geometric route. The Gaussian half-step supplies the same central character directly. Prime-permutation naturality and the nontrivial-character condition select its uniform action on the prime space. Exterior second quantization gives the occupation sign. The half-spin module restricts to two copies of the character. The arithmetic lift uses one copy per prime direction.

The geometric construction uses the root square spectrum and representation capacity; the Gaussian construction uses the first-order factorization of the correlation profile. Their restrictions to the centre agree, with full actions and multiplicities as defined in each construction. The symmetric-power identification (E.7) realizes a fixed-sum input fibre as a normalized monomial space on the minimal two-dimensional module.

8 Conclusion

The cascade-irreducible modes give the prime family. Their residue classes form the augmentation cotangent basis, and the corresponding one-word vectors span the degree-one harmonic subspace of the decomposition complex. The unit gap on the composite sector gives sharp heat and resolvent approximations to their projection. Gaussian weighting makes the corresponding trace errors summable on the complete mode family.

Both symmetric and exterior occupation retain logarithmic energy and evaluated quadratic energy. The first gives Euler traces; the second retains the source Gaussian response. Symmetric occupation carries Liouville parity, while exterior square-free occupation gives the Möbius sign. The adjoint of conditional readout embeds the value space into history space. Under this embedding, the prime projection selects one source-weighted line at each prime root. The full prime-root event also contains the orthogonal history differences.

The faithful additive extension retains ordered inputs, with transverse fixed-sum directions recording the information lost by scalar addition. Its charge defect relates input Liouville parity to output parity. Common-factor coordinates retain a shared multiplicative scale and primitive coprime legs. Prime-direction transfer, square-divisor identities and Mellin transformation relate the signed responses while preserving the unit mode, cutoff boundaries and Gaussian scale dilations.

Combining additive and decomposition responses defines the endpoint map. Under diagonal self-consistency it has a unique stable fixed point. Its small magnitude is determined by the endpoint separation and the first contributing energy; the higher-shell correction and feedback are bounded separately. The interval enclosure includes the infinite tail and the iteration error, rather than only the finite numerical sum.

For representation fibres, realizing square energy and quadratic capacity as one compact Laplace spectrum gives counting exponent $3/2$ and hence dimension three. Affine energy-capacity compatibility in the round spherical class identifies the three-spherical parent and its antipodal projective compression. The central-sign condition selects the minimal half-spin module in the stated representation class.

The counting applications recover Bertrand's theorem, the prime-number asymptotic and square-free sign densities in this carrier. Their boundary and summation arguments use the occupation traces with the same arithmetic normalization as the decomposition and additive operators.

A Further representation extensions

These representations use the arithmetic value obtained from the expression source. Composing their value-basis maps with $(Q_b^{\text{hist}})^*$ embeds them in the source-weighted history subspace. Their scalar and group-completion formulas are those of the value carrier.

The group completion admits bilateral Fourier characters. Logarithmic characters diagonalize multiplicative change. These two representations lead to a positive correlation kernel on the natural-mode semigroup.

A.1 Bilateral Fourier representation of the group completion

For $Z = [u, v] \in \mathbb{Z}_b$ put

$$k_b(Z) := \deg_b(u) - \deg_b(v) \in \mathbb{Z} \quad (\text{A.1})$$

and define

$$\chi_Z(\theta) := e^{ik_b(Z)\theta}, \quad \theta \in [0, 2\pi). \quad (\text{A.2})$$

By [1, Theorem 12.1](i) the map k_b is additive and injective. Thus $Z \mapsto \chi_Z$ embeds $(\mathbb{Z}_b, +)$ into the character group of $\mathbb{T}$. The characters satisfy the usual orthogonality relation

$$\frac{1}{2\pi} \int_0^{2\pi} \chi_Z(\theta) \overline{\chi_W(\theta)} d\theta = \delta_{Z,W}. \quad (\text{A.3})$$

The linear extension of this embedding is the standard bilateral Fourier transform in the intrinsic degree coordinate.

A.2 Logarithmic change characters

For positive natural modes define

$$\chi_t(u) := e^{it \log \deg_b(u)}, \quad t \in \mathbb{R}. \quad (\text{A.4})$$

Then

$$\chi_t(u \boxtimes_b v) = \chi_t(u) \chi_t(v). \quad (\text{A.5})$$

The long-time mean of logarithmic characters at positive natural modes u, v satisfies

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \chi_t(u) \overline{\chi_t(v)} dt = \delta_{u,v}. \quad (\text{A.6})$$

The integrand is one when $u = v$. Otherwise put $\omega = \log(\deg_b u / \deg_b v) \neq 0$. The average equals $\sin(T\omega)/(T\omega)$ and tends to zero.

For distinct prime modes $\pi_1, \dots, \pi_K$ the numbers $\log \deg_b(\pi_j)$ are linearly independent over $\mathbb{Z}$. An integer relation exponentiates to an equality between two prime factorizations. It follows that

$$t \mapsto (\chi_t(\pi_1), \dots, \chi_t(\pi_K)) \quad (\text{A.7})$$

is equidistributed on $\mathbb{T}^K$. The time average of every nonconstant torus character is $\sin(T\omega)/(T\omega)$ with $\omega \neq 0$. Density of trigonometric polynomials in $C(\mathbb{T}^K)$ extends this equality of limiting averages to every continuous function. This strengthens (A.6) from pairwise characters to every finite prime-coordinate cylinder.

A.3 Positive correlations on the natural-mode semigroup

Let $a : \mathcal{U}_b^+ \rightarrow \mathbb{C}$ have finite support and define the two-variable correlation kernel

$$C_a(u, v) := \sum_{w \in \mathcal{U}_b^+} a(w \boxtimes_b u) \overline{a(w \boxtimes_b v)}, \quad (\text{A.8})$$

where absent coefficients are read as zero.

For positive modes $u_1, \dots, u_N$ and coefficients $c_1, \dots, c_N$ regrouping the finite sum gives

$$\sum_{j,k=1}^N c_j \overline{c_k} C_a(u_j, u_k) = \sum_{w \in \mathcal{U}_b^+} \left| \sum_{j=1}^N c_j a(w \boxtimes_b u_j) \right|^2 \geq 0. \quad (\text{A.9})$$

C_a is the Gram kernel of the translated coefficient vectors. Its two-variable form supplies the Hermitian symmetry required for arbitrary complex coefficients.

A.4 Kernel dependence

The algebraic semiring proof uses the monomial analytic lift z^n . A different correlation law changes the endpoint window and spectral generator before any analytic lift is chosen. If its oriented spectrum admits the same monomial realization then the modulation–cascade semiring persists together with its internally defined irreducibles and occupation representations. The results use the Gaussian analytic class and are invariant under changes of its scale b .

B Joint-response identities and estimates

The two-leg source embedding V_2 of (4.15) preserves positive trace-class responses: $\text{Tr}(V_2 T V_2^*) = \text{Tr} T$. Thus the residual traces have the same normalization on the value-input and source subspaces. On each fixed-sum fibre the additive projections and decomposition response have a commutator determined by the variation of decomposition energy. Their joint traces give the full-spectrum capacity of Section 6.8. The spatial kernel and compact charge response reconstruct the same endpoint residual operator.

B.1 Channel curvature and Wilson holonomy

The two-input decomposition response is diagonal on $\mathcal{F}_{b,k}$. We write

$$H_{b,k}^{\text{dec}} |m, k - m\rangle_b = h_{k,m} |m, k - m\rangle_b, \quad h_{k,m} = \widehat{r}_b(m) + \widehat{r}_b(k - m),$$

where $\widehat{r}_b$ is the transported one-input response in Section 6.8 and satisfies $\widehat{r}_b(1) = 0$.

Theorem B.1 (Additive–multiplicative curvature). *On the fixed-sum fibre $\mathcal{F}_{b,k}$ define*

$$\Omega_{b,k} := [H_{b,k}^{\text{dec}}, P_L^{(k)}] = -[H_{b,k}^{\text{dec}}, P_T^{(k)}]. \quad (\text{B.1})$$

In the ordered basis $e_m = |m, k - m\rangle_b$ its entries and Hilbert–Schmidt norm satisfy

$$\langle e_m, \Omega_{b,k} e_n \rangle = \frac{h_{k,m} - h_{k,n}}{k - 1}, \quad (\text{B.2})$$

and

$$\|\Omega_{b,k}\|_{\text{HS}}^2 = 2 \left[\frac{1}{k-1} \sum_{m=1}^{k-1} h_{k,m}^2 - \left(\frac{1}{k-1} \sum_{m=1}^{k-1} h_{k,m} \right)^2 \right]. \quad (\text{B.3})$$

The identity $\Omega_{b,k} = 0$ holds if and only if the decomposition response is constant across the complete fixed-sum fibre. The first nonflat fibre is $k = 5$ and gives

$$(h_{5,1}, h_{5,2}, h_{5,3}, h_{5,4}) = (1, 0, 0, 1), \quad \|\Omega_{b,5}\|_{\text{HS}}^2 = \frac{1}{2}. \quad (\text{B.4})$$

For every $\beta > 0$ the full Gaussian curvature response

$$\mathfrak{C}_b(\beta) := \sum_{k=3}^{\infty} e^{-\beta k^2} \|\Omega_{b,k}\|_{\text{HS}}^2 \quad (\text{B.5})$$

is finite and strictly positive. It is invariant under the scale equivalences $b \mapsto b'$.

Proof. The operator $H_{b,k}^{\text{dec}}$ is diagonal with entries $h_{k,m}$. Every matrix entry of $P_L^{(k)}$ equals $(k-1)^{-1}$. Their commutator therefore gives (B.2). Summing its squared matrix entries yields

$$\frac{1}{(k-1)^2} \sum_{m,n=1}^{k-1} (h_{k,m} - h_{k,n})^2,$$

which is exactly (B.3). The vanishing criterion follows from vanishing of a finite variance. The values in (B.4) follow from $\hat{r}_b(1) = \hat{r}_b(2) = \hat{r}_b(3) = \hat{r}_b(5) = 0$ and $\hat{r}_b(4) = d(4) - 2 = 1$. The bound $0 \leq h_{k,m} \leq k$ gives $\|\Omega_{b,k}\|_{\text{HS}}^2 \leq 2k^2$ and hence convergence of the Gaussian series. Its $k = 5$ term proves strict positivity. Both $h_{k,m}$ and the dimensionless Gaussian exponent are preserved by the canonical scale equivalences. This proves scale invariance. $\square$

The reciprocal normalization (4.63) remains exact on fibres with nonzero curvature. Vanishing of $\Omega_{b,k}$ means that the decomposition response preserves the additive longitudinal and transverse subspaces separately.

The curvature has a closed-path realization. For $u, v \in \mathbb{R}$ put

$$\text{Hol}_{b,k}(u, v) := e^{iuH_{b,k}^{\text{dec}}} e^{ivP_L^{(k)}} e^{-iuH_{b,k}^{\text{dec}}} e^{-ivP_L^{(k)}}. \quad (\text{B.6})$$

This unitary holonomy is formed from the two channel operations. Its expansion is

$$\text{Hol}_{b,k}(u, v) = I - uv \Omega_{b,k} + O(|u|^2|v| + |u||v|^2) \quad \text{as } (u, v) \rightarrow (0, 0). \quad (\text{B.7})$$

The $k = 5$ loop has nontrivial holonomy when u and v are both sufficiently small and nonzero. A common unitary change of basis conjugates both generators and their holonomy. The trace and spectrum of the holonomy are therefore unchanged.

The complete holonomy trace is explicit. Put $d_k := k - 1$ and define the normalized decomposition characteristic function

$$\chi_{b,k}(u) := \frac{1}{d_k} \sum_{m=1}^{d_k} e^{iuh_{k,m}}. \quad (\text{B.8})$$

Theorem B.2 (Gaussian Wilson generating function). *For every $k \geq 3$ and $u, v \in \mathbb{R}$ one has*

$$d_k - \text{Tr Hol}_{b,k}(u, v) = 4 \sin^2 \frac{v}{2} (1 - |\chi_{b,k}(u)|^2) \geq 0. \quad (\text{B.9})$$

On the first nonflat fibre this becomes

$$4 - \text{Tr Hol}_{b,5}(u, v) = 4 \sin^2 \frac{u}{2} \sin^2 \frac{v}{2}. \quad (\text{B.10})$$

The full Gaussian Wilson response

$$\mathfrak{W}_b(\beta; u, v) := \sum_{k=3}^{\infty} e^{-\beta k^2} [d_k - \text{Tr Hol}_{b,k}(u, v)] \quad (\text{B.11})$$

converges absolutely and locally uniformly for $\beta > 0$. It is nonnegative and satisfies

$$\mathfrak{W}_b(\beta; u, v) \geq 4e^{-25\beta} \sin^2 \frac{u}{2} \sin^2 \frac{v}{2}. \quad (\text{B.12})$$

For $\Re s > 1$ its Mellin transform is the joint Dirichlet generator

$$\int_0^{\infty} \beta^{s/2-1} \mathfrak{W}_b(\beta; u, v) d\beta = 4\Gamma\left(\frac{s}{2}\right) \sin^2 \frac{v}{2} \sum_{k=3}^{\infty} \frac{1 - |\chi_{b,k}(u)|^2}{k^s}. \quad (\text{B.13})$$

This generating function records the additive fixed-sum geometry and multiplicative decomposition data together with the Gaussian scale flow.

Proof. Let $P = P_L^{(k)}$ and $P_u = e^{iuH_{b,k}^{\text{dec}}} P e^{-iuH_{b,k}^{\text{dec}}}$. The rank-one identities for P and P_u give

$$e^{ivP_u} = I + (e^{iv} - 1)P_u, \quad e^{-ivP} = I + (e^{-iv} - 1)P.$$

Moreover $\text{Tr}(P_u P) = |\chi_{b,k}(u)|^2$. Multiplication and taking the trace give (B.9). At $k = 5$ the identity $\chi_{b,5}(u) = (1 + e^{iu})/2$ proves (B.10). Since $0 \leq 1 - |\chi_{b,k}(u)|^2 \leq 1$ Gaussian domination proves absolute local uniform convergence. Retaining the $k = 5$ term gives (B.12). Termwise Mellin integration is justified for $\Re s > 1$ by the same domination and

$$\int_0^{\infty} \beta^{s/2-1} e^{-\beta k^2} d\beta = \Gamma\left(\frac{s}{2}\right) k^{-s}.$$

This proves (B.13). □

B.2 Hodge transfer and additive entropy

The Hodge flow sends part of a longitudinal state into the transverse space whenever the decomposition response varies within the fibre. For $\tau \geq 0$ put

$$s_{b,k}(\tau) := \langle v_{b,k}, e^{-\tau H_{b,k}^{\text{dec}}} v_{b,k} \rangle = \frac{1}{k-1} \sum_{m=1}^{k-1} e^{-\tau h_{k,m}}. \quad (\text{B.14})$$

Theorem B.3 (Longitudinal Hodge transfer and transverse defect). *For every $k \geq 3$ and $\tau \geq 0$ one has*

$$P_L^{(k)} e^{-\tau H_{b,k}^{\text{dec}}} P_L^{(k)} = s_{b,k}(\tau) P_L^{(k)}, \quad (\text{B.15})$$

whereas the exact transverse leakage of the normalized longitudinal state is

$$\begin{aligned} \mathfrak{d}_{b,k}(\tau) &:= \left\| P_T^{(k)} e^{-\tau H_{b,k}^{\text{dec}}} v_{b,k} \right\|^2 \\ &= s_{b,k}(2\tau) - s_{b,k}(\tau)^2 = \text{Var}_{1 \leq m < k} (e^{-\tau h_{k,m}}) \geq 0. \end{aligned} \quad (\text{B.16})$$

For $\tau > 0$ it is positive exactly when the additive–multiplicative curvature is nonzero. On the first nonflat fibre

$$\mathfrak{d}_{b,5}(\tau) = \frac{(1 - e^{-\tau})^2}{4}. \quad (\text{B.17})$$

Let

$$g_b(k) := \# \{1 \leq m \leq k-1 : \hat{r}_b(m) = \hat{r}_b(k-m) = 0\}. \quad (\text{B.18})$$

Then the complete Hodge localization is

$$\lim_{\tau \rightarrow \infty} s_{b,k}(\tau) = \frac{g_b(k)}{k-1}, \quad \lim_{\tau \rightarrow \infty} \mathfrak{d}_{b,k}(\tau) = \frac{g_b(k)}{k-1} \left(1 - \frac{g_b(k)}{k-1}\right). \quad (\text{B.19})$$

For $\beta, \tau > 0$ the full joint defect

$$\mathfrak{D}_b(\beta, \tau) := \sum_{k=3}^{\infty} e^{-\beta k^2} \mathfrak{d}_{b,k}(\tau) \quad (\text{B.20})$$

is finite and satisfies the strict internal lower bound

$$\mathfrak{D}_b(\beta, \tau) \geq \frac{1}{4} e^{-25\beta} (1 - e^{-\tau})^2 > 0. \quad (\text{B.21})$$

Proof. The first identity follows because $P_L^{(k)}$ is the rank-one projector onto $v_{b,k}$. Pythagoras gives

$$\|P_T^{(k)} e^{-\tau H_{b,k}^{\text{dec}}} v_{b,k}\|^2 = \langle v_{b,k}, e^{-2\tau H_{b,k}^{\text{dec}}} v_{b,k} \rangle - |\langle v_{b,k}, e^{-\tau H_{b,k}^{\text{dec}}} v_{b,k} \rangle|^2,$$

which is (B.16). A finite variance vanishes exactly when its entries are constant. Because $x \mapsto e^{-\tau x}$ is injective for $\tau > 0$ this is equivalent to $\Omega_{b,k} = 0$. The $k = 5$ formula is immediate from $(h_{5,m}) = (1, 0, 0, 1)$. In the limit $\tau \rightarrow \infty$ precisely the terms with $h_{k,m} = 0$ survive. Nonnegativity of the two one-leg responses makes this equivalent to the two simultaneous zero-mode conditions in (B.18). Gaussian domination proves convergence of (B.20). Its $k = 5$ term proves the strict lower bound. $\square$

Orthogonal blocks satisfy the entropy identity

$$S\left(\bigoplus_j p_j \sigma_j\right) = - \sum_j p_j \log p_j + \sum_j p_j S(\sigma_j), \quad (\text{B.22})$$

where each σ_j has unit trace on its support. The identity follows by writing each block eigenvalue as $p_j \lambda$ in $-\text{Tr}(\rho \log \rho)$. All terms are finite for these Gaussian states by Lemma 3.4.

Define the additive channel state by

$$\rho_{+,b}(\beta) := \frac{e^{-\beta N_+^2}}{Z_{+,b}(\beta)}, \quad Z_{+,b}(\beta) := \text{Tr} e^{-\beta N_+^2} = \sum_{k=2}^{\infty} (k-1) e^{-\beta k^2}. \quad (\text{B.23})$$

Its weight depends only on the total output degree k . Thus all internal prime-charge realizations in the same fixed-sum fibre couple with the same coefficient.

