

The Spectrum of a Compact Internal Space I: Gauge Structure and a Scalar Mass Gap

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ABSTRACT: From minimal unbiased compositional change we construct compact internal geometry and gauge–matter representations, and prove a mass gap for the generated Yang–Mills response. On the least-dimensional nontrivial compact connected faithful irreducible orthogonal carrier, independent composition and square-root scaling with unit covariance fix the centred Gaussian source. Ladder operations and coherent comparisons generate the round $SU(2)$ process spectrum; complete centre-invariant observations give $\mathbb{RP}^3$. Within complete continuations preserving adjoints, exchange structures, source phase and reversal, three theorems characterize the faithfully acting group $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$, a sixteen-dimensional full matter module with a fifteen-dimensional charged quotient, and an elementary scalar connector parametrized by one complex doublet. The colour connection determines a complete spectral response with a unique positive equilibrium at the source-calibrated radius. Its normalized radial and common-frame coexact responses define the generated Yang–Mills mass operator. The Gaussian equilibrium completion has a unique vacuum and an attained positive gap. Its representation preserves the original connection state and polynomial products and intertwines the update. The curvature susceptibility detects the collective radial response. The same source algebra generates a Lorentzian event carrier with a phase clock. Osterwalder–Schrader reconstruction gives free local fields. For the renormalizable gauge–matter realization, causal perturbation theory constructs the interacting net as a formal series at gauge-compatible finite internal resolution.

KEYWORDS: Standard Model gauge group; Chiral fermions; Higgs doublet; Yang–Mills mass gap; Spectral geometry; Vacuum stability

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1 Introduction

Minimal unbiased compositional change describes a reversible source action together with the independent accumulation of its unresolved increments. The problem studied here is to construct the response modes and their couplings from this source and to determine the restoring scale of their collective equilibrium. The elementary carrier distinguishes continuous reversible changes. Comparing their complete processes determines the internal

geometry. Matter carries the resulting responses, and connections compare their local frames.

The terminology is operational. A carrier is a vector space on which source changes act; the elementary carrier has the least dimension among the admissible faithful irreducible orthogonal actions. Operations compose these changes and compare their effects in prepared states. A response records the change detected by an observation. The commutative algebra of complete process observations records the distinguishable operations, and its character space is the geometry resolved by those observations. The process heat spectrum measures how resolved modes decay under unbiased composition. Local fields represent these responses on the event carrier constructed from the same source algebra.

Minimal real dimension selects the standard $SO(2)$ carrier among nontrivial compact connected faithful irreducible orthogonal change data. Independent composition and exact quadratic scaling with normalized finite variance fix a centred Gaussian law on its one-dimensional Lie algebra. The finite correlation window gives the reflecting square spectrum; the full-line ladder supplies matrix operations comparing source levels. The first noncommuting matrix corner has two relevant real forms. Compact unitary operations generate the internal process geometry, while self-adjoint differences generate the Lorentzian event carrier. Internal response and event localization are two observations of one source algebra. The scale radius measures internal resolution.

The source construction realizes relative frames and least faithful matter together with an elementary chiral connector. Three continuation theorems characterize these realizations among complete continuations preserving the generated adjoint and exchange structures together with source phase and reversal. Relative frames express the generated carriers in local bases and turn their comparison into a connection. The generated tensor carrier realizes every simple response block once. The least-matter theorem proves that this adjoint-preserving action attains the minimum faithful dimension and is unique there up to unitary equivalence. The elementary-connector theorem identifies the odd Clifford variation that is natural under independent source comparisons and real under reversal. Basis phases and changes of local trivialization remain gauge descriptions of the same response. Additional multiplicity spaces describe extensions by repeated families. The transported source vacuum fixes the relative phase of the right frame and colour scalar. Its stabilizer determines the charge direction. Primitivity fixes the charge unit and the common centre fixes the faithful global group.

The Yang–Mills problem in this construction is to determine whether the generated colour connection has a stable equilibrium and a positive restoring spectrum. Its mean squared strength resolves collective changes; coclosed probes resolve transverse changes at fixed intensity. Their spectral feedback and information metric are computed from the same source law.

Kaluza–Klein theory associates lower-dimensional fields with modes of a chosen compact internal geometry [1–3]. The spectral action of Chamseddine and Connes assigns an action to noncommutative geometric data $(\mathcal{A}, \mathcal{H}, D)$ [4]. On a spin manifold times a finite noncommutative geometry it produces the gauge and matter sectors with Higgs terms [5]. These constructions establish how internal spectral data enter field dynamics on a chosen

base. The present construction begins with the source operation algebra. From this algebra it obtains both the internal carrier and the affine Lorentzian event carrier before passing to a field representation.

Finite noncommutative geometry already provides classifications of Standard Model content. Chamseddine and Connes characterize the finite geometry under algebraic conditions including KO -dimension six and quaternion linearity [6]. The classification in Ref. [7, Section III] starts from an irreducible symplectic geometry of KO -dimension six combined with a four-dimensional spin manifold. At the smallest graded size its algebra is $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$. Its Pati–Salam realization carries sixteen fermions in $(2, 1, 4) \oplus (1, 2, 4)$. Here the quadratic source response resolves a triplet–singlet carrier $C = W \oplus L$. The colour frames preserve the two summands separately. A full $SU(4)$ frame mixes them. The split-preserving response algebra together with the transported neutral line and common centre determines which operators act on the full sixteen-dimensional module and which descend to the charged quotient. The comparison therefore tracks both the origin of the carriers and the operators retained by their physical realization.

Minimal unbiased compositional change is also the source of the spectral arithmetic developed in Ref. [8]. The two constructions share the Gaussian law, intrinsic correlation endpoints and reflecting spectrum in a common normalization. Spectral arithmetic studies the native operations of the resulting modes. The physical branch uses the Gaussian ladder and complete process comparisons to construct internal geometry, frame transport, matter response and connection dynamics. The source and reflecting-spectrum arguments required for this construction are included here.

The first principal result is the internal structure construction. Coherent comparison of the ladder matrix units generates the round $SU(2)$ operation space and its heat spectrum. Complete centre-invariant observations give $X_3 = \mathbb{RP}^3$. The trace comparison gives the four-real-dimensional response cone. The Weyl growth of the complete process spectrum fixes dimension three. The shell capacity relation identifies the round spherical parent. Applying the three continuation theorems to these carriers gives

$$G_{\text{int}} = \frac{SU(3) \times SU(2)_W \times U(1)}{\mathbb{Z}_6}, \quad \dim_{\mathbb{C}} \mathcal{M}_{\text{full}} = 16, \quad \dim_{\mathbb{C}} \mathcal{M}_{\text{ch}} = 15. \quad (1.1)$$

The group is the range of the source action on the full matter carrier; the six-element subgroup is the kernel of its product presentation. The neutral right line remains in $\mathcal{M}_{\text{full}}$. An operator descends to $\mathcal{M}_{\text{ch}}$ precisely when it preserves that line. The elementary reversal–real connector has one complex coordinate of type $(1, 2)_{1/2}$. Its determinant is its positive gauge-invariant amplitude square. The source number flow has a unique continuous lift to the right frames. Its image fixes the transported source vacuum pointwise. The resulting stabilizer locks the relative phases of the right frame and colour scalar to the source reference and fixes their primitive charge.

The second result is the generated Yang–Mills mass-gap theorem. Let $q = \mathbb{E}\|B\|^2$ be the invariant intensity of the Gaussian colour connection. Its adjoint Casimir fixes the feedback coefficient and the long-root spectrum fixes the relative drive. The unit-covariance source fixes a width b_0 and the length unit $\ell_* = b_0$. The reference profile whose UV endpoint

equals this unit has width $b_0/\sqrt{2}$. Its IR radius gives $L_c = \sqrt{\pi}$. In this endpoint calibration the canonical source has $L_{\text{src}} = \sqrt{2\pi} > L_c$. The complete relative spectral response is C^2 and coercive with strict convexity. It has a unique positive minimum at this source-defined radius and more generally at every $L > L_c$. The source Legendre metric normalizes the radial fluctuation. The exact coclosed connection response gives

$$\mathcal{M}_{\text{YM}}^2 = m_{\text{YM,rad}}^2 \oplus (\Delta_1^\perp + \delta_{q_L} I), \quad \Delta_{\text{YM}}^2 = \min \left\{ m_{\text{YM,rad}}^2, \frac{4}{L^2} + \delta_{q_L} \right\} > 0. \quad (1.2)$$

The connection, source law and generated coexact domain transform together under changes of frame. Their common-frame quotient gives the displayed one-response operator. Independent Gaussian composition of its linear restoring map gives an equilibrium response completion with a unique vacuum and first positive mass-squared Δ_{YM}^2 . Its gauge-invariant cyclic sector inherits this positive lower bound. The original connection polynomials retain their full source inner products and multiplication in this completion. Their equilibrium update is intertwined explicitly with its generator. The intensity score retains its second Hermite degree. The radial linear readout is the derivative of the curvature mean along the source family. Its positive coefficient makes the collective excitation visible in the joint curvature susceptibility. It contains both the direct source response and the additional change caused by relaxation of the mean intensity. The fixed-source curvature correlation is computed separately with its full Hermite weights. Uniform tail estimates provide one finite spectral cutoff controlling both q_L and the gap on compact condensed scale intervals.

The third result supplies the field representation. The self-adjoint real form of the same first matrix corner has determinant $t^2 - |\mathbf{x}|^2$. Algebraic positivity selects its future cone and the determinant-one action gives $SO^+(1,3)$. Translated Gaussian comparisons generate the affine event algebra. A translation character read by the source quadratures calibrates proper time. Osterwalder–Schrader reconstruction gives the free local net. Mode products, gauge curvature, matter transport and the elementary connector define local functionals on this carrier. Causal reconstruction gives the gauge-compatible finite-resolution perturbative net as quantum BRST cohomology in the renormalizable gauge–matter realization. In its quadratic realization, integrating the internal one-response semigroup against the event heat kernel gives the field covariance. The one-particle Poincaré Casimir retains the internal mass spectrum. The colour intensity and the radial curvature polynomial have an isolated positive scalar pole and explicit multiparticle continua, with weights fixed by the observation.

The accompanying Lean development provides machine-checked proofs for the common source structure and finite physical interfaces identified in its theorem correspondence [9]. The archived Coarse Graining Genesis Framework supplies independent computational checks of finite responses and mode products, including their joint resolution corrections [10]. The theorem correspondence and computational records specify which results are checked. The analytic arguments are given in the paper.

The appendices give the source-operation constructions, spectral formulae, convergence and truncation estimates, and field-reconstruction proofs used in the main text.

2 Preliminaries

Complex Hilbert-space inner products are antilinear in the first argument and linear in the second. Real inner products are symmetric bilinear forms.

The structural implications have the form

$$\begin{aligned}
& \begin{array}{c} \text{minimal unbiased} \\ \text{compositional change} \end{array} \longrightarrow \text{Gaussian} \\
& \longrightarrow (\text{finite window, full-line ladder}) \\
& \xrightarrow{\text{source operations}} (X_3, E_4) \\
& \xrightarrow{\text{source continuation theorems}} (G_{\text{int}}, \mathcal{M}_{\text{full}}, \mathcal{M}_{\text{ch}}, \mathcal{K}_{\text{ch}}),
\end{aligned} \tag{2.1}$$

The source operations resolve complete processes into orthogonal heat shells. Their multiplicities supply the quadratic capacity used in the dimension count and the spherical comparison. The relative-frame, least-matter and elementary-connector theorems characterize continuations of the generated carriers that preserve their adjoint and exchange structures, source phase and reversal. Local frames, matter and the scalar connector are successive realizations in this class. The stated operation and observation classes fix the scope of the uniqueness results, and each completion is taken in its specified norm or operator domain. The entropy and representation results recalled here are standard tools for this construction.

2.1 Maximum entropy principle

Let ρ be a probability density on $\mathbb{R}^d$ with respect to Lebesgue measure. Its differential entropy is

$$H(\rho) = - \int_{\mathbb{R}^d} \rho(x) \log \rho(x) dx.$$

The reference measure is fixed in this definition. Jaynes’s maximum-entropy principle [11] selects the Gaussian density ρ_G at fixed mean μ and positive-definite covariance Σ . In these conventions the fixed-moment identity is

$$H(\rho_G) - H(\rho) = D(\rho \| \rho_G) \geq 0.$$

It follows by integrating the quadratic polynomial $\log \rho_G$ against the two densities. Relative-entropy positivity [12] gives uniqueness up to equality almost everywhere. An infinite entropy deficit corresponds to $H(\rho) = -\infty$.

The entropy deficit measures the additional statistical structure carried by a density with the same moments. The compositional source fixes the Gaussian through independent accumulation, exact quadratic scaling and normalized finite variance; maximum entropy describes the uncertainty retained by that law. The Fourier-conjugate entropy inequality of Białynicki-Birula and Mycielski [13] follows from the sharp Fourier norm inequality [14]. Unchirped Gaussian wavefunctions attain the entropy bound. Here the optimized quantity is the total entropy of two conjugate readouts. A self-dual normalization balances their spreads in fixed Fourier units. The profile $e^{-\pi x^2}$ is self-dual for the Fourier kernel $e^{-2\pi i x \xi}$. These variational and harmonic statements use fixed measure and normalization conventions.

2.2 Representation theory preliminaries

Let $\text{Spin}(4) \cong SU(2)_L \times SU(2)_R$. Its irreducible representations are labelled by (j_L, j_R) and have dimension $(2j_L + 1)(2j_R + 1)$.

The standard representation classification [15] gives the following dimensions and kernels. The least nontrivial complex irreducible representations of $\text{Spin}(4)$ are $(1/2, 0)$ and $(0, 1/2)$. Both have dimension two. Neither is faithful. The least faithful complex representation is the four-dimensional Dirac sum $(1/2, 0) \oplus (0, 1/2)$ and is unique up to equivalence. A representation of dimension at most three omits one factor. The four-dimensional vector representation $(1/2, 1/2)$ has a diagonal central kernel.

For even $n = 2m$ the complex Clifford classification [16, Chapter 3] gives an irreducible module of dimension 2^m . Restriction to the even Clifford subalgebra or to $\text{Spin}(2m)$ gives two semispinor modules of complex dimension 2^{m-1} . Odd Clifford multiplication interchanges them. For odd n there is no chiral decomposition.

3 Minimal Change and Gaussian Coarse Graining

The elementary datum specifies how a reversible change is distinguished and how independent increments of it accumulate. A real inner-product space records its response directions. Orthogonality expresses comparison at fixed strength. The direction can change while the comparison norm is preserved. Faithfulness allows distinct source operations to be resolved through their action. Irreducibility makes the elementary response indecomposable into proper invariant sectors. Minimality selects the fewest real directions that support such a continuous reversible action. These conditions specify the elementary carrier before spatial localization or clock calibration enters. The accumulation law belongs to the same source datum. Independent composition and exact scaling determine its Gaussian realization under a finite normalized second moment. This section specializes the common source construction of [8] to the internal response problem.

The two Gaussian constructions answer complementary physical questions. The finite window identifies the range over which changes remain correlated at a fixed resolution and fixes the endpoint scale. The full-line ladder resolves how amplitudes are transferred between source levels and supplies the matrix operations whose finite corners generate the response algebra. The window operator and ladder Hamiltonian act on different Hilbert spaces. Their common square normalization joins the correlation scale to the operation spectrum.

3.1 The minimal real change carrier

Elementary changes are nonidentity operations in a group that also contains the null change. Connectedness makes each group operation continuously accessible from the null change. Compactness allows the family of response orientations to carry a normalized invariant average. Together with orthogonality these requirements describe reversible comparisons at fixed response strength.

Definition 3.1 (Orthogonal change datum and unbiasedness). An orthogonal change datum is a triple (V, G, ρ) with a nonzero finite-dimensional real inner-product space V and a nontrivial compact connected Lie group G . Its action

$$\rho : G \longrightarrow O(V) \quad (3.1)$$

is faithful and continuous and irreducible over $\mathbb{R}$. A set $\mathcal{C}_0 \subset G \setminus \{e\}$ of elementary changes is admissible if it topologically generates G . The identity e represents the null change. The elements of $\mathcal{C}_0$ represent elementary nonzero changes. Two choices of $\mathcal{C}_0$ are physically equivalent when an orthogonal intertwiner of the corresponding data carries their generated actions into one another.

Structural unbiasedness means that the real representation ρ has no nonzero proper invariant subspace. Statistical unbiasedness of a Borel probability measure on a Euclidean tangent space means invariance under its full orthogonal group. Structural unbiasedness concerns the response representation whereas statistical unbiasedness concerns its probability law. Statistical unbiasedness of the compositional source follows from composition and scaling with normalized finite variance.

An orthogonal change datum is minimal if $\dim_{\mathbb{R}} V$ is least among such data.

The source datum combines this minimal carrier with a probability law for the accumulation of independent changes. Its composition and scale dependence are specified together in the following axiom.

Axiom 3.2 (Minimal unbiased compositional change). There exists a minimal orthogonal change datum (V, G, ρ) and a family of Borel probability measures $(\mu_t)_{t>0}$ on its Lie algebra $\mathfrak{g}$ such that

$$(C) \quad \mu_s * \mu_t = \mu_{s+t} \text{ for } s, t > 0.$$

$$(S) \quad \mu_t = (D_{\sqrt{t}})_{\#} \mu_1 \text{ with } D_r X = rX.$$

$$(V) \quad \mu_1 \text{ has finite second moment and } \text{Cov}(\mu_1) = I_{\mathfrak{g}} \text{ in the representation metric.}$$

Condition (C) makes the accumulated law independent of how successive independent increments are grouped. Condition (S) closes this law under changes of composition scale. Division by the square root of the composition parameter recovers the reference law. Condition (V) allows the strength of unresolved changes to be calibrated through a finite second moment and fixes the covariance unit at $t = 1$. The Gaussian uniqueness theorem establishes centring and orthogonal invariance with covariance $tI_{\mathfrak{g}}$. The parameter t measures accumulated variance. Physical time requires a clock calibration.

For an orthogonal change datum the differential representation equips $\mathfrak{g}$ with the inner product

$$\langle X, Y \rangle_{\rho} := -\frac{1}{\dim_{\mathbb{R}} V} \text{Tr}_V(d\rho(X)d\rho(Y)). \quad (3.2)$$

Faithfulness makes $d\rho$ injective. Hence (3.2) is positive definite.

Theorem 3.3 (Minimal real carrier and complex structure). Every minimal orthogonal change datum has

$$\dim_{\mathbb{R}} V = 2, \quad G \cong SO(2), \quad (3.3)$$

and ρ is orthogonally equivalent to the standard rotation action. If X_0 is either unit generator of $\mathfrak{g}$ and $J = d\rho(X_0)$ then

$$J^T = -J, \quad J^2 = -I_V, \quad J^T J = I_V. \quad (3.4)$$

Choosing an orientation of V selects one of the two operators J . The rule

$$(a + ic)v := av + cJv, \quad a, c \in \mathbb{R}, \quad (3.5)$$

makes V a one-dimensional complex Hilbert space with

$$\rho(\exp(\theta X_0))v = e^{i\theta}v, \quad SO(2) \cong U(1). \quad (3.6)$$

Reversing the orientation sends J to $-J$ and complex conjugates this coordinate.

Proof. In real dimension one $O(V) \cong \{\pm 1\}$ has no nontrivial connected subgroup. The standard action of $SO(2)$ on $\mathbb{R}^2$ is faithful and real irreducible. Thus the minimum is two. Connectedness and compactness then make $\rho(G)$ a nontrivial connected closed subgroup of $SO(2)$. It is therefore the whole circle. Faithfulness identifies G with $SO(2)$. In an orthonormal basis the two unit generators are $\pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. They satisfy (3.4).

Equation (3.4) supplies a real-linear complex structure. The compatible Hermitian form is

$$h_J(v, w) = \langle v, w \rangle_V + i \langle Jv, w \rangle_V. \quad (3.7)$$

It is positive definite and satisfies $h_J(v, Jw) = i h_J(v, w)$ in the convention of Section 2. The identity $e^{\theta J} = \cos \theta I + \sin \theta J$ gives (3.6). $\square$

The complex coordinate in Theorem 3.3 organizes the two real comparison directions into an amplitude and a phase. The phase records a reversible rotation of the elementary response. The two-level complex module spans two Gaussian levels and uses these same complex scalars to compare their amplitudes.

3.2 The compositional Gaussian law

Theorem 3.4 (Gaussian uniqueness under composition). The minimal carrier of Theorem 3.3 has one-dimensional Lie algebra $\mathfrak{g} \cong \mathbb{R}$. On this Lie algebra the family in Axiom 3.2 is uniquely

$$d\mu_t(a) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{a^2}{2t}\right) da, \quad a \in \mathfrak{g} \cong \mathbb{R}. \quad (3.8)$$

The family is centred and invariant under the full orthogonal group of $\mathfrak{g}$ with covariance $tI_{\mathfrak{g}}$. Setting $\mu_0 = \delta_0$ gives a weakly continuous convolution semigroup on $t \geq 0$.

Proof. For a bounded continuous function f condition (S) gives

$$\int f d\mu_t = \int f(\sqrt{t}a) d\mu_1(a).$$

Dominated convergence proves weak continuity for $t \geq 0$ with $\mu_0 = \delta_0$. Conditions (V) and (S) give finite second moments at every $t > 0$. Writing $m_t = \int a d\mu_t(a)$ gives by composition and scaling $m_2 = 2m_1 = \sqrt{2}m_1$. Thus $m_1 = 0$ and $m_t = 0$ for all t . In a unit coordinate on $\mathfrak{g} \cong \mathbb{R}$ condition (V) now reads $\int a^2 d\mu_1(a) = 1$.

Let $\varphi_t(k) = \int e^{ika} d\mu_t(a)$. The finite second moment gives the expansion $\varphi_1(h) = 1 - h^2/2 + o(h^2)$. For each positive integer N conditions (C) and (S) imply

$$\varphi_t(k) = \left[\varphi_1\left(k\sqrt{t/N}\right) \right]^N = \left[1 - \frac{tk^2}{2N} + o(N^{-1}) \right]^N \longrightarrow e^{-tk^2/2}.$$

The left-hand side is independent of N . Uniqueness of characteristic functions therefore gives (3.8). Orthogonal invariance and covariance $tI_{\mathfrak{g}}$ follow from this density. $\square$

At its derived mean and covariance this composition law also maximizes entropy as recalled in Section 2.1 [11]. It therefore leaves the greatest uncertainty about unresolved increments consistent with their mean and variance. Composition fixes the law. The entropy characterization explains its lack of statistical preference.

Set $t = b^2$ and distinguish the unit-mass density from the unit-peak correlation profile by

$$G_b(d) = \frac{1}{\sqrt{2\pi}b} e^{-d^2/(2b^2)}, \quad \int_{\mathbb{R}} G_b(d) dd = 1, \quad (3.9)$$

$$K_b(d) = \sqrt{2\pi}b G_b(d) = e^{-d^2/(2b^2)}, \quad K_b(0) = 1. \quad (3.10)$$

The convolution parameter is quadratic with

$$G_{b_1} * G_{b_2} = G_{\sqrt{b_1^2 + b_2^2}}. \quad (3.11)$$

Reversible changes and coarse graining. Let $(U(a)f)(y) = f(y - a)$ be the unitary representation of an individual tangent change on $L^2(\mathbb{R})$. Fourier transformation gives

$$\int_{\mathbb{R}} U(a) d\mu_t(a) = \exp\left[-\frac{t}{2} \left(-\frac{d^2}{dy^2}\right)\right]. \quad (3.12)$$

Each microscopic change remains reversible. Averaging over unresolved independent changes produces a positive contraction with a preferred coarse-graining direction. The average retains the probability-weighted response while discarding which individual displacement occurred. Its Fourier multipliers suppress rapidly varying modes most strongly. The parameter t measures the accumulated variance of these tangent changes.

3.3 The finite correlation window

The two boundaries are local and global readouts of the generated correlation kernel in the displacement coordinate fixed by its covariance. The local readout is the first separation carrying one unit of additive information. The global readout is the length of a constant profile with the same peak and total correlation mass. The information unit and mass normalization specify these readouts and fix their endpoint values.

Let $F : (0, 1] \rightarrow [0, \infty)$ be continuous and satisfy $F(rs) = F(r) + F(s)$ with normalization $F(e^{-1}) = 1$. Then $F(r) = -\log r$. The substitution $H(x) = F(e^{-x})$ gives the continuous Cauchy equation whose slope is fixed to one by the normalization. Consequently the local information of K_b is

$$\mathcal{D}_b(d) := -\log \frac{K_b(d)}{K_b(0)} = \frac{d^2}{2b^2}. \quad (3.13)$$

The signed displacement coordinate is the additive locally compact group $(\mathbb{R}, +)$. In the displacement coordinate fixed by the covariance normalization in Axiom 3.2 normalize Haar measure by assigning mass one to a unit interval. The resulting positive locally finite translation-invariant Borel measure is uniquely Lebesgue measure dd . For a nonzero nonnegative profile $K \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ define its correlation mass and peak-equivalent length by

$$\mathcal{C}(K) := \int_{\mathbb{R}} K(d) dd, \quad \mathcal{L}(K) := \frac{\mathcal{C}(K)}{\|K\|_\infty}. \quad (3.14)$$

The quotient is invariant under translations and positive amplitude rescalings and scales as a length under $K(d) \mapsto K(d/a)$. Thus it is the constant-profile width carrying the same normalized translation-invariant correlation mass.

Correlation boundaries. The first unit-resolvable separation and the translation-invariant peak-equivalent width are uniquely

$$\ell_{\text{UV}}(b) := \inf\{d > 0 : \mathcal{D}_b(d) \geq 1\} = \sqrt{2}b, \quad (3.15)$$

and

$$L_{\text{IR}}(b) := \mathcal{L}(K_b) = \frac{\int_{\mathbb{R}} K_b(\delta) d\delta}{K_b(0)} = \sqrt{2\pi}b. \quad (3.16)$$

They satisfy

$$\mathcal{D}_b(\ell_{\text{UV}}) = 1, \quad \mathcal{D}_b(L_{\text{IR}}) = \pi. \quad (3.17)$$

The function in (3.13) is continuous and strictly increasing for $d > 0$. Its first unit level is therefore $\sqrt{2}b$. The normalized Haar measure fixes the correlation mass in (3.14). The second identity is the elementary Gaussian integral. The two information levels follow by substitution.

Both boundaries are obtained from the same Gaussian profile. The width b sets their common length scale while the local information and global capacity readouts give $L_{\text{IR}}/\ell_{\text{UV}} = \sqrt{\pi}$. Their numerical values supply the two ends of the correlation window

without an independent cutoff parameter. The infrared length also has an entropy interpretation as the exponential of the limiting Rényi entropy of the normalized correlation profile.

Let $K \geq 0$ be nonzero in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Put $p_K = K / \int K$ and define for $\alpha > 1$

$$H_\alpha^R(p_K) = \frac{1}{1-\alpha} \log \int_{\mathbb{R}} p_K(d)^\alpha dd. \quad (3.18)$$

With Lebesgue measure normalized in the same length units as the endpoint

$$e^{H_\infty^R(p_K)} := \exp\left(\lim_{\alpha \rightarrow \infty} H_\alpha^R(p_K)\right) = \frac{\int K(d) dd}{\|K\|_\infty}. \quad (3.19)$$

For K_b this is $\sqrt{2\pi}b = L_{\text{IR}}(b)$. The L^α norms of a probability density in $L^1 \cap L^\infty$ converge to its essential supremum. Since $H_\alpha^R = \alpha(1-\alpha)^{-1} \log \|p_K\|_\alpha$ the limit is $-\log \|p_K\|_\infty$. Normalization of p_K gives (3.19). The Gaussian value follows from (3.16).

The active window is therefore

$$\mathcal{W}_b = [\ell_{\text{UV}}(b), L_{\text{IR}}(b)], \quad \Delta_b = L_{\text{IR}}(b) - \ell_{\text{UV}}(b) = b(\sqrt{2\pi} - \sqrt{2}). \quad (3.20)$$

Reflecting closure. On $\mathcal{H}_b = L^2(\mathcal{W}_b, dd)$ consider the local Laplacian $A_b f = -f''$. Among nondegenerate separated homogeneous self-adjoint boundary conditions conservation of the constant state uniquely selects

$$\mathcal{D}(A_b) = \{f \in H^2(\mathcal{W}_b) : f'(\ell_{\text{UV}}) = f'(L_{\text{IR}}) = 0\}. \quad (3.21)$$

The endpoint flux vanishes on this domain. The zero eigenspace consists precisely of the constant functions. The remaining eigenvalues are

$$E_{b,n} = \left(\frac{n\pi}{\Delta_b}\right)^2, \quad n = 1, 2, \dots \quad (3.22)$$

A separated homogeneous condition at an endpoint has the form $\alpha f + \beta f' = 0$ with $(\alpha, \beta) \neq (0, 0)$. Requiring the constant function to lie in the domain forces $\alpha = 0$. Hence $\beta \neq 0$ and $f' = 0$. The Neumann condition at each endpoint gives zero flux. Integration by parts shows that A_b is nonnegative and self-adjoint on (3.21). Its kernel consists of functions with $\int_{\mathcal{W}_b} |f'|^2 = 0$. Separation of variables then gives the Neumann spectrum of the interval of length Δ_b .

Put $t_b(d) = d^2/2$. Then the endpoint scales are $t_{\text{UV}} = b^2$ and $t_{\text{IR}} = \pi b^2$.

For a candidate outer length L_{trial} write the squared ratio as $c = (L_{\text{trial}}/\ell_{\text{UV}})^2$. Equal-capacity closure is the equation

$$\sqrt{c} - \sqrt{\pi} = 0. \quad (3.23)$$

It has the unique dilation-invariant solution

$$c_{\text{end}} = \pi = \frac{t_{\text{IR}}}{t_{\text{UV}}}. \quad (3.24)$$

The only continuous positive function $R(x, y)$ on $x, y > 0$ satisfying common scale invariance and composition $R(x, z) = R(x, y)R(y, z)$ with unit logarithmic differential at $x = y$

is $R(x, y) = y/x$. The endpoint formulas give $L_{\text{IR}}/\ell_{\text{UV}} = \sqrt{\pi}$. For the last assertion scale invariance gives $R(x, y) = f(y/x)$ and composition gives $f(rs) = f(r)f(s)$. Continuity implies $f(r) = r^q$. The differential normalization fixes $q = 1$.

The Gaussian also reconstructs the metric used by the local generator. On $[0, L_{\text{IR}}]$ define

$$d_{K_b}(x, y) := b \sqrt{2 \log \frac{K_b(0)}{K_b(x-y)}}. \quad (3.25)$$

The Dirichlet form of A_b on the active window is

$$\mathcal{E}_b(f) = \int_{\mathcal{W}_b} |f'(d)|^2 dd, \quad \mathcal{D}(\mathcal{E}_b) = H^1(\mathcal{W}_b). \quad (3.26)$$

Substituting the Gaussian profile shows that its correlation metric is the Euclidean separation $d_{K_b}(x, y) = |x - y|$. On $\mathcal{W}_b$ it equals the intrinsic metric of the form (3.26). If $|f'| \leq 1$ almost everywhere the fundamental theorem for Sobolev functions gives $|f(x) - f(y)| \leq |x - y|$. The affine function $f(d) = d$ saturates this bound.

Translate $\mathcal{W}_b$ to $[0, \Delta_b]$ and reflect it across both endpoints. Its double is

$$\widetilde{\mathcal{W}}_b = \mathbb{R}/(2\Delta_b\mathbb{Z}), \quad \theta = \frac{\pi(d - \ell_{\text{UV}})}{\Delta_b}. \quad (3.27)$$

Theorem 3.5 (Reflecting double and carrier compatibility). Even reflection gives a unitary map from $L^2(\mathcal{W}_b, dd)$ onto the reflection-even subspace of $L^2(S^1, d\theta/2\pi)$. The map is

$$(\mathcal{R}_b f)(e^{i\theta}) = \sqrt{\Delta_b} f\left(\ell_{\text{UV}} + \frac{\Delta_b}{\pi}|\theta|\right), \quad -\pi \leq \theta \leq \pi. \quad (3.28)$$

It takes normalized Neumann modes to 1 and $\sqrt{2}\cos(n\theta)$ and intertwines A_b with $(\pi/\Delta_b)^2(-\partial_\theta^2)$ on that subspace.

After choosing the orientation in Theorem 3.3

$$\mathfrak{C}_b(\exp(\theta J)) = \left[\frac{\Delta_b}{\pi} \theta \right]_{2\Delta_b\mathbb{Z}} \quad (3.29)$$

is the unique orientation-preserving Lie-group isomorphism $G \rightarrow \widetilde{\mathcal{W}}_b$ whose differential sends the unit angular generator to Δ_b/π . Thus the Gaussian tangent line and the reflected spectral circle are respectively a universal cover and compact quotient of one normalized change carrier.

Proof. Even extension doubles the integral on $[0, \Delta_b]$. Multiplication by $\sqrt{\Delta_b}$ gives the unitary normalization. Even H^2 functions have zero derivative at the fixed points and every Neumann function extends evenly. This proves the spectral statement. The exponential coordinate of $G \cong SO(2)$ has kernel $2\pi\mathbb{Z}$ while the lift $\theta \mapsto (\Delta_b/\pi)\theta$ maps this kernel onto $2\Delta_b\mathbb{Z}$. It induces (3.29). The orientation and differential give uniqueness. $\square$

The reflecting operator has the corresponding finite-window interpretation. For $\tau \geq 0$ the semigroup is

$$C_b(\tau) := e^{-\tau A_b/2}, \quad C_b(\tau)\phi_n = e^{-\tau E_{b,n}/2}\phi_n, \quad (3.30)$$

where $\phi_0 = \Delta_b^{-1/2}$ is the constant mode. The Neumann heat semigroup preserves positivity and total probability. Thus every probability density $\varrho_0 \in L^1(\mathcal{W}_b) \cap L^2(\mathcal{W}_b)$ satisfies

$$\left\| C_b(\tau)\varrho_0 - \frac{1}{\Delta_b} \right\|_2 \leq e^{-\tau E_{b,1}/2} \left\| \varrho_0 - \frac{1}{\Delta_b} \right\|_2. \quad (3.31)$$

The uniform state is the equilibrium of this diffusion and is approached exponentially at a rate fixed by the first Neumann eigenvalue. Here τ is the internal diffusion parameter on the correlation window.

3.4 The Gaussian ladder and its central sign

On the full tangent line the Gaussian ladder acts on the normalized $L^2(\mathbb{R})$ vector proportional to the correlation profile K_b given by

$$g_b^G(y) := (\pi b^2)^{-1/4} e^{-y^2/(2b^2)}. \quad (3.32)$$

The reflecting window organizes finite correlation scales. The ladder organizes operations that resolve and connect source modes and their relative phases. Both constructions use the same Gaussian profile with the full tangent line retaining the modes needed for the operator algebra.