The same state also has a coherent longitudinal–transverse resolution. Its two partition functions and fractions are

$$Z_{L,b}(\beta) := \text{Tr} \left(P_L e^{-\beta N_+^2} \right) = \sum_{k=2}^{\infty} e^{-\beta k^2}, \quad Z_{T,b}(\beta) := \text{Tr} \left(P_T e^{-\beta N_+^2} \right) = \sum_{k=3}^{\infty} (k-2) e^{-\beta k^2}, \quad (\text{B.24})$$

$$\Omega_{L,b}(\beta) := \frac{Z_{L,b}(\beta)}{Z_{+,b}(\beta)}, \quad \Omega_{T,b}(\beta) := \frac{Z_{T,b}(\beta)}{Z_{+,b}(\beta)}. \quad (\text{B.25})$$

For $X \in \{L, T\}$ let $\rho_{X,b}(\beta) := P_X e^{-\beta N_+^2} / Z_{X,b}(\beta)$.

Grouping inputs by their sum measures the information within each fixed-sum fibre. Resolving the longitudinal line and its orthogonal complement gives a different decomposition of the same state. Both entropy formulas follow from (B.22).

Theorem B.4 (Additive entropy under output and transverse resolution). (i) *Let*

$$p_{b,k}(\beta) := \frac{(k-1)e^{-\beta k^2}}{Z_{+,b}(\beta)}, \quad k \geq 2. \quad (\text{B.26})$$

Then

$$S(\rho_{+,b}(\beta)) = - \sum_{k \geq 2} p_{b,k}(\beta) \log p_{b,k}(\beta) + \sum_{k \geq 2} p_{b,k}(\beta) \log(k-1). \quad (\text{B.27})$$

Let (M, N) denote the ordered input pair and put $K = M + N$. The equivalent Shannon identities are

$$H(M, N) = H(K) + H(M, N \mid K), \quad H(M, N \mid K) = H(\delta_{+,b}(M, N), \epsilon(M, N) \mid K). \quad (\text{B.28})$$

The compact prime-charge description and the ordered natural-mode description have the same input entropy. The second term in (B.27) is the conditional information carried by the fixed-sum fibre and compressed by the one-output addition map.

(ii) *For every $\beta > 0$ one has*

$$\Omega_{L,b}(\beta) + \Omega_{T,b}(\beta) = 1, \quad \Omega_{L,b}(\beta), \Omega_{T,b}(\beta) > 0, \quad (\text{B.29})$$

and

$$S(\rho_{+,b}) = h_2(\Omega_{L,b}) + \Omega_{L,b} S(\rho_{L,b}) + \Omega_{T,b} S(\rho_{T,b}), \quad (\text{B.30})$$

where $h_2(x) := -x \log x - (1-x) \log(1-x)$. On each fibre with $k \geq 3$ this gives

$$\log(k-1) = h_2\left(\frac{1}{k-1}\right) + \frac{k-2}{k-1} \log(k-2). \quad (\text{B.31})$$

The transverse fraction is therefore a strictly positive component of every finite Gaussian endpoint state.

Proof. Apply (B.22) with $\sigma_k = I_{\mathcal{F}_{b,k}}/(k-1)$ and weights $p_{b,k}$. The sum K is a deterministic function of (M, N) . Thus the first equality in (B.28) is the Shannon chain rule. Theorem 4.4 gives a bijection at fixed K between ordered pairs and admissible defect-orientation labels. This proves the second equality. Unique occupation factorization is a bijection between a natural mode and its prime-charge vector and therefore preserves entropy.

The partition identities follow from $\dim P_L^{(k)} \mathcal{F}_{b,k} = 1$ and $\dim P_T^{(k)} \mathcal{F}_{b,k} = k-2$. Both sums are strictly positive and $Z_{+,b} = Z_{L,b} + Z_{T,b}$. Since P_L and P_T commute with N_+ the state is block diagonal. Equation (B.22) gives (B.30). Equation (B.31) is its specialization for $I_{\mathcal{F}_{b,k}}/(k-1)$. Its longitudinal and transverse weights are $1/(k-1)$ and $(k-2)/(k-1)$. $\square$

The defect identity also measures information lost under generated compression. Suppose a discrete generated label X is mapped deterministically to an output Y and a residual label M such that (Y, M) recovers X . Whenever $H(X) < \infty$ the entropy satisfies

$$H(X) = H(Y, M) = H(Y) + H(M | Y). \quad (\text{B.32})$$

The encoding is a bijection onto its image. The last equality is the chain rule. Discarding the residual loses $H(M | Y)$. Replacing this conditional entropy by $H(M)$ adds the mutual information $I(Y; M)$ to the joint entropy. The additive defect and orientation labels give an instance in which the residual is generated by the same operation as the output.

For the rank-two response impose tensor multiplicativity on survival. If $y \in [0, 1]$ is the longitudinal fraction and $S_r(y)$ the survival fraction of an r -leg transverse carrier then the rule is

$$S_1(y) = 1 - y, \quad S_{r+r'}(y) = S_r(y) S_{r'}(y) \quad (r, r' \geq 1). \quad (\text{B.33})$$

Induction gives

$$S_r(y) = (1 - y)^r. \quad (\text{B.34})$$

The identity $dy/ds = y(1-y)$ for $s = \log(y/(1-y))$ gives the response per leg

$$B_r(y) := -\frac{1}{r} \frac{dS_r}{ds} = y(1-y)^r. \quad (\text{B.35})$$

Its derivative is $(1-y)^{r-1}[1 - (r+1)y]$. Hence its unique maximum is $r^r/(r+1)^{r+1}$ at $y = 1/(r+1)$. For the two additive input legs this gives

$$B(y) := B_2(y) = y(1-y)^2. \quad (\text{B.36})$$

By (B.35) the function B is half the loss rate of the squared transverse fraction along the log-odds flow. For the Gaussian sector probabilities it becomes

$$\mathcal{B}_{LT,b}(\beta) := B(\Omega_{L,b}(\beta)) = \Omega_{L,b}(\beta)\Omega_{T,b}(\beta)^2. \quad (\text{B.37})$$

Its maximizer and value are determined by the spectral probabilities.

Theorem B.5 (Rank-two longitudinal–transverse balance). *The longitudinal fraction $\Omega_{L,b}(\beta)$ is strictly increasing on $(0, \infty)$ and satisfies*

$$\Omega_{L,b}(\beta) \sim \sqrt{\pi\beta} \quad (\beta \downarrow 0), \quad \Omega_{L,b}(\beta) \rightarrow 1 \quad (\beta \rightarrow \infty). \quad (\text{B.38})$$

There is therefore a unique $\beta_{L,*} > 0$ such that

$$\Omega_{L,b}(\beta_{L,*}) = \frac{1}{3}, \quad \max_{\beta > 0} \Omega_{L,b}(\beta)\Omega_{T,b}(\beta)^2 = \frac{4}{27}. \quad (\text{B.39})$$

The balancing parameter is dimensionless and scale invariant. Monotone evaluation gives

$$\beta_{L,*} = 0.0361234412094753\dots \quad (\text{B.40})$$

Proof. Let $\mathbb{E}_L$ denote expectation with weights proportional to $e^{-\beta k^2}$. Let $\mathbb{E}_+$ denote expectation with weights proportional to $(k-1)e^{-\beta k^2}$. Logarithmic differentiation gives

$$\frac{d}{d\beta} \log \Omega_{L,b}(\beta) = \mathbb{E}_+[k^2] - \mathbb{E}_L[k^2] = \frac{\text{Cov}_L(k^2, k-1)}{\mathbb{E}_L[k-1]} > 0, \quad (\text{B.41})$$

because k^2 and $k-1$ are strictly increasing on $k \geq 2$. Gaussian sum–integral comparison gives $Z_{L,b}(\beta) \sim \sqrt{\pi}/(2\sqrt{\beta})$ and $Z_{+,b}(\beta) \sim 1/(2\beta)$ as $\beta \downarrow 0$. This proves the first limit. The $k=2$ term is the unique lowest shell and proves the second. The intermediate-value theorem and strict monotonicity give a unique point where the fraction is $1/3$. The $r=2$ extremum calculated in (B.35) gives the maximum $4/27$. Monotone bisection of the two positive Gaussian series with their integral tail bounds gives (B.40). $\square$

B.3 Free-energy balance and exchange sectors

The connected response also gives a direct entropy–energy comparison. Let

$$\rho_{0,b}(\beta) := \frac{e^{-\beta(N_1^2 + N_2^2)}}{Z_{0,b}(\beta)}, \quad Z_{0,b}(\beta) := \left(\sum_{n \geq 1} e^{-\beta n^2} \right)^2. \quad (\text{B.42})$$

Relative entropy compares the connected and disconnected states through the following identities for every $\beta > 0$.

$$D(\rho_{+,b}(\beta) \| \rho_{0,b}(\beta)) = \log \frac{Z_{0,b}(\beta)}{Z_{+,b}(\beta)} - \beta \text{Tr}(\rho_{+,b}(\beta) K_{+,b}) \geq 0, \quad (\text{B.43})$$

$$S(\rho_{+,b}(\beta)) = \log Z_{+,b}(\beta) + \beta \text{Tr}(\rho_{+,b}(\beta) N_+^2),$$

$$\frac{d}{d\beta} S(\rho_{+,b}(\beta)) = -\beta \text{Var}_{\rho_{+,b}(\beta)}(N_+^2) \leq 0. \quad (\text{B.44})$$

Equation (4.47) gives $N_+^2 = N_1^2 + N_2^2 + K_{+,b}$. Substitution into the definition of relative entropy proves (B.43). Positivity of relative entropy gives the inequality. The Gibbs entropy formula and differentiation give (B.44). The ordered two-input carrier has a canonical finite twist given by exchange of its two legs. Define

$$\Sigma_{12}(|m\rangle_b \otimes |n\rangle_b) := |n\rangle_b \otimes |m\rangle_b, \quad P_{\pm} := \frac{1}{2}(I \pm \Sigma_{12}). \quad (\text{B.45})$$

Theorem B.6 (Two-leg exchange sectors). *The exchange involution commutes with N_+ and $K_{+,b}$ as well as P_L and P_T . It also satisfies*

$$\tilde{A}_{+,b}\Sigma_{12} = \tilde{A}_{+,b}, \quad P_- \leq P_T. \quad (\text{B.46})$$

On every fixed-sum fibre one has

$$P_T \mathcal{F}_{b,k} = P_- \mathcal{F}_{b,k} \oplus (P_+ \mathcal{F}_{b,k} \ominus \mathbb{C}v_{b,k}), \quad (\text{B.47})$$

with

$$\dim P_+ \mathcal{F}_{b,k} = \left\lfloor \frac{k}{2} \right\rfloor, \quad \dim P_- \mathcal{F}_{b,k} = \left\lfloor \frac{k-1}{2} \right\rfloor. \quad (\text{B.48})$$

In particular $P_T \mathcal{F}_{b,2} = 0$ and $P_T \mathcal{F}_{b,3} = P_- \mathcal{F}_{b,3}$ is the first nonzero transverse sector. The two-input additive carrier therefore contains both an exchange-untwisted relative response and an exchange-twisted response. The normalized merge removes the common longitudinal line.

Proof. The total degree and connected Gaussian response are symmetric in the two legs. Exchange invariance of the uniform fibre vector $v_{b,k}$ implies that its rank-one projector $P_L^{(k)}$ commutes with Σ_{12} . The same holds for $P_T^{(k)}$. The merge identity in (B.46) shows that every exchange-antisymmetric vector lies in its kernel and hence in the transverse space. Removing the invariant line $\mathbb{C}v_{b,k}$ from the symmetric sector gives (B.47). Exchange pairs m with $k-m$. There is one fixed pair exactly when k is even. Counting the symmetric and antisymmetric combinations gives (B.48) and the two threshold statements. $\square$

The two smallest fixed-sum fibres determine the minimal nontrivial additive threshold. The $k=2$ fibre is one-dimensional and entirely longitudinal. Hence $P_T^{(2)} = 0$. The $k=3$ fibre is the first with a nonzero transverse direction and satisfies

$$\dim P_T^{(3)} \mathcal{F}_{b,3} = 1, \quad \text{Tr}_{P_T^{(3)} \mathcal{F}_{b,3}} K_{+,b}^T = 4. \quad (\text{B.49})$$

Thus three is the first fixed sum supporting a transverse direction. The calculation depends only on the dimension of the additive fibre.

B.4 Prime and transverse overlap

Let $\hat{R}_b$ be the nonnegative degree-one decomposition response transported from R_b in (6.45) and extended by $\hat{R}_b|1\rangle_b = 0$. The complete additive response kernel

$$\mathcal{K}_{+,b}^T(\beta, \tau) := P_T \exp \left[-\beta N_+^2 - \tau (\hat{R}_b \otimes I + I \otimes \hat{R}_b) \right] P_T \succeq 0 \quad (\text{B.50})$$

is trace class for $\beta > 0$ and uses the full two-input spectrum.

Let

$$\hat{P}_b^{\text{pr}} := \mathbf{1}_{\{0\}}(\hat{R}_b) - |1\rangle_b \langle 1|_b \quad (\text{B.51})$$

be the transported Hodge prime projector on $\mathcal{L}_b^+$. The subtraction removes the separately adjoined unit mode. The response $\hat{R}_b$ vanishes on that mode by definition. For $\pi \in \mathcal{P}_b$ let $\hat{P}_{b,\pi}$ denote its rank-one summand. Define the collective prime-to-transverse overlap map

$$J_b^{\text{pr},+} := P_T(\hat{P}_b^{\text{pr}} \otimes \hat{P}_b^{\text{pr}}). \quad (\text{B.52})$$

Theorem B.7 (Prime-zero-mode and additive-transverse overlap). *For $\beta > 0$ the operator*

$$B_b^{\text{pr},+}(\beta) := (K_{+,b}^T)^{1/2} J_b^{\text{pr},+} e^{-\beta N_+^2/2} \quad (\text{B.53})$$

is Hilbert–Schmidt. Its squared norm

$$\mathcal{O}_b^{\text{pr},+}(\beta) := \|B_b^{\text{pr},+}(\beta)\|_{\text{HS}}^2 = \text{Tr} \left[e^{-\beta N_+^2/2} (J_b^{\text{pr},+})^* K_{+,b}^T J_b^{\text{pr},+} e^{-\beta N_+^2/2} \right] \quad (\text{B.54})$$

is finite and nonnegative. The charge-resolved operators

$$B_{b;\pi,\rho}(\beta) := (K_{+,b}^T)^{1/2} P_T(\hat{P}_{b,\pi} \otimes \hat{P}_{b,\rho}) e^{-\beta N_+^2/2} \quad (\text{B.55})$$

have the positive-semidefinite Gram matrix

$$\mathcal{G}_{b;(\pi,\rho),(\pi',\rho')}(\beta) := \text{Tr} [B_{b;\pi,\rho}(\beta)^* B_{b;\pi',\rho'}(\beta)]. \quad (\text{B.56})$$

The effective coupling between multiplicative prime-charge zero modes and the additive transverse channel is the overlap of the two internally constructed sectors. Its charge resolution is inherited from the Hodge prime projectors.

Proof. The projectors and $J_b^{\text{pr},+}$ are contractions. On the k th fixed-sum fibre the operator $(K_{+,b}^T)^{1/2}$ has squared Hilbert–Schmidt norm $k(k+1)(k-2)/3$ by Theorem 4.8(ii). Its contribution to the squared norm is multiplied by $e^{-\beta k^2}$. The resulting polynomially weighted Gaussian series converges by Lemma 3.4. This proves the Hilbert–Schmidt assertion. Equation (B.54) is the identity $\|B\|_{\text{HS}}^2 = \text{Tr}(B^*B)$. For every finitely supported scalar family $c_{\pi,\rho}$ one has

$$\sum_{\pi,\rho,\pi',\rho'} \overline{c_{\pi,\rho}} c_{\pi',\rho'} \mathcal{G}_{b;(\pi,\rho),(\pi',\rho')}(\beta) = \left\| \sum_{\pi,\rho} c_{\pi,\rho} B_{b;\pi,\rho}(\beta) \right\|_{\text{HS}}^2 \geq 0, \quad (\text{B.57})$$

which proves Gram positivity. $\square$

B.5 Full-spectrum closure and spectral stability

The one-input joint kernel of Theorem 6.3 is

$$\mathcal{K}_b(\beta, \tau) := e^{-\beta \mathbf{H}_b - \tau R_b}, \quad \beta > 0, \quad \tau \geq 0. \quad (\text{B.58})$$

It is positive and trace class. The Hessian of the logarithm of its trace is the positive semidefinite covariance matrix of the two energies. Equations (B.58) and (B.50) are the

complete positive kernels of the multiplicative-decomposition and additive-transverse channels. Both are defined directly on the complete spectral index set.

The transverse projector generally does not commute with decomposition attenuation. We therefore keep the two projections around the joint heat operator. This compression is positive. Inserting the square root of the transverse response on both sides gives a positive trace. The finite-fibre calculation reduces it to scalar weights.

Theorem B.8 (Operator realization of the full-spectrum response). *The response defined in (6.65) has the operator representation*

$$\Pi_{+,b}(\beta, \tau) = \text{Tr}_{\mathcal{T}_{+,b}} \left[(K_{+,b}^T)^{1/2} \mathcal{K}_{+,b}^T(\beta, \tau) (K_{+,b}^T)^{1/2} \right], \quad \beta > 0, \quad \tau \geq 0. \quad (\text{B.59})$$

It is finite and strictly positive. It is continuous and nonincreasing in each variable on $(0, \infty) \times [0, \infty)$ and satisfies

$$\Pi_{+,b}(\beta, 0) = \Pi_{+,b}^T(\beta) = \sum_{k=3}^{\infty} \frac{k(k+1)(k-2)}{3} e^{-\beta k^2}. \quad (\text{B.60})$$

The endpoint ratio satisfies

$$0 < \Re_b(\kappa) < 1 \quad (0 \leq \kappa \leq 1). \quad (\text{B.61})$$

Proof. The diagonal entries of $\widehat{R}_b$ are the values $\widehat{r}_b(n) = {}_b\langle n | \widehat{R}_b | n \rangle_b$ in (6.63). On a fixed-sum fibre put $e_m := |m, k-m\rangle_b$ and $d_{k,m} := 2m(k-m)$ together with $D_k := \text{diag}(d_{k,1}, \dots, d_{k,k-1})$. Then $K_{+,b}|_{\mathcal{F}_{b,k}} = D_k$ and $P_T^{(k)} = I - |v_{b,k}\rangle\langle v_{b,k}|$. Thus

$$\begin{aligned} \langle e_m, P_T^{(k)} D_k P_T^{(k)} e_m \rangle &= d_{k,m} \left(1 - \frac{2}{k-1} \right) + \frac{1}{(k-1)^2} \sum_{j=1}^{k-1} d_{k,j} \\ &= 2m(k-m) \left(1 - \frac{2}{k-1} \right) + \frac{k(k^2-1)}{3(k-1)^2} = a_{k,m}, \end{aligned} \quad (\text{B.62})$$

where $\sum_{j=1}^{k-1} 2j(k-j) = k(k^2-1)/3$. This gives the weights in (6.64). Cyclicity of the finite-fibre trace moves the compressed response next to the diagonal heat operator. Its diagonal entries are $e^{-\beta k^2 - \tau[\widehat{r}_b(m) + \widehat{r}_b(k-m)]}$. Summing therefore gives (6.65).

The operators N_+ and $\widehat{R}_b \otimes I + I \otimes \widehat{R}_b$ are commuting nonnegative diagonal operators before transverse compression. Also P_T commutes with N_+ . On every fixed-sum fibre the trace in (B.59) is nonnegative and bounded by the corresponding trace at $\tau = 0$. Equation (4.66) supplies a summable Gaussian majorant. Lemma 3.4 therefore proves finiteness and local uniform convergence. It also justifies parameter differentiation and interchange of limits. Increasing either parameter decreases the positive fibrewise heat response. At $\tau = 0$ the definition reduces to (4.66). This proves (B.60). Positivity is strict because the $k = 3$ Hodge zero-mode sector has a nonzero transverse component.

Since $\beta_{\text{IR}} > \beta_{\text{UV}}$ every nonzero fixed-sum fibre is suppressed strictly more strongly at the IR endpoint. This proves (B.61). $\square$

Theorem 6.5 proves that the diagonal condition of Definition 6.4 selects a unique scale-invariant capacity κ_b^* . The geometric ratio $t_{\text{IR}}/t_{\text{UV}} = \pi$ fixes the evaluation points. The full additive-decomposition spectrum determines the retained fraction.

Write

$$h_{k,m} := \widehat{r}_b(m) + \widehat{r}_b(k-m), \quad (\text{B.63})$$

and let $\langle h \rangle_{\beta, \kappa}$ denote the expectation of $h_{k,m}$ with respect to the positive weights in the response at parameters (β, κ) . Logarithmic differentiation gives

$$\mathfrak{R}'_b(\kappa) = \mathfrak{R}_b(\kappa) (\langle h \rangle_{\beta_{\text{UV}}, \kappa} - \langle h \rangle_{\beta_{\text{IR}}, \kappa}). \quad (\text{B.64})$$

The divisor estimate $d(n) - 2 \leq n$ gives $0 \leq h_{k,m} \leq k$. The $k = 3$ fibre has zero decomposition response and contributes $4e^{-9\beta}$ to the denominator. Hence

$$\langle h \rangle_{\beta, \kappa} \leq B(\beta) := \frac{1}{4} \sum_{k=4}^{\infty} k \frac{k(k+1)(k-2)}{3} e^{-\beta(k^2-9)}. \quad (\text{B.65})$$

The ratio $\mathfrak{R}_b(\kappa)$ is also a positive weighted average of $e^{-(\beta_{\text{IR}} - \beta_{\text{UV}})k^2}$. Therefore

$$|\mathfrak{R}'_b(\kappa)| \leq e^{-9(\beta_{\text{IR}} - \beta_{\text{UV}})} [B(\beta_{\text{UV}}) + B(\beta_{\text{IR}})]. \quad (\text{B.66})$$

Summing the positive series B and bounding its remaining tail by the geometric estimates below gives

$$\begin{aligned} B(\beta_{\text{UV}}) &< 3.579 \times 10^{-12}, & B(\beta_{\text{IR}}) &< 4.279 \times 10^{-39}, \\ \mathfrak{q}_b^{\text{ctr}} &:= e^{-9(\beta_{\text{IR}} - \beta_{\text{UV}})} [B(\beta_{\text{UV}}) + B(\beta_{\text{IR}})], \\ \sup_{0 \leq \kappa \leq 1} |\mathfrak{R}'_b(\kappa)| &\leq \mathfrak{q}_b^{\text{ctr}} < 8.70 \times 10^{-47} < 1. \end{aligned} \quad (\text{B.67})$$

The numerical certificate combines a finite interval evaluation with the contraction estimate (B.67). It bounds the omitted Gaussian tail and the finite-iteration error together with outward rounding in the retained finite sum. Define

$$\Pi_{+,b}^{(N)}(\beta, \tau) := \sum_{k=3}^N e^{-\beta k^2} \sum_{m=1}^{k-1} a_{k,m} e^{-\tau[\widehat{r}_b(m) + \widehat{r}_b(k-m)]}, \quad \mathfrak{R}_b^{(N)} := \frac{\Pi_{+,b}^{(N)}(\beta_{\text{IR}}, \cdot)}{\Pi_{+,b}^{(N)}(\beta_{\text{UV}}, \cdot)}. \quad (\text{B.68})$$

Starting from $\kappa_0 = 0$ and iterating $\kappa_{j+1} = \mathfrak{R}_b^{(30)}(\kappa_j)$ gives $|\kappa_2 - \kappa_1| < 4 \times 10^{-98}$. Set $\mathfrak{q}_b^{\text{ctr}}$ as in (B.67). The same positive-series estimate applies to every truncation. Both the full and truncated maps therefore have contraction constant at most $\mathfrak{q}_b^{\text{ctr}}$. The a posteriori contraction estimate gives

$$|\kappa_{30}^* - \kappa_2| \leq \frac{\mathfrak{q}_b^{\text{ctr}}}{1 - \mathfrak{q}_b^{\text{ctr}}} |\kappa_2 - \kappa_1| < 4 \times 10^{-144}, \quad (\text{B.69})$$

where κ_{30}^* is the fixed point of $\mathfrak{R}_b^{(30)}$. The outward-rounded evaluation with the endpoint parameters in (6.66) gives

$$2.429273782017158 \times 10^{-35} < \kappa_b^* < 2.429273782017160 \times 10^{-35}. \quad (\text{B.70})$$

The displayed decimal approximation has the explicit error bound

$$\kappa_{\text{num}} := 2.429273782017159 \times 10^{-35}, \quad |\kappa_b^* - \kappa_{\text{num}}| < 10^{-50}. \quad (\text{B.71})$$

All fixed points lie below

$$\exp[-9(\beta_{\text{IR}} - \beta_{\text{UV}})] < 2.429273782019334 \times 10^{-35}, \quad (\text{B.72})$$

because (6.67) is a positive weighted average of $e^{-(\beta_{\text{IR}} - \beta_{\text{UV}})k^2}$ over $k \geq 3$. The closeness of (B.71) to this bound is the quantitative expression of the $k = 3$ threshold in (B.49). To bound the omitted tail put

$$T_N(\beta) := \sum_{k>N} \frac{k(k+1)(k-2)}{3} e^{-\beta k^2}, \quad D_0 := 4e^{-9\beta_{\text{UV}}}. \quad (\text{B.73})$$

Write $W_k := k(k+1)(k-2)/3$ to evaluate the tails. For $k \geq 4$ one has

$$\frac{W_{k+1}}{W_k} = \frac{(k+2)(k-1)}{k(k-2)} < 3, \quad \frac{(k+1)W_{k+1}}{kW_k} < 4. \quad (\text{B.74})$$

For $N \geq 4$ and $r_N(\beta) := 4e^{-\beta(2N+3)} < 1$ the two tails therefore satisfy

$$\begin{aligned} T_N(\beta) &\leq \frac{W_{N+1}e^{-\beta(N+1)^2}}{1 - r_N(\beta)}, \\ \frac{1}{4} \sum_{k>N} kW_k e^{-\beta(k^2-9)} &\leq \frac{(N+1)W_{N+1}e^{-\beta((N+1)^2-9)}}{4(1 - r_N(\beta))}. \end{aligned} \quad (\text{B.75})$$

Each estimate follows by bounding the ratio of consecutive positive summands by $r_N(\beta)$ and summing the geometric series. Positivity and $\Pi_{+,b}(\beta_{\text{UV}}, \kappa) \geq D_0$ give the following uniform bound for $0 \leq \kappa \leq 1$.

$$\|\mathfrak{R}_b - \mathfrak{R}_b^{(N)}\|_\infty \leq \frac{T_N(\beta_{\text{IR}}) + T_N(\beta_{\text{UV}})}{D_0} \leq \frac{2T_N(\beta_{\text{UV}})}{D_0}. \quad (\text{B.76})$$

At $N = 30$ one has $T_{30}(\beta_{\text{UV}}) < 10^{-1721}$. The perturbation bound for contractions then gives

$$|\kappa_b^* - \kappa_{30}^*| \leq \frac{\|\mathfrak{R}_b - \mathfrak{R}_b^{(30)}\|_\infty}{1 - \mathfrak{q}_b^{\text{ctr}}} < 10^{-1705}. \quad (\text{B.77})$$

The finite evaluation at $N = 30$ uses interval arithmetic to enclose each operation at decimal precision $p = 220$. Addition and multiplication use the same directed rounding as division. Lower endpoints round towards negative infinity and upper endpoints towards positive infinity. Scaled integer-square-root bounds enclose square roots. Machin's identity

$$\pi = 16 \arctan(1/5) - 4 \arctan(1/239)$$

is evaluated using alternating-series remainder bounds. With $\eta_p = 10^{-(p+15)}$ each arctan-gent series is stopped when the first omitted term has upper bound below η_p . At $p = 220$

the two retained series have 166 and 49 terms respectively. Exponentials use dyadic reduction to $0 \leq h \leq 1/8$ with the remainder bound

$$0 \leq e^h - \sum_{j=0}^M \frac{h^j}{j!} \leq \frac{h^{M+1}}{(M+1)!} \frac{1}{1 - h/(M+2)}. \quad (\text{B.78})$$

Outward-rounded squaring reverses the reduction. Reciprocation treats negative arguments. The Taylor degree is the first M for which the outward-rounded bound in (B.78) is below η_p . The largest degree used in the $N = 30$ evaluation is 95.

Let $[\underline{\kappa}_2, \bar{\kappa}_2]$ be the resulting enclosure of the exact second iterate. Its width is at most 3.671×10^{-249} and hence less than 10^{-230} . If ϵ_{it} and ϵ_{tail} denote the certified bounds in (B.69) and (B.77) respectively then

$$\kappa_b^* \in [\underline{\kappa}_2 - \epsilon_{\text{it}} - \epsilon_{\text{tail}}, \bar{\kappa}_2 + \epsilon_{\text{it}} + \epsilon_{\text{tail}}]. \quad (\text{B.79})$$

This enclosure accounts for both rounding and iteration as well as truncation. It lies strictly inside (B.70). Table 3 separates these three error contributions. The final fixed-point enclosure has width less than 6.16×10^{-144} . The iteration error dominates the other two contributions. The much smaller second-iterate width measures the finite interval evaluation itself.

Repeating the same interval procedure at $N = 32$ and $p = 260$ gives the second row. In this evaluation the arctangent series retain 195 and 57 terms and the largest exponential Taylor degree is 109. The two fixed-point enclosures overlap and each verifies the displayed interval (B.70). The full rational interval endpoints are supplied in the accompanying computational supplement.

There is also a truncation-independent one-step check. Since $\kappa_b^* = \mathfrak{R}_b(\kappa_b^*)$ and $\mathfrak{R}_b(\kappa) \leq e^{-9(\beta_{\text{IR}} - \beta_{\text{UV}})}$ one obtains

$$|\kappa_b^* - \mathfrak{R}_b(0)| \leq \mathfrak{q}_b^{\text{ctr}} \kappa_b^* < 2.12 \times 10^{-81}. \quad (\text{B.80})$$

One full-spectrum iteration from the unattenuated decomposition end therefore has absolute error less than 2.12×10^{-81} .

The integer gap in Proposition 3.9 gives an exact interval of spectral stability. Spectral functional calculus yields the following identity for every $0 < \lambda < 1$.

$$\mathbf{1}_{[0, \lambda)}(R_b) = P_b^{\text{pr}}. \quad (\text{B.81})$$

The selected sector is therefore constant on the open threshold interval $(0, 1)$. Equation (6.62) makes it invariant under every Gaussian scale equivalence $b \mapsto b'$.

Every threshold in the displayed gap selects exactly the same prime subspace. A finite-energy cutoff approximates individual states in that subspace with a tail error. The algebraic and geometric identifications used to construct the response are compared in Appendix F.

N	p	second-iterate width	ϵ_{it}	ϵ_{tail}
30	220	3.671×10^{-249}	3.076×10^{-144}	9.919×10^{-1707}
32	260	4.255×10^{-289}	3.076×10^{-144}	1.605×10^{-1936}

Table 3. Upper bounds from two outward-rounded evaluations of the endpoint-retention map. Here N is the last retained shell and p is the decimal working precision. The iteration and tail bounds enlarge the second-iterate interval to enclose the fixed point of the full series.

B.6 Global endpoint budget and permanent residual

For $n \geq 1$ let

$$\omega_b(n) := \text{card}\{\pi \in \mathcal{P}_b : \nu_{u_b, n}(\pi) > 0\} \quad (\text{B.82})$$

be the number of occupied prime directions. On $\mathcal{H}_b^+$ define mutually orthogonal projections

$$\begin{aligned} P_{b,0}\phi_{b,n} &:= \mathbf{1}_{\{n=1\}}\phi_{b,n}, \\ P_b^{\text{pp}}\phi_{b,n} &:= \mathbf{1}_{\{\omega_b(n)=1\}}\phi_{b,n}, \\ P_b^{\text{res}}\phi_{b,n} &:= \mathbf{1}_{\{\omega_b(n)\geq 2\}}\phi_{b,n}. \end{aligned} \quad (\text{B.83})$$

They correspond under transport to the Fock vacuum and the one-coordinate axis (3.72) together with its multi-charge complement. Hence

$$\mathcal{H}_b^+ = P_{b,0}\mathcal{H}_b^+ \oplus P_b^{\text{pp}}\mathcal{H}_b^+ \oplus P_b^{\text{res}}\mathcal{H}_b^+. \quad (\text{B.84})$$

Let $T_b^+(t) := e^{-tA_b/2}|_{\mathcal{H}_b^+}$. All three projectors are diagonal in the complete correlation basis and therefore commute with $T_b^+(t)$.

For $t > 0$ define the sector operators by

$$T_{b,0}(t) := P_{b,0}T_b^+(t), \quad T_b^{\text{pp}}(t) := P_b^{\text{pp}}T_b^+(t), \quad T_b^{\text{res}}(t) := P_b^{\text{res}}T_b^+(t). \quad (\text{B.85})$$

Write $Z_{b,j}(t) := \text{Tr } T_{b,j}(t)$ with $T_{b,\text{pp}} := T_b^{\text{pp}}$ and $T_{b,\text{res}} := T_b^{\text{res}}$. Define their trace fractions by

$$\Omega_{b,j}(t) := \frac{Z_{b,j}(t)}{Z_b^+(t)}, \quad j \in \{0, \text{pp}, \text{res}\}. \quad (\text{B.86})$$

Normalize the complete positive-mode state and all three sector states by

$$\begin{aligned} \rho_b^+(t) &:= \frac{T_b^+(t)}{Z_b^+(t)}, & \rho_{b,0}(t) &:= \frac{T_{b,0}(t)}{Z_{b,0}(t)}, \\ \rho_b^{\text{pp}}(t) &:= \frac{T_b^{\text{pp}}(t)}{Z_b^{\text{pp}}(t)}, & \rho_b^{\text{res}}(t) &:= \frac{T_b^{\text{res}}(t)}{Z_b^{\text{res}}(t)}. \end{aligned} \quad (\text{B.87})$$

Use the indexed notation $\rho_{b,\text{pp}} := \rho_b^{\text{pp}}$ and $\rho_{b,\text{res}} := \rho_b^{\text{res}}$.

Theorem B.9 (Endpoint sector budget and entropy decomposition). (i) *The sector operators are positive and trace class. They satisfy*

$$T_b^+(t) = T_{b,0}(t) + T_b^{\text{pp}}(t) + T_b^{\text{res}}(t). \quad (\text{B.88})$$

Taking traces gives

$$Z_b^+(t) = Z_{b,0}(t) + Z_b^{\text{pp}}(t) + Z_b^{\text{res}}(t), \quad (\text{B.89})$$

where each term is the trace of the corresponding positive kernel. At both finite endpoints $t_{\text{UV}} = b^2$ and $t_{\text{IR}} = \pi b^2$ one has

$$Z_b^{\text{res}}(t) > 0. \quad (\text{B.90})$$

(ii) For every $t > 0$ one has

$$\rho_b^+(t) = \Omega_{b,0}(t)\rho_{b,0}(t) + \Omega_{b,\text{pp}}(t)\rho_b^{\text{pp}}(t) + \Omega_{b,\text{res}}(t)\rho_b^{\text{res}}(t) \quad (\text{B.91})$$

on mutually orthogonal supports. The entropy satisfies

$$\begin{aligned} S(\rho_b^+(t)) &= H(\Omega_{b,0}(t), \Omega_{b,\text{pp}}(t), \Omega_{b,\text{res}}(t)) \\ &\quad + \Omega_{b,\text{pp}}(t)S(\rho_b^{\text{pp}}(t)) + \Omega_{b,\text{res}}(t)S(\rho_b^{\text{res}}(t)). \end{aligned} \quad (\text{B.92})$$

The vacuum conditional state is pure and hence contributes no conditional entropy. At the two permanent endpoints every conditional sector obeys the escort law

$$\rho_{b,j}(\pi b^2) = \frac{\rho_{b,j}(b^2)^\pi}{\text{Tr } \rho_{b,j}(b^2)^\pi}, \quad j \in \{0, \text{pp}, \text{res}\}, \quad (\text{B.93})$$

and the sector fractions transform as

$$\Omega_{b,j}(\pi b^2) = \frac{\Omega_{b,j}(b^2)^\pi \text{Tr } \rho_{b,j}(b^2)^\pi}{\sum_{\ell \in \{0, \text{pp}, \text{res}\}} \Omega_{b,\ell}(b^2)^\pi \text{Tr } \rho_{b,\ell}(b^2)^\pi}. \quad (\text{B.94})$$

Proof. The orthogonal decomposition (B.84) and commutation with A_b give the operator and trace budgets. Positivity and trace class are inherited from $T_b^+(t)$. There are at least two spectral primes by [1, Section 5]. Their cascade product has two occupied prime directions and contributes a strictly positive Gaussian eigenvalue to $Z_b^{\text{res}}(t)$ for every finite $t > 0$.