Proposition 3.6 (Gaussian ladder, order-neutral closure and central sign). Let

$$A = c_1 b \frac{d}{dy} + c_2 \frac{y}{b} + c_0 \quad (3.33)$$

be a first-order operator with complex affine coefficients on the Schwartz core of $L^2(\mathbb{R})$. If $Ag_b^G = 0$ and $[A, A^*] = I$ there is some $e^{i\gamma} \in U(1)$ such that

$$A = e^{i\gamma} a_b, \quad a_b := \frac{1}{\sqrt{2}} \left(b \frac{d}{dy} + \frac{y}{b} \right). \quad (3.34)$$

The ordered products A^*A and AA^* are independent of the residual phase. The two phase-independent products differ by I . Reversal exchanges their orderings. In the normalized affine class

$$\mathfrak{Q}_b = \{H_{b,\alpha} = \alpha a_b^* a_b + (1 - \alpha) a_b a_b^* : 0 \leq \alpha \leq 1\}, \quad (3.35)$$

the unique element invariant under interchange of the two orderings is

$$H_b^G = \frac{1}{2} (a_b^* a_b + a_b a_b^*) = a_b^* a_b + \frac{1}{2} I. \quad (3.36)$$

The self-adjoint closure of (3.36) has an orthonormal Hermite basis $(h_{b,n})_{n \geq 0}$ with

$$H_b^G h_{b,n} = \left(n + \frac{1}{2} \right) h_{b,n}. \quad (3.37)$$

Its unitary phase flow satisfies

$$e^{-2\pi i H_b^G} = -I, \quad e^{-4\pi i H_b^G} = I. \quad (3.38)$$

Within the class (3.35) the half-unit and the order-two central sign are therefore fixed by order reversal.

Proof. The logarithmic derivative of g_b^G is $-y/b^2$. Annihilation therefore forces $c_2 = c_1$ and $c_0 = 0$. The commutator is $2|c_1|^2 I$ and fixes $|c_1| = 1/\sqrt{2}$. The ordering involution sends $H_{b,\alpha}$ to $H_{b,1-\alpha}$ whose unique fixed point is $\alpha = 1/2$.

The equation $a_b g_b^G = 0$ gives the ground-state eigenvalue $1/2$. Successive applications of a_b^* give the Hermite basis and raise the eigenvalue by one. Equation (3.38) follows by spectral calculus. The closure has domain

$$\left\{ \sum_{n \geq 0} c_n h_{b,n} : \sum_{n \geq 0} \left(n + \frac{1}{2}\right)^2 |c_n|^2 < \infty \right\}. \quad (3.39)$$

□

This central sign belongs to the metaplectic phase lift on the full tangent Hilbert space. On a finite carrier compressed source operations generate $SU(2)$. The full-cycle operator restricts to its nontrivial central element (Proposition G.1). This identifies the two central actions through a common carrier.

4 Emergent Response Structures

Coherent comparison of source modes retains both intensities and relative phases. The Gaussian ladder supplies matrix units that transfer amplitudes between resolved levels. Their products describe successive transfers. The first two-level corner supports noncommuting comparisons. Complete observations distinguish the resulting operations and their observation algebra determines the compact process space. Unbiased composition determines its heat evolution. The internal response geometry thus records operations compared in the same source preparation. Peter–Weyl decomposition resolves their process spectrum and the $SU(2)$ classification fixes the least-dimensional centre-sensitive spin module. Appendix G proves the spectral and central properties and fixes the metric normalization used here. Relative-frame comparison gives local frames, the least faithful adjoint-preserving representation supplies matter, and the natural reversal-real variation supplies the scalar connector. The three continuation theorems characterize these constructions among complete realizations preserving the generated adjoint and exchange structures, source phase and reversal.

Write $\mathcal{D}_0$ for the preceding source data comprising the minimal real phase plane and its Gaussian convolution semigroup together with the finite Neumann window of normalized positive spectrum n^2 and the metaplectic central character. The real phase plane supplies the complex scalar structure. Here V denotes the two-dimensional complex span of the first two Gaussian modes. Its coordinates are mode amplitudes and the ladder compares and mixes these amplitudes through its matrix units.

Theorem 4.1 (Generated process algebra, round response, and real amplitude). The matrix units obtained from the Gaussian ladder generate the following operational constructions without a spatial dimension as input.

- (i) The first noncommuting finite corner acts on $V = \text{span}\{h_0, h_1\}$. Its determinant-one reversible operations form $SU(V)$. Coherent comparison with the same source level h_2 distinguishes their matrix entries and the uniform closure of the resulting process-observation algebra is $C(SU(V))$.
- (ii) Independent steps $\exp(i\epsilon \sum_a \xi_a J_a)$ formed from the compressed ladder with a fixed centred law and covariance cI for $0 < c < \infty$ converge after $\lfloor 8t/(c\epsilon^2) \rfloor$ compositions to the round group heat evolution. The generated equal frame average realizes these moment conditions. The six coordinate directions have $c = 1/3$. Independent unit-variance Gaussian coefficients have $c = 1$. Both yield the same spectrum

$$\lambda_n = n^2 - 1, \quad m_n = n^2, \quad n = 1, 2, \dots \quad (4.1)$$

The complete centre-averaged algebra has character space $SU(V)/\{\pm I\} \cong \mathbb{RP}^3$.

- (iii) The smallest real algebra containing these reversible operations is $\mathbb{H}_V = \mathbb{R}I \oplus i\text{Herm}_0(V)$. Every nonzero amplitude has a unique polar factorization $h = RU$. The full comparison in the generated trace preparation gives

$$\frac{1}{2} \text{ReTr}(dh^*dh) = dR^2 + R^2 g_{\text{round}}. \quad (4.2)$$

Proof. The ladder polar decomposition gives all E_{ij} and hence the first noncommuting corner $M_2(\mathbb{C})$. Its compressed raising and lowering operators satisfy the $\mathfrak{su}(2)$ relations by Proposition G.1. Theorem G.2 includes the coherent comparison and finite-control constructions and proves point separation. Its finite-variance Fourier limit gives the scalar multiplier $e^{-4tj(j+1)}$ on the spin- j representation. Peter–Weyl decomposition supplies its $(2j+1)^2$ matrix coefficients. Putting $n = 2j+1$ proves (4.1). The central conditional expectation retains the complete invariant algebra. So its character space is the stated quotient. Finally the finite-control preparation in the same appendix and Proposition G.3 establish the trace preparation together with the real algebra and polar metric on the same source-derived carrier. $\square$

Observable rigidity of the process geometry. For the generated process system the complete process-observation algebra together with its source trace metric determines the compact parent as $SU(V)$ with its normalized round metric up to an isometry preserving the source observations. Its complete centre-invariant subalgebra determines the quotient $SU(V)/\{\pm I\} \cong \mathbb{RP}^3$ with the descended metric. The observation class and source normalization thus fix both geometries.

Proof. The complete observations separate the operations and generate $C(SU(V))$. The trace preparation fixes the normalized bi-invariant metric. Passing to the complete centre-invariant subalgebra identifies precisely the central orbits and the trace metric descends through the quotient. $\square$

The comparison algebra distinguishes complete operations on V . An input amplitude in h_0 or h_1 is coherently paired with the reference level h_2 which the controlled operation leaves unchanged. Comparing the transformed amplitude with this reference in two

quadratures recovers the real and imaginary parts of each matrix entry of U . These comparisons also distinguish U from $-U$ through their relative sign on the operated branch. The preparations and probes use the same generated source algebra. The generated J_a are traceless controls of the two-level response. So their exponentials generate relative determinant-one operations. The central source phase remains available in $U(V)$. The finite-variance composition limit is independent of the microscopic step law within the stated class. An unbiased law is realized by conjugating each step with an independently redrawn equally weighted choice of the 24 generated frame controls. This removes the mean and makes the covariance scalar. The quadratic term in the small-step expansion is then proportional to $\sum_a (J_a^{(j)})^2 = j(j+1)I$. Repeated independent composition gives the round heat multiplier. The three control directions are the generated traceless Hermitian responses on V . The trace comparison makes $2iJ_a$ orthonormal tangent generators and the accumulated microsteps have the Laplace–Beltrami operator of this same angular metric as their positive generator. The response geometry and its diffusion therefore share the normalization fixed by the source comparison. A point of the compact response space labels a complete reversible operation. The heat kernel describes the distribution produced by many independent small operations. Its spectrum measures the rates at which process information is lost under averaging. The local cycle (G.13) makes the angular centre detectable. Centre averaging then identifies U and $-U$ and retains every distinction between their central orbits. The spectrum of this complete invariant observation algebra is $\mathbb{RP}^3$. The full second moment retains the scalar amplitude direction which a connected response removes. The resulting four coordinates describe angular and amplitude responses.

The continuation of this process algebra to matter uses recursive minimality. Each generated response algebra has a least-dimensional faithful adjoint-preserving representation. Reversal pairs the source-derived chiral modules and naturality fixes how the generated elementary odd variation acts on the composite response. The three continuation theorems and their universal properties are distinguished in Figure 1.

4.1 Response shells and spherical classification

The reflecting interval already supplies a dimensionless square spectrum. If $E_{b,1}$ is the first positive Neumann eigenvalue in (3.22) then on the positive-mode subspace

$$\hat{A}_b := \frac{A_b}{E_{b,1}}, \quad \hat{A}_b \phi_{b,n} = n^2 \phi_{b,n}, \quad n = 1, 2, \dots \quad (4.3)$$

Each positive Neumann eigenvalue has one interval mode. The Gaussian ladder also generates finite matrix responses whose capacity describes the operations available on a retained set of source levels.

Source-generated finite response capacity. On the Gaussian ladder let $H_n = \text{span}\{h_{b,0}, \dots, h_{b,n-1}\}$ and let P_n be its orthogonal projection and put $a_n = P_n a_b P_n$. Closure under complex linear combination and composition with adjunction gives

$$\mathcal{A}_n := \text{Alg}_{\mathbb{C}}^*(I_n, a_n) = \text{End}_{\mathbb{C}}(H_n), \quad \dim_{\mathbb{C}} \mathcal{A}_n = n^2. \quad (4.4)$$

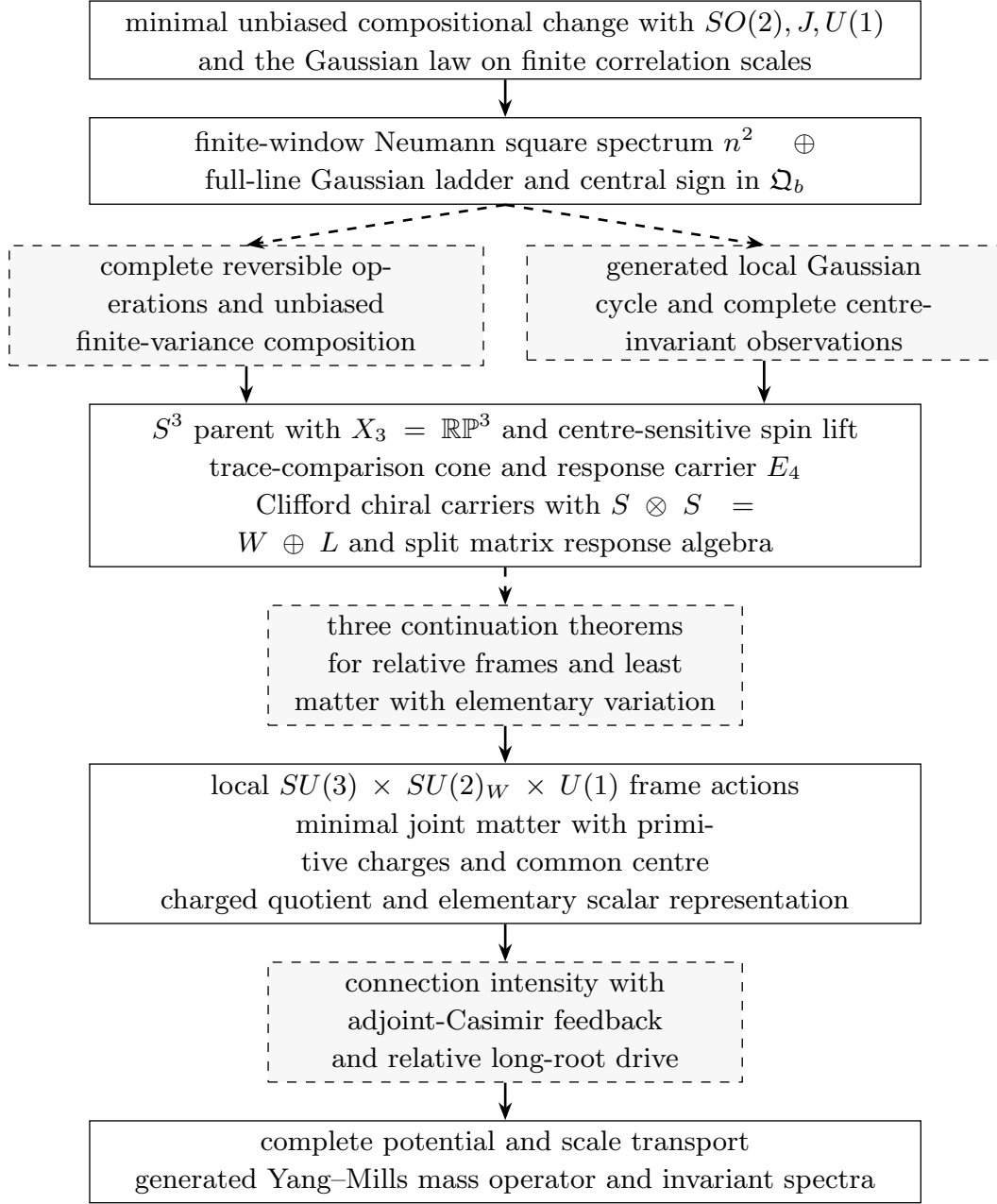


Figure 1. From source operations to internal structure and Yang–Mills response. Solid arrows are deductions within the stated classes. The upper dashed arrows specify the operation and observation classes. The lower dashed arrow marks the change of physical role from generated carriers to their relative-frame and least-matter realizations with the elementary connector. The three continuation theorems record these constructions and the universal properties that retain the source adjoint and exchange sectors together with the phase and reversal actions. The established spectral and central properties and the normalized metric are recorded alongside them. The connection intensity and Casimir feedback together with the long-root drive are derived from the generated colour source. Their complete response gives the mass operator of Theorem 8.4. The figure follows the internal branch. The self-adjoint real form of the first matrix corner supplies the Lorentzian event branch. Their tensor-product representation carries the local field algebra.

The Hermitian response space has real dimension n^2 . Its traceless part has dimension $n^2 - 1$. These are also respectively the ranks of the full exponential response and its connected susceptibility in Proposition G.1.

Proof. The simple spectrum of $N_n = a_n^* a_n$ gives its rank-one spectral projections. Multiplying these by a_n and a_n^* generates every matrix unit. So $\mathcal{A}_n = M_n(\mathbb{C})$. The scalar and traceless Hermitian parts have real dimensions 1 and $n^2 - 1$. Appendix G constructs all corners simultaneously in (G.1) and proves the two response ranks by differentiation. $\square$

Heat shells and finite response spaces. The nested corners add $n^2 - (n-1)^2 = 2n - 1$ independent matrix responses at stage n . The degeneracies of the process generator are obtained by resolving functions of complete operations into orthogonal modes. Theorem 4.1 gives the full decomposition

$$L^2(SU(V)) = \widehat{\bigoplus_{j=0,1/2,\dots}} V_j \otimes V_j^*, \quad \lambda_j = 4j(j+1), \quad m_j = (2j+1)^2. \quad (4.5)$$

Each spin- j block contains $(2j+1)^2$ independent matrix coefficients resolving both input and output indices of the operation. Thus the quadratic shell multiplicity follows from complete process comparison. The match $n = 2j+1$ relates the normalized Neumann eigenvalue to λ_j+1 with the window and process Hilbert spaces remaining distinct. Reading only the final vacuum-density state identifies operations with the same output ray and retains the smaller orbit $SU(V)/U(1) \cong S^2$.

Dimension from the complete process spectrum. At the unit normalization above the number of scalar process modes below a spectral cutoff Λ is

$$N_G(\Lambda) = \sum_{n^2-1 \leq \Lambda} n^2 = \frac{1}{3} \Lambda^{3/2} + O(\Lambda).$$

Indeed with $K = \lfloor \sqrt{\Lambda+1} \rfloor$ the sum equals $K(K+1)(2K+1)/6$. The Peter–Weyl decomposition includes every scalar mode of the generated Laplace–Beltrami operator. For the complete scalar spectrum of a closed smooth d -dimensional Riemannian manifold Weyl’s law gives $N(\Lambda) \sim C \Lambda^{d/2}$ with $C > 0$ [17]. The exponent therefore fixes $d = 3$ for any such realization of this complete process spectrum. This Weyl argument determines the dimension. The exact parent and quotient geometries are fixed separately by the complete-observation rigidity argument above. The leading coefficient agrees with the scalar Weyl constant for the unit trace metric $\text{Vol}(S^3, g_{\text{round}})/(6\pi^2) = 2\pi^2/(6\pi^2) = 1/3$. Increasing the cutoff resolves more independent responses and their cumulative number measures the dimension of the internal process space. The trace comparison has already fixed its round metric. This spectral count reads the same dimension from the density of modes.

The spherical capacity comparison sharpens this asymptotic dimension statement to an exact relation in every shell. Its geometric conditions are satisfied by the trace metric of the process construction.

Definition 4.2 (Isotropic spectral-capacity class). The abstract response fibre is a compact connected complete Riemannian manifold of dimension $d \geq 2$. Its simply connected cover has constant positive sectional curvature normalized to one and is locally fully isotropic. The point stabilizer is transitive on orthonormal two-frames. The order-preserving lift assigns source mode n to angular degree $l = n - 1$ on this cover with full scalar shell capacity

$$\dim \mathcal{F}_n = n^2. \quad (4.6)$$

This is the spectral-capacity property satisfied by the complete-operation realization above.

The normalized cover in this property is S^d with $d \geq 2$ by the constant-curvature space-form theorem. Its scalar shell data are

$$\lambda_l^{(d)} = l(l + d - 1), \quad m_l^{(d)} = \frac{(2l + d - 1)(l + d - 2)!}{l!(d - 1)!}. \quad (4.7)$$

For these sphere data or their even- l antipodal sector the spherical affine response class consists of members with constants $a > 0, c$ independent of the retained degree satisfying

$$m_l^{(d)} = a\lambda_l^{(d)} + c. \quad (4.8)$$

Theorem 4.3 (Capacity and affine characterization within the isotropic spherical class). Within the isotropic spherical class defined above the spectral-capacity property selects $d = 3$. Either of the identities

$$m_l^{(d)} = (l + 1)^2, \quad (4.9)$$

$$m_l^{(d)} = a\lambda_l^{(d)} + c, \quad a > 0, \quad (4.10)$$

on every degree or on all even degrees forces $d = 3$. The second then has $a = c = 1$. The normalized parent is the compact round S^3 .

Proof. The multiplicity is a polynomial in l of degree $d - 1$ with nonzero leading coefficient. Each proposed right-hand side has degree two. So equality forces $d - 1 = 2$. Equality on the infinite even subsequence is also a polynomial identity. For $d = 3$ one has $m_l = (l + 1)^2 = l(l + 2) + 1$. This proves the converse and $a = c = 1$. On the parent sphere the first scalar harmonics are the $d + 1$ ambient coordinate functions. Matching their multiplicity to the generated first nonconstant shell gives $d + 1 = 4$. The space-form condition fixes the normalized simply connected parent as the unit round sphere. $\square$

For the process realization and more generally under this property write

$$\mathcal{F}_n = V_{(n-1)/2} \otimes V_{(n-1)/2}^* \cong \text{End}(V_{(n-1)/2}), \quad n \geq 1. \quad (4.11)$$

Its excitation energy is $n^2 - 1$ and its capacity is n^2 . Under diagonal conjugation the trace gives the invariant splitting

$$\mathcal{F}_n = \mathbb{C}I_n \oplus \text{End}_0(V_{(n-1)/2}), \quad \dim \text{End}_0(V_{(n-1)/2}) = n^2 - 1. \quad (4.12)$$

The scalar–traceless splitting concerns matrix responses within one Laplace eigenspace. Under the matrix-coefficient identification the line $\mathbb{C}I_n$ is spanned by the character $\chi_j(U) = \text{Tr } \pi_j(U)$ where $j = (n-1)/2$. It is invariant under conjugation and retains the shell energy $4j(j+1)$. The global constant function occurs at $j = 0$. Pointwise multiplication of mode functions generally mixes these eigenspaces. Every finite free quotient of the parent S^3 is compact and has discrete elliptic spectra of finite multiplicity.

4.2 Central character and projective compression

A source-derived common carrier. Proposition G.1 realizes both central actions on $V = H_2$ where

$$J_+ = a_2^*, \quad J_- = a_2, \quad J_3 = a_2^* a_2 - \frac{1}{2} I_V, \quad H_b^G|_V = I_V + J_3.$$

These operators satisfy the spin- $\frac{1}{2}$ commutation relations and the Gaussian full-cycle operator is $e^{-2\pi i H_b^G}|_V = -I_V \in SU(V)$. Moreover (G.13) implements this restriction as the generated local cycle $R_V(2\pi) = C(-I_V)$ with an unchanged complement. Hence the central correspondence is realized by one local operation on the common carrier. It is antipodal on the parent operation group and acts trivially by conjugation on $\text{Herm}_0(V)$. These actions and their $\text{Spin}(4)$ lifts remain distinct. The local cycle acts on V with the identity on its complement. The full-source cycle acts on the entire ladder. The single-reversal property records this common reversal in an abstract form.

Single-reversal covariance. Let $Z_2^{\text{rev}} = \{1, r\}$ be generated by reversal of the ordered elementary response. The Gaussian phase lift and the phase-sensitive geometric lift are required to be functorial under this same operation and faithful on it. Thus there are injective homomorphisms

$$\iota_G : Z_2^{\text{rev}} \longrightarrow Z_2^G, \quad \iota_{\text{geom}} : Z_2^{\text{rev}} \longrightarrow SU(2), \quad (4.13)$$

where $\iota_G(r) = \sigma_G$ and the geometric reversal commutes with every connected frame rotation. This is single-reversal covariance. Transporting the reversed response through either lift preserves its composition law.

Proposition 4.4 (Central transfer and projective response geometry). Under single-reversal covariance $\iota_{\text{geom}}(r)$ is the nonidentity element of $Z(SU(2))$ and the map

$$\chi := \iota_{\text{geom}} \circ \iota_G^{-1} : Z_2^G \longrightarrow Z(SU(2)) \quad (4.14)$$

exists is unique and sends σ_G to the antipodal element $-I \in SU(2)$. If ε denotes the nontrivial one-dimensional character of $\mathbb{Z}_2$ then the minimal spin module satisfies

$$S|_{Z_2^{\text{geom}}} \cong \varepsilon \oplus \varepsilon. \quad (4.15)$$

The identification concerns the restricted central actions. With the spectral-capacity property retain the complete centre-invariant observable algebra $\mathcal{A}_{\text{inv}} := C(S^3)^{\{\pm I\}}$. The compact Hausdorff space representing this commutative C^* -algebra is up to homeomorphism

$$X_3 := SU(2)/\{\pm I\} \cong SO(3) \cong \mathbb{RP}^3, \quad (4.16)$$

whereas the centre-sensitive response retains the spin lift $SU(2) \cong S^3$. In the source-spectrum index $q = l + 1$ scalar functions on X_3 are the odd-index compression

$$q = 1, 3, 5, \dots, \quad \lambda_q = q^2 - 1, \quad m_q = q^2. \quad (4.17)$$

Proof. The commutation clause puts the image of ι_{geom} in $Z(SU(2))$. Faithfulness gives $\iota_{\text{geom}}(r) = -I$. So both maps in (4.13) are isomorphisms of groups of order two. Their composite is the unique χ . The defining two-dimensional module has central action $-I$ giving (4.15).

The nontrivial central element acts on $S^3 \cong SU(2)$ by the antipodal map. The quotient map $p : S^3 \rightarrow S^3/\{\pm I\}$ induces an isometric $*$ -isomorphism $p^* : C(S^3/\{\pm I\}) \rightarrow \mathcal{A}_{\text{inv}}$. Invariant continuous functions factor uniquely through this quotient. They separate distinct orbits. The character space of $\mathcal{A}_{\text{inv}}$ is consequently this quotient uniquely up to homeomorphism. The adjoint double cover identifies it with $SO(3) \cong \mathbb{RP}^3$ and the round metric descends from the sphere. Pullback identifies scalar functions on the quotient with the even- l sector on the cover. Since $q = l + 1$ this is exactly the odd-index compression in (4.17). A centre-sensitive module does not descend as an ordinary scalar module and therefore retains the spin lift. $\square$

Centre averaging retains all scalar modes with odd q . Their complete count at the same unit normalization is

$$N_X(\Lambda) = \sum_{\substack{q^2 - 1 \leq \Lambda \\ q \text{ odd}}} q^2 = \frac{1}{6} \Lambda^{3/2} + O(\Lambda).$$

The sum of the first r odd squares is $r(4r^2 - 1)/3$ with $r = \lfloor (\sqrt{\Lambda + 1} + 1)/2 \rfloor$. The quotient therefore preserves the three-dimensional Weyl exponent and halves the leading mode count in agreement with $\text{Vol}(X_3)/(6\pi^2) = \pi^2/(6\pi^2) = 1/6$ at unit radius. The local comparison metric is unchanged while the global observation identifies U with $-U$. The complete invariant algebra separates all these central orbits and fixes the projective space. A proper invariant subalgebra may identify additional points.

The two spin structures on $\mathbb{RP}^3$ have the same central character. We fix the representative τ_- whose antipodal deck lift is $-\omega \in \text{Spin}(4)$ as the spin convention throughout. The alternative τ_+ interchanges the positive and negative Dirac parity subsequences. Appendix A lists both choices. Bosonic spectra and the spectrum of D^2 are independent of this convention.

Physical interpretation. Centre-blind observables descend to X_3 while phase-sensitive amplitudes retain the spin lift. This distinction is expressed by writing an associated field on the universal cover as

$$\psi(-u) = \rho_E(-I)\psi(u). \quad (4.18)$$

The trivial character gives an ordinary field. The nontrivial character gives a twisted associated bundle. The universal twofold cover is distinct from the principal spin-frame bundle. Dirac fields require the spin structure and deck lift specified in Appendix A.

The projective quotient thus expresses the resolution of the observation. Keeping the complete invariant algebra retains every distinction between central orbits while phase-sensitive responses retain the additional sign.

The unit-radius response is expressed in the length units of its generated source. We write R_b for the global correlation length $L_{\text{IR}}(b)$ when using it as the geometric radius. These are two notations for the same quantity. Explicitly the response amplitude $h_b(U) = R_b U$ has normalized trace norm $(\text{Tr}(h_b^* h_b)/2)^{1/2} = R_b$. Restricting (4.2) to this amplitude shell gives the angular metric $R_b^2 g_1$ which descends through the central quotient. Likewise $\ell_{\text{UV}}(b)$ is the unit-information resolution length itself and $\Delta_b = L_{\text{IR}}(b) - \ell_{\text{UV}}(b)$ is the intervening window length. All three lengths are determined by the same source width b .

Endpoint-scaled metric and spectrum. Let g_1 be the bi-invariant metric on X_3 normalized to unit sectional curvature. Put

$$R_b := L_{\text{IR}}(b) = \sqrt{2\pi} b, \quad g_{3,b} := R_b^2 g_1. \quad (4.19)$$

Here R_b denotes the global correlation length defined above. Within the bi-invariant metric class the unit-radius normalization and this source scale fix the metric uniquely. Its sectional and scalar curvatures are R_b^{-2} and $6R_b^{-2}$ and the projective scalar spectrum is

$$\text{Spec}(\Delta_{3,b}) = \left\{ \frac{q^2 - 1}{R_b^2} : q = 1, 3, 5, \dots \right\}, \quad m_q = q^2. \quad (4.20)$$

For the heat parameter $a_b(t) = t/(2R_b^2)$ the two correlation times give $a_{\text{UV}} = 1/(4\pi)$ and $a_{\text{IR}} = 1/4$ with $a_{\text{IR}}/a_{\text{UV}} = \pi$.

Proof. An Ad-invariant inner product on the simple algebra $\mathfrak{su}(2)$ is unique up to a positive scalar. The amplitude shell $h_b(U) = L_{\text{IR}}(b)U$ fixes that scalar through the source trace metric. Constant metric rescaling gives the stated curvature and Laplacian factors while (4.17) supplies the unit-radius spectrum. Finally $t_{\text{UV}} = b^2$ and $t_{\text{IR}} = \pi b^2$ together with $R_b^2 = 2\pi b^2$ give the two dimensionless heat parameters. $\square$

The radius converts the dimensionless spectrum to response energies. For a wave-number cutoff k let $N_b(k^2)$ count the scalar eigenvalues of $\Delta_{3,b}$ up to k^2 including multiplicities. Rescaling gives

$$N_b(k^2) = N_X((kR_b)^2) = \frac{(kR_b)^3}{6} + O((kR_b)^2), \quad kR_b \longrightarrow \infty.$$

Since $\text{Vol}(X_3, g_{3,b}) = \pi^2 R_b^3$ the leading term is precisely $\text{Vol}(X_3, g_{3,b}) k^3 / (6\pi^2)$. Writing L for the internal radius at a chosen scale gives $L = R_b$ and the same dimensionless cutoff $k_{\text{res}} L = kR_b$. It measures the resolution in units of the internal size.

For $E_3 = T_e SU(2)$ differentiation of conjugation gives the double cover $SU(2) \rightarrow SO(E_3)$ identified with $\text{Spin}(E_3)$ after choosing an oriented frame. Its centre acts trivially on E_3 whereas left multiplication by $-I$ is antipodal on the parent.

The required centre-sensitive lift is obtained from the standard $SU(2)$ representation classification [15].

Theorem 4.5 (Minimal phase-sensitive spin lift). Let S be a nonzero finite-dimensional irreducible unitary representation of $\text{Spin}(3) \cong SU(2)$ on which the nontrivial central element acts as $-I_S$. The least possible complex dimension is two. Up to unitary equivalence the minimal representation is the defining module

$$S \cong \mathbb{C}^2, \quad j = \frac{1}{2}. \quad (4.21)$$

Equivalently it is the unique irreducible module of minimal dimension for $\text{Cl}_3^0(\mathbb{C}) \cong M_2(\mathbb{C})$.

Proof. Every finite-dimensional irreducible complex representation of $SU(2)$ is $\text{Sym}^m(\mathbb{C}^2)$ for a unique $m \geq 0$. It has dimension $m + 1$ with spin $m/2$ and central action $(-1)^m I$. Phase sensitivity forces m to be odd whose least value is $m = 1$. The Clifford formulation is the standard isomorphism $\text{Cl}_3^0(\mathbb{C}) \cong \text{Cl}_2(\mathbb{C}) \cong M_2(\mathbb{C})$. $\square$

Unit-sphere realization. For the minimal module $S \cong \mathbb{C}^2$ its unit sphere is $S(S) \cong S^3$. The defining $SU(2)$ action is free and transitive on this sphere and normalized Haar measure pushes forward to the round probability measure. Thus the selected spin module realizes the same round parent geometrically.

4.3 The four-dimensional response carrier

The generated real amplitude algebra has the polar metric (4.2) obtained from the prepared full trace comparison in Proposition G.3. The amplitude space is the smallest real associative closure of the relative reversible operations. It consists of the quaternion amplitudes commuting with the pseudoreal reversal. For $h = RU$ the trace preparation reads $\text{Tr}(h^*h)/2 = R^2$ as total response strength. Changing R alters this strength while preserving the normalized pattern U . Angular variation changes the pattern at fixed strength. The full second-moment readout retains both responses yielding the radial and angular terms of the metric on this real amplitude space. At source width b the global correlation length is the radius of the shell $R = R_b = L_{\text{IR}}(b) = \sqrt{2\pi}b$. The scale-covariant metric property records the unit-speed radial coordinate and unit round orbit together with homothety under Gaussian rescaling. These conditions also characterize the trace metric among invariant metrics.

Scale-covariant cone completion. Let (S^3, g_1) be the unit round parent selected by the spectral-capacity property. Its positive-scale completion $(0, \infty) \times S^3$ carries an $SO(4)$ -invariant metric G for which the endpoint radius R is unit speed and the Gaussian rescaling $\delta_c(R, u) = (cR, u)$ is a homothety $\delta_c^*G = c^2G$. The normalization at $R = 1$ restricts G to g_1 on the angular orbit. This is the scale-covariant metric property.

Theorem 4.6 (Scale completion and the four-dimensional response carrier). The metric conditions in this definition uniquely characterize the cone

$$G = dR^2 + R^2 g_1. \quad (4.22)$$

It is isometric to $\mathbb{R}^4 \setminus \{0\}$ by $(R, u) \mapsto Ru$ and its tangent space has the radial-angular orthogonal splitting

$$T_{(R,u)} \text{Cone}(S^3) = T_u S_R^3 \oplus \mathbb{R} \partial_R. \quad (4.23)$$

Let $e \in X_3$ be a reference frame. Put $E_3 = T_e X_3$ and let $\mathbb{R}_{\text{sc}} = \mathbb{R} \partial_R$ at $R = R_b$. Then

$$E_4 := E_3 \oplus \mathbb{R}_{\text{sc}}, \quad g_4 := g_{3,b} \oplus dR^2 \quad (4.24)$$

is an oriented four-dimensional Euclidean response carrier. The complexified Clifford algebra of (E_4, g_4) is $M_4(\mathbb{C})$ with even subalgebra $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. Under the even algebra or the $\text{Spin}(4)$ action its Dirac module decomposes into two two-dimensional Weyl modules $S_L \oplus S_R$. Odd Clifford multiplication connects the two summands. The polar isometry identifies E_3 with the tangent hyperplane of the standard sphere and $\mathbb{R}_{\text{sc}}$ with its oriented radial normal. It intertwines the metric and orientation with the defining $SO(4)$ action.

Proof. An $SO(4)$ -invariant cohomogeneity-one metric has the form $a(R)^2 dR^2 + f(R)^2 g_1$. There is no invariant cross term because the isotropy $SO(3)$ has no fixed covector in the tangent representation. Unit-speed normalization gives $a = 1$. Homothety gives $f(cR) = cf(R)$. Hence $f(R) = \kappa R$. The unit-orbit normalization fixes $\kappa = 1$. This is the Euclidean polar-coordinate metric whose differential gives the orthogonal tangent splitting and the $SO(4)$ intertwiner. The three angular directions and one radial direction give $\dim_{\mathbb{R}} E_4 = 4$. The complex Clifford classification recalled in Section 2 then gives the stated matrix algebras and chiral modules [16, Chapter 3]. $\square$

The source representation realizes the chiral split concretely. Products of two generated odd Clifford operators span a real even algebra isomorphic to $\mathbb{H} \oplus \mathbb{H}$. Its grading is an ordered product of the four source directions and

$$e_L = \frac{1 + \Gamma}{2}, \quad e_R = \frac{1 - \Gamma}{2}$$

are its only nontrivial primitive central idempotents. Complexification gives $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ while the ranges of the two source projections are precisely the two complex two-dimensional Weyl carriers. After tensoring with the generated $W \oplus L$ response their Hermite ranges have dimension eight on each side. Thus the two chiral source sectors and their dimensions are properties of the generated operator representation.

The Clifford action gives an operator readout of the generated metric. One odd multiplication changes chirality. Repeating the same multiplication returns to the original sector with scalar strength equal to the squared norm of its response direction. The anticommutator of two directions reads their metric pairing. Thus the linear connector and the quadratic trace comparison encode the same positive response norm.

The angular directions of E_3 describe response at fixed resolution. $\mathbb{R}_{\text{sc}}$ describes a change of resolution. The full trace comparison measures both angular variation and amplitude variation and the Gaussian correlation length calibrates the radial coordinate. A smooth scale field in a localized realization gives an equi-scale $3 + 1$ decomposition. At a regular point its level surface has three tangent directions and its gradient gives a normal. This is the geometry of resolution responses. Physical time belongs to the event branch. The self-adjoint real form of the same generated matrix algebra supplies Lorentzian event differences. Its determinant supplies the causal quadratic form and its algebraic positive cone supplies the time orientation. The resulting Clifford algebra also

satisfies the semispinor dimension equation $2^{m-1} = 2$ with $m = 2$ for the four-dimensional carrier.

4.4 Spin and frame actions

The rank-one groups have distinct carriers and actions

$$\begin{aligned}
SU(2)_{\text{geom}} &= \text{Spin}(E_3), \\
SU(2)_{\text{iso,L}} \times SU(2)_{\text{iso,R}} &= \widetilde{\text{Isom}}_0(X_3), \\
SU(2)_{\text{resp,L}} \times SU(2)_{\text{resp,R}} &= \text{Spin}(E_4), \\
SU(2)_S &= \text{the action on the defining spin-lift module } S, \\
SU(2)_W &= \text{the weak action on the left response factor.}
\end{aligned} \tag{4.25}$$

The local group $SU(2)_{\text{geom}}$ embeds diagonally in the response spin group after a base frame is chosen. The isometry and response pairs are identified by the cone intertwiner of Theorem 4.6. The notation $SU(2)_S$ records the representation used in the quadratic response. $SU(2)_W$ denotes the resulting gauge action after local frame covariance has been imposed.

The connected isometry algebra of the unit round $X_3 = \mathbb{RP}^3$ is $\mathfrak{so}(4)$.

Identification of the two $\text{Spin}(4)$ actions. Under the scale-covariant metric property the standard ambient action on the round S^3 parent identifies

$$SU(2)_{\text{iso,L}} \times SU(2)_{\text{iso,R}} \longrightarrow SU(2)_{\text{resp,L}} \times SU(2)_{\text{resp,R}} \tag{4.26}$$

as an isomorphism of simply connected covering groups up to conjugation by the chosen oriented base frame. The covering maps to the sphere and to the projective isometry group have different kernels.

Proof. The connected isometry group of the round sphere is induced by the defining $SO(4)$ action on its ambient $\mathbb{R}^4$. At a base point that representation decomposes as tangent hyperplane plus radial normal. The intertwiner in Theorem 4.6 carries this decomposition to $E_3 \oplus \mathbb{R}_{\text{sc}}$. Passing to spin covers gives (4.26). Changing the oriented base frame conjugates the identification. In quaternionic coordinates the common cover acts by $u \mapsto aub^{-1}$. Consequently

$$\ker[\text{Spin}(4) \rightarrow SO(4)] = \{(I, I), (-I, -I)\}, \tag{4.27}$$

$$\ker[\text{Spin}(4) \rightarrow \text{Isom}_0(\mathbb{RP}^3)] = \{(\epsilon_L I, \epsilon_R I) : \epsilon_L, \epsilon_R = \pm 1\}. \tag{4.28}$$

The latter target is $SO(4)/\{\pm I_4\}$. The elements $(-I, I)$ and $(I, -I)$ induce the antipodal map on S^3 . The diagonal element $(-I, -I)$ acts trivially there. In particular the centre of the diagonal tangent-spin subgroup is not the antipodal transformation. $\square$

Choosing the orientation of E_4 also fixes the standard Hodge decomposition

$$\mathfrak{so}(4) \cong \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \tag{4.29}$$

into the self-dual and anti-self-dual three-dimensional summands of $\Lambda^2 E_4^*$ [16]. The group $SU(2)_R$ has a unique conjugacy class of maximal tori. Choosing a representative gives $U(1)_R \subset SU(2)_R$ but no preferred Cartan direction is fixed by the group alone. A right-sector reference ray fixes a representative through its ray stabilizer. The weight spectra are invariant under a change of representative by conjugation [15]. Localizing the left action and choosing a right-torus representative gives the chiral response algebra

$$\mathfrak{g}_{\text{ch}} = \mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_R. \quad (4.30)$$

Here $SU(2)_W$ is the localized image of $SU(2)_{\text{resp,L}}$ under the $\text{Spin}(4)$ identification above. The transported source vacuum fixes the representative through the least-matter continuation Theorem 4.9 and Corollary G.1. This reference-vector construction fixes the charge convention. The connection condensate and a Higgs expectation value when present instead characterize dynamical backgrounds of their respective response channels.

4.5 Quadratic response and colour

Let $S \cong \mathbb{C}^2$ be the minimal phase-sensitive module generated in Theorem 4.5. We use the quadratic layer as the first composite response of two copies of S . Resolving exchange separates amplitudes unchanged by interchange from amplitudes that reverse sign. The resulting triplet and scalar line retain these two ways in which a pair responds. Its quadratic response decomposes canonically as

$$S \otimes S = \text{Sym}^2 S \oplus \Lambda^2 S =: W \oplus L, \quad \dim_{\mathbb{C}} W = 3, \quad \dim_{\mathbb{C}} L = 1. \quad (4.31)$$

The decomposition $V_{1/2} \otimes V_{1/2} = V_1 \oplus V_0$ is the Clebsch–Gordan decomposition of the generated phase-sensitive module. The same exchange decomposition occurs at the first additive threshold of the construction in Ref. [8].

Full quadratic response algebra. The triplet response carrier is $W = \text{Sym}^2 S$. Let $\mathcal{A}_W$ be the unital complex $*$ -algebra generated by the induced operators $\rho_W(g)$ with $g \in SU(2)_S$. Finite linear combination together with composition and adjunction defines its algebraic closure. Its spectral projections belong to the same finite-dimensional C^* -algebra by functional calculus. This closure is

$$\mathcal{A}_W = \text{End}_{\mathbb{C}}(W). \quad (4.32)$$

Proof. The symmetric-square representation of $SU(2)_S$ on W is irreducible. Burnside’s theorem therefore identifies the generated complex unital algebra with the full endomorphism algebra. $\square$

The source $SU(2)_S$ image and the reversible frame group of the algebra it generates are different objects. The former supplies an irreducible set of controls. Algebraic closure resolves every matrix response on W . The completeness principle for the generated block retains every matrix response as an observable. It is the same closure used for the source algebra not an additional dynamical input. Once the complete algebra is observable no orthonormal basis of W is distinguished by its matrix elements.

Complete response frames. The adjoint-preserving orthonormal frames of the generated algebra $\mathcal{A}_W = M_3(\mathbb{C})$ form $U(W)$. Its scalar centre acts by the common phase character and the determinant-one relative frames form $SU(W) \cong SU(3)$. This is the full connected frame action preserving the Hermitian structure and all matrix-response probabilities on W .

Proof. A unitary change of orthonormal basis conjugates every matrix response and preserves its adjoint and trace together with transition probabilities. Conversely an orthonormal frame change of the defining carrier is unitary. So the complete frame group is $U(W)$. Its determinant gives the central phase character. The kernel is $SU(W)$. Since $\dim_{\mathbb{C}} W = 3$ this kernel is $SU(3)$. The full matrix algebra distinguishes no smaller preferred family of orthonormal frames. $\square$

The exchange projections are observable elements of the source algebra. A frame change preserving their meaning commutes with these projections and hence preserves $W \oplus L$. A transformation mixing the two summands would identify responses with different exchange parity. On each summand the Hermitian form fixes the complete unitary frame group. The common phase records its scalar character and leaves the determinant-one relative action. Thus both the size of the colour frame group and the separation of its phase follow from retaining the resolved source comparisons.

Exchange parity is already a resolved source distinction. So frame comparison preserves $W \oplus L$. On overlaps two local descriptions of the same complete response differ by the complete frames constructed above. A connection records their relative change. The least faithful source representation carries every simple response block once while the common reversal-compatible odd source variation supplies the elementary scalar connector. These three continuation theorems state the universal properties under which their minimality or naturality is proved.

Their physical roles are distinct. Relative frames express the fact that only comparisons of local response bases are observable. Their connection is the rule for transporting such comparisons. The least matter carrier is the smallest Hilbert space on which every generated simple response block acts faithfully. So no generated channel is discarded and no duplicate copy is introduced. The elementary connector is the least odd source variation that changes chirality while preserving reversal reality and independent-source naturality. The three universal properties turn these operational requirements into unique continuations of the already generated carriers.

Source-compatible class. The continuations are compared among realizations that preserve the source adjoint and resolved symmetric–antisymmetric splitting. They also preserve the common phase character and pseudoreal reversal with the calibrated trace metric. Local descriptions must intertwine the same generated source action. Matter representations must be faithful nondegenerate and adjoint-preserving. Elementary odd maps must be natural for the independent pre-exchange source responses. These requirements express what it means to continue the same resolved response. The uniqueness statements com-

pare realizations with these properties. The remaining freedom consists of basis phases and changes of local trivialization.

Definition 4.7 (Generated response structure and continuation theorems). The complete-operation construction establishes the spectrum and common central action together with the calibrated trace metric. These responses continue to relative frames and least faithful matter with an elementary connector. Classification compares alternatives preserving the source adjoint and resolved exchange sectors with the common phase and reversal actions. Their logical content is therefore a construction followed by a uniqueness statement in its stated universal property

- (i) Isotropic spectral-capacity saturation. The constructed response satisfies Definition 4.2 by Theorem 4.1.
- (ii) Single-reversal covariance. The same physical reversal acts faithfully and functorially on the Gaussian and phase-sensitive geometric lifts under the single-reversal covariance stated above realized by the local cycle (G.13) on the common carrier.
- (iii) Scale-covariant metric completion. The endpoint radius is a unit-speed coordinate and Gaussian rescaling acts by metric homotheties as in the scale-covariant cone construction above using the trace metric (4.2) and the global correlation length above.
- (iv) Generated relative frames. The split carrier $C = W \oplus L$ and the Clifford modules S_L, S_R with the phase line have been constructed above. The complete-frame calculation determines the complete unitary frame group of each matrix-response block. On any smooth localization M_{loc} carrying local expressions of these carriers the source action defines relative transitions $g_{ij} = f_i^{-1} f_j$. They satisfy $g_{ij} g_{jk} = g_{ik}$ and are the unique transitions intertwining the two local source actions. Splitting-preserving frames of C pseudoreal Hermitian frames of S_L, S_R and unit frames of the phase line therefore act as relative response frames. The determinant and line characters are identified with the phase response by their joint action on matter. This local expression of the generated source action is the relative-frame construction. It is natural with respect to the localization base. The internal frame and matter results use only relative transitions. The source-generated affine carrier of Section 10 supplies $M_{\text{loc}} = \mathbb{M}_V \cong \mathbb{R}^4$. This base is contractible. So the transitions admit a global trivialization while their connections may retain nonzero local curvature and holonomy.
- (v) Least matter realization. Every generated finite-dimensional response $*$ -algebra acts on its source carrier by a faithful nondegenerate $*$ -representation of least finite dimension. The algebra carries the adjoint inherited from its source inner product and the representation satisfies $\rho(a^*) = \rho(a)^\dagger$. For these unital algebras nondegeneracy is equivalent to $\rho(1) = I$. Every competing least faithful adjoint-preserving realization is unitarily equivalent to this source carrier. Once a realizing unitary is fixed the

transport of every source operator is unique. The unitary itself may differ by block phases. Reversal pairs the two chiral realizations. This realization and its universal property are the least-matter construction. A nonzero right colour-scalar vector v_0 represents the transported Gaussian annihilation line $\mathbb{C}g_b^G$ of Proposition 3.6. The transported source number flow has a unique continuous lift from the identity to the right frames. Its image is the pointwise stabilizer $gv_0 = v_0$ inside $SU(2)_R \times U(1)_{B_0}$ with its primitive character on the faithful joint module. This lifted flow gives the residual connected frame group and carries the source vacuum's phase reference into the matter response.

- (vi) Generated elementary chiral variation. The source matrix units realize the nonzero odd Clifford variations of Corollary G.4 on the common carrier. Their colour-natural lift is elementary precisely when it is natural under independent responses on the two quadratic source factors and its spin component intertwines the pseudoreal reversal operators on S_R and S_L . In this class every connector has a unique complex-doublet coordinate and the source matrix units give the canonical family. This generated family and its naturality class are the connector construction.

Appendix B.4 treats the realization in which these bundles carry a local reflection-positive action on the generated event carrier of Section 10.

Common-carrier realization. On the common carrier the map $F_\omega(v) = (h_{b,0} \wedge h_{b,1}) \otimes v$ transports the Gaussian vacuum to $F_\omega h_{b,0}$. It intertwines the $SU(V)$ actions. The target carries the determinant-twisted $U(V)$ representation. Identification with $L \otimes S_R$ requires the shared diagonal spin action. On the source corner $N_G = a_b^* a_b$ acts as $\text{diag}(0, 1)$. So the flow e^{-itN_G} fixes $h_{b,0}$ and measures the first excitation's phase relative to it. Transport gives $\tilde{U}_t = F_\omega e^{-itN_G}|_V F_\omega^*$. The natural tensor action on $L \otimes S_R$ multiplies this flow by its determinant. Continuing the original source operation therefore requires the inverse determinant character. This scalar correction is unique and restores both source eigenvalues including the fixed vacuum amplitude. Equation (G.24) gives the intertwining identity. The right scalar action $z^{-3}g$ with $g \in SU(2)_R$ must reproduce this operator on the common carrier. Since $\det \tilde{U}_t = e^{-it}$ and $\det(z^{-3}g) = z^{-6}$ its lift satisfies $z^6 = e^{it}$. Continuity from the identity fixes $z = e^{it/6}$ and then fixes g uniquely. Corollary G.1 identifies this unique lift with the full pointwise stabilizer of the transported vacuum vector. The residual phase action thus continues the original number flow including its zero vacuum eigenvalue. The compensating right rotation and relative colour-scalar phase preserve this same reference amplitude. The determinant twist lifts one source cycle through six sheets of the residual phase action. On the full matter module the lifted circle has period 12π while the source circle has period 2π .

Theorem 4.8 (Relative-frame continuation). Under frame localization unitary frames of the W bundle define a $U(W)$ principal bundle. Local determinant-one changes give relative colour frames $SU(W) \cong SU(3)$. The determinant character joins the common Abelian phase response leaving $SU(W)$ as the relative colour-frame group. Use the passive frame

convention $e' = eg$. So matter components transform as $\psi' = \rho(g^{-1})\psi$. The connection then transforms as

$$A_\mu \mapsto g^{-1}A_\mu g + g^{-1}(\partial_\mu g). \quad (4.33)$$

Frames of S_L preserving its Hermitian and pseudoreal structures form a principal $USp(2) \cong SU(2)_W$ bundle while unit frames of the generated phase line form a principal $U(1)$ bundle. Their local connections obey

$$A^{(i)} \mapsto g_i^{-1}A^{(i)}g_i + g_i^{-1}dg_i, \quad i = W, 1. \quad (4.34)$$

These furnish the non-Abelian local frame factors and the phase line. The pointwise stabilizer of the transported source vacuum selects the primitive $U(1)$ character on the joint matter module giving the local factors $SU(3) \times SU(2)_W \times U(1)$.

Proof. Orthonormal frames of a rank-three Hermitian bundle form a principal $U(3)$ bundle whose determinant kernel is $SU(3)$. The joint source action assigns its determinant character to the common phase response and retains the relative $W \oplus L$ character for the joint module. The invariant Hermitian and alternating forms on S_L give the frame group $USp(2) \cong SU(2)$. A Hermitian line has frame group $U(1)$. Covariance of $d + \rho_*(A)$ under the stated passive component transformation gives both connection laws. $\square$

A general $U(3)$ bundle admits a global $SU(3)$ reduction when its determinant line has a compatible trivialization. In general the colour factors are local lifts with Abelian transition data. Globally the construction uses a compatible G_{int} -bundle and its associated representations.

The connection compares response bases on neighbouring patches of M_{loc} . Absolute frame orientation drops out of the relative source comparison while relative transport determines how a matter response in one frame is compared with a response in another. Its curvature measures the failure to return a frame unchanged around a closed loop. The same comparison rule thus supplies both gauge response and matter transport.

For the three-dimensional shell W the determinant-one frame algebra has the real decomposition $\mathfrak{su}(3) = \mathfrak{so}(3) \oplus i\text{Sym}_0(3, \mathbb{R})$. Real rotations account for three generators and imaginary symmetric traceless transformations for five. Thus the generated traceless anti-Hermitian response has real dimension $\dim \mathfrak{su}(3) = 8$.

4.6 Joint response module and charge

Put $C = W \oplus L$ and retain the generated $3 + 1$ splitting. Its split-preserving response algebra is

$$\mathcal{A}_C := \text{End}_{\mathbb{C}}(W) \oplus \text{End}_{\mathbb{C}}(L) \cong M_3(\mathbb{C}) \oplus \mathbb{C}. \quad (4.35)$$

It contains the full quadratic response on W and the scalar response on L but no operator that erases the generated distinction between the two summands.

For $\chi = L, R$ define

$$\begin{aligned} \mathcal{A}_{\text{joint}, \chi} &:= \mathcal{A}_C \hat{\otimes} \text{End}_{\mathbb{C}}(S_\chi) \\ &\cong \text{End}_{\mathbb{C}}(W \otimes S_\chi) \oplus \text{End}_{\mathbb{C}}(L \otimes S_\chi). \end{aligned} \quad (4.36)$$

The symbol $\widehat{\otimes}$ denotes closure under finite compositions of the commuting split-colour and chiral responses. Since both algebras are finite-dimensional it is the ordinary algebraic tensor product. Each $\text{End}(S_\chi)$ factor is itself the complex algebra generated by the irreducible defining $SU(2)$ action again by Burnside's theorem.

The two factors

$$\mathcal{A}_C \otimes I \quad \text{and} \quad I \otimes \text{End}(S_{L/R}) \quad (4.37)$$

commute. Their joint action is already defined on $(W \otimes S_\chi) \oplus (L \otimes S_\chi)$. Each full matrix block acts on its own source tensor factor. This action preserves the source adjoint and is faithful since an operator vanishing on the carrier vanishes on both blocks. Its dimension is $6 + 2 = 8$. The classification below shows that every faithful Hilbert realization has dimension at least eight and that equality determines its unitary-equivalence class. Thus the actual source carrier attains the least dimension. Physically it retains each resolved response block once. Independent repetitions are recorded by additional multiplicity spaces.

Theorem 4.9 (Least-matter continuation). For the split colour carrier $C = W \oplus L$ and the source-derived chiral modules S_L, S_R the source acts faithfully and preserves adjoints on the canonical full modules

$$M_L = (W \oplus L) \otimes S_L, \quad M_R^{\text{full}} = (W \oplus L) \otimes S_R. \quad (4.38)$$

For either chirality the least-dimensional faithful nondegenerate $*$ -representation of $\mathcal{A}_{\text{joint}, \chi}$ on a finite-dimensional complex Hilbert space is unique up to unitary equivalence with carrier

$$(W \otimes S_\chi) \oplus (L \otimes S_\chi) \cong (W \oplus L) \otimes S_\chi. \quad (4.39)$$

In particular

$$M_L = (3, 2) \oplus (1, 2), \quad \dim_{\mathbb{C}} M_L = 8. \quad (4.40)$$

Pointwise fixation of the transported source-vacuum vector gives the primitive character $q = 6T_R + B_0$. The full left-right carrier has complex dimension sixteen. Its unique neutral right line gives a fifteen-dimensional charged quotient and an operator descends precisely when it preserves that line. The image of the colour and weak actions with the residual phase action is $(SU(3) \times SU(2)_W \times U(1))/\mathbb{Z}_6$.

Proof. Equation (4.36) identifies the joint algebra with $M_6(\mathbb{C}) \oplus M_2(\mathbb{C})$ with the adjoints induced by the source inner products. In a nondegenerate $*$ -representation the two central units act as orthogonal projections whose ranges decompose the Hilbert space. For the $M_n(\mathbb{C})$ block put $K_n = \rho(e_{11})H$. The matrix-unit relations give

$$\langle \rho(e_{i1})v, \rho(e_{j1})w \rangle = \delta_{ij} \langle v, w \rangle, \quad v, w \in K_n,$$

because $\rho(e_{i1})^\dagger = \rho(e_{1i})$. Since the diagonal matrix units sum to the central unit the map $e_i \otimes v \mapsto \rho(e_{i1})v$ is a unitary intertwiner from $\mathbb{C}^n \otimes K_n$ onto this block. Hence

$$H \simeq_{\text{u}} (\mathbb{C}^6 \otimes K_6) \oplus (\mathbb{C}^2 \otimes K_2), \quad \dim H = 6 \dim K_6 + 2 \dim K_2.$$

Faithfulness requires both multiplicities to be positive. The least dimension is 8 attained precisely when $\dim K_6 = \dim K_2 = 1$. If either multiplicity vanishes the corresponding central response projection acts as zero and an entire generated block becomes undetectable. A larger multiplicity gives independent copies of its defining carrier. Faithfulness retains all resolved channels while least dimension removes precisely this additional copy freedom. This gives (4.39) up to unitary equivalence. Reversal pairs the left and right realizations and retains both simple blocks. $\square$

Rigidity of the realizing unitary. Let U_1 and U_2 be unitary intertwiners between two least realizations in Proposition 4.9. On the standard carrier their relative intertwiner has the form

$$U_2 U_1^{-1} = e^{i\alpha} I_{W \otimes S_\chi} \oplus e^{i\beta} I_{L \otimes S_\chi}. \quad (4.41)$$

Consequently the response representation is unique up to unitary equivalence. The two displayed phases are changes of block basis not adjustable physical parameters.

Proof. The relative intertwiner commutes with $M_6(\mathbb{C}) \oplus M_2(\mathbb{C})$. The commutant of each defining full-matrix block consists of its scalar operators. Unitarity makes both scalars have modulus one which gives (4.41). $\square$

Let

$$B_0 = \text{diag}(1, 1, 1, -3) \quad (4.42)$$

on $C = W \oplus L$. This is the primitive traceless relative character of the $3 + 1$ splitting unique up to sign.

Before reducing by the transported source-vacuum stabilizer the right frames and relative phase form $SU(2)_R \times U(1)_{B_0}$. The residual circle follows by computing the pointwise stabilizer without first selecting a Cartan direction.

Pointwise vacuum stabilizer and primitive charge. The transported source vacuum v_0 provides a common phase reference for the right frames and the colour scalar. Their relative phases are locked by the requirement that their joint action fix this vector. This vacuum stabilizer fixes a direction in the two phase generators. Primitivity then fixes its charge unit. Write $v_0 = \ell \otimes s_0 \neq 0$ with $\ell \in L$ and $s_0 \in S_R$. On this colour-scalar sector let (g, z) act by $\ell \otimes s \mapsto z^{-3} \ell \otimes gs$. Choose a unitary pseudoreal frame with s_0 proportional to e_+ . The stabilizer is the connected circle

$$H_{v_0} = \{(\text{diag}(z^3, z^{-3}), z) : z \in U(1)\}. \quad (4.43)$$

Its generator on the full joint module is $6T_R + B_0$. With T_R normalized to weights $\pm 1/2$ on S_R and zero on S_L the primitive character is up to the Weyl reflection and common orientation

$$q = 6T_R + B_0. \quad (4.44)$$

In the standard $SU(2)_R$ weight normalization used above put $Y = q/6$. Then

$$Y = T_R + \frac{1}{6} B_0. \quad (4.45)$$

The resulting weights are

$$\frac{Q_L}{Y} \left| \begin{array}{cc|cc} L_L & u_R & d_R & (\nu_R, e_R) \\ \hline \frac{1}{6} & -\frac{1}{2} & \frac{2}{3} & -\frac{1}{3} \end{array} \right. (0, -1). \quad (4.46)$$

Proof. Pointwise fixation gives $ge_+ = z^3 e_+$. Unitarity forces the second column of g to lie along e_- and determinant one fixes its eigenvalue to z^{-3} . Conversely every displayed pair fixes v_0 . Projection to the second factor identifies the full stabilizer with $U(1)$. Setting $z = e^{it}$ gives $g = e^{6itT_R}$.

For a generator $\alpha T_R + \beta B_0$ pointwise fixation of either right scalar weight gives $\pm\alpha/2 - 3\beta = 0$ since $B_0 = -3$ on L . The two signs are exchanged by the Weyl group of $SU(2)_R$. Hence the integer weight vector is proportional to $6T_R + B_0$. Primitivity fixes the proportionality to one up to common sign. The other scalar weight is nonzero. So the fixed scalar line is unique. The character values on the joint module are $1, -3, 4, -2, 0, -6$ whose nonzero entries have greatest common divisor one. This proves primitivity. The normalization $Y = q/6$ retains the standard $SU(2)_R$ generator with weights $\pm 1/2$. Substitution on W and L with S_L and the two weights of S_R gives (4.46). Reversing both primitive generators changes all charges by a common sign and gives an equivalent orientation. $\square$

The source number flow fixes $h_{b,0}$. So its transported flow fixes $v_0 = F_\omega h_{b,0}$ pointwise. The common-carrier lift constructed above has image exactly H_{v_0} . Its determinant character locks the right rotation to the relative colour-scalar phase. Multiplying F_ω by a constant phase leaves this conjugated flow unchanged. The resulting charge circle therefore retains the phase reference already carried by the source vacuum. A ray-preserving action can multiply v_0 by an additional phase whereas the transported number flow has vacuum eigenvalue zero. Reproducing that full source action fixes the vector and relates the two phase factors in $U(1)_R \times U(1)_{B_0}$. Primitivity removes the redundant multiple covering of the residual circle.

The compact response space has a discrete elliptic spectrum and the residual compact phase group has a discrete character lattice. The first organizes internal modes. The second records their transformation under phase rotations. Pointwise source-vacuum fixation and global centre compatibility fix the relative normalization on the primitive lattice in (4.44). The notation $Y = q/6$ additionally uses the inherited normalization of T_R inside $SU(2)_R$. Simultaneous inverse rescalings of an Abelian generator and its gauge coupling leave their product unchanged.

Let C_2^{col} be the positive colour Casimir extended by zero on L and define on M_R^{full}

$$Q_R = C_2^{\text{col}} + Y^2. \quad (4.47)$$

Charged response quotient. The operator Q_R is positive and

$$\ker Q_R = (1, 1)_0. \quad (4.48)$$

The quotient

$$M_R = M_R^{\text{full}} / \ker Q_R \quad (4.49)$$

Table 1. Localized response modules. The neutral right line is shown before forming the charged quotient. The scalar row is the generated elementary connector of the least matter carrier.

Label	Response origin	$SU(3) \times SU(2)_W \times U(1)_Y$
Q_L	$W \otimes S_L$	$(3, 2)_{1/6}$
L_L	$L \otimes S_L$	$(1, 2)_{-1/2}$
u_R	$W \otimes S_R^+$	$(3, 1)_{2/3}$
d_R	$W \otimes S_R^-$	$(3, 1)_{-1/3}$
ν_R	$L \otimes S_R^+$	$(1, 1)_0$
e_R	$L \otimes S_R^-$	$(1, 1)_{-1}$
H	$\mathcal{K}_{\text{ch}}$	$(1, 2)_{1/2}$

has complex dimension seven. Hence the generated charged response sector has $8 + 7 = 15$ complex directions.

Proof. Both terms in (4.47) are positive. The colour Casimir vanishes only on the colour scalar L . On its two right weights (4.46) gives charges 0 and -1 . Their common kernel is therefore the unique neutral scalar line. The dimension count follows from $\dim M_R^{\text{full}} = 8$. $\square$

The quotient resolves the sector detected by the internal charges. The neutral right line remains in M_R^{full} where the full response algebra and the Higgs connector can act on it. This distinction determines which operators descend to the charged sector.

Descent to the charged quotient. Let

$$\mathcal{M}_{\text{full}} = M_L \oplus M_R^{\text{full}}, \quad N = \{0\} \oplus \ker Q_R, \quad \mathcal{M}_{\text{ch}} = \mathcal{M}_{\text{full}}/N, \quad \pi_{\text{ch}} : \mathcal{M}_{\text{full}} \rightarrow \mathcal{M}_{\text{ch}}. \quad (4.50)$$

An endomorphism A of $\mathcal{M}_{\text{full}}$ induces a quotient endomorphism $\bar{A}$ with $\pi_{\text{ch}} A = \bar{A} \pi_{\text{ch}}$ if and only if $AN \subset N$. The reduced gauge representation descends. The full joint algebra does not. For self-adjoint A descent is equivalent to $[A, P_N] = 0$ where P_N is the orthogonal projection onto N .

Proof. The prescription $[v] \mapsto [Av]$ is independent of the representative exactly when $A(v + n) - Av = An$ lies in N . The neutral line is invariant under the reduced gauge group. In contrast the right scalar matrix unit $|e_- \rangle \langle e_+| \in \text{End}(L \otimes S_R)$ maps its neutral vector to the charged scalar line. Hence invariance fails for the full $M_6(\mathbb{C}) \oplus M_2(\mathbb{C})$ algebra. If $A = A^\dagger$ and N is invariant $N^\perp$ is invariant as well which is equivalent to commutation with P_N . $\square$

Accordingly fifteen is a charged representation count while sixteen is the full-space count. A dynamical trace must specify its operator and space. Compression $P_{N^\perp} A P_{N^\perp}$ always defines a new operator on $N^\perp$ but agrees with an induced quotient operator only under the descent condition.

Table 1 gives the internal transformation laws in conventional particle notation. Here $S_R^\pm$ are the T_R eigenspaces of weights $\pm 1/2$.

Let ρ_{src} be the joint source action of the colour and weak factors with the lifted phase factor on $\mathcal{M}_{\text{full}}$ and define the intrinsic operator group

$$\widehat{G}_{\text{src}} := \text{im}(\rho_{\text{src}}) \subset U(\mathcal{M}_{\text{full}}). \quad (4.51)$$

This definition refers only to transformations already acting on the source carrier. The following product notation is its faithful presentation.

Intrinsic generated and faithfully acting group. The group $\widehat{G}_{\text{src}}$ is generated by the images of the colour frames the two source-spin axes generating the weak factor and the unique vacuum phase lift. It is the least subgroup of $U(\mathcal{M}_{\text{full}})$ containing these three source families. With integer charge $q = 6Y$ their product presentation has kernel generated by

$$(\omega_3, -I, e^{i\pi/3}), \quad \omega_3 = e^{2\pi i/3} I_3. \quad (4.52)$$

Consequently the intrinsic group has the faithful presentation

$$G_{\text{int}} = \frac{SU(3) \times SU(2)_W \times U(1)}{\mathbb{Z}_6}. \quad (4.53)$$

Proof. Every element of the product presentation factors into its colour and weak coordinates with its phase coordinate. Its image under ρ_{src} is therefore a product of elements in the three source families. Conversely each family belongs to the image. Hence their subgroup closure is exactly $\widehat{G}_{\text{src}}$ and any subgroup containing all three families contains that image.

Any kernel element $(g_3, g_L, e^{i\theta})$ acts trivially on e_R and L_L as well as d_R . These conditions give $e^{6i\theta} = 1$ and $g_L = e^{3i\theta} I_2$ together with $g_3 = e^{2i\theta} I_3$. Hence $\theta = \pi n/3$ modulo 2π and every kernel element is the n th power of (4.52). Conversely this generator acts trivially on Q_L, L_L, u_R, d_R, e_R with their integer charges $q = 6Y$ and on the neutral line before quotienting. Its order is six proving the asserted kernel and the quotient presentation. $\square$

The generated module also passes the gauge-anomaly checks. We write the anomaly sums using left-handed fields or equivalently subtract the right-handed contributions following the chiral cancellation criterion of [18]. With the charges in (4.46) the mixed and Abelian coefficients are

$$\begin{aligned} [SU(3)]^2 U(1) : \quad & 2Y_Q - Y_u - Y_d = 0, \\ [SU(2)]^2 U(1) : \quad & 3Y_Q + Y_L = 0, \\ \text{grav}^2 U(1) : \quad & 6Y_Q + 2Y_L - 3Y_u - 3Y_d - Y_e = 0, \\ [U(1)]^3 : \quad & 6Y_Q^3 + 2Y_L^3 - 3Y_u^3 - 3Y_d^3 - Y_e^3 = 0. \end{aligned} \quad (4.54)$$

The cubic colour coefficient is $2\mathcal{A}(3) - \mathcal{A}(3) - \mathcal{A}(3) = 0$. There are three coloured weak doublets and one colourless weak doublet. So the number of left-handed $SU(2)$ doublets is four and the mod-two global anomaly vanishes by Witten's criterion [19].

4.7 Scalar response and Higgs multiplicity

Let e_L, e_R be the chiral projections in the total response algebra. A nonzero connector in $e_L \mathcal{A} e_R$ acts on the common carrier. Proposition G.4 constructs the internal Clifford blocks $h(r, x) = rI + ix$ on the common carrier. The scale normal gives $h(1, 0) = I$ and the four real coordinates exhaust $\text{Hom}_J(S_R, S_L)$. Corollary G.4 realizes these blocks explicitly in the generated source matrix units including their colour-natural lift $I_C \otimes h$ on the full carrier and the compatible left-right controls. The connector construction records this generated reversal-real variation and the naturality class selected by independent source responses before exchange resolution. The connector space is the reversal-real natural-intertwiner space defined by these properties.