Divide the operator budget (B.88) by $Z_b^+(t)$ to obtain (B.91). Apply (B.22) with zero conditional entropy on the one-dimensional vacuum support. Since every sector projector commutes with A_b one has

$$T_{b,j}(\pi b^2) = T_{b,j}(b^2)^\pi.$$

Normalization proves (B.93). Taking traces of the same identity and using $Z_{b,j}(b^2) = \Omega_{b,j}(b^2)Z_b^+(b^2)$ gives (B.94). □

The normalized fractions satisfy the exact closure

$$\Omega_{b,0}(t) + \Omega_{b,\text{pp}}(t) + \Omega_{b,\text{res}}(t) = 1. \quad (\text{B.95})$$

The residual is the sector with at least two occupied prime directions. Its weight is fixed by the endpoint state and the orthogonal decomposition.

B.7 Three representations of the endpoint residual

The scalar trace of the residual is

$$\mathcal{E}_b^N(t) := \text{Tr } T_b^{\text{res}}(t) = Z_b^{\text{res}}(t). \quad (\text{B.96})$$

Its spatial representation is the integral kernel

$$K_b^{\text{res}}(d, d'; t) := \sum_{\substack{n \geq 1 \\ \omega_b(n) \geq 2}} e^{-tE_{b,n}/2} \phi_{b,n}(d) \overline{\phi_{b,n}(d')}. \quad (\text{B.97})$$

Transport the compact charge action (3.54) to $\mathcal{H}_b^+$ by Fock evaluation. The phase projection is

$$\mathcal{E}_b^S(t; g) := \text{Tr}(U_b^\times(g) T_b^{\text{res}}(t)), \quad g \in G_b^\times. \quad (\text{B.98})$$

Let $\mathfrak{S}_{1,b}^{\text{diag}}$ denote the trace-class operators on $\mathcal{H}_b^+$ that are diagonal in the complete natural-mode basis. For $D \in \mathfrak{S}_{1,b}^{\text{diag}}$ write K_D for its spatial integral kernel and define the completed endpoint readout

$$\mathfrak{D}_b(D) := (\text{Tr } D, K_D, [g \mapsto \text{Tr}(U_b^\times(g) D)]). \quad (\text{B.99})$$

Proposition B.10 (Three representations of the endpoint residual). *For every $t > 0$ the kernel $K_b^{\text{res}}(\cdot, \cdot; t)$ and the function $g \mapsto \mathcal{E}_b^S(t; g)$ are positive definite. Their scalar specializations agree with the residual trace.*

$$\int_{\mathcal{W}_b} K_b^{\text{res}}(d, d; t) dd = \mathcal{E}_b^N(t), \quad (\text{B.100})$$

$$\mathcal{E}_b^S(t; 1) = \mathcal{E}_b^N(t). \quad (\text{B.101})$$

Thus the trace and spatial kernel represent the same positive residual operator as the compact-charge phase response. The completed readout $\mathfrak{D}_b$ is injective on $\mathfrak{S}_{1,b}^{\text{diag}}$. Its scalar component records the sum of the diagonal coefficients. Its compact-charge component recovers every coefficient by Haar–Fourier inversion.

Proof. Positive definiteness of the spatial kernel follows term by term from (B.97). Its diagonal series converges uniformly and integration with $\|\phi_{b,n}\|_2 = 1$ gives (B.100). For $g_1, \dots, g_r \in G_b^\times$ and $c_1, \dots, c_r \in \mathbb{C}$ one has

$$\sum_{i,j} \overline{c_i} c_j \mathcal{E}_b^S(t; g_i^{-1} g_j) = \text{Tr} \left[\left(\sum_i c_i U_b^\times(g_i) \right)^* \left(\sum_j c_j U_b^\times(g_j) \right) T_b^{\text{res}}(t) \right] \geq 0. \quad (\text{B.102})$$

Evaluation at the identity proves (B.101). The third component of (B.99) is the full-charge transform (5.35) after Fock evaluation. Theorem 5.4 recovers every occupation coefficient and proves injectivity. Absolute summability and the uniform bound on $\phi_{b,n}$ also give

$$\text{Tr } D = \int_{\mathcal{W}_b} K_D(d, d) dd.$$

Evaluation of the charge field at the identity gives the same trace. $\square$

Thus scalar observation compresses a charge readout that retains every diagonal coefficient.

Theorem B.11 (Conditional maximum entropy and endpoint amplitude). *For every $t > 0$ the state $\rho_b^{\text{res}}(t)$ is the unique maximizer of $S(\rho) = -\text{Tr}(\rho \log \rho)$ among all density operators ρ supported on $P_b^{\text{res}}\mathcal{H}_b^+$ with the same mean correlation energy as specified by*

$$\text{Tr}(\rho A_b) = \text{Tr}(\rho_b^{\text{res}}(t) A_b). \quad (\text{B.103})$$

The permanent residual therefore has an intrinsic conditional maximum-entropy characterization.

Put $\delta t_b := t_{\text{IR}} - t_{\text{UV}} = (\pi - 1)b^2$ and define on the residual sector

$$Z_b^{\text{UV} \rightarrow \text{IR}, \text{res}} := P_b^{\text{res}} e^{-\delta t_b A_b / 2}, \quad (\text{B.104})$$

$$M_b^{\text{UV} \rightarrow \text{IR}, \text{res}} := \left(Z_b^{\text{UV} \rightarrow \text{IR}, \text{res}} \right)^{1/2} = P_b^{\text{res}} e^{-\delta t_b A_b / 4}. \quad (\text{B.105})$$

Then

$$T_b^{\text{res}}(t_{\text{IR}}) = T_b^{\text{res}}(t_{\text{UV}})^{1/2} Z_b^{\text{UV} \rightarrow \text{IR}, \text{res}} T_b^{\text{res}}(t_{\text{UV}})^{1/2}, \quad (\text{B.106})$$

and every residual component $\phi_{b,n}$ satisfies

$$M_b^{\text{UV} \rightarrow \text{IR}, \text{res}} \phi_{b,n} = \exp \left[-\frac{(\pi - 1)b^2 E_{b,n}}{4} \right] \phi_{b,n}. \quad (\text{B.107})$$

The operator $Z_b^{\text{UV} \rightarrow \text{IR}, \text{res}}$ is the endpoint quadratic weight ratio. Its positive square root $M_b^{\text{UV} \rightarrow \text{IR}, \text{res}}$ is the corresponding response-amplitude transfer. Both commute with the compact prime-charge action.

Proof. Let σ be a density operator satisfying (B.103). Positivity of relative entropy gives

$$0 \leq D(\sigma \| \rho_b^{\text{res}}(t)) = -S(\sigma) + \frac{t}{2} \text{Tr}(\sigma A_b) + \log Z_b^{\text{res}}(t). \quad (\text{B.108})$$

For $\sigma = \rho_b^{\text{res}}(t)$ the right-hand side is zero. Equality of the mean energies therefore implies $S(\sigma) \leq S(\rho_b^{\text{res}}(t))$ with equality only when the two states agree. This proves the variational statement.

All operators in (B.104)–(B.106) are functions of A_b compressed by a commuting spectral projector. Adding their exponents proves the factorization. Spectral functional calculus gives (B.107). The compact charge action is diagonal in the same complete natural-mode basis and hence commutes with both transfer operators. $\square$

B.8 Conservative completion of the permanent endpoint flow

The positive amplitude in (B.105) is a contraction when the defect channel is ignored. Retaining that channel restores a conservative flow. Put $\mathcal{H}_b^{\text{res}} := P_b^{\text{res}}\mathcal{H}_b^+$ and for $k \in \mathbb{Z}$ set

$$\begin{aligned} t_{b,k} &:= \pi^k b^2, & M_{b,k} &:= \exp \left[-\frac{t_{b,k+1} - t_{b,k}}{4} A_b \right] \Big|_{\mathcal{H}_b^{\text{res}}}, \\ D_{b,k} &:= (I - M_{b,k}^2)^{1/2}. \end{aligned} \quad (\text{B.109})$$

Both operators are commuting positive functions of the residual generator. Define the scale-cell operator on $\mathcal{H}_b^{\text{res}} \oplus \mathcal{H}_b^{\text{res}}$ by

$$\mathbb{J}_{b,k} := \begin{pmatrix} M_{b,k} & D_{b,k} \\ D_{b,k} & -M_{b,k} \end{pmatrix}. \quad (\text{B.110})$$

Theorem B.12 (Conservative scale cascade and its unitary gauge). *Every $\mathbb{J}_{b,k}$ is a self-adjoint unitary and*

$$\|M_{b,k}x\|^2 + \|D_{b,k}x\|^2 = \|x\|^2 \quad (x \in \mathcal{H}_b^{\text{res}}). \quad (\text{B.111})$$

Let

$$\mathfrak{H}_b^{\text{sc}} := \ell^2(\mathbb{Z}; \mathcal{H}_b^{\text{res}} \oplus \mathcal{H}_b^{\text{res}}), \quad \mathbb{J}_b := \bigoplus_{k \in \mathbb{Z}} \mathbb{J}_{b,k}, \quad (\text{B.112})$$

and let $\mathbb{S}$ be the bilateral shift in the scale index. Then

$$\mathbb{U}_b^{\text{sc}} := \mathbb{S} \mathbb{J}_b \quad (\text{B.113})$$

is unitary. It is the conservative bilateral realization of the generated endpoint-scale cascade. In the physical continuation this cascade is termed the Gaussian UV/IR flow. The first channel carries retained amplitude and the second carries exactly the complementary endpoint defect. The compact prime-charge action commutes with every cell and with the complete cascade.

Put $\mathcal{K}_b^{\text{sc}} := \mathcal{H}_b^{\text{res}} \oplus \mathcal{H}_b^{\text{res}}$. There is a diagonal unitary

$$\mathbb{V}_b = \bigoplus_{k \in \mathbb{Z}} V_{b,k} \quad \text{on} \quad \ell^2(\mathbb{Z}; \mathcal{K}_b^{\text{sc}}) \quad (\text{B.114})$$

such that

$$\mathbb{V}_b \mathbb{U}_b^{\text{sc}} \mathbb{V}_b^* = \mathbb{S} \otimes I_{\mathcal{K}_b^{\text{sc}}}. \quad (\text{B.115})$$

Unitary equivalence gives

$$\sigma(\mathbb{U}_b^{\text{sc}}) = \mathbb{T}. \quad (\text{B.116})$$

The local Julia cells form a flat unitary connection on the bilateral scale lattice. On the contractible index graph $\mathbb{Z}$ their weights are removed by a unitary gauge. The projections distinguishing retained and defect amplitudes are transported by the same gauge when individual cell responses are compared.

Proof. Functional calculus gives $M_{b,k}D_{b,k} = D_{b,k}M_{b,k}$ and $M_{b,k}^2 + D_{b,k}^2 = I$. Block multiplication proves that every $\mathbb{J}_{b,k}$ is a self-adjoint unitary and gives (B.111). Both its orthogonal direct sum and the bilateral shift are unitary. Their product is therefore unitary. The charge action commutes with every cell entry because these are functions of the same diagonal generator. It consequently commutes with the cascade.

Write $(\mathbb{U}_b^{\text{sc}}x)_k = \mathbb{J}_{b,k-1}x_{k-1}$. Choose $V_{b,0} = I$ and determine the remaining unitary operators recursively by

$$V_{b,k}\mathbb{J}_{b,k-1}V_{b,k-1}^* = I \quad (k \in \mathbb{Z}). \quad (\text{B.117})$$

The recursion is solvable in both directions because every $\mathbb{J}_{b,k}$ is unitary. For $x \in \mathfrak{H}_b^{\text{sc}}$ one has

$$(\mathbb{V}_b \mathbb{U}_b^{\text{sc}} \mathbb{V}_b^* x)_k = V_{b,k} \mathbb{J}_{b,k-1} V_{b,k-1}^* x_{k-1} = x_{k-1}, \quad (\text{B.118})$$

which proves (B.115). The bilateral shift has spectrum $\mathbb{T}$. Unitary equivalence proves (B.116). $\square$

B.9 Integrated UV/IR entropy–energy bridge

The differential identities (6.28)–(6.33) and [1, Theorem 3.6] can be integrated over the complete endpoint interval. Its positive lower endpoint allows Gaussian domination to control all spectral moments uniformly there. With $t_{\text{UV}} = b^2$ and $t_{\text{IR}} = \pi b^2$ one obtains

$$\log \frac{Z_b(t_{\text{UV}})}{Z_b(t_{\text{IR}})} = \frac{1}{2} \int_{t_{\text{UV}}}^{t_{\text{IR}}} \langle E \rangle_t dt, \quad (\text{B.119})$$

$$S_{\text{sp}}(t_{\text{UV}}) - S_{\text{sp}}(t_{\text{IR}}) = \frac{1}{4} \int_{t_{\text{UV}}}^{t_{\text{IR}}} t \text{Var}_t(E) dt, \quad (\text{B.120})$$

$$H_{\text{pos},b}(t_{\text{IR}}) - H_{\text{pos},b}(t_{\text{UV}}) = \frac{1}{2} \int_{t_{\text{UV}}}^{t_{\text{IR}}} \mathcal{I}_{\text{pos},b}(t) dt. \quad (\text{B.121})$$

For the universal additive state functional calculus and $\beta_{\text{IR}} = \pi \beta_{\text{UV}}$ give the endpoint escort. Integrating (B.44) on the same dimensionless interval gives its entropy loss.

$$\rho_{+,b}(\beta_{\text{IR}}) = \frac{\rho_{+,b}(\beta_{\text{UV}})^\pi}{\text{Tr} \rho_{+,b}(\beta_{\text{UV}})^\pi}, \quad (\text{B.122})$$

$$S(\rho_{+,b}(\beta_{\text{UV}})) - S(\rho_{+,b}(\beta_{\text{IR}})) = \int_{\beta_{\text{UV}}}^{\beta_{\text{IR}}} \beta \text{Var}_{\rho_{+,b}(\beta)}(N_+^2) d\beta. \quad (\text{B.123})$$

At both endpoints the exact longitudinal–transverse budget (B.29) has $\Omega_{T,b} > 0$ by Theorem B.4. The window lengths and endpoint times in (2.1) and (2.7) give the spans

$$\Sigma_b := \log \frac{L_{\text{IR}}}{\ell_{\text{UV}}} = \frac{1}{2} \log \pi, \quad \log \frac{t_{\text{IR}}}{t_{\text{UV}}} = \log \pi. \quad (\text{B.124})$$

The logarithmic span compares the two Gaussian endpoints on the same carrier. The vacuum, prime-power and multi-charge sectors in (B.84) partition its normalized state exactly.

The exact response fixed point has a dimensionless endpoint action. Define

$$\Xi_b^{\text{resp}} := \frac{-\log \kappa_b^*}{\Sigma_b}. \quad (\text{B.125})$$

The certified approximation in (B.71) gives

$$\Xi_b^{\text{resp}} = 139.251865234744 \dots \quad (\text{B.126})$$

The first-shell value is

$$\Xi_{b,3}^{\text{resp}} := \frac{9(\beta_{\text{IR}} - \beta_{\text{UV}})}{\Sigma_b} = 139.251865234742 \dots \quad (\text{B.127})$$

Their relative difference is below 10^{-12} . The quantity Ξ_b^{resp} is the normalized action of the endpoint retention response. Shannon entropy is given separately by (B.92). The near saturation of Ξ_b^{resp} by the first-shell square 3^2 is the quantitative fixed-point form of the minimal additive threshold. Appendix C independently selects the same first positive shell index under antipodal geometric lifting.

The occupation basis diagonalizes the finite-order sector projectors, occupation operators and $H_b^\times$. Charge reflection exchanges conjugate sectors and commutes with even Gaussian functions of $L_b^\times$ by Proposition 3.8. In the additive channel the exchange projectors commute with total degree, connected response and the longitudinal–transverse projectors, with $P_- \leq P_T$. Retaining the signed defect gives the compact-charge covariance of Theorem 4.5; uniform-defect compression recovers normalized merge. These sector resolutions preserve the endpoint budgets because their projectors commute with the corresponding Gaussian operators.

C Projective representation-fibre continuation

Representation capacity is attached to the root spectral value recovered by the source-weighted readout of Section 2. It counts linear coefficients between input and output states, rather than histories or phases above that value.

The normalized correlation generator supplies the dimensionless square spectrum

$$\hat{A}_b := \frac{A_b}{E_{b,1}}, \quad \hat{A}_b \phi_{b,n} = n^2 \phi_{b,n}. \quad (\text{C.1})$$

The radial spectrum has one state at each square index. To retain every linear coefficient between an input and an output in the generated rank n , use the spaces

$$W_{b,n} := \mathbb{C}_b^n, \quad \mathcal{C}_{b,n} := \text{End}_{\mathbb{C}_b}(W_{b,n}), \quad \dim_{\mathbb{C}_b} \mathcal{C}_{b,n} = n^2. \quad (\text{C.2})$$

The full coefficient space is determined by this input and output task: it contains every rank-one map.

Lemma C.1 (Complete coefficient space). *For a finite-dimensional vector space W , the smallest linear subspace of $\text{End}(W)$ containing every map $w \mapsto f(w)u$, with $u \in W$ and $f \in W^*$, is $\text{End}(W)$. Its dimension is $(\dim W)^2$.*

Proof. For any basis (e_i) and its dual (e^j) , these maps include the matrix units $w \mapsto e^j(w)e_i$. They form a basis of $\text{End}(W)$. $\square$

Thus retaining all input and output coefficients gives the capacity n^2 independently of the choice of basis. A rotational action and a geometric eigenspace require a representation-fibre lift. Such a lift realizes these coefficient spaces as geometric multiplicity spaces while leaving the natural-mode semiring and its prime factorization unchanged. When one compact Laplace spectrum realizes both the square energy coordinate in (C.1) and the quadratic capacity in (C.2) their common counting law determines the dimension.

C.1 The Weyl dimension of a square-capacity lift

Definition C.2 (Square-capacity Laplace realization). Let X be a closed connected Riemannian manifold. A representation-fibre lift is a square-capacity Laplace realization on X if one common positive normalization yields a finite integer $q_0 \geq 1$ and an index set

$$\mathcal{Q} \in \{\{1, 2, 3, \dots\}, \{1, 3, 5, \dots\}\}$$

such that its complete scalar Laplace spectrum consists of finitely many exceptional shells of finite multiplicity together with one shell for every $q \in \mathcal{Q}$ satisfying $q \geq q_0$ with

$$\lambda_q = q^2 + O(1), \quad m_q = q^2 + O(q), \quad (\text{C.3})$$

where m_q is the multiplicity of the shell with eigenvalue λ_q . The first asymptotic law is the bounded recentering of (C.1). The second is the corresponding geometric realization of (C.2) up to lower-order terms that preserve its quadratic growth.

Proposition C.3 (Weyl dimension of a square-capacity lift). *If the generated square-energy law (C.1) and the coefficient-capacity law (C.2) admit a common Laplace realization in the sense of Definition C.2 then the realizing manifold has dimension three. Its scalar counting function satisfies*

$$N_X(\Lambda) = c_{\mathcal{Q}} \Lambda^{3/2} + O(\Lambda), \quad c_{\mathcal{Q}} = \begin{cases} 1/3, & \mathcal{Q} = \{1, 2, 3, \dots\}, \\ 1/6, & \mathcal{Q} = \{1, 3, 5, \dots\}. \end{cases} \quad (\text{C.4})$$

Proof. Let $Q(\Lambda)$ be the largest retained index with $\lambda_q \leq \Lambda$. Equation (C.3) gives $Q(\Lambda) = \Lambda^{1/2} + O(1)$. The finite initial segment below q_0 and the exceptional shells contribute $O(1)$ to $N_X(\Lambda)$. For the full sequence one has

$$\sum_{q \leq Q} q^2 = \frac{Q^3}{3} + O(Q^2),$$

whereas restriction to odd q gives $Q^3/6 + O(Q^2)$. The accumulated $O(q)$ term in the multiplicities contributes $O(Q^2)$. This proves (C.4). The Weyl law for a closed d -dimensional manifold has leading exponent $d/2$ [15, Chapter VI]. Hence $d/2 = 3/2$ and $d = 3$. $\square$

The proposition determines the dimension from the complete lifted counting law. Within the round spherical class the exact geometry is fixed by requiring the shell capacity to be affine in the shell eigenvalue with positive slope after a common dimensionless normalization. The constant term is the zero-energy shell capacity. This relation selects a spherical parent whose antipodal compression gives the projective realization.

C.2 Exact affine selection in the spherical class

On the unit round d -sphere with $d \geq 2$ the scalar Laplace eigenvalue and multiplicity at degree l are [15]

$$\lambda_l^{(d)} = l(l + d - 1), \quad m_l^{(d)} = \frac{(2l + d - 1)(l + d - 2)!}{l!(d - 1)!}. \quad (\text{C.5})$$

Definition C.4 (Spherical affine energy–capacity class). For a round d -sphere or the even-degree sector determined by its antipodal quotient shellwise affine energy–capacity closure means that there are constants $a > 0$ and c independent of l such that

$$m_l^{(d)} = a\lambda_l^{(d)} + c \quad (\text{C.6})$$

on every retained shell. The coefficient a fixes the common response unit and c is the zero-energy shell capacity.

Theorem C.5 (Classification of spherical affine energy–capacity lifts). *Within the spherical affine energy–capacity class dimension three is the unique dimension satisfying the shellwise relation on all degrees or on all even degrees retained by the antipodal quotient. In that dimension*

$$a = 1, \quad c = 1, \quad m_l^{(3)} = \lambda_l^{(3)} + 1 = (l + 1)^2. \quad (\text{C.7})$$

The one-dimensional spherical family has constant positive-mode multiplicity and hence lies outside this affine positive-slope class.

Proof. For $d \geq 2$ the expression $m_l^{(d)}$ is a polynomial in l of degree $d - 1$ with nonzero leading coefficient whereas $a\lambda_l^{(d)} + c$ is a polynomial of degree two because $a > 0$. Equality on all nonnegative integers or on the infinite even subsequence implies a polynomial identity and therefore $d - 1 = 2$. Thus $d = 3$. Substitution gives

$$\lambda_l^{(3)} = l(l + 2), \quad m_l^{(3)} = (l + 1)^2 = \lambda_l^{(3)} + 1,$$

so $a = c = 1$. On S^1 the zero mode has multiplicity one and every positive scalar eigenvalue has multiplicity two. These multiplicities cannot satisfy the stated positive-slope affine relation. $\square$

Proposition C.3 fixes the dimension from the complete counting law. The theorem identifies the exact shells within the round spherical class and its antipodal quotients. In this class the eigenvalue equals the shell multiplicity minus one, the dimension of the traceless subspace in the matrix realization.

C.3 The full three-spherical parent spectrum

For every $n \geq 1$ set

$$j_n := \frac{n - 1}{2}, \quad \mathcal{F}_n^{\text{sph}} := V_{j_n} \otimes V_{j_n}^* \cong \text{End}(V_{j_n}), \quad \mathcal{F}_n^{\text{sph}} \cong \mathcal{C}_{b,n} \otimes_{\mathbb{C}_b} \mathbb{C}, \quad \dim \mathcal{F}_n^{\text{sph}} = n^2. \quad (\text{C.8})$$

Here V_j denotes the irreducible $SU(2)$ representation of spin j and dimension $2j + 1$. For the selected three-dimensional spherical member the abstract coefficient capacity (C.2) acquires a rotational realization. Write its action as ρ_j . The left–right action on the matrix fibre is

$$(g_L, g_R) \cdot T = \rho_{j_n}(g_L) T \rho_{j_n}(g_R)^{-1}. \quad (\text{C.9})$$

On the diagonal subgroup this is conjugation. The decomposition $T = (\text{tr } T/n)I_n + T_0$ with $\text{tr } T_0 = 0$ is orthogonal for the Hilbert–Schmidt inner product and invariant under that subgroup because conjugation preserves both the trace and I_n . Independent left and right actions send I_n to $\rho_{j_n}(g_L g_R^{-1})$. This matrix need not be scalar when $n > 1$. Thus the scalar–traceless splitting refers specifically to diagonal conjugation.

Define the all-mode parent Hilbert space and its shifted generator by

$$\mathcal{H}_b^{\text{sph},3} := \bigoplus_{n \geq 1} \mathcal{F}_n^{\text{sph}} \otimes \text{span}\{\phi_{b,n}\}, \quad (\text{C.10})$$

$$\mathcal{A}_b^{\text{sph},3} := \bigoplus_{n \geq 1} (n^2 - 1) I_{\mathcal{F}_n^{\text{sph}}} \otimes |\phi_{b,n}\rangle\langle\phi_{b,n}|. \quad (\text{C.11})$$

Theorem C.6 (Full three-spherical parent lift). *The spectrum of (C.11) with its multiplicities is*

$$\lambda_n = n^2 - 1, \quad m_n = n^2, \quad n = 1, 2, 3, \dots \quad (\text{C.12})$$

It is the complete scalar Laplace spectrum of the unit round three-sphere. On every shell restriction to diagonal conjugation gives

$$\mathcal{F}_n^{\text{sph}} \cong \mathbb{C}I_n \oplus \text{End}_0(V_{j_n}), \quad \dim \text{End}_0(V_{j_n}) = n^2 - 1 = \lambda_n. \quad (\text{C.13})$$

Under spherical energy–capacity closure the all-mode natural coordinate defines the selected spherical parent. Parity quotients are subsequent orthogonal compressions of this complete space.

Proof. Scalar harmonics of degree l on the unit S^3 have eigenvalue $l(l+2)$ and multiplicity $(l+1)^2$. Their representation fibre is $V_{l/2} \otimes V_{l/2}^*$ [15–17]. Setting $n = l+1$ proves (C.12) and (C.8). The scalar–traceless matrix splitting proves (C.13). $\square$

The scalar line in a shell is a representation-theoretic notion. Under the matrix-coefficient realization on $SU(2)$ the matrix I_n gives a function proportional to the character χ_{j_n} . This function is nonconstant for $n > 1$. The generator in (C.11) satisfies

$$\mathcal{A}_b^{\text{sph},3}(I_n \otimes \phi_{b,n}) = (n^2 - 1)(I_n \otimes \phi_{b,n}). \quad (\text{C.14})$$

The same eigenvalue acts on every traceless direction in that shell. Only the $n = 1$ shell supplies the constant zero mode.

C.4 Antipodal compression and projective representation fibres

Let

$$P_{\text{odd}}\phi_{b,n} := \mathbf{1}_{\{n \text{ odd}\}}\phi_{b,n} \quad (\text{C.15})$$

on the positive correlation-mode space. Write every retained index as $q = 2r + 1$ and put

$$j_q := \frac{q-1}{2} = r. \quad (\text{C.16})$$

For the irreducible $SU(2)$ module V_{j_q} of dimension $2j_q + 1 = q$ define the fibre

$$\mathcal{F}_q^{\text{proj}} := \mathcal{F}_q^{\text{sph}} = V_{j_q} \otimes V_{j_q}^* \cong \text{End}(V_{j_q}), \quad \dim \mathcal{F}_q^{\text{proj}} = q^2. \quad (\text{C.17})$$

The projectively lifted Hilbert space and its nonnegative generator are

$$\mathcal{H}_b^{\text{proj},3} := \bigoplus_{\substack{q \geq 1 \\ q \text{ odd}}} \mathcal{F}_q^{\text{proj}} \otimes \text{span}\{\phi_{b,q}\}, \quad (\text{C.18})$$

$$\mathcal{A}_b^{\text{proj},3} := \bigoplus_{\substack{q \geq 1 \\ q \text{ odd}}} (q^2 - 1) I_{\mathcal{F}_q^{\text{proj}}} \otimes |\phi_{b,q}\rangle \langle \phi_{b,q}|. \quad (\text{C.19})$$

The parent and projective spaces are related by

$$\mathcal{H}_b^{\text{proj},3} = P_{\text{odd}} \mathcal{H}_b^{\text{sph},3}, \quad \mathcal{A}_b^{\text{proj},3} = P_{\text{odd}} \mathcal{A}_b^{\text{sph},3} P_{\text{odd}}. \quad (\text{C.20})$$

The first positive natural square $q^2 = 1$ is thereby recentered as the geometric zero mode. All higher retained squares become positive excitations.

Theorem C.7 (Projective spectrum with quotient topology and normalized metric). (i) *The spectrum of (C.19) with its multiplicities is*

$$\lambda_q = q^2 - 1, \quad m_q = q^2, \quad q = 1, 3, 5, \dots \quad (\text{C.21})$$

It is the scalar Laplace spectrum of the unit round real projective three-space. It is obtained from the scalar spectrum of S^3 by retaining the antipodally even harmonics. On every fibre diagonal conjugation preserves the scalar–traceless splitting

$$\mathcal{F}_q^{\text{proj}} \cong \mathbb{C} I_q \oplus \text{End}_0(V_{j_q}), \quad \dim \text{End}_0(V_{j_q}) = q^2 - 1 = \lambda_q. \quad (\text{C.22})$$

The normalized shell eigenvalue equals the traceless fibre dimension. Both summands carry that eigenvalue under $\mathcal{A}_b^{\text{proj},3}$.

(ii) *The antipodal compression in (C.20) is the spatial quotient*

$$X_3 := SU(2)/\{\pm I\} \cong SO(3) \cong \mathbb{RP}^3. \quad (\text{C.23})$$

It is a compact connected orientable three-manifold with

$$\tilde{X}_3 \cong SU(2) \cong S^3, \quad \pi_1(X_3) \cong \mathbb{Z}/2\mathbb{Z}. \quad (\text{C.24})$$

The bi-invariant Riemannian metric on X_3 is unique up to an overall positive factor. The normalization for which its scalar Laplacian has spectrum

$$q^2 - 1 \quad \text{with multiplicity } q^2, \quad q = 1, 3, 5, \dots, \quad (\text{C.25})$$

fixes that factor uniquely. Denote the resulting metric by g_1 . It is the unit round projective metric and has constant sectional curvature one.

Proof. The antipodal map acts on degree- l harmonics by $(-1)^l$ [15]. The quotient therefore retains exactly the even l and hence the odd $q = l + 1$ in Theorem C.6. Restricting that theorem to these shells proves both the spectrum and the diagonal-conjugation splitting together with the eigenvalue on each summand.

The standard covering $S^3 \cong SU(2) \rightarrow SO(3)$ has kernel $\{\pm I\}$ [16]. It gives the stated topology. Its pullback on functions is the even compression in part (i). An invariant inner product on the simple algebra $\mathfrak{su}(2)$ is a positive multiple of the negative Killing form. The bi-invariant metric therefore has only a scale freedom. Replacing g by $R^2 g$ multiplies its Laplacian by R^{-2} . Thus (C.25) fixes the unit round projective metric with sectional curvature one [15].

□

The same square coordinate has two compatible roles. Before the lift q^2 is the normalized one-component correlation energy. After the projective lift it is the full shell capacity while $q^2 - 1$ is the eigenvalue of the entire shell. The identity direction and the traceless directions are degenerate in energy for each q . The original correlation coordinate therefore fixes both the multiplicity and the eigenvalue of every lifted shell. The first three retained shells and their central characters are displayed in Table 4.

D Spectral calculations for the projective lift

The shell multiplicities of Appendix C determine finite capacities, heat–Mellin transforms and the lift from radial probabilities to fibre states. The resulting entropy correction counts representation directions within a shell. Conditional history entropy is the separate term in [1, equation (8.21)]; the root probabilities are the marginals of that history source.

D.1 Finite capacity and the three-dimensional Weyl exponent

For an odd cutoff $Q = 2N + 1$ the lifted finite-band capacity is

$$D_3(Q) := \sum_{\substack{1 \leq q \leq Q \\ q \text{ odd}}} q^2. \quad (\text{D.1})$$

Summing $(2r + 1)^2$ for $0 \leq r \leq N$ gives $(N + 1)(4N^2 + 8N + 3)/3$. Substitution of $N = (Q - 1)/2$ yields

$$D_3(Q) = \frac{Q(Q + 1)(Q + 2)}{6} = \binom{Q + 2}{3} \quad (\text{D.2})$$

for every odd $Q \geq 1$. Thus $D_3(Q) \sim Q^3/6$. Since the geometric energy is $q^2 - 1$ the number of states below energy E grows as $E^{3/2}/6$.

This exact finite sum is the odd-shell instance of Proposition C.3. It supplies the Weyl coefficient as well as the exponent for the projective realization.

The base correlation spectrum has one state per square index and therefore linear finite-band growth in Q . The representation-fibre lift changes this count to cubic growth without changing the underlying square coordinate.

Table 4. The first projective shells. Odd square indices correspond to antipodally even scalar harmonics and integer-spin fibres. The scalar-line and traceless dimensions refer to diagonal conjugation. The listed eigenvalue acts on both summands. The half-spin lift detects the opposite central character on a separate two-dimensional carrier.

q	j_q	eigenvalue $q^2 - 1$	shell capacity q^2	scalar line	traceless capacity	centre on V_{j_q}
1	0	0	1	1	0	$+I$
3	1	8	9	1	8	$+I$
5	2	24	25	1	24	$+I$

D.2 Scale covariance and stable shell platforms

The scale equivalence $U_{b,b'}$ of Theorem 6.2 preserves every normalized square coordinate. Its fibre lift is

$$\tilde{U}_{b,b'}^{\text{proj},3} := \bigoplus_{q \text{ odd}} I_{\mathcal{F}_q^{\text{proj}}} \otimes U_{b,b'}. \quad (\text{D.3})$$

The identity $U_{b,b'}\phi_{b,q} = \phi_{b',q}$ and the scale-independent fibre identity show that this lift intertwines the normalized generators

$$\tilde{U}_{b,b'}^{\text{proj},3} \mathcal{A}_b^{\text{proj},3} = \mathcal{A}_{b'}^{\text{proj},3} \tilde{U}_{b,b'}^{\text{proj},3}. \quad (\text{D.4})$$

The retained eigenvalues are $0, 8, 24, 48, \dots$ and correspond to $q = 1, 3, 5, 7, \dots$. If P_q^{proj} is the q th fibre projector then spectral functional calculus gives

$$\mathbf{1}_{[0,\lambda)}(\mathcal{A}_b^{\text{proj},3}) = P_1^{\text{proj}}, \quad 0 < \lambda < 8, \quad (\text{D.5})$$

$$\mathbf{1}_{(0,\lambda)}(\mathcal{A}_b^{\text{proj},3}) = P_3^{\text{proj}}, \quad 8 < \lambda < 24. \quad (\text{D.6})$$

Both the geometric zero sector and the first positive shell are constant on open threshold intervals and are independent of b .

D.3 Heat and Mellin transforms of the lifted spectrum

Let

$$\Theta_{\text{sq}}(a) := \sum_{n \geq 1} e^{-an^2}, \quad \Theta_{\text{odd}}(a) := \text{Tr}(P_{\text{odd}} e^{-a\hat{A}_b}) = \Theta_{\text{sq}}(a) - \Theta_{\text{sq}}(4a). \quad (\text{D.7})$$

Here $\Theta_{\text{sq}}(a) = \Theta_+(a/\pi)$ in the normalization of (5.26). The subscript distinguishes the square-energy parameter used for the geometric generator.

Theorem D.1 (Projective heat–Mellin lift). *For every $a > 0$ the lifted heat trace is*

$$Z_{\text{proj},3}(a) := \text{Tr}_{\mathcal{H}_b^{\text{proj},3}} e^{-a\mathcal{A}_b^{\text{proj},3}} = -e^a \frac{d}{da} \Theta_{\text{odd}}(a). \quad (\text{D.8})$$

As $a \downarrow 0$ one has

$$\Theta_{\text{odd}}(a) \sim \frac{\sqrt{\pi}}{4} a^{-1/2}, \quad Z_{\text{proj},3}(a) \sim \frac{\sqrt{\pi}}{8} a^{-3/2}. \quad (\text{D.9})$$

For $\Re s > 3$ one also has

$$\mathrm{Tr}\left(P_{\mathrm{odd}}\widehat{A}_b^{1-s/2}\right) = \sum_{\substack{q \geq 1 \\ q \text{ odd}}} q^{2-s} = (1 - 2^{2-s})\zeta(s-2). \quad (\text{D.10})$$

Fibre multiplication raises the heat exponent from $1/2$ to $3/2$ and shifts the Mellin zeta coordinate by two.

Proof. The definition gives

$$Z_{\mathrm{proj},3}(a) = \sum_{q \text{ odd}} q^2 e^{-a(q^2-1)} = -e^a \partial_a \sum_{q \text{ odd}} e^{-aq^2},$$

which proves (D.8). Poisson summation gives $\Theta_{\mathrm{sq}}(a) = \sqrt{\pi}/(2\sqrt{a}) - 1/2 + O(a^{-1/2}e^{-\pi^2/a})$ as $a \downarrow 0$. Differentiation of the convergent Poisson series shows that the derivative of the remainder is $O(a^{-5/2}e^{-\pi^2/a})$. Subtracting the value at $4a$ proves the first asymptotic in (D.9). Differentiation of the controlled Poisson expansion proves the second. Removal of the even indices from the defining Dirichlet series gives

$$\sum_{q \text{ odd}} q^{-(s-2)} = (1 - 2^{-(s-2)})\zeta(s-2),$$

which is (D.10). □

Let (X_3, g_1) be the normalized central quotient of Theorem C.7 and choose an orthonormal scalar-harmonic basis $\{Y_{q,\alpha} : 1 \leq \alpha \leq q^2\}$ in each retained shell. The identification

$$\mathcal{U}_b^{\mathrm{proj},3}(e_{q,\alpha} \otimes \phi_{b,q}) := Y_{q,\alpha} \quad (\text{D.11})$$

is unitary from $\mathcal{H}_b^{\mathrm{proj},3}$ to $L^2(X_3)$ and intertwines $\mathcal{A}_b^{\mathrm{proj},3}$ with the nonnegative scalar Laplacian on X_3 .