Before exchange resolution independent responses on the two factors of $C = S \otimes S$ generate

$$\mathcal{A}_{\text{pre}} := \text{Alg}_{\mathbb{C}}\{\rho(g) \otimes I, I \otimes \rho(h) : g, h \in SU(2)_S\} = \text{End}_{\mathbb{C}}(C). \quad (4.55)$$

Indeed Burnside's theorem gives $\text{End}(S)$ on each factor and their tensor product is $\text{End}(C)$. This complete comparison algebra tests how a chiral variation responds to independent changes of either source factor. Exchange resolution subsequently retains $W \oplus L$. The generated elementary chiral variation commutes with these independent source responses before the pair is resolved into its exchange sectors. This naturality condition makes one variation act on both the coloured and colourless sectors. Reversal reality makes the variation compatible with reversing either chiral response. Reversing on S_R before the variation has the same effect as reversing its image on S_L . Together these conditions determine the common scalar doublet coordinate.

Chiral scalar response module. Let J_L, J_R be the invariant antiunitary pseudoreal structures on S_L, S_R normalized by $J_L^2 = J_R^2 = -I$. First take natural complex-linear intertwiners

$$\text{Hom}_{\mathcal{A}_{\text{pre}}}(C \otimes S_R, C \otimes S_L) = I_C \otimes \text{Hom}_{\mathbb{C}}(S_R, S_L). \quad (4.56)$$

For each such T write its unique reduced map as $h : S_R \rightarrow S_L$. Impose reversal reality on this map. The elementary connector space is

$$\mathcal{C}_{\text{elem}} := \{I_C \otimes h : h \in \text{Hom}_{\mathbb{C}}(S_R, S_L), hJ_R = J_L h\}. \quad (4.57)$$

Both sides of the reality equation are antilinear maps from S_R to S_L . The condition is real-linear in h . So $\mathcal{C}_{\text{elem}}$ is a four-dimensional real response space. The complex scalar coordinate extracted from it is denoted $\mathcal{K}_{\text{ch}}$.

Theorem 4.10 (Elementary-connector continuation). On the least matter modules the generated elementary connector space has one complex doublet coordinate and

$$\mathcal{K}_{\text{ch}} \cong (1, 2)_{1/2}, \quad \text{mult}_{\mathcal{K}_{\text{ch}}}(1, 2)_{1/2} = 1. \quad (4.58)$$

Its pseudoreal conjugate $\tilde{H} = i\sigma_2 H^*$ is the conjugate coordinate in the same generated field module. Its intrinsic invariant is $\det h_H = \|H\|^2$ the positive amplitude square of this connector.

Proof. Since $\mathcal{A}_{\text{pre}} = \text{End}(C)$ its commutant on $C \otimes S_\chi$ is $I_C \otimes \text{End}(S_\chi)$. Hence every natural connector is $I_C \otimes h$. The same h acts on W and L . So their distinct Yukawa coefficients do not create field copies.

Choose pseudoreal bases with $J_\chi = i\sigma_2 K$ where K is complex conjugation. Solving $hJ_R = J_L h$ gives

$$h = h_H = (\tilde{H}, H) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad H = \begin{pmatrix} b \\ \bar{a} \end{pmatrix}, \quad \tilde{H} = i\sigma_2 H^*. \quad (4.59)$$

Thus $H \mapsto h_H$ is a real-linear bijection from one complex doublet $\mathbb{C}^2$ onto $\mathcal{C}_{\text{elem}}$. The first column is fixed by the second. So both are coordinates of the same doublet. Under $T \mapsto \pi_L(g)T\pi_R(g)^{-1}$ the coordinate H is a defining $SU(2)_W$ doublet. The relative colour character cancels between source and target while $q = 6T_R + B_0$ gives the two columns charges -3 and $+3$. Choosing the displayed orientation therefore gives $Y_H = 3/6 = 1/2$. Its complex dimension is two. So this representation occurs once in $\mathcal{K}_{\text{ch}}$. $\square$

Intrinsic scalar intensity. For the generated connector h_H

$$\det h_H = \|H\|^2 = |H_0|^2 + |H_1|^2. \quad (4.60)$$

This quadratic intensity is nonnegative and vanishes only for $H = 0$. It is invariant under the complete generated gauge action. It is therefore the intrinsic amplitude square of the elementary scalar response.

Proof. The determinant of the matrix in (4.59) is $|a|^2 + |b|^2 = \|H\|^2$. The colour action cancels between the two common colour factors while the left and right spin-frame matrices have determinant one. Hence $h_H \mapsto g_L h_H g_R^{-1}$ preserves the determinant. Its positive-definite expression proves the remaining statements. $\square$

The canonical elementary continuation has one doublet coordinate in $\mathcal{C}_{\text{elem}}$. Several independent scalar fields require a further copy space carrying their independent labels. The faithful matter module is unchanged by that extension.

The connector couples the neutral right line to left matter. For $H \neq 0$

$$h_H^\dagger h_H = \|H\|^2 I_{S_R}, \quad T_H(\ell \otimes e_+) = \ell \otimes \tilde{H} \neq 0. \quad (4.61)$$

Thus $T_H : M_R^{\text{full}} \rightarrow M_L$ does not factor through $M_R^{\text{full}} / \ker Q_R$. The self-adjoint odd operator $\mathcal{D}_H = \begin{pmatrix} 0 & T_H \\ T_H^\dagger & 0 \end{pmatrix}$ on $\mathcal{M}_{\text{full}}$ fails the descent test in Theorem 4.9. Removing neutral couplings by projection defines a modified dynamical map not the induced quotient of $\mathcal{D}_H$.

The least module gives one elementary family. Repeated families have the same simple response blocks with nontrivial multiplicity spaces K_n . Flavour indices and mixing operators act on these spaces. The quark and lepton colour intertwiners I_W and I_L may carry different numerical Yukawa coefficients multiplying the same coordinate in $\mathcal{K}_{\text{ch}}$. These coefficients are additional dynamical data. The weighted Yukawa operator need not obey the pre-exchange naturality or reversal-reality condition of the elementary T_H . Those conditions classify the field coordinate not every action-level coupling.

For the generated charges covariance of the three standard chiral contractions is visible directly

$$-Y_Q - Y_H + Y_u = 0, \quad -Y_Q + Y_H + Y_d = 0, \quad -Y_L + Y_H + Y_e = 0, \quad (4.62)$$

where $Y_H = 1/2$ and the first relation uses $\tilde{H} = i\sigma_2 H^*$. Their numerical Yukawa matrices remain dynamical parameters.

4.8 Content identities

Complete source exchange and capacity preservation. In common source coordinates the full matter carrier is $K \oplus K$ with $\dim_{\mathbb{C}} K = 8$. Its charge decomposition is $8_L + (7_R + 1_{\text{neutral}})$. Let $E(x_L, x_R) = (x_R, x_L)$ and $\Gamma(x_L, x_R) = (x_L, -x_R)$. Exchange permutes the complete basis preserving its total weighted capacity. For the same response $A \in \text{End}(K)$ on both chiral copies

$$\text{Tr}[\Gamma(A \oplus A)] = \text{Tr } A - \text{Tr } A = 0. \quad (4.63)$$

Separately the charged quotient removes the kernel of the charge response namely the neutral right line giving $8_L + 7_R$ directions. This count follows from the kernel calculation above. The line remains in the full carrier where its scalar connections are retained. Its removal from a charge trace and the descent of an operator are governed by the preceding quotient construction.

The generated charged module satisfies

$$N_L = 6 + 2 = 8, \quad N_R = 3 + 3 + 1 = 7, \quad N_L - N_R = 1. \quad (4.64)$$

Thus $N_L = \dim \mathfrak{su}(3) = 8$ for the colour algebra and matter module constructed above. Both counts refer to the generated carriers. Colour transformations act within the three-dimensional W while left matter combines W and L with a two-dimensional chiral response.

The same content data fix the charged-quotient normalization. The charged module has $N_f = 8 + 7 = 15$ complex directions and $\text{Tr}(Y^2) = 10/3$ in the fractional hypercharge convention of (4.46). The primitive integer generator $q = 6Y$ instead has $\text{Tr}(q^2) = 120$. Using Y throughout gives

$$\tau = \frac{N_L - N_R}{N_f \text{Tr}(Y^2)} = \frac{1}{15 \cdot 10/3} = \frac{1}{50}. \quad (4.65)$$

This is the normalization of the gauge-active charged quotient. It is not an additional parameter and does not enter the scalar gap formula.

For a scalar conformally coupled to a three-dimensional round factor the conformal Laplacian uses the coefficient [20, p. 22, equation (1.10)]

$$\xi_3 = \left. \frac{d-2}{4(d-1)} \right|_{d=3} = \frac{1}{8}. \quad (4.66)$$

Since the already generated colour algebra has $N_g := \dim \mathfrak{su}(3) = 8$ one obtains

$$N_g \xi_3 = 8 \times \frac{1}{8} = 1. \quad (4.67)$$

The identity relates the dimension of the colour algebra to the standard conformal coefficient on the three-dimensional response factor.

The statistical equi-scale-surface interpretation additionally uses a Gaussian field and regularized level sets on the generated Euclidean event carrier (Appendix D).

4.9 Internal structure and collective response

The source operations supply the quadratic carriers $W \oplus L$ and the two primitive chiral sectors with their matrix responses. Relative frame localization turns comparisons of these carriers into gauge and matter transport. The least faithful matter realization retains every simple response block and pointwise fixation of the transported source vacuum determines the common residual phase action. Its weights determine the charges. The range of the full matter action is the intrinsic generated gauge group. The kernel of its product presentation determines the global centre quotient. The fifteen-dimensional charged quotient retains precisely the directions detected by these charges while the full sixteen-dimensional module also carries the neutral response and its scalar connections.

The shared reversal-real chiral variation supplies the elementary Higgs-type channel and its determinant supplies the invariant amplitude square used by the scalar response. The long-root coordinate instead describes collective right-sector response. A stable nonzero spin-one background has a maximal-torus stabilizer. This dynamical stabilizer depends on the condensation potential and is distinct from the source-vacuum stabilizer that fixes charge.

Resulting internal response structure. The complete-operation construction yields $X_3 = \mathbb{RP}^3$, its least centre-sensitive spin module and the four-real-dimensional carrier E_4 . Within the source-compatible classes, the three continuation theorems determine the intrinsic group (4.51) with faithful presentation (4.53), the sixteen-dimensional full matter space and its fifteen-dimensional charged quotient. The elementary connector contains one $(1, 2)_{1/2}$ module whose determinant is its positive invariant amplitude square.

5 Generated Modes, Scale Composition, and States

The generated geometry and matter modules specify the internal response channels. Mode products couple these channels, heat evolution changes their resolution, and positive states assign probabilities to observations. Matrix-valued modes combine the geometric product with operations on the matter carrier. The full carrier retains the neutral line, whose contribution to the connector resolvent and spectral traces survives even when the source and observable descend to charged matter.

5.1 The internal mode algebra and Gaussian composition

On the round response factor $X_3 = \mathbb{RP}_L^3$ obtained above use the normalized Haar measure $d\bar{v} = dv/(\pi^2 L^3)$. Let D_{mn}^j be the matrix coefficients of the unitary spin- j representation

of the parent $SU(2)$ and write

$$e_{mn}^j = \sqrt{r_j} D_{mn}^j, \quad r_j = 2j + 1, \quad \Delta_0 e_{mn}^j = \lambda_j e_{mn}^j, \quad \lambda_j = \frac{4j(j+1)}{L^2}, \quad j = 0, 1, \dots \quad (5.1)$$

Integer j is precisely the central-even condition. The representation dimension r_j is not the shell multiplicity r_j^2 .

Proposition 5.1 (Generated mode products and scale maps). The finite span $\mathcal{R}(X_3)$ of these modes is a unital conjugation closed algebra dense in $C(X_3)$ and in every finite-order Sobolev space. For $\alpha = (j, m, n)$ and $\beta = (k, a, b)$ with $\gamma = (J, M, N)$ its multiplication coefficients are

$$e_\alpha e_\beta = \sum_\gamma C_{\alpha\beta}^\gamma e_\gamma, \quad C_{\alpha\beta}^\gamma = \sqrt{\frac{r_j r_k}{r_J}} C_{jm,ka}^{JM} \overline{C_{jn,kb}^{JN}}, \quad |j - k| \leq J \leq j + k. \quad (5.2)$$

The two coefficients on the right are Clebsch–Gordan coefficients in a common unitary basis convention. The representation fixes the allowed product channels and their relative coefficients. No separate multiplication coefficient is assigned to each modal channel. The scale maps

$$G_t = e^{-t\Delta_0}, \quad G_t e_\alpha = e^{-t\lambda_j} e_\alpha, \quad G_{t+u} = G_t G_u, \quad t, u \geq 0, \quad (5.3)$$

preserve constants and positivity together with central parity. No cutoff retaining all shells $j \leq J$ with $J \geq 1$ is closed under multiplication.

Proof. The Clebsch–Gordan decomposition and Peter–Weyl theorem with normalized Haar measure [15] give (5.2) and the stated density. The same round operation diffusion of Theorem G.2 in the heat parameter normalized by (5.3) gives the positive scale map. In particular $(e_{JJ}^J)^2 = r_J e_{2J,2J}^{2J} / \sqrt{4J+1}$ outside the cutoff. Appendix A.6 supplies the normalization and completion with scale-product estimates. $\square$

Matrix-valued modes multiply by $(e_\alpha \otimes B)(e_\beta \otimes C) = \sum_\gamma C_{\alpha\beta}^\gamma e_\gamma \otimes BC$ on the stated full carrier. Their noncommutative coefficients and their geometric mode couplings are different factors of one product. The product descends to charged matter when its matrix factors preserve the neutral line.

The parameter t in (5.3) measures internal heat resolution. The difference $G_t(fg) - (G_t f)(G_t g)$ measures the failure of heat resolution to preserve products. The estimates in Appendix A.6 control this difference.

5.2 The source system and its positive states

Put $\mathcal{H}_G = L^2(\mathbb{R})$ with the Gaussian ladder basis h_j and number operator $N_G h_j = j h_j$ and the matrix units $E_{ij} = |h_i\rangle\langle h_j|$ constructed in (G.1). The subscript distinguishes this number operator from the neutral matter line N . The source algebra is the unitization

$$\mathcal{A}_{\text{src}} = \mathcal{K}(\mathcal{H}_G) + \mathbb{C}I. \quad (5.4)$$

Unitization includes the identity operation I together with the compact source responses.

Theorem 5.2 (Positive source completion). Let $t > 0$ be the dimensionless ladder reference parameter distinct from the internal heat parameter. The source reference and its finite resolutions are

$$\rho_t = (1 - e^{-t})e^{-tN_G}, \quad \rho_{t,n} = \frac{P_n e^{-tN_G} P_n}{\text{Tr}(P_n e^{-tN_G})}, \quad P_n = \sum_{j=0}^{n-1} E_{jj}. \quad (5.5)$$

They define normalized positive functionals on $\mathcal{A}_{\text{src}}$ and

$$\|\rho_{t,n} - \rho_t\|_1 = 2e^{-nt}, \quad \frac{P_n \rho_{t,m} P_n}{\text{Tr}(P_n \rho_{t,m})} = \rho_{t,n} \quad (m \geq n). \quad (5.6)$$

Keeping an unresolved scalar tail gives a unital state-preserving system $M_n(\mathbb{C}) \oplus \mathbb{C}$ whose norm completion is $\mathcal{A}_{\text{src}}$.

Appendix G constructs the unital inclusions and proves these identities. The trace-norm error is twice the unresolved geometric tail mass and bounds the error of every bounded observation by duality. The common-domain product completion in Ref. [21] provides a related construction. Here the explicit tail mass controls the source-state completion.

The vacuum functional is instead $\omega_0(X) = \langle h_0, X h_0 \rangle$. Indeed $\|\rho_t - E_{00}\|_1 = 2e^{-t}$ as $t \rightarrow \infty$. The pure vacuum and the faithful finite- t states are source density operators. Haar measure is the stationary law on operations.

The geometric weights of ρ_t give the entropy

$$S_{\text{src}}(t) = -\text{Tr}(\rho_t \log \rho_t) = -\log(1 - e^{-t}) + \frac{t}{e^t - 1}, \quad S'_{\text{src}}(t) = -\frac{te^t}{(e^t - 1)^2}. \quad (5.7)$$

The geometric source weights have mean ladder degree $(e^t - 1)^{-1}$. Substituting their logarithms gives the formula. Thus $S_{\text{src}} \rightarrow 0$ as $t \rightarrow \infty$ and $S_{\text{src}} = 1 - \log t + o(1)$ as $t \downarrow 0$. The divergence at $t = 0$ reflects the infinite ladder. The unit operator has infinite trace. So this endpoint admits no normalized uniform density.

On the generated scalar modes the entropy includes the shell multiplicities. For $\beta > 0$ put

$$Z_0(\beta) = \sum_{j \geq 0} (2j+1)^2 e^{-\beta \lambda_j}, \quad \rho_{0,\beta} = Z_0(\beta)^{-1} e^{-\beta \Delta_0}, \quad (5.8)$$

$$S_0(\beta) = \log Z_0(\beta) + \beta \text{Tr}(\Delta_0 \rho_{0,\beta}), \quad S'_0(\beta) = -\beta \text{Var}_{\rho_{0,\beta}}(\Delta_0).$$

The quadratic eigenvalue growth makes the heat trace and every required moment finite at positive β . So differentiation is justified. Writing $p_j = (2j+1)^2 e^{-\beta \lambda_j} / Z_0$ gives $S_0 = H(p) + \sum_j p_j \log(2j+1)^2$. The second term is the entropy within the unbiased shell. The scalar ground mode is simple and $S_0 \rightarrow 0$ at large β . These formulas use positive state multiplicities. Signed coefficients in an effective trace require a separate interpretation. Here β has units of length squared unlike the dimensionless ladder parameter t . Both label reference families at fixed operator. A scalar shift cancels in the normalized state while it remains present in the mass spectrum.

Generated reversible and averaged source updates. The Gaussian ladder generates the strongly continuous unitary group $U_\theta = e^{-i\theta N_G}$ and the point-norm continuous automorphisms

$$\alpha_\theta(X) = U_\theta^* X U_\theta, \quad \alpha_\theta(E_{ij}) = e^{i\theta(i-j)} E_{ij}. \quad (5.9)$$

Composing these operations with the already generated Gaussian change law defines a unital completely positive semigroup

$$\mathcal{C}_s(X) = \int_{\mathbb{R}} \alpha_\theta(X) d\mu_s(\theta), \quad \mathcal{C}_s(E_{ij}) = e^{-s(i-j)^2/2} E_{ij}. \quad (5.10)$$

Both updates preserve ω_0 and ρ_t and every diagonal source state is stationary under $\mathcal{C}_s$. Spectral calculus gives (5.9). Density of finite matrices extends its continuity to $\mathcal{A}_{\text{src}}$. Gaussian averaging multiplies each matrix unit by the characteristic function at $i - j$ giving (5.10). A probability average of unitary automorphisms is unital and completely positive. The convolution law gives the semigroup law and density and contractivity give strong continuity. Diagonal states are fixed by every phase rotation and hence by the average.

Here θ is an operation parameter and s its accumulated variance. The averaged update acts on $\mathcal{A}_{\text{src}}$. The Neumann window evolution acts on interval functions and the group diffusion acts on process functions. The last diffusion induces the internal heat maps on mode functions.

5.3 Gaussian fields on the generated spectrum

On the same round $X_3 = \mathbb{RP}_L^3$ use the positive conformal scalar operator $A = \Delta_0 + \mathcal{R}/8$ where $\mathcal{R} = 6/L^2$ with domain $H^2(X_3)$. Its shell eigenvalues are $a_j = [(2j)(2j+2) + 3/4]/L^2$ with real multiplicity $(2j+1)^2$. This operator supplies the reference stiffness of the scalar response. The conformal curvature term gives its constant harmonic a positive stiffness while the Laplace spectrum resolves the higher harmonics. Fix a real orthonormal eigenbasis $Ae_i = a_i e_i$ with i enumerating these shells and their multiplicities. At fixed L the field covariance is the integral of the heat operator over the resolution parameter. A mode with stiffness a_i contributes variance $\int_0^\tau e^{-ua_i} du = (1 - e^{-\tau a_i})/a_i$. Independent Gaussian source coordinates with these variances give for $\tau > 0$ a distribution-valued field with the fixed-parameter law

$$X_\tau = \sum_i \sqrt{\frac{1 - e^{-\tau a_i}}{a_i}} Z_i e_i, \quad C_\tau = A^{-1}(I - e^{-\tau A}), \quad Z_i \sim N(0, 1). \quad (5.11)$$

Theorem G.8 proves mean-square completion in H_A^{-q} for $q > 1/2$ retaining all real multiplicities and constructs the reference law $N(0, A^{-1})$ on that space. Neither field is L^2 -valued. The associated regulator and the scalar feedback law $N(0, (A + gE^2 I)^{-1})$ are proved on this same field space in Appendix G.9. Here τ sets the upper resolution limit in the accumulated covariance on the fixed internal spectrum. For $g > 0$ the scalar amplitude changes each mode's stiffness from a_i to $a_i + gE^2$ giving variance $(a_i + gE^2)^{-1}$ in the corresponding reference law. The common shift makes every harmonic respond to the same collective intensity

while preserving its mode projector. Increasing E^2 therefore suppresses these fluctuations. On this common field space the mean and variance of the reference-subtracted intensity determine the first and second derivatives of the relative spectral response. The restoring response thus reads the change in fluctuations caused by the scalar amplitude.

5.4 Operators and matter spaces

Spectral traces depend on the eigenvalues and operator domain as well as the carrier and its multiplicities. The spectral data therefore include the spin structure and the maps between the full and charged matter spaces.

Definition 5.3 (Spectral data). The geometry and operators with the matter spaces are collected as

$$\mathcal{D} = (X_3, g_{3,b}, R_b, \tau_-, \text{Spec}_m \Delta_0, \text{Spec}_m D\tau_-, \text{Spec}_m \Delta_1^\perp, \text{Spec}_m \Delta_{TT}, E_4, G_{\text{int}}, \mathcal{M}_{\text{full}}, N, \mathcal{M}_{\text{ch}}, Q_R, \pi_{\text{ch}}, P_N, \mathcal{K}_{\text{ch}}), \quad (5.12)$$

where

- (i) $(X_3, g_{3,b})$ is the endpoint-scaled compact response geometry and $R_b = L_{\text{IR}}(b)$ is its radius.
- (ii) τ_- is the spin structure whose lift of the antipodal deck transformation is $-\omega \in \text{Spin}(4)$.
- (iii) Spec_m records eigenvalues together with multiplicities. The operators are the nonnegative scalar Laplacian and self-adjoint Dirac operator together with the Hodge Laplacian restricted to coclosed one-forms and the positive rough Laplacian on TT tensors. Their domains are respectively the corresponding constrained H^2, H^1, H^2, H^2 Sobolev domains.
- (iv) E_4 is the four-dimensional response carrier.
- (v) G_{int} is the faithfully acting reduced internal group. The matter spaces are $\mathcal{M}_{\text{full}} = M_L \oplus M_R^{\text{full}}$ of dimension sixteen and $\mathcal{M}_{\text{ch}} = \mathcal{M}_{\text{full}}/N$ of dimension fifteen where $N = \{0\} \oplus \ker Q_R$. Here $Q_R = C_2^{\text{col}} + Y^2$ acts on M_R^{full} . The map π_{ch} is the quotient map and P_N is the orthogonal neutral projection.
- (vi) $\mathcal{K}_{\text{ch}}$ is the elementary chiral-response module of the least-matter and reversal-real connector constructions.

For the compact response construction of Section 4 the factor is $X_3 = \mathbb{RP}^3$ with $g_{3,b} = R_b^2 g_1$

and the four spectra are

$$\text{Spec}_m(\Delta_0) = \left\{ \left(\frac{l(l+2)}{R_b^2} \right)^{[(l+1)^2]} : l = 0, 2, 4, \dots \right\}, \quad (5.13)$$

$$\begin{aligned} \text{Spec}_m(D\tau_-) = & \left\{ \left(+\frac{k+3/2}{R_b} \right)^{[(k+1)(k+2)]} : k = 0, 2, 4, \dots \right\} \\ & \cup \left\{ \left(-\frac{k+3/2}{R_b} \right)^{[(k+1)(k+2)]} : k = 1, 3, 5, \dots \right\}, \end{aligned} \quad (5.14)$$

$$\text{Spec}_m(\Delta_1^\perp) = \left\{ \left(\frac{(l+1)^2}{R_b^2} \right)^{[2l(l+2)]} : l = 1, 3, 5, \dots \right\}, \quad (5.15)$$

$$\text{Spec}_m(\Delta_{TT}) = \left\{ \left(\frac{k(k+2)-2}{R_b^2} \right)^{[2(k-1)(k+3)]} : k = 2, 4, 6, \dots \right\}. \quad (5.16)$$

Here $\lambda^{[m]}$ denotes multiplicity m and $\Delta_{TT} = -\nabla^a \nabla_a$. A curvature-shifted Lichnerowicz operator has a different spectrum. The scalar formula follows from the spectral-capacity and single-reversal properties and the Dirac formula uses the spin convention following Proposition 4.4. The one-form and tensor spectra follow by harmonic analysis on the round quotient with conventions and derivations in Appendix A. The scale extension E_4 is the metric completion. The matter spaces and charged quotient are the relative-frame and least-matter constructions and the elementary connector is the connector continuation.

Radius and reference units. The dimensional radius is $R_b = L_{\text{IR}}(b) = \sqrt{2\pi} b$. Let b_0 be the width of the unit-covariance source μ_1 . In its normalized coordinates $b_0 = 1$. Use $\ell_* = b_0$ as the fixed length unit. The reference profile whose local unit-information endpoint equals this unit has width $b_* = b_0/\sqrt{2}$ and is the member $\mu_{1/2}$ of the same Gaussian semigroup. Thus $\ell_* = \ell_{\text{UV}}(b_*) = \sqrt{2} b_*$ and the reference profile is fixed by this endpoint calibration. In the scale-dependent response formulas write

$$L(b) = \frac{R_b}{\ell_*}, \quad L_c = \frac{R_{b_*}}{\ell_*} = \sqrt{\pi}, \quad \frac{L(b)}{L_c} = \frac{b}{b_*}. \quad (5.17)$$

The IR radius of the canonical source is consequently

$$L_{\text{src}} := L(b_0) = \sqrt{2\pi}, \quad \frac{L_{\text{src}}}{L_c} = \sqrt{2} > 1. \quad (5.18)$$

All spectral and feedback coefficients in those formulas are expressed in the same fixed units. The instantaneous ratio $R_b/\ell_{\text{UV}}(b) = \sqrt{\pi}$ instead compares the two endpoints of one source state. Scale transport varies b relative to the reference state. It keeps ℓ_* fixed. The reference prescription fixes the zero of the relative drive and the response equations determine the equilibrium at each relative scale b/b_* . The canonical member $b = b_0$ and the variable-width family therefore describe respectively the response at the generated IR edge and its dependence on resolution. The finite tensor-response endpoint is determined by its own spectral response equation.

Quotient operators and traces. The matter entries fit the gauge-equivariant exact sequence

$$0 \longrightarrow N \longrightarrow \mathcal{M}_{\text{full}} \xrightarrow{\pi_{\text{ch}}} \mathcal{M}_{\text{ch}} \longrightarrow 0. \quad (5.19)$$

Only operators satisfying $AN \subset N$ induce operators on the quotient. For a self-adjoint bundle operator A on a Hilbert realization require also $P_N \mathcal{D}(A) \subset \mathcal{D}(A)$ and $AP_N = P_N A$ on $\mathcal{D}(A)$. Then N and its orthogonal complement reduce A and whenever the traces exist

$$\text{Tr}_{\text{full}} f(A) = \text{Tr}_{\text{ch}} f(\bar{A}) + \text{Tr}_N f(A_N). \quad (5.20)$$

Finite-dimensional descent is part of Theorem 4.9. In the self-adjoint case the stated domain conditions make P_N commute with the resolvent and hence with the spectral projections. Functional calculus then gives the trace splitting. A regulated trace uses the same decomposition and a common subtraction prescription.

The elementary connector can take a charged response into the neutral line and return it to the detected sector. Eliminating that intermediate direction therefore leaves a contribution to the retained response. For this connector squaring and charged compression differ by a rank-one contribution from the neutral line.

Spectrum and neutral feedback of the elementary connector. For a fixed nonzero elementary coordinate H let $r = \|H\|$ and $T_H = I_C \otimes h_H$ with $\mathcal{D}_H = \begin{pmatrix} 0 & T_H \\ T_H^\dagger & 0 \end{pmatrix}$ on the generated sixteen-dimensional full module. Choose a unit $n \in N$ and put $P = I - |n\rangle\langle n|$ and $v = \mathcal{D}_H n$ with $D_{\text{ch}} = P\mathcal{D}_H P|_{N^\perp}$. Then

$$\mathcal{D}_H^2 = r^2 I_{16}, \quad P\mathcal{D}_H^2 P|_{N^\perp} = r^2 I_{15}, \quad r^2 I_{15} - D_{\text{ch}}^2 = |v\rangle\langle v|, \quad \|v\|^2 = r^2. \quad (5.21)$$

Thus the square descends although the odd connector does not. The spectra of $\mathcal{D}_H$ and D_{ch} are respectively $\{+r, -r\}$ with multiplicities $(8, 8)$ and $\{+r, -r, 0\}$ with multiplicities $(7, 7, 1)$. The zero direction of the compression is v not a zero mode of the full operator. For $z \notin \{0, r, -r\}$ its exact retained resolvent is

$$P(\mathcal{D}_H - zI)^{-1}P|_{N^\perp} = \left(D_{\text{ch}} - zI + \frac{|v\rangle\langle v|}{z} \right)^{-1}. \quad (5.22)$$

The Schur formula eliminates the neutral block for $z \neq 0$. The explicit generated matrix (4.59) satisfies $h_H^\dagger h_H = h_H h_H^\dagger = r^2 I_2$. Hence $\mathcal{D}_H^2 = r^2 I_{16}$. The image v is left-handed and orthogonal to the right-handed neutral line. Inserting $I = P + |n\rangle\langle n|$ between the two factors of $\mathcal{D}_H$ proves the rank-one identity. Moreover $\mathcal{D}_H v = r^2 n$. So $D_{\text{ch}} v = 0$. The plane spanned by n, v reduces $\mathcal{D}_H$ and has eigenvalues $\pm r$. On its fourteen-dimensional orthogonal complement the square is $r^2 I$ and chiral grading pairs the two signs giving seven of each. The full operator similarly has eight of each sign. In the splitting $N^\perp \oplus N$ the lower block of $\mathcal{D}_H - zI$ is $-z$ and the off-diagonal block is v . Eliminating this one scalar block gives (5.22). Rephasing n leaves $|v\rangle\langle v|$ unchanged.

The same calculation gives the connector feedback without an independent scalar ansatz. On the full generated matter carrier

$$A \otimes I_{16} + I \otimes \mathcal{D}_H^2 = (A + \|H\|^2 I) \otimes I_{16}. \quad (5.23)$$

Thus the elementary reversal-real connector changes every eigenvalue of a scalar response block by its intrinsic determinant $\det h_H = \|H\|^2$. The coefficient is fixed by the normalization of the generated connector matrix. This electroweak connector response and the colour-Casimir feedback of Section 8.8 are two applications of the same rule—square the generated odd or connection comparison and retain its invariant intensity—but they act on different response sectors.

Effective sources, linear observations and the full trace. Keep the same generated connector and $z \notin \{0, r, -r\}$ and set $S_H(z) = D_{\text{ch}} - zI + vv^\dagger/z$. For a source j on the full module and a complex-linear row covector ℓ write $j_c = Pj$ and $j_n = n^\dagger j$. Likewise set $\ell_c = \ell|_{N^\perp}$ and $\ell_n = \ell(n)$. The linear source pairing $x \mapsto j^\dagger x$ descends to the charged quotient exactly when $j_n = 0$. The covector ℓ descends exactly when $\ell_n = 0$. This follows by replacing x with $x + \alpha n$. Their exact eliminated counterparts are

$$j_{\text{eff}}(z) = j_c + \frac{vj_n}{z}, \quad \ell_{\text{eff}}(z) = \ell_c + \frac{\ell_n v^\dagger}{z}. \quad (5.24)$$

Indeed the two blocks of $(\mathcal{D}_H - zI)x = j$ give

$$x_c = S_H(z)^{-1} j_{\text{eff}}(z), \quad x_n = \frac{v^\dagger x_c - j_n}{z}.$$

Substitution into the full-space linear response gives

$$\ell(\mathcal{D}_H - zI)^{-1} j = \ell_{\text{eff}}(z) S_H(z)^{-1} j_{\text{eff}}(z) - \frac{\ell_n j_n}{z}. \quad (5.25)$$

The last term is the direct neutral contribution to the response. For complex z the quantity ℓ_{eff} is the displayed analytic row covector. Taking the adjoint of an effective column would instead conjugate z . The neutral feedback $vv^\dagger/z$ remains when both pairings descend. Rephasing n changes v, j_n, ℓ_n with cancelling phases. The same block solution gives the trace over all sixteen internal directions

$$\text{Tr}_{16}(\mathcal{D}_H - zI)^{-1} = \text{Tr}_{15} S_H(z)^{-1} + \frac{v^\dagger S_H(z)^{-1} v}{z^2} - \frac{1}{z} = \frac{16z}{r^2 - z^2}. \quad (5.26)$$

Here $S_H(z)^{-1} v = zv/(r^2 - z^2)$ and the seven remaining pairs give $14z/(r^2 - z^2)$ to its trace. The apparent singularity at $z = 0$ cancels in the full response.

For a self-adjoint scalar mode operator A bounded below with $\text{Tr} e^{-tA} < \infty$ for every $t > 0$ define the squared-response operator $K_{\text{full}} = A \otimes I_{16} + I \otimes \mathcal{D}_H^2$ on $D(A) \otimes \mathcal{M}_{\text{full}}$. Its charged restriction and neutral block are $(A + r^2) \otimes I_{15}$ and $A + r^2$. So for $t > 0$

$$\begin{aligned} \text{Tr} e^{-tK_{\text{full}}} &= 16e^{-tr^2} \text{Tr} e^{-tA} = (15 + 1)e^{-tr^2} \text{Tr} e^{-tA}, \\ \text{Tr} e^{-t(A \otimes I_{15} + I \otimes \mathcal{D}_{\text{ch}}^2)} &= (14e^{-tr^2} + 1) \text{Tr} e^{-tA}. \end{aligned} \quad (5.27)$$

These formulas follow by the displayed orthogonal spectral decomposition. For $A = \Delta_0$ the operator K_{full} has lower bound r^2 whereas compression before squaring introduces one constant zero mode. The elementary mass block shifts the whole tower. The bound

concerns the colour-trivial matter operator K_{full} at fixed $H \neq 0$. A Yukawa-weighted operator uses its weighted square. A chiral Dirac operator may also contribute cross terms.

For complex sources and complex-linear row covectors the same block decomposition gives the finite-dimensional response explicitly. For three independent copies the full and charged dimensions are 48 and 45 with a three-dimensional neutral block. Specified Yukawa matrices and positive kinetic forms define an analogous response through the same block decomposition with the weighted square in place of $\mathcal{D}_H^2 = r^2 I$. Flavour hierarchies then depend on these additional matrices.