The spatial realization also makes the local and global heat descriptions identical. For every $a > 0$ the positive trace-class operator $e^{-a\mathcal{A}_b^{\mathrm{proj},3}}$ has the smooth spectral kernel

$$K_{\mathrm{proj},3}(a; x, y) = \sum_{q \text{ odd}} e^{-a(q^2-1)} \sum_{\alpha=1}^{q^2} Y_{q,\alpha}(x) \overline{Y_{q,\alpha}(y)}. \quad (\text{D.12})$$

Its operator trace is the integral of its spatial diagonal

$$Z_{\mathrm{proj},3}(a) = \mathrm{Tr}(e^{-a\mathcal{A}_b^{\mathrm{proj},3}}) = \int_{X_3} K_{\mathrm{proj},3}(a; x, x) dx. \quad (\text{D.13})$$

Homogeneity gives

$$K_{\mathrm{proj},3}(a; x, x) = \frac{Z_{\mathrm{proj},3}(a)}{\mathrm{Vol}(X_3)}, \quad \mathrm{Vol}(X_3) = \pi^2. \quad (\text{D.14})$$

The local heat-kernel coefficient independently reproduces

$$\frac{\mathrm{Vol}(X_3)}{(4\pi a)^{3/2}} = \frac{\sqrt{\pi}}{8} a^{-3/2}, \quad (\text{D.15})$$

in agreement with (D.9). The spectral theorem gives (D.12) and the trace identity. The diagonal of every spectral projector is constant on the homogeneous round projective space. Its integral is the projector rank q^2 . This proves (D.14). The unit three-sphere has volume $2\pi^2$ and its antipodal quotient has volume π^2 . The standard leading heat coefficient on a closed three-manifold [15, Chapter VI] then gives (D.15).

D.4 Endpoint states and the entropy chain rule

The fibre multiplicities determine a canonical lift of shell probabilities to complete fibre states. Let I be countable and let $\mathcal{K} = \bigoplus_{j \in I} \mathcal{F}_j$ with $0 < d_j := \dim \mathcal{F}_j < \infty$. Let P_j be the projection onto $\mathcal{F}_j$. For a probability vector $p = (p_j)_{j \in I}$ put

$$\rho_{\text{rad}}(p) := \sum_{j \in I} p_j |j\rangle\langle j| \quad \text{on } \ell^2(I), \quad (\text{D.16})$$

$$\mathfrak{L}_d(\rho_{\text{rad}}(p)) := \bigoplus_{j \in I} \frac{p_j}{d_j} I_{\mathcal{F}_j} \quad \text{on } \mathcal{K}, \quad (\text{D.17})$$

$$\mathfrak{C}_d(\sigma) := \sum_{j \in I} \text{Tr}(P_j \sigma) |j\rangle\langle j|. \quad (\text{D.18})$$

Here $\mathfrak{L}_d$ is the capacity lift and $\mathfrak{C}_d$ is shell coarse graining.

Theorem D.2 (Canonical capacity–entropy bridge). *Suppose*

$$H(p) := - \sum_j p_j \log p_j < \infty, \quad \sum_j p_j \log d_j < \infty. \quad (\text{D.19})$$

Then

(i) $\mathfrak{C}_d \mathfrak{L}_d(\rho_{\text{rad}}(p)) = \rho_{\text{rad}}(p)$. *The lifted state is the unique density operator with shell weights p_j that is invariant under every block unitary $\bigoplus_j U_j$ with $U_j \in U(\mathcal{F}_j)$.*

(ii) *The lift has the exact entropy*

$$S(\mathfrak{L}_d(\rho_{\text{rad}}(p))) = H(p) + \sum_j p_j \log d_j. \quad (\text{D.20})$$

It is the unique entropy maximizer among all density operators σ on $\mathcal{K}$ satisfying $\text{Tr}(P_j \sigma) = p_j$ for every j .

(iii) *For $\theta > 0$ define the capacity-corrected escort*

$$\mathcal{E}_\theta^{(d)}(p)_j := \frac{d_j^{1-\theta} p_j^\theta}{\sum_{k \in I} d_k^{1-\theta} p_k^\theta}. \quad (\text{D.21})$$

Whenever its denominator is finite one has

$$\frac{\mathfrak{L}_d(\rho_{\text{rad}}(p))^\theta}{\text{Tr } \mathfrak{L}_d(\rho_{\text{rad}}(p))^\theta} = \mathfrak{L}_d\left(\rho_{\text{rad}}(\mathcal{E}_\theta^{(d)}(p))\right). \quad (\text{D.22})$$

Shell entropy and fibre-capacity entropy are the two terms of one state entropy. Endpoint escorting commutes with the lift when the derived multiplicity is retained.

Proof. Equations (D.17) and (D.18) give the first identity. Independent scalar phase changes on the blocks force all off-diagonal blocks to vanish. By Schur's lemma invariance under all $U(\mathcal{F}_j)$ forces every diagonal block to be a scalar multiple of $I_{\mathcal{F}_j}$. The prescribed shell trace fixes that scalar to p_j/d_j . This proves uniqueness in part (i).

The lifted state has the eigenvalue p_j/d_j with multiplicity d_j . Direct substitution gives (D.20). If σ has the same shell weights then the relative entropy satisfies

$$D(\sigma \| \mathfrak{L}_d(\rho_{\text{rad}}(p))) = -S(\sigma) + H(p) + \sum_j p_j \log d_j \geq 0. \quad (\text{D.23})$$

Equality holds only for the lifted state. This proves part (ii). The j th block of the normalized θ th power has scalar eigenvalue proportional to p_j^θ/d_j^θ and block trace proportional to $d_j^{1-\theta}p_j^\theta$. This is (D.21) and proves part (iii). $\square$

Apply the theorem to the full spherical fibres with $I = \mathbb{N}_+$ and $d_n = n^2$. Put

$$Z_{\text{sph},3}(a) := \sum_{n \geq 1} n^2 e^{-a(n^2-1)}.$$

The normalized radial law and its capacity lift are

$$r_n(a) := \frac{n^2 e^{-a(n^2-1)}}{Z_{\text{sph},3}(a)}, \quad \rho_{\text{rad},3}(a) := \sum_{n \geq 1} r_n(a) |n\rangle\langle n|, \quad (\text{D.24})$$

and

$$\rho_{\text{sph},3}(a) := \mathfrak{L}_{n^2}(\rho_{\text{rad},3}(a)) = \frac{1}{Z_{\text{sph},3}(a)} \bigoplus_{n \geq 1} e^{-a(n^2-1)} I_{\mathcal{F}_n^{\text{sph}}}. \quad (\text{D.25})$$

At the specified Gibbs parameter a multiplying the weight of one microstate by the derived shell capacity gives the radial measure. Taking the trace within each shell recovers this same measure from the complete state.

The global correlation radius obtained in Section 2.1 is also the infrared endpoint length

$$R_b^{\text{geom}} = L_{\text{IR}}(b) = \sqrt{2\pi} b. \quad (\text{D.26})$$

This radius defines the geometric generators

$$\Delta_b^{\text{sph},3} := (R_b^{\text{geom}})^{-2} \mathcal{A}_b^{\text{sph},3}, \quad \Delta_b^{\text{proj},3} := (R_b^{\text{geom}})^{-2} \mathcal{A}_b^{\text{proj},3}. \quad (\text{D.27})$$

The dimensionless parameter multiplying either unit-radius generator in the Gaussian semigroup is

$$a_b(t) := \frac{t}{2(R_b^{\text{geom}})^2}, \quad e^{-t\Delta_b^{\text{proj},3}/2} = e^{-a_b(t)\mathcal{A}_b^{\text{proj},3}}, \quad (\text{D.28})$$

and similarly for the spherical parent.

The infrared length fixes the remaining metric scale. The metric

$$g_b := (R_b^{\text{geom}})^2 g_1 \quad (\text{D.29})$$

is the unique bi-invariant metric on X_3 whose radius is the intrinsic infrared correlation distance $R_b^{\text{geom}} = L_{\text{IR}}(b)$. Its distance and angle are

$$d_b(x, y) = R_b^{\text{geom}} d_1(x, y), \quad (\text{D.30})$$

$$\cos \angle_b(v, w) = \frac{g_b(v, w)}{\sqrt{g_b(v, v)g_b(w, w)}} = \cos \angle_1(v, w), \quad (\text{D.31})$$

and its curvature tensors satisfy

$$\text{Sec}(g_b) = (R_b^{\text{geom}})^{-2}, \quad \text{Ric}(g_b) = 2 (R_b^{\text{geom}})^{-2} g_b, \quad \text{Scal}(g_b) = 6 (R_b^{\text{geom}})^{-2}. \quad (\text{D.32})$$

The nonnegative scalar Laplacian of (X_3, g_b) is precisely $\Delta_b^{\text{proj}, 3}$ under (D.11). Hence

$$\text{Spec}(\Delta_b^{\text{proj}, 3}) = \left\{ \frac{q^2 - 1}{(R_b^{\text{geom}})^2} : q = 1, 3, 5, \dots \right\}, \quad m_q = q^2. \quad (\text{D.33})$$

The chosen infrared metric scale fixes the distance and angle together with the curvature and the dimensional square spectrum.

At the two correlation times of (2.7) the geometric heat parameters are

$$a_{\text{UV}} := a_b(t_{\text{UV}}) = \frac{1}{4\pi}, \quad a_{\text{IR}} := a_b(t_{\text{IR}}) = \frac{1}{4}, \quad \frac{a_{\text{IR}}}{a_{\text{UV}}} = \pi. \quad (\text{D.34})$$

They are independent of b . The interval coefficient $\beta_{\text{UV}} = t_{\text{UV}} E_{b,1}/2$ and the geometric coefficient $a_{\text{UV}} = t_{\text{UV}}/(2(R_b^{\text{geom}})^2)$ are the dimensionless products associated with two different generators. Each is fixed by the same dimensional endpoint time and the natural unit of its own spectral channel. Theorem C.7 leaves one positive scale in the bi-invariant metric. Equation (D.26) fixes its radius to R_b^{geom} . Scaling the unit metric by its square multiplies lengths by R_b^{geom} and preserves angles. It multiplies sectional curvature and the scalar Laplacian by $(R_b^{\text{geom}})^{-2}$. The constant-curvature identities give the Ricci and scalar-curvature formulas. Transport of the unit-radius spectrum through (D.11) gives (D.33).

To obtain the endpoint parameters substitute $(R_b^{\text{geom}})^2 = 2\pi b^2$ into $a_b(t) = t/(2(R_b^{\text{geom}})^2)$ at $t = b^2$ and $t = \pi b^2$. This gives $1/(4\pi)$ and $1/4$ after the common factor b^2 cancels. The distinction between β and a follows from their generators $\hat{A}_b = A_b/E_{b,1}$ and (D.27). Extend P_{odd} fibrewise to the spherical parent and put $P_{\text{even}} := I - P_{\text{odd}}$. The normalized projective Gibbs state is the odd conditional state

$$\rho_{\text{proj}, 3}(a) := \frac{P_{\text{odd}} \rho_{\text{sph}, 3}(a) P_{\text{odd}}}{\text{Tr}(P_{\text{odd}} \rho_{\text{sph}, 3}(a))} = \frac{1}{Z_{\text{proj}, 3}(a)} \bigoplus_{q \text{ odd}} e^{-a(q^2-1)} I_{\mathcal{F}_q^{\text{proj}}}. \quad (\text{D.35})$$

Let $\rho_{\text{even}, 3}(a)$ be the corresponding normalized even compression and define

$$\Omega_{\text{odd}}(a) := \text{Tr}(P_{\text{odd}} \rho_{\text{sph}, 3}(a)), \quad \Omega_{\text{even}}(a) := 1 - \Omega_{\text{odd}}(a). \quad (\text{D.36})$$

Then the parent entropy has the exact parity chain

$$\begin{aligned} S(\rho_{\text{sph},3}(a)) &= h_2(\Omega_{\text{odd}}(a)) + \Omega_{\text{odd}}(a)S(\rho_{\text{proj},3}(a)) \\ &\quad + \Omega_{\text{even}}(a)S(\rho_{\text{even},3}(a)). \end{aligned} \quad (\text{D.37})$$

The projective state's probability of occupying shell q is

$$p_q(a) := \frac{q^2 e^{-a(q^2-1)}}{Z_{\text{proj},3}(a)}. \quad (\text{D.38})$$

With

$$\rho_{\text{rad},\text{proj},3}(a) := \sum_{q \text{ odd}} p_q(a) |q\rangle\langle q|, \quad (\text{D.39})$$

the projective state is its odd-shell capacity lift

$$\rho_{\text{proj},3}(a) = \mathfrak{L}_{q^2}(\rho_{\text{rad},\text{proj},3}(a)), \quad p_q(a) = \text{Tr}(P_q^{\text{proj}} \rho_{\text{proj},3}(a)). \quad (\text{D.40})$$

The parent and projected states have the same shell energy $q^2 - 1$ and multiplicity q^2 . They differ in the allowed shells and their normalization. Write $\star = \text{sph}$ or proj with $(\mathcal{I}_{\text{sph}}, w^{\text{sph}}) = (\mathbb{N}_+, r)$ and $(\mathcal{I}_{\text{proj}}, w^{\text{proj}}) = (\{1, 3, 5, \dots\}, p)$.

The capacity-entropy bridge of Theorem D.2 with $d_q = q^2$ gives for either state and every $a > 0$

$$S(\rho_{\star,3}(a)) = H(w^\star(a)) + \sum_{q \in \mathcal{I}_\star} w_q^\star(a) \log q^2. \quad (\text{D.41})$$

The two terms are the shell entropy and the mean entropy within a fibre. For both index sets every energy moment is locally uniformly summable for $a > 0$. Substituting $w_q^\star = q^2 e^{-a(q^2-1)} / Z_{\star,3}$ into (6.35) therefore gives

$$\begin{aligned} S(\rho_{\star,3}(a)) &= \log Z_{\star,3}(a) + a \text{Tr}(\rho_{\star,3}(a) \mathcal{A}_b^{\star,3}), \\ \frac{dS}{da} &= -a \text{Var}_{\rho_{\star,3}(a)}(\mathcal{A}_b^{\star,3}) \leq 0. \end{aligned} \quad (\text{D.42})$$

At the fixed dimensionless endpoint parameters of (D.34) functional calculus gives an operator escort law with the same intrinsic endpoint exponent π both before and after parity compression

$$\rho_{\text{sph},3}(a_{\text{IR}}) = \frac{\rho_{\text{sph},3}(a_{\text{UV}})^\pi}{\text{Tr} \rho_{\text{sph},3}(a_{\text{UV}})^\pi}, \quad \rho_{\text{proj},3}(a_{\text{IR}}) = \frac{\rho_{\text{proj},3}(a_{\text{UV}})^\pi}{\text{Tr} \rho_{\text{proj},3}(a_{\text{UV}})^\pi}. \quad (\text{D.43})$$

The shell probabilities at the two endpoints are

$$\begin{aligned} p_q^{\text{UV}} &= \frac{q^2 e^{-(q^2-1)/(4\pi)}}{\sum_{r \text{ odd}} r^2 e^{-(r^2-1)/(4\pi)}}, \\ p_q^{\text{IR}} &= \frac{q^2 e^{-(q^2-1)/4}}{\sum_{r \text{ odd}} r^2 e^{-(r^2-1)/4}}. \end{aligned} \quad (\text{D.44})$$

Grouping the microstates into shells gives the fibre-corrected level law

$$p_q(a_{\text{IR}}) = \frac{q^{2(1-\pi)} p_q(a_{\text{UV}})^\pi}{\sum_{r \text{ odd}} r^{2(1-\pi)} p_r(a_{\text{UV}})^\pi}. \quad (\text{D.45})$$

Equation (D.45) is (D.22) with $d_q = q^2$ and $\theta = \pi$. The complete state and its radial coarse graining use one probability measure at each endpoint. The factor $q^{2(1-\pi)}$ is the pushforward of the derived fibre capacity. The entropy chain rule then counts shell uncertainty and conditional fibre uncertainty as the two components of one state entropy.

For each of the two states integration of the entropy derivative gives

$$\begin{aligned} S(\rho_{\star,3}(a_{\text{UV}})) - S(\rho_{\star,3}(a_{\text{IR}})) &= H(w^\star(a_{\text{UV}})) - H(w^\star(a_{\text{IR}})) \\ &\quad + \sum_{q \in \mathcal{I}_\star} (w_q^\star(a_{\text{UV}}) - w_q^\star(a_{\text{IR}})) \log q^2 \\ &= \int_{a_{\text{UV}}}^{a_{\text{IR}}} a \operatorname{Var}_{\rho_{\star,3}(a)}(\mathcal{A}_b^{\star,3}) da. \end{aligned} \quad (\text{D.46})$$

Every term is fixed by the same normalized endpoint state. The radial Shannon contribution and the conditional fibre-capacity contribution form the two components of the total entropy loss under this measure.

D.5 The first nontrivial shell

The projective zero mode is $q = 1$. The first nonzero antipodally retained shell is therefore

$$q = 3, \quad \dim \mathcal{F}_3^{\text{proj}} = 9, \quad \lambda_3 = 8, \quad 9 = 1 + 8. \quad (\text{D.47})$$

Under the diagonal action one has the exact matrix decomposition

$$\operatorname{End}(V_1) = \mathbb{C}I_3 \oplus \operatorname{End}_0(V_1), \quad \dim \operatorname{End}_0(V_1) = 8. \quad (\text{D.48})$$

All nine directions have eigenvalue eight. The diagonal conjugation singlet $\mathbb{C}I_3$ belongs to this positive-energy shell while the constant zero mode lies in the separate $q = 1$ shell.

The first odd shell above the scalar zero mode also has the same index as the first nonzero additive transverse quotient in (B.49)

$$q_{\min}^{\text{proj},+} = 3, \quad \min\{k : \dim P_T^{(k)} \mathcal{F}_{b,k} > 0\} = 3. \quad (\text{D.49})$$

The corresponding spaces have dimensions nine and one respectively. Their common shell index expresses compatibility of two selection rules while their dimensions and representation actions distinguish the two spaces.

The projective lift equips the square-spectrum shells with geometric representation spaces and their multiplicities. Its square coordinate and endpoint semigroup agree with the arithmetic carrier, while its geometric action resolves the additional directions within each shell.

E Central-sign and half-spin continuation

The projective realization obtained in Appendix C has a nontrivial central double cover. Its dimension is fixed by the square-capacity Weyl law while the exact spherical parent and antipodal compression follow from the affine shell identity. The minimal phase-sensitive rotational representation carries the resulting central sign. Its central character also occurs in the exterior occupation lift. The same minimal carrier gives an additive two-input realization.