The scalar mass assignment combines the Laplace spectrum with a common positive radial restoring coefficient m^2 obtained from the stationary collective response after kinetic normalization. The assigned masses are

$$M_l^2 = \frac{l(l+2)}{R_b^2} + m^2. \quad (5.28)$$

The complete scalar mass tower. On $\mathcal{H}_{\text{sc}} = L^2(X_3, d\bar{v})$ define

$$\mathcal{M}_{\text{sc}}^2 = \Delta_0 + m^2 I, \quad D(\mathcal{M}_{\text{sc}}^2) = H^2(X_3). \quad (5.29)$$

This is a positive self-adjoint operator with compact resolvent. Its full spectrum is (5.28) for $l = 0, 2, 4, \dots$ with multiplicity $(l+1)^2$. In particular

$$\inf \text{Spec } \mathcal{M}_{\text{sc}}^2 = m^2, \quad \langle \psi, \mathcal{M}_{\text{sc}}^2 \psi \rangle = \|d\psi\|_2^2 + m^2 \|\psi\|_2^2 \geq m^2 \|\psi\|_2^2. \quad (5.30)$$

The equality case is the constant scalar mode. The inverse powers satisfy $\text{Tr}(\mathcal{M}_{\text{sc}}^2)^{-p} < \infty$ exactly for $p > 3/2$. These statements follow directly from the complete scalar decomposition (5.13). The bounded shift changes neither the eigenspaces nor the domain. Integration by parts gives (5.30) on the form domain $H^1(X_3)$ and the trace terms grow as l^{2-2p} along the even subsequence.

Spectral physical quantities. For the generated operator and carrier with their multiplicities and normalization a spectral physical quantity is an absolutely convergent expression

$$Q_f = \sum_l d_l f(\varepsilon_l), \quad (5.31)$$

or a renormalized relative trace with a specified subtraction, operator domain and renormalization prescription. Finite closed expressions in these data and their scale relations define physical quantities on the stated domains and branches, with units inherited from the spectral normalization.

Collective quantities are derived from these spectral sums and their scale dependence. Each trace includes every mode of its specified carrier with the corresponding multiplicity. Probabilities additionally require a positive state and an allowed observation. Positive functionals on the generated source algebra supply this information. A field representation gives the same quantities when it preserves the operators and their units and normalizations. For relative traces Appendix E proves the estimates for the specified subtractions.

5.5 States, composition, and observables

The state construction uses positive functionals and their cyclic representations [22] on the generated source algebra. The source operations specify the available effects and updates. In the cyclic representation an allowed preparation operation A produces the unnormalized vector $\pi(A)\Omega$. The source comparison $\omega(A^*B)$ gives the overlap of preparations produced by A and B . A difference of zero norm has zero overlap with every generated preparation and is identified with zero. Completion includes norm limits of these preparations. Cyclicity means that the generated vectors exhaust the representation. The source comparisons therefore fix its Hilbert-space geometry.

States and operations. A state is a normalized positive functional ω on the generated observable algebra. Its cyclic Hilbert representation is obtained by quotienting the form $\omega(A^*B)$ by its null space and completing it with left multiplication as the algebra action. This is the representation of the source states in Theorem 5.2.

An allowed effect $0 \leq E \leq I$ has probability $\omega(E)$. A finite allowed partition $\sum_a E_a = I$ therefore has nonnegative probabilities summing to one. In the cyclic representation these are $\langle \Omega, \pi(E_a)\Omega \rangle$. The available effects and instruments are fixed by the source operations.

Evolution is a composition-compatible update of the generated states and observables as in (5.9) and (5.10). A nonautonomous update has a two-parameter composition law. A strongly continuous unitary group has a self-adjoint generator by Stone's theorem. The operation parameter and accumulated variance with the heat resolution label the corresponding updates defined above.

Independence means commuting operations and factorized expectations of all allowed product observables. For full finite corners this gives the product state on $M_n \otimes M_m \cong M_{nm}$.

Local response operations act through the relative-frame bundles over M_{loc} . Their composition is the mode and matrix product above.

The spectral quantities defined above are evaluated with fixed units on their stated parameter domains. A state-dependent representation intertwines the state and its operations with the sources and observables so that their probabilities and response normalizations are preserved.

For the complete source functional $\omega_t(A) = \text{Tr}(\rho_t A)$ at $t > 0$ in (5.5) the uniqueness part of the GNS construction [22] relates any two cyclic representations $(\pi_i, \mathcal{H}_i, \Omega_i)$ by the unique unitary satisfying

$$U\pi_1(A)\Omega_1 = \pi_2(A)\Omega_2, \quad U\pi_1(A)U^* = \pi_2(A), \quad U\Omega_1 = \Omega_2.$$

Consequently their finite source correlations and common continuous extensions of the specified operations agree on their stated domains and in the prescribed norms. The inner products $\omega_t(A^*B)$ fix the isometry on generated vectors. Cyclicity extends it to the Hilbert space. Common bounded operations extend from their specified dense cores by continuity.

The source algebra and the internal factor X_3 carry these states and spectral quantities. Their field realization uses the generated event carrier for localization and the internal modes for field content. Appendices D and C give the Gaussian covariance and its positive-

energy graded-local reconstruction. The collective potential depends on the internal spectra and their scale dependence.

6 Source Events, Composition, and Readout

A positive source state assigns probabilities to the allowed preparations and readouts. Mutually exclusive outcomes, unread comparisons and conditional states are defined within the same source functional. Semigroups and connections provide the updates whose responses are measured. This event calculus applies before a spacetime field representation is chosen. On the affine Lorentz carrier, local fields represent the internal responses and the phase clock calibrates timelike separations between events.

6.1 Quantum events and composition

The source construction supplies positive states on its generated observable algebra. A finite observation resolves this algebra into mutually exclusive quantum events. The unresolved remainder is retained as one outcome. An allowed orthogonal partition describes these distinctions. The same source state assigns their probabilities and projection gives the Lüders conditional update [23].

For a generated orthogonal partition $\sum_{a=0}^{m-1} P_a = I$ and a positive normalized state ω put

$$p_a = \omega(P_a), \quad \omega_a(X) = \omega(P_a X P_a) / p_a \quad (p_a > 0). \quad (6.1)$$

These are normalized probabilities and states. The unread map is

$$\mathcal{L}(X) = \sum_a P_a X P_a = \frac{1}{m} \sum_{k=0}^{m-1} V^k X (V^k)^*, \quad V = \sum_a e^{2\pi i a / m} P_a. \quad (6.2)$$

It is unital and completely positive and $\omega \circ \mathcal{L} = \sum_a p_a \omega_a$. The partition identity normalizes the probabilities and positivity normalizes each conditional state. Zero-probability blocks vanish by Cauchy–Schwarz. Character orthogonality cancels all off-diagonal event blocks in the finite phase average. This average of unitary conjugations is completely positive. Equal-mass quantiles of a nondegenerate source Gaussian implement the random choice of V^k when such control is available.

The last event may contain the unresolved tail and retain its probability in the partition. In the cyclic representation of ω the identity $P^* P = P$ gives $\omega(P) = \|\pi(P)\Omega\|^2$ [22]. Thus the projected vector’s squared norm expresses the original source probability. The same rule reads $p_\psi(P) = \langle \psi, P \psi \rangle$ for a unit vector ψ . For $P = |\phi\rangle\langle\phi|$ with $\|\phi\| = 1$ it is $|\langle\phi, \psi\rangle|^2$. This is the usual Born expression. Gleason’s uniqueness argument additionally requires noncontextual countable additivity and dimension at least three [24].

The unread map retains each diagonal event block. Averaging the finite relative phases cancels the blocks connecting distinct events. The distinction is visible in the allowed two-level preparations $\psi_\pm = (h_0 \pm h_1)/\sqrt{2}$. They have the same level probabilities $(1/2, 1/2)$. The generated observable $X = E_{01} + E_{10}$ has expectation $+1$ in ψ_+ and -1 in ψ_- . It reads their relative sign. After the unread comparison in the h_0, h_1 basis both preparations

give the same equal mixture and expectation zero. Level occupations and coherent source comparisons therefore convey different information within the same generated algebra.

Independent systems use factorized states on the product algebra. In a Hilbert realization superposition is vector addition. A pure bipartite state is entangled when it does not factor into two vectors. Separability of density operators means membership in the trace-norm closed convex hull of product density operators. For independent Euclidean measures and bosonic free fields the standard identifications are

$$L^2(\mu_1 \otimes \mu_2) \cong L^2(\mu_1) \otimes L^2(\mu_2), \quad \mathcal{F}_+(h_1 \oplus h_2) \cong \mathcal{F}_+(h_1) \otimes \mathcal{F}_+(h_2). \quad (6.3)$$

The first identity follows by completing products of simple functions. The second follows from the symmetric tensor decomposition. Factorization specifies independence. Tensor symmetry specifies the exchange statistics.

6.2 Measurement and dephasing

For a self-adjoint observable O and apparatus generator Q the unitary evolution of the specified interaction $gO \otimes Q$ maps $\sum_k c_k \psi_k \otimes \phi$ to $\sum_k c_k \psi_k \otimes e^{-itgO_k Q/\hbar} \phi$ when $O\psi_k = o_k \psi_k$. Two occupied branches with nonproportional apparatus factors are entangled.

For pure dephasing let a finite pointer have commuting H_S, A and use $H_\xi(t) = H_S + gA\xi(t)$. The real Gaussian process is centred and stationary with mean-square continuity. It is initially independent of the pointer and has covariance $C(s - u)$ with $C(0) < \infty$. In a joint eigenbasis the noise-averaged reduced density matrix satisfies

$$\rho_{ij}(t) = e^{-i(E_i - E_j)t/\hbar} \rho_{ij}(0) \exp \left[-\frac{g^2(a_i - a_j)^2}{2\hbar^2} \int_0^t \int_0^t C(s - u) ds du \right]. \quad (6.4)$$

All Hamiltonians commute so the noise enters only through $X_t = \int_0^t \xi(s) ds$. Its Gaussian characteristic function proves the formula. The integral is a variance and populations are unchanged. Its growth determines the decay law. A variance proportional to t gives exponential dephasing. The pointer probabilities $\text{Tr}(\rho P_k)$ remain those of the state.

7 Fields, Correlations, and Spectral Description

A spectrum lists the available excitations; a correlation weights them by the transition overlaps of a particular preparation and observation. An internal trace instead counts the full multiplicity of the specified operator. The weights determine both the modes visible to an observation and the convergence of the corresponding sums.

The source generator resolves changes between ladder levels, while the mass-squared operator measures the normalized restoring response of physical fluctuations. Their observable spectral measures use the same overlap construction. Passing between the two representations requires the stated response maps and intertwiners. Eigenvalues determine where an excitation occurs; the observation weights determine whether it is seen.

7.1 Spectral measures of source observables

For a bounded self-adjoint generated observable O let $O_c = O - \omega_0(O)I$ and $v = O_c h_0$. Its connected source spectral measure is

$$\nu_O = \sum_{j \geq 1} |\langle h_j, O h_0 \rangle|^2 \delta_j, \quad \nu_O([0, \infty)) = \|v\|^2. \quad (7.1)$$

It is finite and positive and satisfies

$$\begin{aligned} \omega_0(\alpha_\theta(O_c)O_c) &= \int e^{-i\theta\lambda} d\nu_O(\lambda), \\ \langle v, e^{-sN_G} v \rangle &= \int e^{-s\lambda} d\nu_O(\lambda) \leq e^{-s\|v\|^2}, \quad s \geq 0. \end{aligned} \quad (7.2)$$

Whenever a positive level is visible the first is the least $j \geq 1$ with $\langle h_j, O h_0 \rangle \neq 0$. Centering removes the vacuum coefficient. Parseval gives the total weight $\|v\|^2$. The spectral theorem applied to N_G gives both transforms with this absolute convergence bound.

These weights are observable-dependent. Physical scalar masses are determined by the complete scalar mass operator (5.29) with sharp lower bound (5.30). An excitation with zero transition overlap is absent from that source correlation even though it remains in the source spectrum.

For the fixed-covariance Gaussian field of Theorem G.8 real test vectors $f, g \in H_A^q = (H_A^{-q})'$ give continuous evaluations and

$$\mathbb{E} e^{iX_\tau(f)} = \exp[-\tfrac{1}{2}\langle f, C_\tau f \rangle]. \quad (7.3)$$

In the eigenbasis $Ae_i = a_i e_i$ its static covariance is

$$\mathbb{E}[X_\tau(f)X_\tau(g)] = \sum_i c_\tau(a_i) \overline{f_i} g_i = \langle f, C_\tau g \rangle, \quad c_\tau(a) = \frac{1 - e^{-\tau a}}{a}. \quad (7.4)$$

These follow from finite Gaussian coordinates and the distribution-space tail estimate in Appendix G.9. The bounded observables $e^{iX_\tau(f)}$ generate a commutative algebra. Equations (7.3) and (7.4) determine its static correlations at fixed τ .

7.2 Analytic control

The three-dimensional Weyl exponent fixes the threshold for finite resolvent traces. The zero modes determine the additional dependence on the lower mass bound. At high levels convergence balances the growing number of modes against the decay of the response weight. Near the bottom of the spectrum a positive mass shift controls the contribution of the constant and other zero modes.

Resolvent traces. Let $A \geq 0$ be a self-adjoint Laplace-type operator on a finite-rank Hermitian bundle over a closed compact three-manifold M . Write its eigenvalues as ε_l with multiplicities d_l and set $\nu_0 = \dim \ker A$. For $p > 3/2$ the trace

$$Q_p(m^2) = \sum_l d_l (\varepsilon_l + m^2)^{-p} \quad (7.5)$$

is positive and real analytic for $m^2 > 0$. Choose C_A with $N_+(\lambda) = \sum_{0 < \varepsilon_l \leq \lambda} d_l \leq C_A \lambda^{3/2}$ for all $\lambda > 0$. Then

$$Q_p(m^2) \leq \nu_0(m^2)^{-p} + C_A \frac{\Gamma(5/2)\Gamma(p-3/2)}{\Gamma(p)} (m^2)^{3/2-p}. \quad (7.6)$$

On $m^2 \geq m_0^2 > 0$ a bound $Q_p(m^2) \leq C_{p,A,m_0} (m^2)^{3/2-p}$ holds with

$$C_{p,A,m_0} = \nu_0(m_0^2)^{-3/2} + C_A \frac{\Gamma(5/2)\Gamma(p-3/2)}{\Gamma(p)}. \quad (7.7)$$

Thus the half-line constant depends on the chosen operator. Zero modes introduce additional dependence on m_0 . Weyl asymptotics for compact elliptic operators [25] bound $N_+(\lambda)/\lambda^{3/2}$ above a sufficiently large threshold. Below that threshold the positive spectral gap and the finite number of eigenvalues make the ratio bounded as well. This proves the existence of C_A . A numerical value follows from a quantitative high-energy bound and the finite low spectrum. An explicit spectrum also determines this constant directly. Separate the zero modes and put $x = m^2$. Stieltjes integration by parts yields

$$\begin{aligned} Q_p(x) - \nu_0 x^{-p} &= p \int_0^\infty \frac{N_+(\lambda)}{(\lambda+x)^{p+1}} d\lambda \\ &\leq C_A p x^{3/2-p} \int_0^\infty \frac{t^{3/2}}{(1+t)^{p+1}} dt \\ &= C_A \frac{\Gamma(5/2)\Gamma(p-3/2)}{\Gamma(p)} x^{3/2-p}. \end{aligned} \quad (7.8)$$

The boundary terms vanish for $p > 3/2$ and the integral is $B(5/2, p-3/2)$. On any compact subset of $\text{Re } z > 0$ with $\text{Re } z \geq \delta$ the principal-branch series is dominated by $Q_p(\delta)$. Normal convergence proves analyticity and

$$\frac{d^j Q_p}{dx^j} = (-1)^j (p)_j \sum_l d_l (\varepsilon_l + x)^{-p-j}, \quad j \geq 0, \quad (7.9)$$

where $(p)_j = \Gamma(p+j)/\Gamma(p)$.

For the scalar Laplacian on the generated X_3 of radius L the odd-shell spectrum gives the explicit constant $C_{\Delta_0} = 9L^3/(16\sqrt{2})$. The count $N_+(\lambda) = 0$ below $8/L^2$. Above this threshold let $q \geq 3$ be the largest odd integer with $q^2 - 1 \leq L^2 \lambda$. Then

$$N_+(\lambda) = \frac{q(q+1)(q+2)}{6} - 1 \leq \frac{q^3}{3} \leq \frac{9}{16\sqrt{2}} (q^2 - 1)^{3/2} \leq \frac{9L^3}{16\sqrt{2}} \lambda^{3/2}.$$

The first inequality uses $(q-3)(q^2-2) \geq 0$ and the second uses $q^2/(q^2-1) \leq 9/8$. Equality holds at $\lambda = 8/L^2$ so this global constant is sharp. Together with $\nu_0 = 1$ it makes (7.6) explicit for the complete scalar tower.

Green operators. For $A = \Delta_M$ an internal source j produces the response $u = (\Delta_M + m^2)^{-1}j$. Each harmonic is weighted by the inverse of its stiffness $\varepsilon_l + m^2$. The Green kernel gives the response at y to a source at y' . Spectral calculus gives

$$Q_p(m^2) = \text{Tr}_{L^2(M)}(\Delta_M + m^2)^{-p}, \quad p > 3/2. \quad (7.10)$$

In particular $Q_2 = \int_{M \times M} |G_{m^2}(y, y')|^2 dy dy'$ is the squared Hilbert–Schmidt norm of the compact Green operator. It sums the squared response over source and observation positions with the stated internal measure. The ordinary Q_1 trace diverges in three dimensions. This contrast is the analytic hinge of the condensation mechanism. The linear trace response is not summable over the complete three-dimensional tower. The compensated logarithm cancels that linear term and leaves the Hilbert–Schmidt quantity Q_2 . The same Q_2 controls two derivatives of the relative potential and its local quartic coefficient together with the spectral tail entering the condensate and mass estimates of Section 8. In the specified product realization (C.1) Fourier transformation in the external coordinates gives $(p_{\text{ext}}^2 + \Delta_M + m^2)^{-1}$. Here p_{ext} is Euclidean four-momentum. At $p_{\text{ext}} = 0$ this is the same Green operator. Setting $m^2 = m_\delta^2$ therefore reads the internal response as the zero-momentum component of the condensed quadratic propagator.

The internal trace weights each level by its multiplicity and gives the positive locally finite measure

$$\rho(d\mu^2) = \sum_l d_l \delta_{M_l^2}(d\mu^2). \quad (7.11)$$

If $M_l^2 = \varepsilon_l + m_\delta^2$ with $m_\delta^2 > 0$ its support has that lower bound. For a field evaluated against an internal test vector each atom carries the squared norm of that vector’s eigenspace projection as in (C.9). The mass parameter is the condensate’s normalized radial curvature.

8 Gauge Fields, Scale Transport, and Stable Spectra

Internal mode multiplication and the matrix action on matter fix the nonlinear coupling. Changing resolution smooths the fields and produces a product correction, which the joint scale map retains. This correction makes successive resolutions compatible with the original interaction.

The colour connection supplies its own collective coordinate: its basis-independent mean squared strength. Casimir contraction shifts the complete conformal response tower, the relative long-root eigenvalue provides the drive, and the changed Gaussian fluctuations give the restoring response. The complete relative potential balances these contributions and has a unique nonzero equilibrium beyond the reference radius. The source Legendre metric normalizes intensity fluctuations; the exact curvature response controls coclosed connection variations. On the condensed branch these two blocks form a positive Yang–Mills mass operator, measuring restoration of the collective intensity and transverse non-Abelian deformations at fixed intensity. Appendix B.4 develops the corresponding action-level realization.

8.1 Gauge interaction from mode multiplication and common transport

The local frames of Section 4 fix the gauge transformation law of a connection A . Its field strength satisfies

$$F_A = dA + A \wedge A, \quad D_A F_A = dF_A + A \wedge F_A - F_A \wedge A = 0. \quad (8.1)$$

The second identity follows by substitution and $d^2 = 0$. Here A includes the coupling in the skew-adjoint connection convention. Restoring it as $A = g_{\text{conn}} \mathcal{A}^A T_A$ gives $F_A = g_{\text{conn}} d\mathcal{A} + g_{\text{conn}}^2 \mathcal{A} \wedge \mathcal{A}$ and $\nabla = d + g_{\text{conn}} \rho_*(\mathcal{A})$. The quadratic field-strength response consequently has cubic and quartic terms proportional to $g_{\text{conn}} f_{AB}^C$ and $g_{\text{conn}}^2 f_{AB}^C f_{DE}^C$ after factoring out its quadratic normalization. These differential formulas describe the interaction in the generated internal frames.

The mode product in Section 5.1 acts directly on the fields of the generated internal geometry. In a fixed trivialization write a connection as $A = \sum_a A_a \theta^a$ in the left-invariant orthonormal coframe on round X_3 and let $[T_A, T_B] = f_{AB}^C T_C$ be the matrix commutator in the generated internal Lie algebra. With $A_a = \sum_{\alpha, A} a_a^{\alpha A} e_\alpha T_A$ the field strength is

$$F_{ab}(A) = E_a A_b - E_b A_a - \frac{2}{L} \epsilon_{abc} A_c + [A_a, A_b],$$

$$[A_a, A_b]_\gamma^C = \sum_{\alpha, \beta, A, B} C_{\alpha\beta}^\gamma f_{AB}^C a_a^{\alpha A} a_b^{\beta B}. \quad (8.2)$$

The coefficients $C_{\alpha\beta}^\gamma$ express pointwise mode multiplication while f_{AB}^C are the structure constants of the internal representation. Their product determines which internal harmonics and gauge components couple. Thus the geometric modes and the noncommuting matrices enter one nonlinear response. Applying the same connection in the matter representation fixes the corresponding transport coupling. The connection normalization g_{conn} sets the overall strength of this transport. The scalar feedback coefficient g in $A_E = A + gE^2 I$ measures the change of scalar precision per unit collective intensity.

Nonlinear field strength and its scale correction. For $A_a \in H^\eta(X_3)$ with $\eta > 5/2$ apply $G_t = e^{-t\Delta_0}$ componentwise in this same frame. Then

$$G_t F_{ab}(A) = F_{ab}(G_t A) + \mathfrak{C}_{ab,t}(A), \quad \mathfrak{C}_{ab,t}(A) = 2 \int_0^t G_{t-u} \sum_c [E_c G_u A_a, E_c G_u A_b] du. \quad (8.3)$$

The integral is in $H^{\eta-1}$ with norm at most $C_{\eta, L, \mathfrak{g}} t \|A_a\|_{H^\eta} \|A_b\|_{H^\eta}$. Its modal coefficient is the second line of (8.2) multiplied termwise by $e^{-t\lambda_\gamma} - e^{-t(\lambda_\alpha + \lambda_\beta)}$. There are real connections at every shell cutoff $j \leq J$ with $J \geq 1$ whose curvature has a nonzero $j = 2J$ component and for which this scale correction is nonzero.

Proof. Round scalar heat commutes with the invariant derivatives. So the linear terms cancel in the difference. The heat product rule of Proposition A.1 gives the integral and its norm estimate. Take $A_1 = fT_A$ and $A_2 = fT_B$ with $A_3 = 0$ where $f = e_{JJ}^J + e_{JJ}^{\overline{J}}$ and $[T_A, T_B] \neq 0$. Equation (A.22) gives the nonzero $2J$ component and $\lambda_{2J} - 2\lambda_J = 8J^2/L^2 > 0$ proves that its scale correction cannot vanish for $t > 0$. $\square$

The proposition uses scalar heat on coefficients in the fixed trivialization and coframe. Its frame dependence follows from the commutator of G_t with a position-dependent gauge transformation.

For the $(2, 1)$ long-root source Proposition G.7 sharpens this identity. It computes the connection–field-strength composition law and a strictly positive mean-square remainder

from the same noncommutative product. Its coefficient heat rate is $24/L^2$ whereas its Hodge eigenvalue is $16/L^2$. The joint response therefore retains both the discarded modes and the difference $G_t F(A) - F(G_t A)$. The matter response requires the corresponding product correction. Changing the internal metric requires the derivative commutators of Appendix A.6.

8.2 The same connection acts on matter

The localized principal bundle and the representation ρ on the full matter module were constructed in Section 4 with the stated local frame and matter realization. Applying that representation to principal transport determines the matter coupling. We use the skew-adjoint connection convention of (8.2).

Theorem 8.1 (Generated gauge interaction and common matter transport). Transport by the same principal connection A induces on its associated matter bundle the covariant derivative and curvature action

$$\nabla_A^\rho \psi = d\psi + \rho_*(A)\psi, \quad (\nabla_A^\rho)^2 \psi = \rho_*(F_A)\psi. \quad (8.4)$$

In a compatible trivialization over X_3 for scalar coefficient fields $A_a = \sum_{\alpha,A} a_a^{\alpha A} e_\alpha T_A$ and $\psi = \sum_\beta e_\beta \psi_\beta$ the coupling coefficients are

$$[\rho_*(A_a)\psi]_\gamma = \sum_{\alpha,\beta,A} C_{\alpha\beta}^\gamma a_a^{\alpha A} \rho_*(T_A)\psi_\beta. \quad (8.5)$$

For finite modal fields this is an exact finite product and it extends continuously to $H^\eta(X_3)$ for $\eta > 3/2$. The tensors $C_{\alpha\beta}^\gamma$ and $\rho_*(T_A)$ fix the modal selection rules and relative matrix coefficients of this interaction in the chosen connection normalization.

Proof. Applying ρ to infinitesimal principal transport $I - A(\dot{\gamma})ds$ gives $I - \rho_*(A(\dot{\gamma}))ds$. The induced derivative transforms with $\rho(g^{-1})$ under a frame change. Since ρ_* preserves brackets its square is $\rho_*(F_A)$. Equation (5.2) gives the displayed coefficients and Sobolev multiplication on the three-dimensional factor gives the completion for $\eta > 3/2$. $\square$

On a spinor-valued associated bundle the derivative is $\nabla^{\text{spin}} \otimes I + I \otimes \rho_*(A)$ and its coefficients must use the selected spinor modes of Appendix A. The internal connection acting on them is the same one. Its matter action descends to the charged quotient under the [quotient and trace conditions](#) of Section 5.4. A general Higgs connector still acts on the full matter module.

Joint resolution of gauge and matter coefficients. Keep the same fixed trivialization for the connection and associated matter coefficients and set $Q_a = E_a \psi + \rho_*(A_a)\psi$. Applying G_t to both fields produces the difference $C_t^Q = G_t Q(A, \psi) - Q(G_t A, G_t \psi)$ just as for the field strength. Proposition A.2 proves the exact integral composition law and Sobolev completion for both responses. In particular with $U = (A, \psi)$ and the unresolved responses $R = (R_F, R_Q)$

$$(U, R) \longmapsto (G_t U, G_t R + (C_t^F, C_t^Q)) \quad (8.6)$$

is a strongly continuous semigroup. Starting from $R = 0$ every later residual is fixed by the initial fields. The complete responses evolve as $G_t(F + R_F)$ and $G_t(Q + R_Q)$ while the resolved fields retain their nonlinear coupling. The parameter t resolves the gauge and matter coefficients together with their unresolved responses. The residual therefore records interaction information carried by the initial fields at the new resolution. Successive scale changes preserve this complete response by the same composition law. The resolved fields and their residuals thus represent the same coupled response at successive resolutions including the high harmonics created by its mode products.

8.3 Resolution and scale transport

The spectral quantities defined in Section 5 depend on the operator and its coefficients with the Gaussian resolution parameter. For a normalized source Gaussian of width σ_{src} the additive variance is $s = \sigma_{\text{src}}^2$ and the heat parameter is $s/2$. Equation (5.10) acts by diagonal phase averaging on the source algebra and its positive states. Here s parametrizes resolution. Homothetic variation of the internal radius transports the spectral weights on a common mode space and determines the scale dependence of stationary quantities.

Homothetic scale transport of the spectral window. Identify the round factors by their normalized Haar coordinates. For the scalar Laplacian or its conformal shift homothety gives $A_\sigma = L(\sigma)^{-2} \hat{A}$ on the same modal Hilbert space. For positive C^1 functions L, τ put $q = \tau/L^2$. The Gaussian multiplier and its derivative are

$$e^{-\tau A_\sigma} = e^{-q \hat{A}}, \quad \frac{d}{d\sigma} e^{-\tau A_\sigma} = -q' \hat{A} e^{-q \hat{A}}. \quad (8.7)$$

The derivative is in operator norm on compact parameter intervals. Thus a constant ratio τ/L^2 leaves every dimensionless modal weight unchanged. In particular this holds for $L = c/k$ and $\tau = a/k^2$ with fixed $a, c > 0$.

Proof. The metric scales by L^2 . So both the Laplacian and the conformal curvature term scale by L^{-2} . Spectral calculus gives the first identity. On a compact interval $q \geq q_0 > 0$ the multipliers $x e^{-qx}$ and $x^2 e^{-qx}$ are uniformly bounded for $x \geq 0$. Taylor's formula therefore proves the operator-norm derivative. $\square$

At constant q the observation retains the same relative resolution of the internal harmonics while their dimensional eigenvalues change with L^{-2} . A scale comparison first identifies the modes and their resolution weights then specifies the path of the geometry and response coefficients. The stationary amplitude is determined at each point using the same feedback rule and generated relative drive. Thus the resolution parameter describes how the modes are observed while the path of L and g with the kinetic coefficient Z determines their condensed amplitudes and normalized masses. Homothety makes these comparisons on one modal space. Changes in the shape of $\hat{A}$ enter through its Duhamel derivative and integrated responses also include the change in volume. Appendix B.5 gives the action-based FRG realization in which the regulated functional measure supplies the running response.

8.4 Complete spectral response and condensation

The analytic mechanism concerns a positive scalar shift of the generated spectrum. Increasing the collective intensity suppresses Gaussian fluctuations while preserving colour symmetry. Its restoring response competes with the drive to determine the stationary amplitude. In the colour realization, E^2 is the invariant connection intensity q , and the adjoint Casimir fixes g . The complete-tower analysis separates convergence and stability from the representation supplying this shift.

Let E be the collective condensate coordinate in the specified scalar prescription $A_E = A + gE^2I$ with $g > 0$. The dependence on E^2 gives the same response for opposite amplitude signs. Multiplication by I implements a scalar shift uniform over the internal factor. It preserves the mode projectors while making every mode respond to the same intensity. Increasing the amplitude raises each conformal precision eigenvalue and reduces the corresponding Gaussian fluctuation. The potential measures the response of this whole tower to the collective amplitude. The latter is held fixed while the scalar response coordinates are traced including the constant conformal mode. That traced mode is distinct from E . For the connection realization the mean precision has a direct statistical meaning. Averaging the log likelihood of the conditional Gaussian tower and normalizing its exponential gives the Gaussian law with this precision. It uniquely minimizes the mean relative entropy to the conditional laws. The potential below measures relative information between these comparison laws. This specifies which comparison the spectral feedback represents. Retaining instead the field integral $E_{\text{int}} = \int \Phi$ defines a different scalar realization. The reference slope specifies the driving term for the chosen feedback.

Relative entropy measures the mean log-likelihood ratio of two laws with the averaging law fixing the direction of comparison [12]. For a real Gaussian precision deformation $A_y = A + yB$ whitening gives $T = A^{-1/2}BA^{-1/2}$ and the finite identity

$$2D_{\text{KL}}(\nu_0\|\nu_y) = I_T(y) = \text{Tr}\{yT - \log(I + yT)\}. \quad (8.8)$$

For positive Hilbert–Schmidt relative operators quadratic summability controls the potential and its first two derivatives. This extends the finite identity and permits a common spectral cutoff for stationary amplitudes and radial curvatures (Theorem E.1).

This relative information compares two generated Gaussian field laws whereas (5.8) is the entropy of a normalized heat operator. In finite real dimension the differential entropy change of the field law itself is $H(\nu_y) - H(\nu_0) = -\frac{1}{2} \log \det(I + yT)$. The additional term $y \text{Tr } T$ in (8.8) comes from evaluating the changed precision against the reference covariance. It cancels the linear part and produces the convergent positive restoring response. This cancellation also permits the Hilbert–Schmidt limit when the two separate differential entropies have no finite limit. The stability criterion therefore concerns this relative functional together with the specified drive.

Theorem 8.2 (Complete relative-response stability). Let $\beta_j \geq 0$ and $t_j > 0$ with at least

one $\beta_j > 0$ and $Q_2 = \sum_j \beta_j t_j^2 < \infty$. For fixed weights define

$$\begin{aligned} F_\mu(y) &= \frac{\mu}{2}y + I(y), \quad I(y) = \sum_j \beta_j [yt_j - \log(1 + yt_j)], \\ S &= \sum_j \beta_j t_j \in (0, \infty], \quad y \geq 0. \end{aligned} \tag{8.9}$$

For $\mu \geq 0$ the unique minimizer is $y = 0$ and the potential is coercive. For $\mu < 0$ there is a unique positive global minimizer exactly when $\mu/2 + S > 0$. The potential is then coercive and strictly convex in y . If $S < \infty$ and $\mu/2 + S \leq 0$ the potential is unbounded below including the equality case. The conclusion applies equally to finite and infinite index sets.

Proof. The respective bounds by $Q_2 y^2/2$ and $Q_2 y$ and Q_2 justify local uniform convergence of the potential and its first two derivatives. Thus $F'_\mu(y) = \mu/2 + \sum_j \beta_j y t_j^2 / (1 + y t_j)$ and $F''_\mu(y) = \sum_j \beta_j t_j^2 / (1 + y t_j)^2 > 0$. The derivative increases from $\mu/2$ to $\mu/2 + S$ by monotone convergence. Its zero and its strictly positive limiting slope prove the stable cases. For finite S write $F_\mu(y) = (\mu/2 + S)y - \sum_j \beta_j \log(1 + y t_j)$. A single positive weight proves unboundedness in the remaining cases. $\square$

Source-normalized complete restoring response. For the generated even scalar shells put

$$c_j = (2j+1)^2 - \frac{1}{4}, \quad d_j = (2j+1)^2, \quad \mathcal{R}(z) = \sum_{j \geq 0} \frac{d_j z}{c_j(c_j + z)}, \quad \mathcal{C}(z) = \sum_{j \geq 0} \frac{d_j}{(c_j + z)^2}. \tag{8.10}$$

On $z \geq 0$ the function $\mathcal{R}$ is continuous and strictly increasing with $\mathcal{R}(0) = 0$ and $\mathcal{R}(z) \rightarrow \infty$. The curvature $\mathcal{C}(z)$ is finite and strictly positive. Hence every dimensionless drive $\eta > 0$ determines a unique $z_\eta > 0$ satisfying $\mathcal{R}(z_\eta) = \eta$. At the same amplitude

$$\gamma_\eta = 4z_\eta \mathcal{C}(z_\eta) > 0, \quad M_j^2 = 4j(j+1) + \gamma_\eta \geq \gamma_\eta, \quad M_0^2 = \gamma_\eta. \tag{8.11}$$

The unit-drive member $\eta = 1$ defines a canonical normalized response. Here γ_η is the radial curvature in the coordinate $\sqrt{z}$ with unit kinetic coefficient. For an original amplitude E set $z = \kappa E^2$ with $\kappa = gL^2$. A negative drive $\mu < 0$ and kinetic coefficient Z then give

$$\eta = -\frac{\mu}{2\kappa}, \quad m_{\text{rad}}^2 = \frac{\kappa}{Z} \gamma_\eta.$$

Thus the normalized family supplies the restoring response. The source drive and metric determine its member and its mass units.

Proof. The curvature summand is bounded uniformly for $z \geq 0$ by a constant multiple of $(j+1)^{-2}$ which gives convergence and compact-uniform continuity. For $0 \leq x < y$ each response summand obeys

$$\frac{d_j y}{c_j(c_j + y)} - \frac{d_j x}{c_j(c_j + x)} = \frac{d_j(y - x)}{(c_j + x)(c_j + y)} > 0,$$

so the complete response is strictly increasing and $\mathcal{C}(z) > 0$. At $z = c_N$ every shell with $j \leq N$ contributes at least $1/2$. Hence $\mathcal{R}(c_N) \geq (N+1)/2$ proving unboundedness. Existence and uniqueness follow from continuity and strict monotonicity. Finally $4j(j+1) \geq 0$ with equality exactly at $j = 0$ which gives (8.11). $\square$

Generated critical scale and long-root drive. Use the fixed reference-endpoint units of (5.17). The correlation window of the reference source state supplies the critical radius

$$L_c := \frac{L_{\text{IR}}(b_*)}{\ell_{\text{UV}}(b_*)} = \sqrt{\pi}. \quad (8.12)$$

Let $\lambda_R(L) = 16/L^2$ be the generated long-root eigenvalue. Among drives affine in λ_R normalized to unit response to a change of λ_R and vanishing at $L = L_c$ the unique drive is

$$\mu_L = \lambda_R(L) - \lambda_R(L_c) = 16(L^{-2} - L_c^{-2}). \quad (8.13)$$

Proof. Equation (8.12) is the endpoint ratio of Proposition 3.3. An affine function of λ_R with derivative one is $\lambda_R + c$. Its zero at L_c fixes $c = -\lambda_R(L_c)$. $\square$

This comparison assigns a relative spectral cost to the current source state. The ratio $L/L_c = b/b_*$ locates it with respect to the reference state. Increasing the correlation radius lowers the long-root cost and makes μ_L negative. Minimization below determines the connection intensity at that scale. The reference state sets the comparison origin. The equilibrium condition fixes the amplitude rather than varying the reference radius. For the unit-covariance source (5.18) gives $L = L_{\text{src}} = \sqrt{2\pi}$ and hence $\mu_{L_{\text{src}}} = -8/\pi < 0$. The relative long-root cost is therefore negative at the generated IR edge. Varying L describes how this response changes with source width.

Theorem 8.3 (Condensation, scale transport, and scalar mass gap). Use the conformal spectrum $a_j = [(2j+1)^2 - 1/4]/L^2$ with $d_j = (2j+1)^2$ for $j \geq 0$ and its feedback $a_j \mapsto a_j + gE^2$ with $g > 0$ together with the generated source drive

$$\mu_L = 16(L^{-2} - L_c^{-2}) = \frac{8}{3}(\mathcal{R} - \mathcal{R}_c), \quad \mathcal{R} = 6/L^2, \quad \mathcal{R}_c = 6/L_c^2. \quad (8.14)$$

Here $L_c = \sqrt{\pi}$ is fixed by Proposition 8.3. With $y = E^2$ the cutoff negative logarithm normalized by $F_{N,L}(0) = 0$ and $2F'_{N,L}(0) = \mu_L$ has the unique constant-and-linear completion

$$F_{N,L}(y) = - \sum_{j=0}^N d_j \log(1 + gy/a_j) + \left(\frac{\mu_L}{2} + g \sum_{j=0}^N \frac{d_j}{a_j} \right) y. \quad (8.15)$$

It converges in C^2 on compact y intervals to $F_L = F_{\mu_L}$ of (8.9) with $t_j = g/a_j$ and $\beta_j = d_j$. This response is coercive and has a unique positive minimizing amplitude square exactly when $L > L_c$. For $L \leq L_c$ its minimizer is $y = 0$. At $L > L_c$ its radial curvature is strictly positive. With any positive kinetic normalization this curvature gives a strictly positive bottom for the assigned complete scalar mass operator. The exact scale-composition law transports the stationary branch which has a uniformly positive spectral bottom on every compact path contained in $L > L_c$ with positive bounded feedback and kinetic coefficients.

Proof. The two normalization conditions fix the constant and linear terms of the cutoff expression. Rearrangement gives $\mu_L y/2 + I_N(y)$. Proposition E.3 proves C^2 convergence with $Q_2 < \infty$ and $S = \infty$. The preceding stability theorem gives the stated phase criterion. At its positive root $\frac{d^2}{dE^2} F_L(E^2) = 4y_* F''_L(y_*) > 0$. $\square$

In particular the canonical source radius L_{src} has a unique positive equilibrium. Its location on the condensed branch follows from the source endpoints and their calibration. The minimization determines the intensity at that radius. The same theorem describes the phase boundary of the variable-width family. In the symmetric phase $V(E) = F_L(E^2)$ has $V''(0) = \mu_L > 0$ for $L < L_c$ with a vanishing quadratic coefficient at $L = L_c$.

Put $\kappa = gL^2$ and $z = \kappa y$. The stationary equation of the dimensional potential is exactly

$$\mathcal{R}(z) = -\frac{\mu_L}{2\kappa}. \quad (8.16)$$

Thus $L > L_c$ makes the right-hand side positive and Proposition 8.3 supplies its unique positive solution. The factors L, g, Z_L carry the generated dimensionless response into the chosen amplitude and mass units.

The long-root interpretation uses the relative spectral coefficient $\varepsilon_{\text{lr}}(L) - \varepsilon_{\text{lr}}(L_c)$ with $\varepsilon_{\text{lr}}(L) = 16/L^2$. Equation (A.16) proves its quadratic comparison in common mode units. Proposition 8.3 fixes its normalized difference as μ_L . The critical radius is the zero of this relative drive. For $L > L_c$ the long-root quadratic cost lies below its critical value, favouring a nonzero collective amplitude. The conformal operator remains positive, and the precision of its fluctuations increases with gE^2 . This increase supplies the restoring response that balances the softened collective direction and selects a finite condensed amplitude.

The unbounded limiting restoring slope has a simple spectral origin. The potential I and its derivatives I' and I'' are finite at every finite amplitude. Progressively more shells respond as the amplitude grows. For this unweighted three-dimensional tower the exact conformal sum gives

$$I'(y) \sim \frac{\pi}{4}\kappa^{3/2}\sqrt{y}, \quad I(y) \sim \frac{\pi}{6}\kappa^{3/2}y^{3/2}, \quad \kappa = gL^2.$$

Thus the large-amplitude restoring term grows as $|E|^3$ and dominates the negative quadratic drive. This explains both the finite equilibrium and the departure from a quartic Landau approximation at large amplitude. The exact expression and its asymptotic expansion are given in Appendix E.

The response has a realization on a distribution-valued Gaussian field. Proposition G.8 proves $I(y) = 2D_{\text{KL}}(\nu_0\|\nu_y)$ for $\nu_y = N(0, (A + gyI)^{-1})$ and constructs its reference-subtracted intensity W_A . Consequently the same potential has the exact restoring response

$$F'_L(y) = \frac{\mu_L}{2} - g\mathbb{E}_{\nu_y} W_A, \quad F''_L(y) = \frac{g^2}{2} \text{Var}_{\nu_y}(W_A). \quad (8.17)$$

The mean of W_A measures the change in intensity from the reference field law. Its contribution opposes the negative drive while its strictly positive variance supplies the increasing restoring response. Both moments belong to the same scalar field family ν_y and the same reference subtraction. This gives the complete potential a statistical realization at every amplitude. Regulators approximating this unweighted trace give the same F_L when their compressed responses converge in Hilbert–Schmidt norm and retain the prescription. With fixed Gaussian weights $S = \sum_j \beta_j t_j$ is the limiting restoring slope. The condition $S > -\mu/2$ states that the response tower can overcome the imposed negative drive. Thus

the weights determine how much restoring response remains available at large amplitude as quantified by Theorem 8.2.

Condensation and the Landau approximation. For $V(E) = F_L(E^2)$ the uncondensed radial coefficient is $V''(0) = \mu_L$. Increasing L through L_c changes its sign. So a small collective amplitude lowers the potential. For the complete unweighted conformal tower the restoring slope increases without bound. It therefore balances the negative drive at exactly one $y_L > 0$. Coercivity makes this stationary amplitude square the global minimum. This is the condensation mechanism of the full negative-logarithmic response with all conformal modes retained. The positive variance above describes its resistance to further radial displacement. At fixed spectrum $\partial_\mu y_* = -1/[2F''_\mu(y_*)] < 0$. A stronger negative drive increases the stationary amplitude. The Landau potential describes the leading quartic approximation to this balance. Its error is quantified in Proposition E.4.

Exact scale transport of the complete potential. In fixed response units write the unweighted potential at scale s as $F_s(y) = \mu_s y/2 + J(\kappa_s y)$ with $\kappa_s = g_s L_s^2 > 0$ and the same J as in Proposition E.4. For two scales put $a_{21} = \kappa_2/\kappa_1$ and $b_{21} = (\mu_2 - a_{21}\mu_1)/2$. For all $y \geq 0$ the potential obeys

$$F_2(y) = F_1(a_{21}y) + b_{21}y, \quad a_{31} = a_{21}a_{32}, \quad b_{31} = a_{32}b_{21} + b_{32}. \quad (8.18)$$

For C^1 parameters this gives the exact differential identity

$$\partial_s F_s(y) = \frac{\kappa'_s}{\kappa_s} y \partial_y F_s(y) + \frac{1}{2} \left(\mu'_s - \frac{\kappa'_s}{\kappa_s} \mu_s \right) y. \quad (8.19)$$

Proof. Substitute $a_{21}y$ in F_1 . The nonlinear part becomes $J(\kappa_2 y)$ and the displayed linear correction supplies $\mu_2 y/2$. Substituting successively at three scales gives the two composition identities. The proved local convergence of J and its derivatives allows the chain rule which yields the differential formula. $\square$

The identity transports the complete potential along a specified parameter path including its coupling g_s . For the distinct weighted response J_q with $q = \tau/L^2$ a changing weight adds the term $q' \partial_q J_q$. The weighted response and its stationary-branch formula are given in Theorem E.2. The calculation following that theorem gives its stability boundary.

8.5 Condensation and the scalar response spectrum

The minimizing amplitude supplies the background for radial excitations. For a signed coordinate r the minimum at $y = r^2 > 0$ occurs at $r = \pm\sqrt{y}$. The observable algebra determines their physical identification. The collective colour-singlet interpretation preserves colour symmetry.

For an identified radial coordinate with positive kinetic normalization Z_* the stable amplitude response U gives the radial mass

$$m_{\text{curv}}^2 = \frac{4y_* U''(y_*)}{Z_*}. \quad (8.20)$$

This is the second derivative of $U(r^2)$ at its stationary point divided by the kinetic coefficient of the same coordinate.

Scalar gap for the assigned mass operator. For $g > 0$ and $L > L_c$ let $y_L > 0$ be the unique stationary amplitude square of F_L . Use the scalar mass assignment (5.28) with $R_b = L$ and a specified kinetic normalization $Z_L > 0$. Unit kinetic normalization corresponds to $Z_L = 1$. Then

$$\begin{aligned} m_\delta^2(L) &= \frac{4y_L F_L''(y_L)}{Z_L}, & \mathcal{M}_{\text{sc}}^2 &= \Delta_0 + m_\delta^2(L)I, \\ \inf \text{Spec } \mathcal{M}_{\text{sc}}^2 &= m_\delta^2(L), & 0 < m_\delta^2(L) &< \frac{-2\mu_L}{Z_L}. \end{aligned} \quad (8.21)$$

The self-adjoint mass-squared operator on $H^2(X_3)$ contains every even scalar shell including the constant mode with multiplicities $(l+1)^2$.

Proof. Theorem 8.3 supplies $y_L > 0$ and $F_L''(y_L) > 0$. The exact correction (E.20) gives the strict upper bound. The operator (5.29) and its complete spectrum (5.28) then give the domain and multiplicities. (5.30) gives the sharp lower bound attained by the constant mode. $\square$

At the stationary amplitude the feedback Gaussian law ν_{y_L} supplies the restoring moment in (8.17). Conversion from intensity $y = E^2$ to the radial coordinate and division by its kinetic coefficient express the mass as

$$m_\delta^2(L) = \frac{2g^2 y_L}{Z_L} \text{Var}_{\nu_{y_L}}(W_A) > 0. \quad (8.22)$$

The restoring curvature is thus fixed by an intensity fluctuation of the same field whose precision responds to the condensate. Kinetic normalization converts it into the radial mass and the mass assignment in (5.28) carries this response through the complete scalar tower. For each L, g, L_c, Z_L in the stated condensed domain the unique equilibrium determines this shift. The constant scalar mode receives the radial mass and attains the spectral bottom. Every higher even shell adds its positive internal eigenvalue. The full operator therefore has a gap including its zero harmonic. In unit kinetic normalization the quartic Landau value $-2\mu_L$ is a strict upper bound on the complete response's mass squared. The gap of $\mathcal{M}_{\text{sc}}^2$ describes the collective colour-singlet excitation in the framework's Yang–Mills channel. Its visibility in a colour observable is determined by that observable's coupling to the mass mode.

8.6 Scale dependence of the scalar spectrum

The same potential can be evaluated along the self-similar radius family of Proposition 8.3. In fixed reference units $L_k = L_0 k_0/k$ crosses the critical radius at $k_c = k_0 L_0/L_c$. The condensed branch occupies $k < k_c$. The uniform estimates below concern compact intervals within this branch.

Scale transport and a uniform scalar gap. In fixed response units let $I = [k_-, k_0]$ with $0 < k_- < k_0$ and use $L_k = L_0 k_0/k$ with $L_0 > L_c$. Let g_k, Z_k be specified positive C^1 functions on I retaining the specified spectral feedback and amplitude normalization. Put

$$\mu_k = 16(L_k^{-2} - L_c^{-2}), \quad \kappa_k = g_k L_k^2, \quad \nu_k = -\frac{\mu_k}{2\kappa_k}.$$

The stationary amplitude square and each scalar mass eigenvalue are uniquely determined C^1 functions along this parameter path. With $\mathcal{H} = J'$ as in Proposition E.4 it is

$$\begin{aligned} z_k &= \mathcal{H}^{-1}(\nu_k), & y_k &= z_k/\kappa_k, \\ \frac{dy_k}{dk} &= \frac{\nu'_k}{\kappa_k \mathcal{H}'(z_k)} - \frac{\kappa'_k}{\kappa_k} y_k, & m_{l,k}^2 &= \frac{l(l+2)}{L_k^2} + \frac{4\kappa_k z_k \mathcal{H}'(z_k)}{Z_k}, \quad l = 0, 2, \dots \end{aligned} \quad (8.23)$$

Set $\kappa_- = \min_I \kappa_k$ and $Z_+ = \max_I Z_k$. Define $z_- = \mathcal{H}^{-1}(\min_I \nu_k)$ and $z_+ = \mathcal{H}^{-1}(\max_I \nu_k)$. Then

$$\inf_{k \in I} \inf \text{Spec } \mathcal{M}_{\text{sc},k}^2 \geq \frac{4\kappa_- z_- \mathcal{H}'(z_+)}{Z_+} > 0. \quad (8.24)$$

Proof. On I one has $L_k \geq L_0 > L_c$. So $\mu_k < 0$. Theorem 8.3 and the smooth strictly increasing function $\mathcal{H}$ give the unique branch above. The inverse-function theorem and differentiation give its derivative. The whole potential obeys Proposition 8.3. So restricting to successive scale intervals gives the same branch and composes exactly. Homothety identifies the scalar modes on a common carrier. Corollary 8.3 gives their masses. Compactness gives $\kappa_-, z_- > 0$ and finite z_+, Z_+ while $\mathcal{H}'(z) = \sum_j d_j/(c_j + z)^2$ is positive and decreasing. These facts prove the uniform bound including the constant scalar mode. $\square$

The same condensed branch is transported on a common internal mode space with continuously varying amplitudes and masses. The bound controls its entire specified scale interval through the positive distance from $L = L_c$ and the bounds on g_k and Z_k . It therefore gives stability along the parameter path as well as at its endpoint. For computation truncate only the unweighted potential retaining its original drive and normalization with the constant conformal mode. Positive lower and finite upper bounds for ν_k and κ_k and Z_k on the entire interval give a common root bracket from finite low-mode sums. A single sufficiently large cutoff then preserves condensation and controls the errors in the amplitude square y_k and scalar mass squared uniformly in k . Both mass operators retain the complete scalar tower. These estimates concern exact truncated roots. Finite-sum evaluation and numerical root-solving errors must also be included. The bracket and error constants with their proof are given in Corollary E.3.

8.7 Scalar realization and colour-singlet interpretation

A scalar zero-mode realization retains a coordinate of the fluctuating field itself. Its direct self-coupling and the response of the remaining modes are both part of its stability problem. On $L^2(X_3, dv)$ write $\Phi = E_0/\sqrt{V} + \eta$ with $\int \eta = 0$. Here $V = \pi^2 L^3$ and $E_{\text{int}} = \sqrt{V} E_0$. Retaining this dependence in the scalar coupling gives $F_g = \gamma E_0^4/2 + \gamma E_0^2 \|\eta\|^2/2$ with $\gamma = gV$. Proposition G.5 proves the full Hessian. Its retained entry is $6\gamma E_0^2$ while the eliminated $l = 2, 4, \dots$ tower has the uniform shift $\gamma E_0^2 I$. The kernel and Schur reduction including mixed fluctuations are (G.50) and (G.51).

For the negative-logarithmic response with fixed weights and $S_\perp = \sum_\perp \beta_j \gamma / a_j < \infty$ this realization has

$$W(y) = \frac{a_0}{2} y + \frac{\gamma}{2} y^2 - \sum_{j \in \perp} \beta_j \log(1 + y t_j), \quad y = E_0^2, \quad t_j = \gamma / a_j. \quad (8.25)$$

Proposition G.5 gives its unique positive minimizer exactly when $2S_\perp > a_0$. There

$$m_{\text{curv}}^2 = \frac{4y_*}{Z_*} \left[\gamma + \sum_{j \in \perp} \frac{\beta_j t_j^2}{(1 + y_* t_j)^2} \right] \geq \frac{4\gamma y_*}{Z_*} > 0, \quad (8.26)$$

$$\inf \text{Spec } A_\perp(\sqrt{y_*}) = \frac{35}{4L^2} + \gamma y_*. \quad (8.27)$$

The second equality uses the first allowed harmonic $l = 2$. The two lines describe the radial response and fixed-amplitude tower separately. The extra direct term and omission of the constant mode together with $\mu_{\text{ret}} = a_0 - 2S_\perp$ distinguish W from F_L . Equation (G.43) gives the condition for agreement of their quadratic and quartic coefficients. Amplitude and scale normalization are fixed in (G.45) and (G.46).

8.8 Generated Yang–Mills response and mass gap

The scalar analysis has determined the radial restoring block. A Yang–Mills response also contains variations of the connection at fixed intensity. On the physical coclosed one-form sector these are measured by the exact non-Abelian curvature functional. The radial intensity and the coclosed probe are orthogonal at the isotropic background. Their Hessians therefore combine into one mass operator with an explicitly computable spectral bottom.

The colour connection already supplies its own invariant collective coordinate. For the complete Gaussian long-root family set

$$q = \mathbb{E} \|B\|_{L^2}^2, \quad E_{\text{YM}} = \sqrt{q}. \quad (8.28)$$

The addition formula makes this intensity independent of the chosen mode basis. On a nontrivial irreducible response block ρ the same source changes the mean comparison operator by

$$\mathbb{E}(\nabla_B^* \nabla_B) = \Delta_0 + g_\rho q I, \quad g_\rho = \frac{C_2(\rho)}{Vh} > 0, \quad (8.29)$$

by Proposition G.5. Thus the uniform spectral feedback used above is the quadratic intensity response of the generated connection. Its Gaussian comparison law is fixed by the averaged log likelihood in (G.32). Comparing that law with the zero-intensity law gives precisely the complete restoring functional (G.33). The adjoint block gives $g_{\text{YM}} = C_{\text{ad}}/(Vh)$. For the generated $SU(W)$ frame take $-2 \text{Tr}_W(T_A T_B) = \delta_{AB}$. Then

$$C_{\text{ad}} = 3, \quad h = 8, \quad V = \text{Vol}(\mathbb{RP}_L^3) = \pi^2 L^3, \quad g_{\text{YM}} = \frac{3}{8\pi^2 L^3}. \quad (8.30)$$

The long-root eigenvalue $\lambda_R(L) = 16/L^2$ also fixes its relative drive

$$\mu_L = \lambda_R(L) - \lambda_R(L_c) = 16(L^{-2} - L_c^{-2}). \quad (8.31)$$

Consequently the amplitude and feedback coefficient with the reference slope in the Yang–Mills response all come from the same connection source and its generated spectrum.

Let F_L^{YM} denote the complete potential of Theorem 8.3 with $y = q$ and $g = g_{\text{YM}}$. The curvature response to an additive connection probe is the exact functional

$$\mathcal{R}_q(a) = \frac{1}{2} \|da + a \wedge a\|^2 + \frac{\delta_q}{2} \|a\|^2, \quad \delta_q = \frac{2qC_{\text{ad}}}{3Vh}, \quad (8.32)$$

proved in Proposition G.7. The generated Yang–Mills response is therefore

$$\Gamma_L^{\text{YM}}(q, a) = F_L^{\text{YM}}(q) + \mathcal{R}_q(a), \quad q \geq 0, \quad a \in H(d) \cap L^4. \quad (8.33)$$

It retains the noncommutative term $a \wedge a$ rather than replacing the connection response by its quadratic approximation.

This expression uses the source frame. Its local-frame meaning is fixed by transporting the connection and its source law together with the probe. For a presentation $U : X_3 \rightarrow SU(3)$ put $A_0 = -(dU)U^{-1}$ and $B_U = UBU^{-1}$ with $a_U = UaU^{-1}$ and push the Gaussian law forward by $B \mapsto B_U$. Then $F_{A_0+B_U+a_U} = UF_{B+a}U^{-1}$. So the translated expectation of the curvature norm is exactly (8.32). The generated coexact space is transported by Ad_U together with its Hodge projector and quadratic-form domain. The common-frame quotient consequently has the source-frame representative $a \in \text{ran } P_\perp$. The calculation in (G.74) proves this at the level of the full response and the source inner product. A pure frame change varies A_0 and the source presentation jointly. The fixed-frame probe in (8.32) changes the relative connection.

The source law also fixes the radial metric. With $M = d_R h = 15 \cdot 8 = 120$ the intensity Legendre transform of Proposition G.6 has for $e = \sqrt{q}$

$$Z_{\text{YM}}(q) := \mathcal{G}_{ee}^{\text{src}}(q) = \frac{2M}{q}. \quad (8.34)$$

This is the information metric of the same finite Gaussian connection family. In the intensity coordinate it is $\mathcal{G}_{qq}^{\text{src}} = M/(2q^2)$ as follows directly from the score $\partial_q \log \gamma_q = M(\mathcal{N} - q)/(2q^2)$. At an equilibrium q_L set

$$\sigma_L^2 = \frac{2q_L^2}{M}, \quad q = q_L + \sigma_L r, \quad V_L(r) = F_L^{\text{YM}}(q_L + \sigma_L r) - F_L^{\text{YM}}(q_L). \quad (8.35)$$

The coordinate r has unit source information metric at the equilibrium. The complete restoring force is $V_L'(r) = \sigma_L (F_L^{\text{YM}})'(q_L + \sigma_L r)$ on the interval where $q_L + \sigma_L r > 0$. Therefore $V_L''(0) = \sigma_L^2 (F_L^{\text{YM}})''(q_L)$. This derives the normalized radial restoring operator from the complete potential. Its linear equilibrium response obeys $\partial_\tau r = -V_L''(0)r$ in the composition constructed below. This fixes the normalization of the response parameter τ . Its drift is minus the normalized restoring force. Physical clock units enter through the time readout of Section 6. At a stationary point the ratio of the potential Hessian to the information metric is invariant under a change of amplitude coordinate. Thus the coordinates q and $e = \sqrt{q}$ give the same radial restoring coefficient. The restoring law fixes the response rate. The information metric fixes how that rate is read from an intensity variation.

Theorem 8.4 (Generated Yang–Mills response Hessian). Let $L > L_c$ and use the generated colour source feedback (8.29) and drive (8.31). Then Γ_L^{YM} has the unique global minimum

$(q_L, 0)$ with $q_L > 0$. On the common-frame quotient of the generated coexact response class its quadratic response acts on $\mathbb{R}_{\text{rad}} \oplus L_{\text{coex}}^2(X_3, \mathfrak{su}(3))$ with form domain $\mathbb{R} \oplus H_{\text{coex}}^1$ as

$$\mathcal{M}_{\text{YM}}^2 = m_{\text{YM,rad}}^2 \oplus (\Delta_1^\perp + \delta_{q_L} I), \quad m_{\text{YM,rad}}^2 = \frac{4q_L (F_L^{\text{YM}})''(q_L)}{Z_{\text{YM}}(q_L)} = \frac{2q_L^2}{M} (F_L^{\text{YM}})''(q_L) > 0. \quad (8.36)$$

It is self-adjoint on $\mathbb{R} \oplus (H^2 \cap L_{\text{coex}}^2)$ and has the strict spectral bottom

$$\Delta_{\text{YM}}^2 := \inf \text{Spec } \mathcal{M}_{\text{YM}}^2 = \min \left\{ m_{\text{YM,rad}}^2, \frac{4}{L^2} + \delta_{q_L} \right\} > 0. \quad (8.37)$$

The source-mean curvature observation $\mathcal{O}(q) = \mathbb{E}_q \mathcal{Q}$ with $\mathcal{Q} = \|F_B\|^2$ has the nonzero normalized radial differential $\sigma_L \mathcal{O}'(q_L)$. Its linear susceptibility therefore detects the radial mass which is the spectral bottom when it realizes the smaller entry in (8.37).

Proof. Theorem 8.3 with the source-derived $g_{\text{YM}} > 0$ and $\mu_L < 0$ gives the unique $q_L > 0$ and the positive curvature $(F_L^{\text{YM}})''(q_L)$. Equation (8.34) is positive there and gives the first block of (8.36). Since $\mathcal{R}_q(a) \geq \delta_q \|a\|^2/2$ for $q > 0$ its unique minimum is $a = 0$. Hence $(q_L, 0)$ is the unique global minimum.

For every q Proposition G.7 gives $D_a \mathcal{R}_q(0) = 0$ and $D_a^2 \mathcal{R}_q(0) = d^* d + \delta_q I$. Differentiating the first identity in q shows that the radial-probe mixed Hessian vanishes at $a = 0$. The common-frame identity (G.74) transports this Hessian and its domain. Its vertical directions have zero joint variation and its source-frame representative is the displayed orthogonal sum. On the generated coexact probe space $d^* d = \Delta_1^\perp$ and its lowest eigenvalue is $4/L^2$. Both blocks are strictly positive proving (8.37).

Finally Proposition G.6 gives

$$\Pi_{\text{rad}}(\mathcal{Q} - \mathbb{E}_q \mathcal{Q}) = \left(\frac{16}{L^2} + 2\alpha_R q \right) (\mathcal{N} - q),$$

whose coefficient is positive at $q = q_L$. Differentiation of the source density gives $\sigma_L \mathcal{O}'(q_L) = \mathbb{E}_{q_L}[(\mathcal{Q} - \mathcal{O}(q_L))s_L]$ where $s_L = (\mathcal{N} - q_L)/\sigma_L$. This is the radial differential of the mean observation with the normalization already fixed by (8.35). $\square$

At $L = L_{\text{src}}$ the source endpoints give $L_{\text{src}} > L_c$. The adjoint contraction fixes g_{YM} and the source information metric fixes Z_{YM} . The theorem consequently supplies a unique positive connection intensity and an attained positive Yang–Mills spectral bottom at this canonical radius. The statements for general $L > L_c$ describe the corresponding family of condensed responses.

Radial tangent and curvature readout. At the generated minimum q_L let

$$s_L = \sqrt{\frac{M}{2}} \frac{\mathcal{N} - q_L}{q_L}, \quad \phi_0 = \mathbf{1}_{X_3}. \quad (8.38)$$

Both vectors have unit norm. The source uses its probability law and the scalar space uses normalized Haar measure. The score line $\mathbb{C}s_L$ represents the information-normalized intensity tangent. Give $\mathcal{H}_{\text{sc}} = L^2(X_3, d\bar{v})$ the mass operator

$$\mathcal{M}_{\text{sc,L}}^2 = \Delta_0 + m_{\text{YM,rad}}^2 I. \quad (8.39)$$

There is a unique unitary

$$\mathcal{J}_c : \mathbb{C}s_L \longrightarrow \mathbb{C}\phi_0, \quad \mathcal{J}_c s_L = \phi_0, \quad (8.40)$$

and it obeys

$$\mathcal{J}_c \left(e^{-\tau m_{Y\text{M},\text{rad}}^2} I_{\mathbb{C}s_L} \right) = e^{-\tau \mathcal{M}_{\text{sc},L}^2} \Big|_{\mathbb{C}\phi_0} \mathcal{J}_c, \quad \tau \geq 0. \quad (8.41)$$

Moreover

$$\Pi_{\text{rad}}(\mathcal{Q} - \mathbb{E}_{q_L} \mathcal{Q}) = (\lambda_R + 2\alpha_R q_L) q_L \sqrt{\frac{2}{M}} s_L, \quad (8.42)$$

This coefficient has a direct response meaning. For any source polynomial A differentiation of the Gaussian density gives

$$\frac{d}{dr} \omega_{q_L + \sigma_L r}(A) \Big|_{r=0} = \omega_{q_L}[(A - \omega_{q_L}(A))s_L]. \quad (8.43)$$

For $A = \mathcal{Q}$ this is $b_{\mathcal{Q}} = \sigma_L(\lambda_R + 2\alpha_R q_L) > 0$. The vector $b_{\mathcal{Q}}\phi_0$ is its radial linear readout with susceptibility $b_{\mathcal{Q}}^2/(m_{Y\text{M},\text{rad}}^2 - z)$. This coefficient also follows by inserting the original curvature polynomial as a source and allowing the mean intensity to relax. The joint-source continuation (G.63) gives the total static response

$$\chi_{\mathcal{Q}}^{\text{stat}} = V_{\perp}(q_L) + \frac{b_{\mathcal{Q}}^2}{m_{Y\text{M},\text{rad}}^2}, \quad V_{\perp}(q_L) = \text{Var}_{q_L} \mathcal{Q} - b_{\mathcal{Q}}^2 > 0. \quad (8.44)$$

The direct term measures fluctuations orthogonal to the intensity score. The second term measures the additional change of the mean curvature as the intensity moves along its stable branch. Equation (G.65) proves both terms from the original joint source.

Proof. The Gaussian intensity satisfies $\text{Var}_{q_L}(\mathcal{N}) = 2q_L^2/M$ which gives $\|s_L\| = 1$. The normalized constant function has unit norm and $\Delta_0\phi_0 = 0$. The radial block of (8.36) and the constant scalar block of (8.39) therefore have the same eigenvalue $m_{Y\text{M},\text{rad}}^2$ proving the intertwining identity. Equation (8.42) is (G.53) written in the normalized score coordinate. Every factor in its coefficient is positive. $\square$

Let $\mathcal{H}_{Y\text{M}}^{(1)}$ be the one-response space in the preceding theorem complexified after taking its real source form. The restoring operator determines $C_{\tau} = \exp(-\tau \mathcal{M}_{Y\text{M}}^2)$. Its equilibrium Gaussian composition has the following explicit realization. Choose an orthonormal eigenbasis (e_{α}) with eigenvalues $\varepsilon_{\alpha} > 0$ and let γ be the product of standard real Gaussian laws in coordinates x_{α} . For a polynomial depending on finitely many coordinates define

$$(\mathsf{T}_{\tau}P)(x) = \int P\left(C_{\tau}x + (I - C_{\tau}^2)^{1/2}y\right) d\gamma(y). \quad (8.45)$$

This update follows from independent Gaussian composition at equilibrium. The conditional mean is the linear restoring map $C_{\tau}x$ and preservation of unit covariance fixes the independent increment to have covariance $I - C_{\tau}^2$. Gaussian addition proves its composition law. The symmetric joint covariance of x and its update proves reversibility. Jensen's inequality proves contraction on $L^2(\gamma)$.

For a finite multi-index ν the normalized Hermite polynomial h_ν is multiplied by $\exp(-\tau \sum_\alpha \nu_\alpha \varepsilon_\alpha)$. Its derivation uses the Gaussian generating function in (G.55) and includes every contraction in an ordinary product. The Hermite unitary $\mathcal{U}$ therefore gives

$$\mathcal{F}_{\text{YM}} = \Gamma_s(\mathcal{H}_{\text{YM}}^{(1)}), \quad K_{\text{YM}} = d\Gamma(\mathcal{M}_{\text{YM}}^2), \quad \Omega_{\text{YM}} \in \Gamma_s^0(\mathcal{H}_{\text{YM}}^{(1)}). \quad (8.46)$$

On the polynomial domain with $\pi(P) = \mathcal{U}M_P\mathcal{U}^{-1}$

$$\begin{aligned} \langle \mathcal{U}P, \mathcal{U}Q \rangle &= \omega_{\text{resp}}(P^*Q), & \omega_{\text{resp}}(P) &= \int P d\gamma, \\ \mathcal{U}(PQ) &= \pi(P)\mathcal{U}Q, & \mathcal{U}\mathsf{T}_\tau P &= e^{-\tau K_{\text{YM}}}\mathcal{U}P. \end{aligned} \quad (8.47)$$

Thus the additive generator is obtained from the independent equilibrium composition of the already derived restoring operator. This is its Gaussian linear-response completion. Its closed quadratic form gives the same construction directly. On cylinder polynomials it is

$$\mathcal{E}_{\text{resp}}(P, Q) = \sum_\alpha \varepsilon_\alpha \int \overline{\partial_\alpha P} \partial_\alpha Q d\gamma, \quad K_{\text{resp}}P = \sum_\alpha \varepsilon_\alpha (x_\alpha \partial_\alpha P - \partial_\alpha^2 P). \quad (8.48)$$

Gaussian integration by parts gives the displayed generator. The Hermite basis makes the form closable and identifies its closure with the form of $d\Gamma(\mathcal{M}_{\text{YM}}^2)$ under $\mathcal{U}$. The second-derivative term retains the contractions of source products. Multiplication is represented by π while T_τ is the reversible averaging operation on that algebra.