E.1 Minimal central-sign representation

For the three-dimensional spherical/projective realization of Theorem C.5 the oriented local frame rotations form $SO(3)$ with universal double cover

$$\text{Spin}(3) \cong SU(2) \longrightarrow SO(3). \quad (\text{E.1})$$

The nontrivial central element $-I \in SU(2)$ maps to the identity rotation. It therefore records information invisible to an ordinary $SO(3)$ vector representation.

A phase-sensitive rotational lift is a finite-dimensional irreducible unitary representation

$$\rho : SU(2) \longrightarrow U(S) \quad (\text{E.2})$$

for which $\rho(-I) = -I_S$.

Theorem E.1 (Minimal half-spin unit). *Up to unitary equivalence the phase-sensitive rotational lift of smallest positive dimension is the defining representation on $S \cong \mathbb{C}^2$. Its highest weight is one and its spin is*

$$j = \frac{1}{2}. \quad (\text{E.3})$$

Equivalently S is the unique minimal irreducible module of the even complex Clifford algebra

$$\text{Cl}_3^0(\mathbb{C}) \cong M_2(\mathbb{C}). \quad (\text{E.4})$$

Proof. Every finite-dimensional irreducible complex representation of $SU(2)$ is $\text{Sym}^m(\mathbb{C}^2)$ for a unique $m \in \mathbb{N}_0$. Its dimension is $m+1$ and its spin is $j = m/2$. The central element $-I$ acts as $(-1)^m I$. Phase sensitivity therefore requires m to be odd. The smallest admissible value is $m = 1$. It gives dimension two and spin $1/2$. The Clifford statement follows from the standard isomorphism $\text{Cl}_3^0(\mathbb{C}) \cong \text{Cl}_2(\mathbb{C}) \cong M_2(\mathbb{C})$ whose irreducible module is $\mathbb{C}^2$ [16, 17]. $\square$

The scalar projective shell $\mathcal{F}_q^{\text{proj}}$ supplies the multiplicity q^2 of a geometric eigenvalue. The module S supplies the minimal phase-sensitive lift of the local three-dimensional rotation group. The central-sign condition selects the spin value $1/2$ after three-dimensional closure.

The Gaussian half-step and the minimal half-spin unit provide compatible realizations of the same central character. Write $Z_2 := \{\pm I\} \subset SU(2)$ and let ε denote its nontrivial one-dimensional character. The restriction of the half-spin module to the center is

$$S|_{Z_2} \cong \varepsilon \oplus \varepsilon. \quad (\text{E.5})$$

Theorem 5.5 gives the nontrivial sign $U_b^G(2\pi) = -I$. The nontrivial center of the spin lift acts as $-I$ on S . Since $\mathbb{Z}/2\mathbb{Z}$ has a unique nontrivial complex character each multiplicity component carries the same character ε . The identification is at the level of central characters. Each representation retains its own carrier and full group action.

Figure 5 relates the projective geometric continuation to the Gaussian construction of Section 5.4 through their common central character. The prime-natural lift of Proposition 5.6(i) then attaches one copy of that character to each prime direction. Exterior second quantization gives its occupation grading.

These constructions realize the representation interfaces of Section 7.

E.2 Additive half-spin fibres

The same minimal two-dimensional carrier also lifts the fixed-sum additive fibres of Section 4. Let x, y be an ordered orthonormal basis of $S \cong \mathbb{C}^2$.

For every $k \geq 2$ the additive fibre has a symmetric-power realization. The basis correspondence

$$|m, k-m\rangle_b \mapsto x^{m-1} \widehat{y^{k-m-1}}, \quad 1 \leq m \leq k-1, \quad (\text{E.6})$$

uses the normalized monomial indicated by the hat and gives a unitary identification

$$\mathcal{F}_{b,k} \cong \text{Sym}^{k-2}(S) = V_{(k-2)/2}. \quad (\text{E.7})$$

For $k \geq 3$ coupling one minimal phase-sensitive unit gives the Clebsch–Gordan splitting

$$\mathcal{F}_{b,k} \otimes S \cong V_{(k-1)/2} \oplus V_{(k-3)/2}, \quad 2(k-1) = k + (k-2). \quad (\text{E.8})$$

At the first nontrivial additive threshold one has

$$\mathcal{F}_{b,3} \cong V_{1/2}, \quad V_{1/2} \otimes V_{1/2} \cong V_1 \oplus V_0. \quad (\text{E.9})$$

The homogeneous monomials of degree $k-2$ form the standard weight basis of $\text{Sym}^{k-2}(S)$ and are in bijection with the $k-1$ ordered positive decompositions of k . Choosing their standard invariant normalization makes (E.6) unitary. The Clebsch–Gordan rule $V_j \otimes V_{1/2} \cong V_{j+1/2} \oplus V_{j-1/2}$ proves (E.8). Setting $k=3$ gives the final identity. Let $\Sigma_S(v \otimes w) := w \otimes v$ on $S \otimes S$ and put $P_{S,\pm} := (I \pm \Sigma_S)/2$. Then

$$S \otimes S = \text{Sym}^2(S) \oplus \wedge^2 S = V_1 \oplus V_0, \quad (\text{E.10})$$

with

$$P_{S,+}(S \otimes S) = V_1, \quad P_{S,-}(S \otimes S) = V_0. \quad (\text{E.11})$$

The highest-weight vector $x \otimes x$ lies in the three-dimensional symmetric sector and identifies it with V_1 . The remaining antisymmetric line is V_0 . Thus the spin-one triplet is exchange-symmetric and the spin-zero singlet is exchange-antisymmetric.

Example E.2 (The first projective and half-spin thresholds). The first three retained projective indices are $q = 1, 3, 5$. Their energy–capacity pairs are $(0, 1)$ followed by $(8, 9)$ and $(24, 25)$ as displayed in Table 4. Under diagonal conjugation the traceless summands

have dimensions 0, 8, 24 respectively. Each dimension equals its shell eigenvalue. This eigenvalue also acts on the scalar line. In the independent additive family $k = 3$ is the first fibre with a nonzero transverse quotient and

$$\mathcal{F}_{b,3} \cong V_{1/2}, \quad \mathcal{F}_{b,3} \otimes S \cong V_1 \oplus V_0.$$

The projective lift first acquires positive excitation energy at index three. The additive lift first resolves a triplet–singlet pair at the same index. The two selection rules therefore reach the same threshold. Each space carries its own representation action.

The projectors $P_{S,+}$ and $P_{S,-}$ are the two-particle instances of the permutation idempotents used by the symmetric and exterior Fock functors in Section 3.2. The common functorial mechanism is exchange symmetry. The arithmetic trace assigns one occupation coordinate to each prime direction and therefore uses the one-dimensional central character ε as in Proposition 5.6(ii).

The lower Clebsch–Gordan branch and the additive transverse quotient both have dimension $k - 2$. This exact rank agreement records their common two-input loss count. The Clebsch–Gordan branch carries an irreducible $SU(2)$ action whereas $P_T^{(k)} \mathcal{F}_{b,k}$ carries the fixed-sum additive representation. These actions are defined on their respective carriers.

The geometric carrier has the shell capacities and heat–Mellin transforms of Appendix D. The same appendix derives its endpoint entropy.

F Occupation, response, and geometric context

The source arithmetic and ordered history construction are established in [1]. The present paper adds occupation energies and operator readouts together with representation multiplicities.

F.1 Relation to arithmetic quantum statistics

Julia’s free Riemann gas assigns one-particle energies $\log p$ so that occupation energies add under prime factorization [2]. Spector relates the square-free Möbius sign to fermion parity and studies its supersymmetric interpretation [3]. These works supply direct precedents for the occupation partition and graded trace formulas. The Bost–Connes construction begins with Hecke algebras associated with ring inclusions. For the inclusion of the integers in the rationals its C^* -dynamical system has the zeta partition function. It also has arithmetic symmetries and a phase transition in its equilibrium states [4]. The Bohr lift instead identifies Dirichlet series with functions of prime coordinates [5].

The present Fock evaluation retains both energies in (3.34). Quadratic correlation energy controls the Gaussian response while logarithmic energy adds over prime factors and gives the Euler traces. On a general occupation state transporting the correlation generator assigns $E_{b,1}$ times the square of the evaluated degree. This differs from the sum of the prime quadratic energies. The full-charge Mellin identity of Theorem 5.4 relates this Gaussian trace to the arithmetic trace without discarding the occupation characters.

F.2 Square spectrum and geometric capacity

The one-component generator and the projective lift retain different data. The former has simple normalized eigenvalues n^2 and constructs the natural-mode semiring. The latter retains odd coordinates q and attaches the representation fibre $\text{End}(V_{(q-1)/2})$. It therefore obeys

$$\underbrace{q^2}_{\text{full fibre capacity}} = \underbrace{1}_{\text{scalar line}} + \underbrace{(q^2 - 1)}_{\text{traceless capacity}}. \quad (\text{F.1})$$

This enriches the generated square spectrum by the selected representation multiplicity spaces. The splitting is invariant under diagonal conjugation. Its traceless dimension equals the normalized shell eigenvalue $\lambda_q = q^2 - 1$. This eigenvalue acts on the entire fibre including its scalar line. The global constant mode belongs to $q = 1$. Theorem C.5 gives the dimension selection within the affine energy-capacity class. Theorem E.1 then selects the smallest central-sign-sensitive rotational module. Scalar shell multiplicity and rotational spin thus arise at different stages of the construction.

G Bertrand's theorem in the generated Hodge shell

Prime counts refer to root modes. The history-source embedding carries the root prime projection to its source-weighted history subspace as in (3.90). The ranks of corresponding finite restrictions agree.

We apply the central-binomial argument of Erdős [18] to the generated Pascal modes. If a Hodge shell is empty then the prime occupations bound the degree of the central mode from above. The lower bound for that degree gives a contradiction at all sufficiently large degrees. Explicit finite certificates cover the remaining shells. The Hodge prime projector and the occupation coordinates thus express the central-binomial argument on the generated carrier.

For $N \geq 0$ and $0 \leq K \leq N$ define the Pascal modes recursively by

$$\mathfrak{B}_{b,0,0} := 1_b, \quad \mathfrak{B}_{b,N,K} := \mathfrak{B}_{b,N-1,K-1} \boxplus_b \mathfrak{B}_{b,N-1,K}, \quad (\text{G.1})$$

with an out-of-range term read as 0_b . The semiring degree map gives

$$\deg_b(\mathfrak{B}_{b,N,K}) = \binom{N}{K}. \quad (\text{G.2})$$

Write $\mathfrak{C}_{b,n} := \mathfrak{B}_{b,2n,n}$ for the central mixing mode.

The central mode is also an exact $U(1)$ phase average. If $m_{U(1)}$ is normalized Haar measure then

$$\deg_b(\mathfrak{C}_{b,n}) = \int_{U(1)} |1 + z|^{2n} dm_{U(1)}(z). \quad (\text{G.3})$$

The integral extracts the zero character from $(1+z)^n(1+z^{-1})^n$. With $z = e^{i\theta}$ the integrand is $4^n \cos^{2n}(\theta/2)$. Near the identity one has

$$4^n \cos^{2n}(\theta/2) = 4^n \exp\left(-\frac{n\theta^2}{4} + O(n\theta^4)\right).$$

This coefficient combines the additive mixing capacity with a compact phase carrier and a Gaussian local profile before its prime occupations are resolved.

Let $\pi \in \mathcal{P}_b$ have degree p . Its occupation in $\mathfrak{C}_{b,n}$ is

$$\nu_{\mathfrak{C}_{b,n}}(\pi) = \sum_{j \geq 1} \left(\left\lfloor \frac{2n}{p^j} \right\rfloor - 2 \left\lfloor \frac{n}{p^j} \right\rfloor \right). \quad (\text{G.4})$$

Every summand is zero or one. The occupation therefore satisfies

$$p^{\nu_{\mathfrak{C}_{b,n}}(\pi)} \leq 2n. \quad (\text{G.5})$$

In the cascade product of the modes of degrees $1, \dots, N$ each multiple of p^j contributes one further unit of π -occupation. Thus its total occupation is $\sum_{j \geq 1} \lfloor N/p^j \rfloor$. The factorial identity for the central Pascal coefficient subtracts twice the value at $N = n$ from the value at $N = 2n$. This gives (G.4). For a real number x the quantity $\lfloor 2x \rfloor - 2\lfloor x \rfloor$ is zero or one. A nonzero term requires $p^j \leq 2n$. The number of nonzero terms is therefore at most $\lfloor \log_p(2n) \rfloor$. This proves (G.5). For $N \geq 1$ let

$$\mathfrak{P}_b(N) := \bigboxtimes_{\substack{\pi \in \mathcal{P}_b \\ \deg_b(\pi) \leq N}} \pi \quad (\text{G.6})$$

be the finite prime-shell product.

For every integer $N \geq 1$ the finite prime-shell product satisfies

$$\deg_b \mathfrak{P}_b(N) \leq 4^N. \quad (\text{G.7})$$

We prove this by strong induction. If $N = 2m$ then every prime mode of degree in $(m, 2m]$ occurs in $\mathfrak{C}_{b,m}$. Hence

$$\deg_b \mathfrak{P}_b(2m) \leq \deg_b \mathfrak{P}_b(m) \binom{2m}{m} \leq 4^m 4^m = 4^{2m}. \quad (\text{G.8})$$

If $N = 2m + 1$ then every prime mode of degree in $(m + 1, 2m + 1]$ occurs in $\mathfrak{B}_{b,2m+1,m}$. The two central coefficients in row $2m + 1$ are equal and the full row has degree sum 2^{2m+1} . Therefore $\binom{2m+1}{m} \leq 2^{2m} = 4^m$. It follows that

$$\deg_b \mathfrak{P}_b(2m + 1) \leq 4^{m+1} 4^m = 4^{2m+1}. \quad (\text{G.9})$$

The cases $N = 1, 2$ follow directly and complete the induction. Let $\mathbf{N}_b[u_{b,m}] := m[u_{b,m}]$ on the degree-one bar space and recall the Hodge prime projector P_b^{pr} from (3.83).

Lemma G.1 (Finite Hodge certificates). *Each degree in*

$$\mathcal{C}_{\text{fin}} := \{3, 5, 7, 13, 23, 43, 83, 163, 317, 631\} \quad (\text{G.10})$$

is a Hodge zero mode. Thus $R_b[u_{b,p}] = 0$ for every $p \in \mathcal{C}_{\text{fin}}$.

Proof. By the internal factorization theorem a composite mode of degree p has a prime divisor mode of degree at most $\sqrt{p}$. The required finite composition tests are

p	$q \leq \sqrt{p}$ to be tested	$p \bmod q$
3	$\emptyset$	$\emptyset$
5	2	1
7	2	1
13	2, 3	1, 1
23	2, 3	1, 2
43	2, 3, 5	1, 1, 3
83	2, 3, 5, 7	1, 2, 3, 6
163	2, 3, 5, 7, 11	1, 1, 3, 2, 9
317	2, 3, 5, 7, 11, 13, 17	1, 2, 2, 2, 9, 5, 11
631	2, 3, 5, 7, 11, 13, 17, 19, 23	1, 1, 1, 1, 4, 7, 2, 4, 10.

(G.11)

Every displayed remainder is nonzero. Hence none of the modes has a nontrivial composition divisor. Each therefore lies in $\ker R_b = \text{Ran } P_b^{\text{pr}}$. $\square$

Theorem G.2 (Bertrand shell theorem). *For every integer $n > 1$ one has*

$$\text{Tr}_{\mathcal{K}_{b,1}}(\mathbf{1}_{(n,2n)}(\mathbf{N}_b)P_b^{\text{pr}}) \geq 1. \quad (\text{G.12})$$

Equivalently there is a spectral-prime mode π with $n < \deg_b(\pi) < 2n$.

Proof. Assume first that $n \geq 512$ and that the trace in (G.12) vanishes. Formula (G.4) separates the Fock occupation of $\mathfrak{E}_{b,n}$ into three ranges. Prime degrees at most $\sqrt{2n}$ contribute at most $(2n)^{\sqrt{2n}}$. In the range $\sqrt{2n} < p \leq 2n/3$ each occupation is at most one. Equation (G.7) bounds their product by $4^{2n/3}$. For $2n/3 < p \leq n$ formula (G.4) gives zero occupation. The assumption excludes any contribution from $n < p < 2n$. Therefore

$$\binom{2n}{n} \leq (2n)^{\sqrt{2n}} 4^{2n/3}. \quad (\text{G.13})$$

The $2n+1$ Pascal modes in row $2n$ have total degree 4^n and the central one is maximal. Therefore

$$\binom{2n}{n} \geq \frac{4^n}{2n+1}. \quad (\text{G.14})$$

Equations (G.13) and (G.14) would imply

$$4^{n/3} \leq (2n+1)(2n)^{\sqrt{2n}}. \quad (\text{G.15})$$

For real $x \geq 512$ put

$$F(x) := \frac{x}{3} \log 4 - \log(2x+1) - \sqrt{2x} \log(2x). \quad (\text{G.16})$$

Direct evaluation gives $F(512) > 7.85$ and

$$F'(x) = \frac{\log 4}{3} - \frac{2}{2x+1} - \frac{\log(2x)+2}{\sqrt{2x}} > 0 \quad (x \geq 512), \quad (\text{G.17})$$

because the two subtracted terms decrease on that interval and $F'(512) > 0.181$. Hence (G.15) is impossible.

For $2 \leq n < 512$ the Hodge zero modes of degrees

$$3, 5, 7, 13, 23, 43, 83, 163, 317, 631 \quad (\text{G.18})$$

give the successive integer covers

p	$p/2 < n < p$
3	2
5	3–4
7	4–6
13	7–12
23	12–22
43	22–42
83	42–82
163	82–162
317	159–316
631	316–511.

(G.19)

Lemma G.1 verifies every listed zero mode by the internal composition tests. This completes the proof. $\square$

H Prime density in the generated spectral carrier

The counting measures are defined on root values. Exact pushforward of the history source and the Fock embeddings of (3.13) identify their spectral realizations. A root contributes once to these measures independently of the number of histories evaluating to it.

The positive prime-power measure defined by the von Mangoldt observable counts the Hodge prime modes through the logarithmic response of the Fock partition function. The boundary phase inequality is the de la Vallée Poussin nonvanishing argument described in [19, Chapter 6]. The distributional Wiener–Ikehara theorem [20, Theorem 1.1], in Mellin form, then gives the counting asymptotic. Removing higher prime powers gives the prime count. Adding a constant to bounded signed coefficients reduces the exterior-trace conclusion to the same positive theorem.