The original connection observables have an exact representation in this space. Let I_R be the M real long-root colour coordinates and define

$$(\mathcal{I}_q A)(x) = A(\sqrt{q/M} (x_i)_{i \in I_R}), \quad \iota_c A = \mathcal{U} \mathcal{I}_q A. \quad (8.49)$$

Its joint law is precisely γ_q . Hence for all original source polynomials

$$\begin{aligned} \langle \iota_c A, \iota_c B \rangle &= \omega_q(A^* B), \\ \iota_c (AB) &= \pi(\mathcal{I}_q A) \iota_c B, & \iota_c \mathsf{T}_\tau^{\text{coeff}} A &= e^{-\tau K_{\text{YM}}} \iota_c A. \end{aligned} \quad (8.50)$$

Here $q = q_L$ and $\omega_R = \lambda_R + \delta_{q_L}$ in (G.75). Every long-root coordinate has this restoring eigenvalue. So the last identity follows on each Hermite polynomial and extends by contraction. In particular the original s_L has second-sector rate $2\omega_R$ in this embedding. The quadratic forms are intertwined as well. For every original source polynomial A the chain rule gives

$$\mathcal{E}_{\text{resp}}(\mathcal{I}_q A, \mathcal{I}_q A) = \frac{q\omega_R}{M} \sum_{i=1}^M \omega_q(|\partial_{c_i} A|^2). \quad (8.51)$$

Thus the source inner product and multiplication and restoring form all use the same Gaussian coefficient law. Equations (G.57) and (G.58) retain its non-Gaussian moments and the full angular part of $\mathcal{Q}$. More explicitly the centred curvature splits into Hermite degrees two through four. Its fixed-law correlation is

$$\omega_{q_L}[(\mathcal{Q} - \mathcal{O}(q_L))\mathsf{T}_\tau^{\text{coeff}}(\mathcal{Q} - \mathcal{O}(q_L))] = \sum_{n=2}^4 W_{\mathcal{Q},n}(q_L) e^{-n\omega_R \tau}. \quad (8.52)$$

The explicit nonnegative weights in (G.60) have $W_{\mathcal{Q},2} = b_{\mathcal{Q}}^2 > 0$ and $W_{\mathcal{Q},4} > 0$. Thus the polynomial representation controls the complete original observation and determines its first supported restoration rate $2\omega_R$. The original coefficient resolution is represented between source laws by (G.70). The equilibrium update above acts within the law at q_L .

The collective mean observation uses the differential (8.43). Its radial linear readout is $\mathcal{Q}_{\text{lin}} = b_{\mathcal{Q}}x_{\text{rad}}$ with $\mathcal{U}\mathcal{Q}_{\text{lin}} = b_{\mathcal{Q}}e_{\text{rad}}$ and restoring eigenvalue $m_{\text{YM,rad}}^2$. The score pairing computes this differential from the original source moments. $\mathcal{J}_c$ expresses the resulting tangent response on the constant scalar harmonic. Source multiplication itself is transported by $\mathcal{I}_q$ and π . These formulas give separately the complete fixed-source observation and its response to a change of the mean intensity. The former follows fluctuations in one connection ensemble. The latter follows the ensemble itself under a curvature source and has the direct and collective terms in (8.44).

Let $\mathfrak{A}_{\text{YM}}^{\text{gen}}$ be the unital $*$ -algebra of finite-mode colour-invariant response polynomials generated by

$$\text{Tr}(D_A^{r_1}F_A \cdots D_A^{r_k}F_A), \quad k \geq 2, \quad r_i \geq 0, \quad (8.53)$$

their adjoints and source-mean linear readouts together with mode products and T_τ images. The finite-mode domain retains the full polynomial expression of each observation. Complete $\mathcal{U}\mathfrak{A}_{\text{YM}}^{\text{gen}}$ in the response norm and denote the resulting gauge-invariant cyclic space by $\mathcal{H}_{\text{YM}}$. The GNS quotient map

$$\iota_{\text{resp}} : \mathfrak{A}_{\text{YM}}^{\text{gen}} / \{X : \omega_{\text{resp}}(X^*X) = 0\} \longrightarrow \mathcal{H}_{\text{YM}}, \quad [X] \longmapsto \mathcal{U}X, \quad (8.54)$$

is isometric and has dense range by (8.47). The common-frame transport carries the source state and polynomial products with the restoring operator unitarily. Thus T_τ preserves colour invariance. Its inclusion among the generating response operations makes $\mathcal{H}_{\text{YM}}$ invariant under $e^{-\tau K_{\text{YM}}}$.

The positive lower edge has two contributions. The restoring potential controls the radial response and the coclosed connection spectrum with its source feedback controls relative connection variations. A common frame change leaves both responses unchanged and gives a zero direction before the quotient. On the physical quotient every nonzero variation has a positive quadratic energy cost.

Theorem 8.5 (Generated Yang–Mills mass gap). Under the hypotheses of Theorem 8.4 the full response completion has vacuum Ω_{YM} and exact first positive mass-squared $\Delta_{\text{YM}}^2 = \min\{m_{\text{YM,rad}}^2, 4/L^2 + \delta_{q_L}\} > 0$. The equilibrium response semigroup leaves $\mathcal{H}_{\text{YM}}$ invariant and obeys

$$\iota_{\text{resp}} \mathsf{T}_\tau = e^{-\tau K_{\text{YM}}} \iota_{\text{resp}}. \quad (8.55)$$

Its invariant subspace at eigenvalue zero is $\mathbb{C}\Omega_{\text{YM}}$. On $\mathcal{H}_{\text{YM}}^0 = \mathcal{H}_{\text{YM}} \ominus \mathbb{C}\Omega_{\text{YM}}$

$$\text{Spec } K_{\text{YM}} \subset [\Delta_{\text{YM}}^2, \infty), \quad \Delta_{\text{YM}}^2 = \min \left\{ m_{\text{YM,rad}}^2, \frac{4}{L^2} + \delta_{q_L} \right\} > 0, \quad (8.56)$$

and every centred generated invariant response satisfies

$$\|X\Omega_{\text{YM}}\|^2 \leq \frac{1}{\Delta_{\text{YM}}^2} \langle X\Omega_{\text{YM}}, K_{\text{YM}}X\Omega_{\text{YM}} \rangle. \quad (8.57)$$

The original centred curvature observation is represented by $\iota_c(\mathcal{Q} - \omega_{q_L}(\mathcal{Q}))$ and has spectral support above the same lower bound. Its source-mean linear readout $\mathcal{Q}_{\text{lin}}$ has radial weight $b_{\mathcal{Q}}^2 > 0$. In particular the invariant response gap is exactly $m_{\text{YM},\text{rad}}^2$ whenever the radial block is the lower block. In all cases it is bounded below by Δ_{YM}^2 .

The lower of the radial and coclosed branches sets the first excitation of the full response completion. Common frame invariance identifies equivalent presentations. The positive restoring coefficients determine the excitation energy. The gap is intrinsic to the generated operator and is expressed in internal units L^{-2} . Its conversion to physical mass units uses the mass unit M_G introduced in the metrological bridge of Section 9.4. The spectral statement itself is independent of that choice of units.

Proof. The identities for the state and multiplication and for the update are (8.47) and (8.50). They establish the original-source embedding before the cyclic completion is taken. The Hermite calculation diagonalizes the equilibrium update and gives $K_{\text{YM}} = d\Gamma(\mathcal{M}_{\text{YM}}^2)$ on its finite-particle core. Writing $E_\nu = \sum_\alpha \nu_\alpha \varepsilon_\alpha$ its closure has domain

$$\left\{ \sum_\nu z_\nu |\nu\rangle : \sum_\nu E_\nu^2 |z_\nu|^2 < \infty \right\}.$$

The real diagonal eigenvalues make this closure self-adjoint and nonnegative. Invariance and contractivity give (8.55) on $\mathcal{H}_{\text{YM}}$.

Theorem 8.4 gives $\mathcal{M}_{\text{YM}}^2 \geq \Delta_{\text{YM}}^2 I$. On every sector with $n \geq 1$ additivity gives $d\Gamma(\mathcal{M}_{\text{YM}}^2) \geq n\Delta_{\text{YM}}^2 I$. The one-response sector contains the radial eigenvector and the lowest coclosed one-form eigenspace. So its first positive mass-squared is exactly Δ_{YM}^2 . Only the zero-response sector has eigenvalue zero and it is spanned by Ω_{YM} . The spectral theorem gives (8.56) and (8.57). Matrix commutators and mode products change the polynomial representing an observation but do not alter this positive grading. So the nonlinear curvature produces no additional zero response. Equation (8.43) gives $\mathcal{U}\mathcal{Q}_{\text{lin}} = b_{\mathcal{Q}} e_{\text{rad}}$ with $b_{\mathcal{Q}} > 0$ proving the radial weight and the equality assertion when the radial block is the lower block. $\square$

The theorem joins the two variations that define the Yang–Mills response. Changing the source intensity moves radially through inequivalent connection laws. Adding a coclosed one-form changes the connection at fixed intensity. The first variation is stabilized by the complete spectral tower and the second by the exact non-Abelian curvature response. Their orthogonality at the isotropic background makes the lower edge the smaller of the two restoring scales. Original curvature polynomials keep their full Hermite grading under the source embedding while source-mean derivatives give the collective linear readouts. Every nonconstant component evolves in a positive response sector. The generated colour response therefore has a unique invariant background and the strict Yang–Mills gap (8.37).

9 Spectral Observables and Physical Normalization

The stable colour intensity has a radial restoring response and a transverse connection response. Their direct sum is the generated Yang–Mills mass operator. A physical readout

retains its source metric, operator domain and scale normalization. Table 2 relates the generated quantities to their normalized observations. The field representations below specify the corresponding amplitudes, overlap weights and units.

The internal scalar curvature $6/L^2$ describes the round response geometry whereas the gauge curvature $F_A = dA + A \wedge A$ describes frame comparison and non-Abelian coupling. The radial curvature of the collective potential is its second variation with respect to the amplitude. Kinetic normalization converts it to the radial mass squared. Macroscopic spacetime curvature describes the geometry of a further long-range-response representation.

Table 2. Internal spectral quantities and their physical normalization.

Generated quantity	Physical meaning and normalization
X_3 with its modes and products	Internal response carrier and allowed couplings on the compact internal space.
$q = \mathbb{E}\ B\ ^2$ and $E_{\text{YM}} = \sqrt{q}$	Basis-independent colour intensity and its radial amplitude.
F_L^{YM} and q_L	Complete relative response and its unique positive stationary connection intensity for $L > L_c$.
$\mathcal{M}_{\text{YM}}^2$	Direct sum of the source-normalized radial response and the coclosed connection response. Its bottom is $\Delta_{\text{YM}}^2 > 0$.
$\mathcal{M}_{\text{full}} \rightarrow \mathcal{M}_{\text{ch}}$	Full and charged carriers. Descending operators act on the charged quotient while the full trace retains the neutral contribution.
$\text{Tr } F_c^2$	Lowest-degree colour-singlet curvature readout. Its source-mean radial differential is nonzero at the condensed intensity.

9.1 Normalization and operator domains

The scalar operator and its condensed mass are fixed in Sections 5.4 and 8.5. For a potential $U(y)$ with $y = r^2$ its normalized radial mass at $U'(y_*) = 0$ with $y_* > 0$ and $Z_* > 0$ is

$$C_* = 4y_*U''(y_*), \quad m_{\text{curv}}^2 = C_*/Z_*. \quad (9.1)$$

For the retained-coordinate potential W this includes the direct term in (8.26). Its coefficient comparison with F_μ is (G.43). The unweighted conformal prescription uses the response of Proposition E.4. The Landau error is bounded in Proposition E.4. An endpoint evaluation must retain the chosen carrier and domain with the weights and subtraction in the same units.

Operator and trace descent use the quotient and trace conditions in Section 5.4. The linear source pairing $x \mapsto j^\dagger x$ and the observable ℓ descend exactly when $j \perp N$ and $\ell|_N = 0$ respectively. For the same generated elementary connector (5.24) and (5.25) give the eliminated source and analytic observable covector with the direct neutral term. Its full-carrier trace is (5.26). When the operator fails to descend the eliminated neutral block contributes to both the charged kernel and its source response.

The exponential regulator satisfies

$$R_k(z) = \frac{z}{e^{z/k^2} - 1}, \quad [A + R_k(A)]^{-1} = C_{k^{-2}}, \quad \frac{\partial_{\log k} R_k(z)}{z + R_k(z)} = \frac{2z/k^2}{e^{z/k^2} - 1}. \quad (9.2)$$

The scalar functions have the continuous zero limits proved in Appendix G.9. The conformal operator used here itself has lower bound $3/(4L^2) > 0$. The covariance is a function of k^{-2} . The last expression is the scale derivative $\partial_{\log k} \log[z + R_k(z)]$ entering the regulated trace. For derivatives along an endpoint path transport the operator on a common carrier and distinguish the radius $L_{\text{phys}}(k)$ from the dimensionless capacity $k_{\text{res}} L_{\text{phys}}(k)$. Constant capacity permits a varying radius. External frequency integrals and signed field supertraces together with channel weights and kinetic normalizations determine the corresponding physical quantities.

9.2 Collective amplitudes and composite observables

The feedback coordinate E , scalar zero coordinate E_0 and curvature observable $\mathcal{O}_c$ describe different aspects of the response. In the Yang–Mills specialization, $E = E_{\text{YM}} = \sqrt{q}$ is the rms amplitude of the generated connection. The coordinate E_0 projects a scalar realization onto its constant harmonic whereas $\mathcal{O}_c$ reads the non-Abelian curvature. Their common source law determines the conversion and the observable overlap.

The construction fixes dimensionless amplitudes, spectral ratios and normalized response operators. Comparison with conventional mass units requires a metrological calibration. The symbols M_G , g_2 , v and Λ_{QCD} specify the readout conventions used in these examples. The condensate q_L and operator $\mathcal{M}_{\text{YM}}^2$ with its gap are fixed before this choice of units. Keeping the two stages separate allows the same generated response to be compared with different physical calibrations without changing its internal dynamics.

Under $E = a\varphi$, the kinetic, source, quadratic and quartic coefficients transform according to (G.45). A scale-dependent a also contributes its derivative to the amplitude evolution as in (G.46). The same conversion therefore fixes both the normalized fluctuation and its coupling to a source.

The endpoint convention used here illustrates this conversion. Here τ_{end} is the dimensionless endpoint shape parameter. The constant $M_G > 0$ is the mass unit converting its amplitude to E_{gen} and g_2 is the dimensionless coupling in the mass readout. Together with the normalized constant-mode coordinate these quantities obey

$$\begin{aligned} \varphi_0 &= 2\tau_{\text{end}}, & E_{\text{gen}} &= \frac{M_G}{\sqrt{2}}\varphi_0, & m_{\text{gen}} &= g_2 E_{\text{gen}}, \\ E_0 &= V^{-1/2} \int \Phi dv, & E_{\text{int}} &= \sqrt{V} E_0, & \mathcal{O}_c &= \text{Tr } F_c^2. \end{aligned} \tag{9.3}$$

Here $V = \pi^2 L^3$ is the internal volume. So E_0 is the coefficient of its unit-norm constant harmonic. The factors $M_G/\sqrt{2}$ and $\sqrt{V}$ implement the endpoint and zero-mode amplitude conversions. Along a scale path their derivatives enter the same conversion law. In this endpoint convention $\Delta E = M_G/8$ denotes a scale difference. The symbol v denotes the electroweak Higgs expectation and $8\Lambda_{\text{QCD}}$ denotes a glueball mass estimate expressed in terms of the strong-interaction scale Λ_{QCD} . Their physical readouts use the corresponding amplitude and coupling in the stated units.

For the amplitude calibration use the internal curvatures $\mathcal{R} = 6/L^2$ and $\mathcal{R}_c = 6/L_c^2$ and choose a reference $\mathcal{R}_{\text{ref}} < \mathcal{R}_c$ with positive amplitude φ_0 . In fixed response units

compare the two quartic potentials

$$V_\varphi = \frac{\xi}{2}(\mathcal{R} - \mathcal{R}_c)\varphi^2 + \frac{\lambda_{\text{cal}}}{4}\varphi^4, \quad V_K = \frac{K}{2}(\mathcal{R} - \mathcal{R}_c)E_K^2 + \frac{\lambda_{\text{cal}}}{4}E_K^4, \quad (9.4)$$

with $K = 8/3$ and $\xi = 1/8$ together with $\lambda_{\text{cal}} = \xi(\mathcal{R}_c - \mathcal{R}_{\text{ref}})/\varphi_0^2$. This calibration fixes the positive stationary amplitude of V_φ to φ_0 at the reference curvature. The subscript K denotes the convention with quadratic coefficient K . Their positive stationary amplitudes satisfy $E_{K,*} = (8/\sqrt{3})\varphi_*$ and at $\mathcal{R} = \mathcal{R}_{\text{ref}}$

$$E_{K,*}^2 = \frac{64}{3}\varphi_0^2, \quad (E_{\text{gen}}/M_G)^2 = \frac{1}{2}\varphi_0^2, \quad E_{\text{gen}} = \frac{\sqrt{6}}{16}M_GE_{K,*}. \quad (9.5)$$

This last relation is a calibration-point comparison. For $a = 8/\sqrt{3}$ the written potentials satisfy $V_K(E) = a^4 V_\varphi(E/a)$. Applying the same factor a^4 to the kinetic functional gives $Z_K = a^2 Z_\varphi$. Thus this comparison includes both a coordinate conversion and an overall normalization of the functional. The spectral coefficient $\lambda_{\text{spec}} = 2Q_2$ is determined by the expansion of the complete potential at $E = 0$ (Proposition E.4). λ_{cal} is fixed by the reference amplitude above. Matching the calibration to the spectral potential requires expressing both coefficients in the same coordinate and functional normalization.

The colour curvature can be measured in a fixed connection ensemble or used as a source which changes its equilibrium intensity. Both procedures start with the original polynomial $\mathcal{Q} = \|F_B\|^2$. At fixed q its mean is $\mathcal{O}(q) = \lambda_R q + \alpha_R q^2$ and its centred correlation is supported on Hermite rates $2\omega_R$ and $3\omega_R$ and $4\omega_R$ with the weights in (G.60). The first weight is positive. This measures restoration of the full curvature observation within that source law including its angular correlations.

To measure the change of the ensemble insert $-J\mathcal{Q}$ in its log likelihood and let the mean intensity relax in the complete potential. The fixed-intensity Legendre response and the potential combine through (G.63). Their stationary branch gives

$$\chi_{\mathcal{Q}}^{\text{stat}} = V_\perp(q_L) + \frac{b_{\mathcal{Q}}^2}{m_{\text{YM,rad}}^2}, \quad b_{\mathcal{Q}} = q_L \sqrt{2/M} (\lambda_R + 2\alpha_R q_L) > 0.$$

The direct term retains the source fluctuations orthogonal to the intensity score. The second term is the restoring response of the collective intensity to the same curvature insertion. Its coefficient is fixed by the exact projection $\Pi_{\text{rad}}(\mathcal{Q} - \mathcal{O}(q)) = \mathcal{O}'(q)(\mathcal{N} - q)$. Under the linear restoring evolution this collective term has susceptibility $b_{\mathcal{Q}}^2/(m_{\text{YM,rad}}^2 - z)$. The measured curvature thus couples to the radial mode through a change of the connection law itself.

Conditioning on a realized intensity instead gives $P(N) = \lambda_R N + \kappa_R N^2$ with $\kappa_R = M\alpha_R/(M + 2)$. It retains the conditional curvature average used in the quadratic field realization below. The mean response and the conditional observation carry the different slopes computed by these two operations.

Once an observable is represented on the scalar mass space its projection onto a mass eigenspace determines the corresponding spectral weight. A nonzero projection makes that excitation visible. The lowest supported mass gives the observable's gap. In a common

positive spectral representation a nonzero cross weight with the collective scalar at the same mass forces a positive colour-observable mass weight. The source-response and overlap formulas are given in Appendix G. These relations connect the internal Yang–Mills channel to an observable collective excitation through its specified colour-singlet observation.

In the free field representation of Appendix C this spectral selection also determines the long-distance response. For a linear field observation with nonzero constant-mode projection the mass m_δ sets the asymptotic correlation length m_δ^{-1} . The constant mode changes the internal amplitude uniformly. Higher harmonics carry the additional cost of internal variation. The first nonconstant shell has $M_2 = \sqrt{m_\delta^2 + 8/L^2}$. So its contribution when observed decays relative to the constant-mode term at rate $M_2 - m_\delta$ up to polynomial factors. The mass gap governs the overall decay while this separation of masses governs how quickly internal spatial structure is suppressed at large distance.

In the condensed quadratic realization on the generated flat event base the centred internally averaged Wick intensity has an isolated pole at $Q^2 = -m_\delta^2$ with positive weight $4y_L/Z_L$. Its two-particle continuum starts at $s = 4m_\delta^2$ where s is the mass-squared spectral variable. Its linear fluctuation about the nonzero amplitude couples to the constant scalar mode while its quadratic part produces the two-particle continuum. The full measure and subtraction prescription with the internal tail bound are given in Appendix C.5.

In the same quadratic realization the radial polynomial $P(N) = \lambda_R N + \kappa_R N^2$ obtained by averaging the colour observable at fixed intensity is evaluated on the internally integrated squared amplitude. The elementary fields in its expansion are jointly Wick ordered with respect to the same free vacuum [26]. Write $y = y_L$ and $Z = Z_L$ and denote the canonically normalized constant harmonic by φ_{00} . Its small fluctuation changes the integrated intensity to first order by $2\sqrt{y}\varphi_{00}/\sqrt{Z}$. The polynomial transmits this variation with slope $P'(y) = \lambda_R + 2\kappa_R y > 0$ giving the linear coefficient $g_1 = 2\sqrt{y}P'(y)/\sqrt{Z}$. Its square is the positive one-particle weight at m_δ^2 . Condensation thus makes an intensity observable sensitive at first order to the collective amplitude. The higher Wick orders create two- three- and four-particle contributions. Their orthogonality separates these positive contributions from the isolated mass pole. This specified quadratic realization retains the colour response’s dependence on intensity and exhibits how it detects the scalar excitation. Appendix C.6 gives their spectral densities and the convergence proof.

9.3 Homothetic scaling of a tensor spectral density

The tensor channel supplies a scale readout for a fixed modal selection and fixed channel weights. Changing the radius and frequency resolution with their dimensionless ratios held fixed preserves the resolved spectral content. The kernel, frequency measure and internal volume then fix the scaling degree of the specified TT density.

Fix a finite modal selection I with multiplicities d_i and nonnegative channel weights w_i . Fix also constants $L_0, \Lambda_0, b_\omega > 0$. The dimensionless b_ω sets the frequency window relative to Λ_0 . On the same internal carrier let $r = k/k_0 > 0$ and

$$L_r = L_0/r, \quad \Lambda_r = r\Lambda_0, \quad \lambda_{i,r} = r^2\lambda_{i,0}, \quad m_{i,r}^2 = r^2m_{i,0}^2, \quad V_r = \pi^2 L_r^3,$$

where $\lambda_{i,0}, m_{i,0}^2 \geq 0$. Define the specified spectral density and its volume-weighted integral by

$$\begin{aligned}\Pi_r &= \frac{1}{V_r} \sum_{i \in I} d_i w_i \int_0^{\sqrt{(b_\omega \Lambda_r^2 - \lambda_{i,r})_+}} \frac{d\omega}{\pi} \frac{(\omega^2 + \lambda_{i,r})^2}{(\omega^2 + \lambda_{i,r} + m_{i,r}^2)^2} e^{-(\omega^2 + \lambda_{i,r})/\Lambda_r^2}, \\ \mathcal{P}_r &= \frac{V_r \Pi_r}{32\pi^2}, \quad x_+ = \max(x, 0).\end{aligned}\tag{9.6}$$

The isolated point with zero numerator and denominator can be assigned any finite value without changing the integral. Then

$$\Pi_r = r^4 \Pi_1, \quad \mathcal{P}_r = r \mathcal{P}_1.\tag{9.7}$$

If $\mathcal{P}_1 > 0$ a prescribed threshold $p_c > 0$ has exactly one positive crossing $k_* = k_0 p_c / \mathcal{P}_1$. It belongs to a restricted scale interval exactly when that interval contains this value. If $\mathcal{P}_1 = 0$ the observable vanishes at every scale and no positive threshold is crossed. Set $\omega = ru$. The integration limit scales by r . The rational kernel has degree zero in the squared spectral variables and the exponential is unchanged. Each integral therefore scales by r whereas $V_r^{-1} = r^3 V_1^{-1}$. This proves both identities and when the coefficient is positive the unique crossing. Positivity holds whenever one positive-weight mode has an integration interval of positive length. Selections made by fixed harmonic bounds and by λ_i/Λ^2 and ΛL are invariant under this rescaling. So the same proof applies to that window selection.

In the same endpoint convention let $\chi_{\text{end}} := k \ell_{\text{end}}(k)$ denote the fixed dimensionless length–resolution product and let M_P be the reference Planck mass used to express the resolution as $\hat{\Lambda} = k/M_P$. Then $\ell_{\text{end}} = \chi_{\text{end}}/k$ and $\hat{\Lambda} \ell_{\text{end}} = \chi_{\text{end}}/M_P$. This product sets the endpoint length normalization. The correlation parameter ratio remains $c_{\text{end}} = t_{\text{IR}}/t_{\text{UV}} = \pi$ of (3.24). The prescribed finite frequency domain corresponds to $b_\omega = 5$. The length ℓ_{end} fixes the internal spectral scale. For fixed species and spin projections the density degree four consists of one frequency degree and three inverse-volume degrees. For this fixed finite modal selection and homothetic cutoff law the volume-weighted response obeys $\mathcal{P}_r = r \mathcal{P}_1$. It crosses each positive threshold exactly once on the positive scale axis when $\mathcal{P}_1 > 0$. The zero case has no crossing. The kernel fixes this scale dependence. Replacing it by $z/(z + m^2)^2$ gives density degree two.

9.4 Quadratic action and physical masses

In a field-theoretic realization expansion about a stable stationary background expresses the free covariance in the internal eigenmodes. Derivatives of the renormalized effective action give its vertices. The mass spectrum is determined jointly by the kinetic and mass forms on the physical fluctuation space as follows.

After gauge constraints have been resolved suppose a finite physical fluctuation multiplet has a translation-invariant Euclidean quadratic action

$$\Gamma^{(2)}[\phi] = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \phi(p)^* (p^2 Z + B) \phi(p), \quad Z = Z^* > 0, \quad B = B^*.\tag{9.8}$$

Its canonically normalized mass-squared operator and generalized eigenvalue equation are

$$M^2 = Z^{-1/2} B Z^{-1/2}, \quad Bv = m^2 Zv. \quad (9.9)$$

For a single channel with constant Hessian entry a and kinetic coefficient $Z > 0$ this gives $m^2 = a/Z$. Thus the kinetic form as well as the Hessian determines the mass spectrum. If $B > 0$ all canonically normalized free masses are positive. Exact interacting pole masses additionally require the poles of the complete momentum-dependent covariance including mixing and self-energy terms. Indeed $\varphi = Z^{1/2}\phi$ changes the kernel to $p^2 I + M^2$. Diagonalizing its Hermitian constant part gives the covariance $(p^2 + m^2)^{-1}$ for each positive mode and the generalized eigenvalue equation above. Congruence by $Z^{-1/2}$ preserves positivity.

Gauge and ghost directions are excluded before this normalization. Theorem B.8 supplies an explicit analytic bound for an isolated radial pole. General momentum-dependent or mixed kernels need their own pole calculation.

9.5 Thermal states and fixed-order amplitudes

The normalized modes also determine which excitations a thermal preparation populates and which transitions an interaction probes. In the quasi-free thermal representation temperature changes the occupation weights of the same energy tower. For scattering at fixed scale derivatives of the potential supply vertices between those modes with their masses fixed by the kinetic normalization above. These are the thermal and scattering uses of the spectral data [27].

The condensed fluctuation picture retains its Landau interpretation. For the quartic approximation with coefficient λ_{spec} the two minima $E = \pm E_{\text{quad}}$ satisfy $E_{\text{quad}}^2 = -\mu_L/\lambda_{\text{spec}}$ and have radial curvature $2\lambda_{\text{spec}}E_{\text{quad}}^2$. Division by the kinetic coefficient gives the squared mass. Proposition E.4 bounds the resulting stationary-amplitude and mass errors relative to the complete potential. The sign symmetry acts on this collective amplitude coordinate.

For a positive normalized scalar mass m_δ^2 represented on the generated event base Appendix C gives the free Fock space and local commutator. At inverse temperature $\beta > 0$ the operator $(e^{\beta\omega} - I)^{-1}$ is the Bose occupation operator in the regular quasi-free KMS state. A positive isolated atom $Z\delta(\mu^2 - M^2)$ in an observable's Källén–Lehmann measure gives a one-particle pole. Interacting asymptotic states additionally require the Haag–Ruelle or LSZ hypotheses.

For the fixed-scale illustration

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial e)^2 - \frac{1}{2}m_\delta^2 e^2 - \frac{\lambda_{\text{eff}}}{4}e^4, \quad (9.10)$$

four vertex derivatives give the tree contact amplitude $-6\lambda_{\text{eff}}$. Matching to the complete potential at $E_* = \sqrt{y_L}$ uses the normalized fluctuation $e = \sqrt{Z_L}(E - E_*)$. Its quartic coefficient is $\lambda_{\text{eff}} = (6Z_L^2)^{-1} \frac{d^4}{dE^4} F_L(E^2) \Big|_{E_*}$. The same expansion generally supplies a cubic vertex. Thus the fluctuation vertices are derivatives at the condensed background whereas λ_{spec} describes the expansion at the origin. With specified vertices and counterterms under

the stated subtraction a finite-order amplitude depends on the masses and couplings with the external invariants. For example

$$B_R(s; s_{\text{sub}}) = \int_{4m_\delta^2}^{\infty} \frac{ds'}{\pi} \text{Im } B(s') \left(\frac{1}{s' - s - i0} - \frac{1}{s' - s_{\text{sub}}} \right), \quad (9.11)$$

where $\text{Im } B(s') = (16\pi)^{-1} \sqrt{1 - 4m_\delta^2/s'}$ on its displayed support and the subtraction point $s_{\text{sub}} < 4m_\delta^2$ is real and below the cut. This is the once-subtracted scalar bubble. Its large- s' integrand is of order $(s')^{-2}$.

These amplitudes are evaluated at fixed order and with ultraviolet and volume cutoffs on $\lambda_{\text{eff}} : e^4 \therefore$. The stated C^2 remainder controls the Landau approximation to the stationary amplitude and radial mass. A vertex approximation uses the derivatives of the complete potential at the chosen background and the corresponding higher-order remainder.

10 Field Theory on the Generated Lorentz Carrier

The internal gauge-matter response and its Yang-Mills gap are defined at the operation level. The same first source corner supplies the affine Lorentz carrier on which local fields represent these responses and their correlations propagate causally.

10.1 The generated Lorentz carrier and its affine completion

The first noncommuting source corner has produced the full matrix algebra $\mathcal{A}_2 = \text{End}_{\mathbb{C}}(V) \cong M_2(\mathbb{C})$ with its adjoint and normalized trace. Two real forms of this one algebra serve different physical questions. The quaternionic form $\mathbb{H}_V$ parametrizes compact reversible response and leads to the internal process space. The self-adjoint form parametrizes ordered differences between localized events. Its determinant supplies the Lorentzian quadratic form while positivity supplies the time orientation. The radius R of Section 4.3 continues to measure internal spectral resolution. In the rest frame selected by I the trace component of an event difference is its time coordinate. A phase clock along a future timelike translation calibrates proper time.

Choose an oriented Pauli triple σ_a in $\text{Herm}_0(V)$. Its trace normalization is $\text{Tr}(\sigma_a \sigma_b) = 2\delta_{ab}$.

Theorem 10.1 (Generated Lorentz carrier and time orientation). The self-adjoint real form

$$\mathbb{M}_V := \mathcal{A}_{2,\text{sa}} = \mathbb{R}I \oplus \text{Herm}_0(V) \quad (10.1)$$

is four-dimensional. For $X = tI + x^a \sigma_a$ the determinant and its polarization are

$$q(X) := \det X = t^2 - |\mathbf{x}|^2, \quad (10.2)$$

$$\eta(X, Y) := \frac{1}{2} \{q(X + Y) - q(X) - q(Y)\} = \frac{1}{2} \{\text{Tr } X \text{Tr } Y - \text{Tr}(XY)\}. \quad (10.3)$$

Thus η has signature $(+---)$. The positive cone of the generated $*$ -algebra is the closed future cone

$$\mathbb{M}_V^+ = \{X \geq 0\} = \{tI + \mathbf{x} \cdot \boldsymbol{\sigma} : t \geq |\mathbf{x}|\}. \quad (10.4)$$

Its nonzero rank-one elements are future null and its positive-definite elements are future timelike. The determinant-one invertible group of $\mathcal{A}_2$ acts by

$$X \mapsto gXg^*, \quad g \in SL(V), \quad (10.5)$$

preserving q and $\mathbb{M}_V^+$. Its kernel is $\{\pm I\}$ and its image is $SO^+(1, 3)$. The stabilizer of I is the previously generated $SU(V)$.

Proof. The Pauli multiplication law gives $\det(tI + \mathbf{x} \cdot \boldsymbol{\sigma}) = t^2 - |\mathbf{x}|^2$ and polarization gives (10.3). The two eigenvalues of X are $t \pm |\mathbf{x}|$ which proves (10.4). Congruence by g preserves self-adjointness and positivity together with the determinant. The differential of this action identifies $\mathfrak{sl}(V)$ with $\mathfrak{so}(1, 3)$. Connectedness and the kernel $\{\pm I\}$ give $SL(V)/\{\pm I\} \cong SO^+(1, 3)$. Finally $gIg^* = I$ is equivalent to $g \in SU(V)$. $\square$

The vector space and involution with the trace in (10.1) all belong to the generated matrix corner. Among connected linear actions induced by its determinant-one invertibles preservation of the determinant and of the algebraic positive cone leaves precisely $SO^+(1, 3)$. The compact group $SU(V)$ is its rest-frame subgroup. Lorentz covariance is the noncompact continuation of the same comparison algebra that gave the compact operation orbit. Complete unitary-operation observations have spectrum $SU(V)/\{\pm I\}$ whereas self-adjoint comparison differences form $\mathbb{M}_V$. The former records an internal response pattern and the latter orders possible separations of localized events.