For $u \in \mathcal{U}_b^+$ define the spectral von Mangoldt observable by

$$\Lambda_b(u) := \begin{cases} \log \deg_b(\pi), & u = \pi^{\boxtimes_b k}, \quad \pi \in \mathcal{P}_b, \quad k \geq 1, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{H.1})$$

Write $v \mid_b u$ when $u = v \boxtimes_b w$ for some $w \in \mathcal{U}_b^+$ and let $\star_b$ denote convolution over this divisor relation.

For every $u \in \mathcal{U}_b^+$ one has

$$\log \deg_b(u) = \sum_{v \mid_b u} \Lambda_b(v). \quad (\text{H.2})$$

Equivalently $\log_b = \mathbf{1} \star_b \Lambda_b$.

Let $\nu_u(\pi)$ be the finite prime-occupation profile. Unique Fock occupation gives

$$\log \deg_b(u) = \sum_{\pi \in \mathcal{P}_b} \nu_u(\pi) \log \deg_b(\pi).$$

For a fixed π exactly the powers $\pi, \dots, \pi^{\boxtimes_b \nu_u(\pi)}$ divide u . Their Λ_b -weights add to the corresponding summand.

Define the positive Hodge prime-power counting function

$$\Psi_b(X) := \sum_{\deg_b(u) \leq X} \Lambda_b(u). \quad (\text{H.3})$$

Its Mellin–Stieltjes transform is the logarithmic marked response.

$$\int_{1^-}^{\infty} x^{-s} d\Psi_b(x) = \sum_{u \in \mathcal{U}_b^+} \frac{\Lambda_b(u)}{\deg_b(u)^s} = \mathcal{T}_b^{\text{pp}}(s, 1) = -\frac{\zeta'(s)}{\zeta(s)}, \quad \Re s > 1. \quad (\text{H.4})$$

The first equality uses only the positive atomic measure determined by (H.1). The last two equalities are Theorem 5.2.

Lemma H.1 (Fock boundary phase positivity). *For every nonzero $t \in \mathbb{R}$ one has*

$$\zeta(1 + it) \neq 0. \quad (\text{H.5})$$

It follows that

$$G_b(s) := -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \quad (\text{H.6})$$

extends continuously to every point of the line $\Re s = 1$.

Proof. For $\sigma > 1$ the positive loop expansion of the symmetric Fock trace is

$$\log \zeta(s) = \sum_{\pi \in \mathcal{P}_b} \sum_{m \geq 1} \frac{\deg_b(\pi)^{-ms}}{m}.$$

Applying $3 + 4 \cos v + \cos(2v) = 2(1 + \cos v)^2 \geq 0$ term by term gives

$$\zeta(\sigma)^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1. \quad (\text{H.7})$$

If $\zeta(1 + it)$ had a zero of order $q \geq 1$ then letting $\sigma \downarrow 1$ would make the left side $O((\sigma - 1)^{4q-3})$. A possible zero at $1 + 2it$ only increases the vanishing order. This contradicts (H.7). Thus (H.5) holds.

The completed Gaussian trace of Proposition 5.3 continues ζ meromorphically across the line. Its only pole there is the simple pole at $s = 1$. Away from $s = 1$ the zero-free result makes $-\zeta'/\zeta$ locally holomorphic. At $s = 1$ the Laurent expansion $\zeta(s) = (s-1)^{-1} + O(1)$ shows that the subtraction in (H.6) removes the pole. Hence G_b has the asserted continuous boundary values. $\square$

The Mellin form of the Wiener–Ikehara theorem turns these analytic boundary values into an asymptotic for an unsmoothed counting function. Positivity enters through monotonicity of the counting function.

Let $A : [1, \infty) \rightarrow [0, \infty)$ be nondecreasing and right-continuous with $A(x) = O(x)$. Suppose

$$F(s) := \int_{1-}^{\infty} x^{-s} dA(x), \quad \Re s > 1, \quad (\text{H.8})$$

and that for some $a \geq 0$ the remainder

$$F(s) - \frac{a}{s-1} \quad (\text{H.9})$$

has distributional boundary values on $\Re s = 1$ that are locally represented by continuous functions. Then

$$\frac{A(x)}{x} \longrightarrow a. \quad (\text{H.10})$$

To reduce to the boundary formulation in [20, Theorem 1.1] put $S(v) := e^{-v} A(e^v)$ for $v \geq 0$ and extend it by zero for $v < 0$. Then write $g(v) := S(v) - a \mathbf{1}_{[0, \infty)}(v)$. The growth assumption makes g bounded. For $\Re z > 0$ Stieltjes integration by parts gives

$$\int_{\mathbb{R}} g(v) e^{-zv} dv = \frac{F(1+z) - a/z}{1+z} - \frac{a}{1+z}. \quad (\text{H.11})$$

The right side has locally continuous boundary values on $\Re z = 0$. After multiplication by a smooth compactly supported cutoff each boundary function is integrable. Its inverse Fourier transform tends to zero by the Riemann–Lebesgue lemma. The boundary function is therefore a local pseudofunction. Thus the nondecreasing function $v \mapsto A(e^v)$ extended by zero to $v < 0$ satisfies the boundary hypothesis of [20, Theorem 1.1]. That theorem gives $e^{-v} A(e^v) \rightarrow a$ and $x = e^v$ proves the assertion. The same positive theorem applies to a bounded signed sequence after adding a constant. Let $(a_n)_{n \geq 1}$ be a bounded real sequence and set

$$A_a(X) := \sum_{n \leq X} a_n, \quad D_a(s) := \sum_{n \geq 1} \frac{a_n}{n^s} \quad (\Re s > 1). \quad (\text{H.12})$$

Suppose D_a has distributional boundary values on $\Re s = 1$ that are locally represented by a continuous function and that this function vanishes at $s = 1$. Then

$$\frac{A_a(X)}{X} \longrightarrow 0. \quad (\text{H.13})$$

Choose $C > \sup_n |a_n|$. The function $A_C(X) := C[X] + A_a(X)$ is nonnegative and nondecreasing. It is also right-continuous and $O(X)$. Its Mellin–Stieltjes transform is $F_C(s) = C\zeta(s) + D_a(s)$. Thus

$$F_C(s) - \frac{C}{s-1} = D_a(s) + C \left(\zeta(s) - \frac{1}{s-1} \right).$$

The right side has locally continuous boundary values by the hypothesis and Proposition 5.3. The positive Mellin conclusion (H.10) therefore gives $A_C(X)/X \rightarrow C$. Subtracting $C[X]/X \rightarrow C$ proves (H.13).

Lemma H.2 (Gaussian Abelian passage). *Let $N : [1, \infty) \rightarrow [0, \infty)$ be nondecreasing and right-continuous. Suppose it vanishes below some $x_0 > 1$ and satisfies*

$$N(X) \sim \frac{X}{\log X}. \quad (\text{H.14})$$

Then

$$\int_{1-}^{\infty} e^{-\beta x^2} dN(x) \sim \frac{\sqrt{\pi}}{2\sqrt{\beta} \log(1/\sqrt{\beta})} \quad (\beta \downarrow 0). \quad (\text{H.15})$$

Proof. Write $L_\beta := \log(1/\sqrt{\beta})$. Stieltjes integration by parts and $y = \sqrt{\beta}x$ give

$$\begin{aligned} \int_{1-}^{\infty} e^{-\beta x^2} dN(x) &= 2\beta \int_{x_0}^{\infty} x e^{-\beta x^2} N(x) dx \\ &= 2 \int_{x_0 \sqrt{\beta}}^{\infty} y e^{-y^2} N(y/\sqrt{\beta}) dy. \end{aligned} \quad (\text{H.16})$$

For each fixed $y > 0$ hypothesis (H.14) implies

$$\sqrt{\beta} L_\beta N(y/\sqrt{\beta}) \rightarrow y, \quad (\text{H.17})$$

because $\log(y/\sqrt{\beta}) = L_\beta + \log y$.

Split the first integral in (H.16) at $x = \beta^{-1/4}$. Its lower part is at most $N(\beta^{-1/4})$ because $2\beta \int_0^\infty x e^{-\beta x^2} dx = 1$. Multiplication by $\sqrt{\beta} L_\beta$ gives $O(\beta^{1/4})$. On the upper part $y \geq \beta^{1/4}$ and hence

$$L_\beta + \log y \geq \frac{1}{2} L_\beta. \quad (\text{H.18})$$

The hypothesis gives $N(x) \leq Cx/\log x$ for sufficiently large x . It follows uniformly on this upper range for small β that

$$\sqrt{\beta} L_\beta N(y/\sqrt{\beta}) \leq 2Cy. \quad (\text{H.19})$$

Thus the normalized integrand is dominated by a constant multiple of $y^2 e^{-y^2}$. Dominated convergence in (H.16) now gives

$$\sqrt{\beta} L_\beta \int_{1-}^{\infty} e^{-\beta x^2} dN(x) \rightarrow 2 \int_0^\infty y^2 e^{-y^2} dy = \frac{\sqrt{\pi}}{2}, \quad (\text{H.20})$$

which proves (H.15). $\square$

For the prime contribution alone write

$$\vartheta_b(X) := \sum_{\substack{\pi \in \mathcal{P}_b \\ \deg_b(\pi) \leq X}} \log \deg_b(\pi). \quad (\text{H.21})$$

Theorem H.3 (Prime-number theorem for the generated spectral primes). *The Hodge prime modes satisfy*

$$\Psi_b(X) \sim X, \quad \#\{\pi \in \mathcal{P}_b : \deg_b(\pi) \leq X\} \sim \frac{X}{\log X}. \quad (\text{H.22})$$

Their Gaussian heat trace consequently satisfies

$$\Theta_{b,\mathcal{P}}(\beta) := \sum_{\pi \in \mathcal{P}_b} e^{-\beta \deg_b(\pi)^2} \sim \frac{\sqrt{\pi}}{2\sqrt{\beta} \log(1/\sqrt{\beta})} \quad (\beta \downarrow 0). \quad (\text{H.23})$$

Proof. The measure $d\Psi_b$ in (H.4) is positive. The finite prime-capacity estimate of Appendix G gives

$$\vartheta_b(Y) \leq Y \log 4.$$

Since every prime power below X has exponent at most $\log_2 X$ one has

$$\Psi_b(X) = \sum_{m \geq 1} \vartheta_b(X^{1/m}) \leq (\log 4) \sum_{1 \leq m \leq \log_2 X} X^{1/m} = O(X).$$

Lemma H.1 verifies the boundary hypothesis of the positive Mellin conclusion (H.10) with $a = 1$. Hence $\Psi_b(X) \sim X$.

Put

$$\pi_b(X) := \#\{\pi \in \mathcal{P}_b : \deg_b(\pi) \leq X\}. \quad (\text{H.24})$$

Prime powers of exponent at least two contribute

$$0 \leq \Psi_b(X) - \vartheta_b(X) = O(\sqrt{X} \log X) = o(X), \quad (\text{H.25})$$

because their bases have degree at most $\sqrt{X}$ and the Hodge capacity bound is linear. Thus $\vartheta_b(X) \sim X$. Stieltjes partial summation then gives

$$\pi_b(X) = \frac{\vartheta_b(X)}{\log X} + \int_2^X \frac{\vartheta_b(t)}{t \log^2 t} dt \sim \frac{X}{\log X}.$$

The integral is $O(X/\log^2 X)$ after splitting it at $\sqrt{X}$.

Applying Lemma H.2 to $N = \pi_b$ gives

$$\Theta_{b,\mathcal{P}}(\beta) = \int_{2^-}^{\infty} e^{-\beta x^2} d\pi_b(x) \sim \frac{\sqrt{\pi}}{2\sqrt{\beta} \log(1/\sqrt{\beta})}, \quad (\text{H.26})$$

which is (H.23). $\square$

For uniform sampling among the prime modes of degree at most x the counting entropy introduced in Section 6 is $\log \pi_b(x)$. The theorem now gives $\log \pi_b(x) = \log x - \log \log x + o(1)$ as $x \rightarrow \infty$.

Theorem H.4 (Asymptotic balance of the central-sign sectors). *Define the intrinsic Möbius sum and square-free capacity together with the two central-sector counts by*

$$M_b(X) := \sum_{\substack{u \in \mathcal{U}_b \setminus \{0_b\} \\ \deg_b(u) \leq X}} \mu_b(u), \quad Q_b^{\text{sf}}(X) := \sum_{\substack{u \in \mathcal{U}_b \setminus \{0_b\} \\ \deg_b(u) \leq X}} \mu_b(u)^2, \quad (\text{H.27})$$

$$N_{b,\pm}^{\text{sf}}(X) := \#\{u \in \mathcal{U}_b \setminus \{0_b\} : \deg_b(u) \leq X, \mu_b(u) = \pm 1\}. \quad (\text{H.28})$$

Then

$$M_b(X) = o(X), \quad (\text{H.29})$$

$$Q_b^{\text{sf}}(X) = \frac{X}{\zeta(2)} + O(\sqrt{X}) = \frac{6}{\pi^2} X + O(\sqrt{X}), \quad (\text{H.30})$$

$$N_{b,+}^{\text{sf}}(X) \sim N_{b,-}^{\text{sf}}(X) \sim \frac{3}{\pi^2} X. \quad (\text{H.31})$$

Thus the two central characters have equal asymptotic density within the square-free occupation sector.

Proof. The exterior-Fock Euler identity gives

$$\sum_{u \in \mathcal{U}_b \setminus \{0_b\}} \frac{\mu_b(u)}{\deg_b(u)^s} = \frac{1}{\zeta(s)} \quad (\Re s > 1). \quad (\text{H.32})$$

Lemma H.1 and the simple pole of ζ at $s = 1$ show that the right-hand side has continuous boundary values on $\Re s = 1$ and vanishes at $s = 1$. Since $|\mu_b(u)| \leq 1$ the signed conclusion (H.13) applies to the generated degree ordering and proves (H.29).

Exterior nilpotence gives the square-divisor identity

$$\mu_b(u_{b,n})^2 = \sum_{d^2 | n} \mu_b(u_{b,d}). \quad (\text{H.33})$$

Summing it up to X yields

$$Q_b^{\text{sf}}(X) = \sum_{d \leq \sqrt{X}} \mu_b(u_{b,d}) \left\lfloor \frac{X}{d^2} \right\rfloor = X \sum_{d=1}^{\infty} \frac{\mu_b(u_{b,d})}{d^2} + O(\sqrt{X}). \quad (\text{H.34})$$

The omitted tail contributes $O(\sqrt{X})$. The exterior Euler product at $s = 2$ identifies the series with $1/\zeta(2) = 6/\pi^2$. The sector counts satisfy

$$Q_b^{\text{sf}}(X) = N_{b,+}^{\text{sf}}(X) + N_{b,-}^{\text{sf}}(X), \quad M_b(X) = N_{b,+}^{\text{sf}}(X) - N_{b,-}^{\text{sf}}(X). \quad (\text{H.35})$$

Equations (H.29) and (H.30) give (H.31). $\square$

The boundary phase inequality supplies the common analytic input for the prime-number asymptotic and central-sign balance. The first uses the positive prime-power measure while the second uses bounded signed occupation coefficients.

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Declarations

Data availability. The interval certificates and code for the endpoint fixed-point enclosure are included in the computational supplement. The formalization of the underlying source arithmetic and history construction accompanies [1] in Lean Formalization of Arithmetic and Basic Physical Interfaces [8]. The archive is available at <https://doi.org/10.5281/zenodo.23254521>.

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References

- [1] J. Guo, “A spectral construction of natural-number arithmetic.” SSRN preprint 7421398, 2026. [10.2139/ssrn.7421398](https://ssrn.com/abstract=7421398).
- [2] B.L. Julia, *Statistical theory of numbers*, in *Number Theory and Physics*, J.-M. Luck, P. Moussa and M. Waldschmidt, eds., vol. 47 of *Springer Proceedings in Physics*, (Berlin), pp. 276–293, Springer (1990), [DOI](#).
- [3] D. Spector, *Supersymmetry and the Möbius inversion function*, *Communications in Mathematical Physics* **127** (1990) 239.
- [4] J.-B. Bost and A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, *Selecta Mathematica, New Series* **1** (1995) 411.
- [5] H. Hedenmalm, P. Lindqvist and K. Seip, *A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0, 1)$* , *Duke Mathematical Journal* **86** (1997) 1.
- [6] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, vol. 150 of *Graduate Texts in Mathematics*, Springer, New York (1995), [10.1007/978-1-4612-5350-1](https://doi.org/10.1007/978-1-4612-5350-1).
- [7] C.A. Weibel, *An Introduction to Homological Algebra*, vol. 38 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge (1994), [10.1017/CBO9781139644136](https://doi.org/10.1017/CBO9781139644136).
- [8] J. Guo, “Lean Formalization of Arithmetic and Basic Physical Interfaces.” Lean proof library, Zenodo, 2026. [10.5281/zenodo.23254521](https://zenodo.org/record/23254521).
- [9] V. Fock, *Konfigurationsraum und zweite Quantelung*, *Zeitschrift für Physik* **75** (1932) 622.
- [10] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 2*, Springer, Berlin, 2 ed. (1997), [10.1007/978-3-662-03444-6](https://doi.org/10.1007/978-3-662-03444-6).
- [11] J. Brüning and M. Lesch, *Hilbert complexes*, *Journal of Functional Analysis* **108** (1992) 88.
- [12] S. Thangavelu, *Lectures on Hermite and Laguerre Expansions*, vol. 42 of *Mathematical Notes*, Princeton University Press, Princeton (1993), Electronic reissue, 2020, [10.1515/9780691213927](https://doi.org/10.1515/9780691213927).
- [13] C. Tsallis, A.R. Plastino and R.F. Alvarez-Estrada, *Escort mean values and the characterization of power-law-decaying probability densities*, *Journal of Mathematical Physics* **50** (2009) 043303 [[0802.1698](#)].
- [14] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 4 ed. (1996), Reprint of the fourth edition (1927), [10.1017/CBO9780511608759](https://doi.org/10.1017/CBO9780511608759).
- [15] I. Chavel, *Eigenvalues in Riemannian Geometry*, vol. 115 of *Pure and Applied Mathematics*, Academic Press, Orlando (1984).

- [16] B.C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, vol. 222 of *Graduate Texts in Mathematics*, Springer, Cham, 2 ed. (2015), [10.1007/978-3-319-13467-3](#).
- [17] H.B. Lawson, Jr. and M.-L. Michelsohn, *Spin Geometry*, vol. 38 of *Princeton Mathematical Series*, Princeton University Press, Princeton (1989), Electronic reissue, 2016, [10.1515/9781400883912](#).
- [18] P. Erdős, *Beweis eines Satzes von Tschebyschef*, *Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae, Sectio Scientiarum Mathematicarum* **5** (1932) 194.
- [19] H.L. Montgomery and R.C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, vol. 97 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge (2006), [10.1017/CBO9780511618314](#).
- [20] J. Korevaar, *Distributional Wiener–Ikehara theorem and twin primes*, *Indagationes Mathematicae* **16** (2005) 37.