For $A \in \mathbb{M}_V$ and $\alpha > 0$ let

$$G_{\alpha, A}(X) = \exp\left[-\frac{\alpha}{2} \text{Tr}(X - A)^2\right]. \quad (10.6)$$

Independent source composition on the four trace-orthonormal matrix coordinates gives the product Gaussian with unit covariance. Translating its centre gives (10.6). These are the finite-resolution comparison profiles on the self-adjoint corner. Let $\mathcal{C}_{\text{ev}}$ be the uniform closure of the complex self-adjoint algebra generated by these real-valued profiles.

Proposition 10.2 (Generated affine event space). One has

$$\mathcal{C}_{\text{ev}} = C_0(\mathbb{M}_V), \quad \text{Spec } \mathcal{C}_{\text{ev}} \cong \mathbb{M}_V. \quad (10.7)$$

Differences of event characters form the translation group $(\mathbb{M}_V, +)$. Together with the determinant-preserving action of Theorem 10.1 they give the generated proper orthochronous Poincaré group

$$\mathcal{P}_V = \mathbb{M}_V \rtimes SO^+(1, 3), \quad (10.8)$$

acting faithfully on $C_0(\mathbb{M}_V)$ by pullback. Its causal order is $p \preceq q$ precisely when $q - p \in \mathbb{M}_V^+$.

Proof. The translated Gaussians are continuous and vanish at infinity. Their algebra is closed under conjugation and has no common zero. They separate points. If $X \neq Y$ then $G_{\alpha, X}(X) = 1 > G_{\alpha, X}(Y)$. The locally compact Stone–Weierstrass theorem therefore gives $\mathcal{C}_{\text{ev}} = C_0(\mathbb{M}_V)$. Gelfand duality identifies its characters with $\mathbb{M}_V$. Translation of

the Gaussian centre gives the additive action and (10.5) gives the linear action preserving both the determinant and the future cone. Their semidirect product proves the remaining statements. $\square$

The generated event carrier is the vector space $\mathbb{M}_V \cong \mathbb{R}^4$ and is therefore contractible. The relative frames of the relative-frame construction consequently form a globally trivial principal bundle on this event completion while a connection on that bundle can have nonzero curvature and holonomy along local loops. Event completions with other global topology define distinct global realizations. The local source algebra and its frame law remain the same.

The Euclidean carrier used for reflection reconstruction is the event copy

$$\mathbb{E}_V = \mathbb{R}I \oplus i \operatorname{Herm}_0(V), \quad \vartheta_E(\tau I + i\mathbf{x} \cdot \boldsymbol{\sigma}) = -\tau I + i\mathbf{x} \cdot \boldsymbol{\sigma}. \quad (10.9)$$

Its trace norm is $\tau^2 + |\mathbf{x}|^2$. The real-linear continuation $\mathscr{W}(\tau I + i\mathbf{x} \cdot \boldsymbol{\sigma}) = \tau I + \mathbf{x} \cdot \boldsymbol{\sigma}$ supplies the Lorentzian carrier and its distinguished time orientation. The symbol τ here is an event coordinate. It is independent of the internal radius R and of the Gaussian resolution parameter.

10.2 Statistics and the phase clock

Once a one-particle representation carrying the generated matter module has been specified its repeated excitations require an exchange rule. Symmetric and antisymmetric tensor powers implement the bosonic and fermionic choices. This is the step from a particle's internal type to the statistics of several indistinguishable particles.

The bosonic realization in Appendix C is Fock space over the positive-energy mode tower. For its specified product covariance the reflection form pairs a positive-time Euclidean preparation with the reflected conjugate of another across the chosen time slice. The mode decomposition gives a nonnegative squared norm for each linear preparation and Gaussian Wick extension preserves this positivity. Identifying null preparations and completing the resulting inner product gives the physical state space. Euclidean correlations thereby become overlaps of physical preparations. The translations of (10.8) supply their positive-energy evolution.

Let h_F be a positive-energy one-particle representation of $\mathcal{P}_V$ carrying $\mathcal{M}_{\text{full}}$. The charged quotient $\mathcal{M}_{\text{ch}}$ is retained as the gauge-charged observation space. The neutral right line remains part of the interacting carrier. Exterior multiplication and its adjoint on antisymmetric tensor powers give the standard fermionic Fock representation [27, 28]

$$\mathcal{H}_F = \mathcal{F}_-(h_F), \quad \{a(f), a^*(g)\} = \langle f, g \rangle, \quad \{a(f), a(g)\} = 0. \quad (10.10)$$

Second quantization preserves the positive translation spectrum. Covariant free-field intertwiners with the Pauli–Jordan support relation give graded-local fields on this space. The spin–statistics theorem applies when positivity and covariance hold together with the spectral condition and locality.

In the reconstructed free sector $H = d\Gamma(\omega) \geq 0$ generates Schrödinger evolution e^{-itH} in natural units. The coordinate t parametrizes translations along a chosen future unit

vector in $\mathbb{M}_V$. Its metrical meaning is fixed by a phase clock. For future timelike momentum P the translation character

$$\chi_P(a) = e^{-i\eta(P,a)}, \quad a \in \mathbb{M}_V, \quad (10.11)$$

obeys $d \arg \chi_P(tU)/dt = -\eta(P,U)$ along a future unit timelike U . In the rest frame this is $-m$. The coherent two-level preparation $(h_0 + e^{-i\eta(P,tU)}h_1)/\sqrt{2}$ is read by the generated quadratures $E_{01} + E_{10}$ and $-iE_{01} + iE_{10}$ as cosine and sine. Consecutive distinguishable phase readings therefore calibrate proper time by $\Delta\vartheta = m\Delta t$. Restoring units gives $e^{-iEt/\hbar}$ and $E_f - E_i = \hbar\omega_\gamma$. $\hbar$ converts the calibrated time and energy units.

The source phase θ rotates relative phases between Gaussian ladder levels. Averaging these rotations with variance s suppresses coherence according to the squared level separation and leaves diagonal populations fixed by the generated source updates (5.9)–(5.10). Internal heat evolution instead weights modes by their internal eigenvalues and sets the spectral resolution. Generated timelike translations organize the energy phases and transition frequencies. A clock reading therefore requires a distinguishable phase change and its calibration. A source state invariant under α_θ carries no such reading in a fixed observable.

10.3 Free reconstruction

The field description uses two factors of the source construction. Complete centre-invariant operation observations give the compact internal factor X_3 with its projectors and multiplicities. Self-adjoint observation of the first matrix corner gives the affine event carrier $\mathbb{M}_V$ with its future cone and Poincaré action. These are different spectra of compatible source-observation algebras. Their tensor product expresses localization in the event variable together with an internal response type.

The event copy $\mathbb{E}_V$ of the quaternionic real form carries the Euclidean trace metric and the reflection ϑ_E of (10.9). For the product covariance Theorem C.1 proves reflection positivity and reconstructs a positive-energy local field on $\mathbb{M}_V$. Its one-particle mass squares are the eigenvalues of the generated internal mass-squared operator with the same projectors and multiplicities. The Laplace integral (C.4) relates the covariance directly to the internal one-response semigroup. The reconstructed one-particle Poincaré Casimir is that internal operator. Multiparticle Lorentz masses follow from addition of future four-momenta and have the thresholds in (C.8). The clock (10.11) calibrates the timelike translation parameter by an observable phase. The scale radius remains the parameter of internal spectral resolution.

Fixing the complete source functional fixes every preparation overlap $\omega_t(A^*B)$. GNS uniqueness in Section 5.5 identifies its cyclic realizations by a unitary preserving the state and operations. For the Gaussian field covariance Wick’s rule fixes the correlation hierarchy and therefore the quasi-free reconstruction. This is the free reference representation for the native interaction. The internal spectrum and its couplings remain those fixed by the source operation algebra.

Haag’s theorem explains why the representation of an interacting field must be examined explicitly. In the scalar form stated in Appendix C the Wightman and sharp-time

hypotheses together with a unitary map intertwining the fields and their vacua force the second field to be free when the first is massive and free [27, 29]. The source states and spectral response of the present framework are constructed before this comparison. Free reconstruction supplies the quadratic reference and its mode space. Local relative S -operators in Appendix C.8 construct the interacting theory by causal comparison. Interaction is encoded in the relative local algebras and their causal composition law. Haag’s theorem fixes the conditions under which a global interaction-picture unitary would instead force freeness.

10.4 Generated and customary field representations

The Yang–Mills result of this paper is formulated in the generated response system. Its collective coordinate is the mean squared strength of the Gaussian colour connection. The adjoint Casimir gives its spectral feedback. The relative long-root eigenvalue gives its drive and the source Legendre transform gives its radial metric. The exact non-Abelian probe response supplies the transverse block. Common transport of the source frame and its law with its coexact domain makes this a frame-independent quadratic response. Theorems 8.4 and 8.5 then give the positive mass-squared operator and the gap on the centred generated response space. These are properties of the source family and its equilibrium response. The polynomial embedding preserves the original connection state and ordinary products while the source-mean differential gives the collective curvature readout. The complete original curvature has its explicit second- through fourth-order Hermite spectrum. A joint curvature source also measures the collective radial susceptibility with its direct source term retained. Their update and representation identities are proved in Section 8.8.

Customary continuum and lattice constructions organize the same physical questions around a prescribed field content and a regulator limit. Scalar triviality for example concerns a specified limit of an elementary field. The theorem of Aizenman and Duminil-Copin provides the relevant four-dimensional comparison.

For four-dimensional zero-field nearest-neighbour ferromagnetic Ising models with $\beta \leq \beta_c$ spin-fluctuation scaling limits on scales no larger than the correlation length are Gaussian [30, Proposition 1.4]. For the lattice φ^4 construction of Ref. [30, Theorem 1.2] the limiting field is Gaussian when its two-point function tends to zero at large separation.

The limiting object in this theorem is an elementary scalar field in the stated lattice families. The internal quartic coefficient is a Taylor coefficient of the complete relative response at fixed spectral data. A continuum scalar realization specifies the field normalization and regulator family on which a scaling limit is taken. For a colour-composite observable the scalar response is carried by correlations of the underlying gauge field. Its source derivatives and spectral measure determine how that response enters a field representation.

The long-root deformation and collective amplitude together with the colour-singlet observable $\text{Tr } F_c^2$ and Higgs radial coordinate have different transformation laws. Their coupling to a common excitation is determined by the source derivatives and spectral projections in Section 9.2. A proposed elementary scalar representation falls under a triviality theorem only if its field content and regulator with the limiting procedure satisfy that the-

orem’s hypotheses. A colour-composite observable has its own correlation functions and spectral measure. The bounds of Refs. [31, 32] concern scalar models in their respective regulator limits.

On a reflection-compatible hypercubic lattice the Wilson plaquette action for a compact gauge group at nonnegative coupling with product Haar measure and reflection-compatible boundary conditions defines a reflection-positive regulated gauge model [33, 34].

In that formulation the continuum Yang–Mills problem combines short-distance construction with a positive long-distance mass scale. Uniform regulator-removal and volume bounds on gauge-invariant correlations carry the main burden [35, Sections 6 and 6.5]. The generated formulation fixes a physical source resolution and constructs the connection law and invariant response algebra together with the positive generator. Its mass-gap theorem is therefore proved at the level of that generated system. The Lorentzian reconstruction and local net then retain its one-response mass projectors on the event carrier and realize its native interaction vertices. In the quadratic representation the reconstructed one-particle Casimir has exactly these mass squares. Local composite spectra are obtained from the specified observable and its field correlations. Reflection positivity is preserved under convergence of reflected polynomial pairings as Proposition B.2 shows.

Appendix F constructs an auxiliary exchange system on the same carrier $S \otimes S$ whose symmetric and antisymmetric parts underlie the earlier quadratic colour splitting. In the stated rational class the Yang–Baxter relation makes the two orders of three pairwise exchanges compatible. With its rational normalization fixed the ordered product over a chosen finite list of retained Dirac levels gives the monodromy. Its RTT relation then makes the traced transfer matrices commute [36]. The resulting commuting family acts on the finite auxiliary space associated with those retained mode labels.

10.5 Local interacting quantum field theory

The native interaction precedes field quantization. Pointwise composition in the event algebra fixes locality, internal mode multiplication fixes the coupled response channels, and matrix commutators give $A \wedge A$. The same connection transports matter. The local density on $\mathbb{M}_V$ therefore contains the gauge, matter and connector vertices constructed in Section 8. Once the connection and representation normalizations are fixed the relative coefficients of the curvature commutator and matter transport follow from the same connection action. The source-natural connector fixes the scalar tensor structure. The numerical Yukawa weights of distinct quark and lepton channels remain action-level dynamical data as stated in Section 4. Internal heat resolution acts on mode labels and preserves event support, making coarse graining compatible with the generated causal order.

Theorem C.4 applies causal renormalization to the gauge-compatible finite-resolution realization of these local functionals. It retains complete associated multiplets and the generated matrix Lie bracket. Relative S -operators obey causal factorization and generate an isotonic Poincaré-covariant local net on $\mathbb{M}_V$. Gauge fixing enlarges the intermediate algebra by ghosts and antifields. The physical local net is its degree-zero quantum BRST cohomology. The anomaly identities (4.54) and even weak-doublet count together with the perturbative Adler–Bardeen theorem allow the Ward identities of this renormalizable

gauge–matter realization to be maintained at every perturbative order. The Lorentzian interaction is thereby realized as a local perturbative QFT in formal power series. Its defining objects are the causally factorized local algebras and their BRST cohomology.

The complete spectral response supplies information that is not obtained by expanding about the quadratic state. Its condensation mechanism balances the reference drive against the restoring response of the same spectral family and the full logarithm controls the stationary amplitude and its radial curvature beyond the quartic approximation. Appendix B separates an action projected onto retained modes from the effective response obtained by eliminating the remaining tower. In a loop representation the internal trace and event momentum integral are evaluated together with the subtractions and kinetic normalization of that representation. In the quasi-free thermal representation of Section 9.5 temperature changes the occupation of the same energy tower. Derivatives of the condensed potential in the normalized fluctuation coordinate determine the local radial response vertices.

The generated Yang–Mills operator of Theorem 8.4 joins the normalized radial intensity response to the coclosed connection response. Its full response completion has first positive mass-squared $\Delta_{\text{YM}}^2 > 0$ and the gauge-invariant cyclic sector inherits the same positive lower bound by Theorem 8.5. The exact source covariance in Section 9.2 gives the mean curvature a nonzero radial differential. The original curvature correlation has the nonnegative Hermite weights in (G.61). When the curvature insertion changes the equilibrium source intensity the same polynomial gives $\chi_{\mathcal{Q}}^{\text{stat}} = V_{\perp} + b_{\mathcal{Q}}^2/m_{\text{YM,rad}}^2$. The direct term retains the remaining source fluctuations and the second term detects the collective restoring mode. The original source moments and multiplication are preserved in both calculations. In the condensed quadratic realization on the generated flat event base used in Appendix C.5 the nonzero background makes the averaged Wick intensity couple linearly to the constant scalar mode. The analytic continuation of its two-point function has an isolated pole at $Q^2 = -m_{\delta}^2$ with positive weight $4y_L/Z_L$ while its quadratic part produces a continuum starting at squared mass $4m_{\delta}^2$. The Wick intensity and the jointly Wick-ordered radial colour polynomial are explicit colour-singlet spectral realizations of the radial part of the generated Yang–Mills gap. The condensed connection intensity opens a linear observation channel for the stable excitation. The local interacting net carries the same source-generated vertices. Its interacting two-point functions organize the self-energy and mixing of these composite observations without changing the source definition of the mass-gap operator.

11 Conclusion

The construction relates internal geometry and the restoring spectrum to the same Gaussian source. Its reflecting window fixes the correlation scale, and its ladder supplies the operations whose complete centre-invariant observations determine the projective internal geometry. The trace metric and heat spectrum agree on the normalization, three-dimensional Weyl growth and round shell capacity.

The same first matrix corner supplies the event representation. Its self-adjoint real form has determinant $t^2 - |\mathbf{x}|^2$ and algebraic positive cone $t \geq |\mathbf{x}|$. These fix the Lorentzian

metric, time orientation and proper orthochronous Lorentz action. Translated Gaussian comparisons generate the affine event algebra; a source phase character calibrates its time-like translation. The phase clock reads physical time, while the radius measures internal resolution.

The relative-frame, least-matter and elementary-connector theorems characterize complete continuations preserving the generated adjoint and exchange structures, source phase and reversal. The lifted source number flow fixes the transported vacuum pointwise and determines the charge circle. The faithful group presentation is

$$\frac{SU(3) \times SU(2)_W \times U(1)}{\mathbb{Z}_6}.$$

It acts on a sixteen-dimensional full module containing a neutral right line. The fifteen-dimensional charged space is its quotient by that line. The elementary connector is parametrized by one complex doublet; its determinant is the positive gauge-invariant amplitude square. This fixes the response content of one family; repeated families are multiplicity extensions of the same blocks.

Mode multiplication and matrix commutators give the non-Abelian interaction, with the same connection transporting matter. Its invariant intensity $q = \mathbb{E}\|B\|^2$ is the collective coordinate. Casimir feedback changes the conditional tower laws. The Gaussian minimizing the mean relative entropy to these laws gives the comparison law. Relative information between comparison laws supplies the restoring functional that balances the long-root drive. Unit covariance and endpoint calibration fix the reference profile within the source semigroup. The canonical IR radius $L_{\text{src}} = \sqrt{2\pi} > L_c = \sqrt{\pi}$ lies on the condensed branch. At this radius the balance fixes a unique positive intensity q_L . The source Legendre metric normalizes the radial mean response, while the exact curvature response controls coexact connection variations. Joint transport of the frame, source law and form domain preserves both blocks of the mass operator,

$$\mathcal{M}_{\text{YM}}^2 = m_{\text{YM,rad}}^2 \oplus (\Delta_1^\perp + \delta_{q_L} I), \quad \inf \text{Spec } \mathcal{M}_{\text{YM}}^2 = \Delta_{\text{YM}}^2 > 0.$$

Independent Gaussian composition of the linear restoring map gives an equilibrium update whose completion has a unique vacuum and attained first positive mass-squared Δ_{YM}^2 . Every centred invariant response satisfies the corresponding coercivity bound. The intertwined representation preserves the inner products and multiplication of the original connection polynomials, including their angular correlations. The intensity score retains its second Hermite degree, while the full curvature observation has second- through fourth-order Hermite weights. Changing the source mean supplies a separate radial readout: the joint curvature susceptibility contains the direct residual variance and the positive restoring contribution $b_Q^2/m_{\text{YM,rad}}^2$. The collective mass thus has a colour-singlet readout retaining the source correlations. Uniform tail bounds control finite computations of the equilibrium intensity and gap throughout compact condensed scale intervals.

Gaussian resolution transports the nonlinear response together with its compensating residual. Its exact composition law makes successive changes of resolution consistent with their combined step.

Osterwalder–Schrader reconstruction gives free local fields on the event carrier. The covariance is the Laplace integral of the internal one-response semigroup, so the one-particle Poincaré Casimir retains the internal mass operator. In the quadratic realization, the averaged colour intensity and radial curvature polynomial have an isolated positive scalar pole and separated multiparticle continua. Their thresholds follow from addition of future four-momenta. The mode products, gauge curvature, matter transport and scalar connector also define local interaction functionals. Causal factorization and quantum BRST cohomology construct their physical net as a formal perturbative series in the gauge-compatible finite-resolution renormalizable realization. These results specify how the internal response spectrum enters local masses, colour-singlet observations and interaction vertices.

A Harmonic spectra and finite spectral windows

The standard round-space harmonics determine the quotient parities and operator normalizations used in Section 4 and in the spectral data of Section 5.4. The long-root calculation compares quadratic responses at two radii. The finite-window count uses a specified Dirac cutoff independently of the Fourier-width convention.

A.1 Round harmonics on the cover and quotient

Let S_L^3 be the round three-sphere of radius L . We use the standard spherical harmonic decomposition [17, 37] for which the nonnegative scalar Laplacian has

$$\varepsilon_l = \frac{l(l+2)}{L^2}, \quad d_l = (l+1)^2, \quad l = 0, 1, 2, \dots \quad (\text{A.1})$$

The degree- l eigenspace is the $(l/2, l/2)$ representation of $\text{Spin}(4) \simeq SU(2)_L \times SU(2)_R$. In the convention

$$C_2(j_L, j_R) = 2\{j_L(j_L + 1) + j_R(j_R + 1)\}, \quad (\text{A.2})$$

one has $C_2(l/2, l/2) = l(l+2)$.

The antipodal map acts on scalar harmonics by $(-1)^l$. Ordinary scalars on $X_3 = \mathbb{RP}^3 = S^3/\{\pm 1\}$ therefore retain

$$\varepsilon_l = \frac{l(l+2)}{L^2}, \quad d_l = (l+1)^2, \quad l = 0, 2, 4, \dots \quad (\text{A.3})$$

The quotient is the one constructed in Proposition 4.4. Equation (A.3) gives its scalar spectrum.

The coclosed one-form tower on the cover decomposes as

$$\bigoplus_{l \geq 1} \left[\left(\frac{l}{2} + \frac{1}{2}, \frac{l}{2} - \frac{1}{2} \right) \oplus \left(\frac{l}{2} - \frac{1}{2}, \frac{l}{2} + \frac{1}{2} \right) \right]. \quad (\text{A.4})$$

For ordinary one-forms pullback by the antipodal map includes the differential parity and the descending tower has odd l . The $l = 1$ sector contains Killing directions. Their Hodge

eigenvalue is positive since $H_{\text{dR}}^1(\mathbb{RP}^3; \mathbb{R}) = 0$. With $\Delta_1^\perp = d\delta + \delta d$ restricted to coclosed one-forms the complete quotient spectrum is

$$\text{Spec}(\Delta_1^\perp) = \left\{ \left(\frac{(l+1)^2}{L^2} \right)^{[2l(l+2)]} : l = 1, 3, 5, \dots \right\}. \quad (\text{A.5})$$

Here $\lambda^{[m]}$ means eigenvalue λ with multiplicity m . In particular the first value is $4/L^2$ with multiplicity six. So this tower has no zero mode.

For tensor Hessians use the positive rough Laplacian $\Delta_{TT} := -\nabla^a \nabla_a$ on divergence-free traceless symmetric two-tensors. On the cover its degree- k eigenvalue and multiplicity are $[k(k+2) - 2]/L^2$ and $2(k-1)(k+3)$ respectively [38, Section II, equations (2.5)–(2.7)] with the sign changed to our nonnegative Laplacian convention. The antipodal action is $(-1)^k$ on an ordinary rank-two tensor. So the quotient retains even k

$$\text{Spec}(\Delta_{TT}) = \left\{ \left(\frac{k(k+2) - 2}{L^2} \right)^{[2(k-1)(k+3)]} : k = 2, 4, 6, \dots \right\}. \quad (\text{A.6})$$

The lowest value is $6/L^2$ with multiplicity ten giving $\Delta_{TT} \geq 6/L^2$ on this carrier. A curvature-shifted Lichnerowicz operator is specified by its shift on the same tensor space.

TT projection and Casimir shifts. Let $\mathcal{H}_{TT}$ be the L^2 space of ordinary transverse-traceless symmetric two-tensors on round $X_3 = \mathbb{RP}_L^3$ and let P_{TT} denote the orthogonal projection onto it. The complexified TT harmonic space on the parent sphere consists of the two branches

$$\mathcal{T}_k^+ = \left(\frac{k+2}{2}, \frac{k-2}{2} \right), \quad \mathcal{T}_k^- = \left(\frac{k-2}{2}, \frac{k+2}{2} \right), \quad k \geq 2. \quad (\text{A.7})$$

Ordinary tensors on X_3 retain $k = 2, 4, 6, \dots$ in both branches with dimension $(k-1)(k+3)$ per branch. In the normalization (A.2)

$$C_2|_{\mathcal{T}_k^\pm} = k(k+2) + 4, \quad \Delta_{TT} = \frac{\mathcal{C}_2 - 6I}{L^2} \quad \text{on } \mathcal{H}_{TT}, \quad (\text{A.8})$$

where $\mathcal{C}_2$ is the dimensionless Casimir operator. Hence for real α, β the Casimir-labelled operator $K_\alpha = (\mathcal{C}_2 + \alpha I)/L^2$ is reduced by P_{TT} and satisfies

$$K_\alpha|_{\mathcal{H}_{TT}} = \Delta_{TT} + \frac{\alpha + 6}{L^2} I, \quad (\text{A.9})$$

$$\left(K_\alpha + \frac{\beta}{L^2} I \right) \Big|_{\mathcal{H}_{TT}} = \Delta_{TT} + \frac{\alpha + \beta + 6}{L^2} I. \quad (\text{A.10})$$

All these self-adjoint operators have the same Sobolev domain $H^2(X_3; \text{Sym}^2 T^* X_3) \cap \mathcal{H}_{TT}$. For the homogeneous presentation $S^3 = \text{Spin}(4)/SU(2)_{\text{diag}}$ the traceless symmetric tensor fibre is the diagonal spin-two module. The representations in (A.7) contain that module once and contain neither diagonal spin zero nor spin one. Their trace and divergence therefore vanish by equivariance. The homogeneous connection Laplacian is the group Casimir minus the fibre Casimir $2(2+1) = 6$ divided by L^2 . Substitution into (A.2)

gives $k(k+2)+4$. Hence the rough eigenvalue $[k(k+2)-2]/L^2$. The combined dimension $2(k-1)(k+3)$ equals the complete TT multiplicity stated above. So these branches exhaust each eigenspace. The antipodal element acts in either branch by $(-1)^k$ giving the stated quotient parity. The operator is block diagonal in the harmonic decomposition. So P_{TT} reduces it. The identities now hold on the common core of finite TT harmonic sums and extend to the stated Sobolev domain. Adding a real constant is a bounded self-adjoint perturbation and does not change that domain.

For a trace over this TT space the retained labels satisfy $|j_L - j_R| = 2$ with both branches and ordinary-tensor parity at the stated multiplicities. Scalar (j, j) representations belong to a different orthogonal summand. Comparing TT traces therefore requires the same projection and regulator in common radius units. Under the endpoint convention $L = R_b$ for example $\alpha = 6$ gives a shift $12/L^2$ from Δ_{TT} . A separately specified $\beta = 6$ raises that total shift to $18/L^2$. A physical Hessian can be compared with these operators by identifying its TT restriction including all of its zeroth-order terms.

A.2 The first non-Killing right-weight line

Write $q_R = 2m_R$ for twice the right weight. The first descending non-Killing coclosed one-form representation containing $j_R = 1$ occurs at $l = 3$

$$(j_L, j_R) = (2, 1), \quad C_2(2, 1) = 16. \quad (\text{A.11})$$

Its Casimir spectral line is therefore

$$\varepsilon_{\text{lr}}(L) = \frac{16}{L^2}. \quad (\text{A.12})$$

This positive eigenvalue enters the quadratic gauge response. A mass assignment also depends on the full response operator and its normalization specified for the action realization in Definition B.5.

A.3 Fourier normalization and the relative long-root response

Equal position and momentum entropies select a width only after the Fourier units have been fixed.

Use the unitary Fourier convention with $\hbar = 1$ and let a normalized unchirped Gaussian wavefunction have position probability density $N(0, b_{\text{ent}}^2)$. If its position and momentum differential entropies are required to be equal then

$$b_{\text{ent}} = \frac{1}{\sqrt{2}} \quad (\text{A.13})$$

in those Fourier units. If one additionally identifies $b_{\text{ent}} = b$ and applies the independent endpoint metric convention $L = R_b = \sqrt{2\pi} b$ the numerical representatives are $L = \sqrt{\pi}$ and $R_{\text{scal}} = 6/\pi$. The sharp Fourier entropy inequality gives this normalization directly through $H_x + H_p \geq 1 + \log \pi$ with equality for this unchirped Gaussian [13]. Equal entropies at equality give $H_x = \frac{1}{2}(1 + \log \pi)$. Since $H_x = \frac{1}{2} \log(2\pi e b_{\text{ent}}^2)$ one obtains $b_{\text{ent}}^2 = 1/2$. The statements about L and scalar curvature are substitutions into the endpoint-scaled

spectrum (4.20). The equal-entropy condition fixes a coordinate representative in the chosen Fourier units. In the dimensionless coordinate $x = d/(\sqrt{2\pi}b)$ the Gaussian is self-dual for every b after the corresponding dual rescaling. The condensation threshold is determined separately by the stationary equation of the spectral potential in Section 8.

For any reference radius $L_* > 0$ with $R = 6/L^2$ and $R_* = 6/L_*^2$ substitution of $\varepsilon_{\text{lr}}(L) = 16/L^2$ gives

$$\varepsilon_{\text{lr}}(L) - \varepsilon_{\text{lr}}(L_*) = \frac{8}{3}(R - R_*). \quad (\text{A.14})$$

It is the relative coefficient of the quadratic gauge response when the mode coefficients are identified isometrically as below.

Quadratic comparison of gauge modes. In the trivial-connection sector of the internal mode construction let B be a coclosed $(2,1)$ long-root mode with values in the fixed gauge-algebra carrier. Use a positive invariant inner product on that carrier and factor out the common quadratic normalization of the response in Section 8.1. The field strength (A.29) gives

$$\frac{1}{2}\|F_{\epsilon B}\|_L^2 = \epsilon^2 Q_L^{(2)}(B) + O(\epsilon^3), \quad Q_L^{(2)}(B) = \frac{1}{2}\|dB\|_L^2 = \frac{8}{L^2}\|B\|_L^2. \quad (\text{A.15})$$

Indeed $F_{\epsilon B} = \epsilon dB + \epsilon^2 B \wedge B$. Integration by parts and $\delta B = 0$ give $\|dB\|_L^2 = \langle B, \Delta_1^\perp B \rangle_L = 16L^{-2}\|B\|_L^2$. Thus $Q_L^{(2)}$ is the quadratic coefficient half the second variation of $\frac{1}{2}\|F\|_L^2$.

For comparison at L and L_* keep the coefficients in orthonormal mode bases fixed. With $g_L = L^2 g_1$ the norm squared of a fixed one-form scales as L . Hence $u_a(L) = (L_*/L)^{1/2} u_a(L_*)$ is the required isometric identification. For an orthonormal gauge basis T_A set $B_L = \sum_{a,A} c_{aA} u_a(L) T_A$ and $E_{\text{lr}}^2 = \sum_{a,A} |c_{aA}|^2$. Then

$$Q_L^{(2)}(B_L) - Q_{L_*}^{(2)}(B_{L_*}) = \frac{1}{2}[\varepsilon_{\text{lr}}(L) - \varepsilon_{\text{lr}}(L_*)] E_{\text{lr}}^2. \quad (\text{A.16})$$

The comparison uses the same gauge-mode coefficients at two internal radii with $R = 6/L^2$ parametrizing their spectrum. If the quadratic normalization is $Z(L)$ rather than one each term in (A.16) retains its own Z . The displayed expansion is at the trivial reference connection. A nonflat reference also contributes background-dependent quadratic terms.

The invariant intensity E_{lr}^2 and the retained scalar coordinate E_0^2 have different definitions. Relating the long-root comparison to the condensation functional requires the amplitude and coefficient matching in Section 9.2. A negative reference difference is compatible with a positive unsubtracted quadratic form. The prescribed spectral response has a stable scalar gap under Corollary 8.3. In an action realization the pole criterion also uses kinetic normalization (Theorem B.8).

A.4 Spin lift and Dirac data

The round-sphere Dirac eigenvalues are

$$\lambda_k^\pm = \pm \frac{k + 3/2}{L}, \quad k = 0, 1, 2, \dots, \quad (\text{A.17})$$

with multiplicity $(k+1)(k+2)$ for each sign [39, Theorem 1]. Write $\omega = e_1 e_2 e_3 e_4 \in \text{Spin}(4)$. The quotient $\mathbb{RP}^3$ has two spin structures $\tau_{\pm}$ defined by lifting the antipodal deck transformation to $\pm\omega$. Bär’s generating-function formula [39, Theorem 2] applied to this two-element deck group gives

$$\begin{aligned} F_{\tau_-}^+(z) &= \frac{1}{(1-z)^3} + \frac{1}{(1+z)^3}, & F_{\tau_-}^-(z) &= \frac{1}{(1-z)^3} - \frac{1}{(1+z)^3}, \\ F_{\tau_+}^+(z) &= F_{\tau_-}^-(z), & F_{\tau_+}^-(z) &= F_{\tau_-}^+(z). \end{aligned} \quad (\text{A.18})$$

Consequently the complete spectrum for the selected spin structure τ_- is

$$\begin{aligned} \lambda_{k,\tau_-}^+ &= +\frac{k+3/2}{L}, & k &= 0, 2, 4, \dots, \\ \lambda_{k,\tau_-}^- &= -\frac{k+3/2}{L}, & k &= 1, 3, 5, \dots, \\ \text{mult } \lambda_{k,\tau_-}^{\pm} &= (k+1)(k+2). \end{aligned} \quad (\text{A.19})$$

For τ_+ the two parity assignments are interchanged. The deck lift thus fixes the spectral asymmetry of the signed Dirac tower.

The window calculation in the main text is therefore formulated for the positive branch of τ_-

$$\lambda_n^+ = \frac{n+3/2}{L}, \quad n = 0, 2, 4, \dots \quad (\text{A.20})$$

Each retained eigenvalue has multiplicity $(n+1)(n+2)$.

A.5 A conditional finite-window count

A sharp window projects onto spectral levels below a dimensionless cutoff c (Figure 2). Heat-kernel coarse graining instead multiplies every mode by a strictly positive weight. The window records which response levels are retained at a specified resolution. Moving its threshold between adjacent Dirac levels leaves that content unchanged. The plateau below therefore expresses stability of the retained level count over a range of resolutions.

For the selected positive Dirac subsequence (A.20) a dimensionless cutoff c retains

$$N_{\text{win}}(c) = \#\{n \in 2\mathbb{N}_0 : n + \tfrac{3}{2} < c\}. \quad (\text{A.21})$$

The count is three throughout $11/2 < c < 15/2$. The levels $n = 0, 2, 4$ lie below c while $n = 6$ does not. This count describes the retained positive Dirac levels at the chosen cutoff. The full finite-response content also includes their multiplicities.

The retention rule supplies c independently of the endpoint ratio $c_{\text{end}} = \pi$ and the radius convention $R_b/b = \sqrt{2\pi}$. A generation interpretation requires these retained levels to represent the fermionic responses that remain coupled at the observation scale.

A.6 Mode multiplication and Gaussian resolution

Schur orthogonality and Clebsch–Gordan decomposition fix the mode-product coefficients used in Section 8.1. The heat product defect measures the difference between resolving a product and multiplying resolved coefficients. Its Sobolev bound controls the corresponding gauge and matter responses.

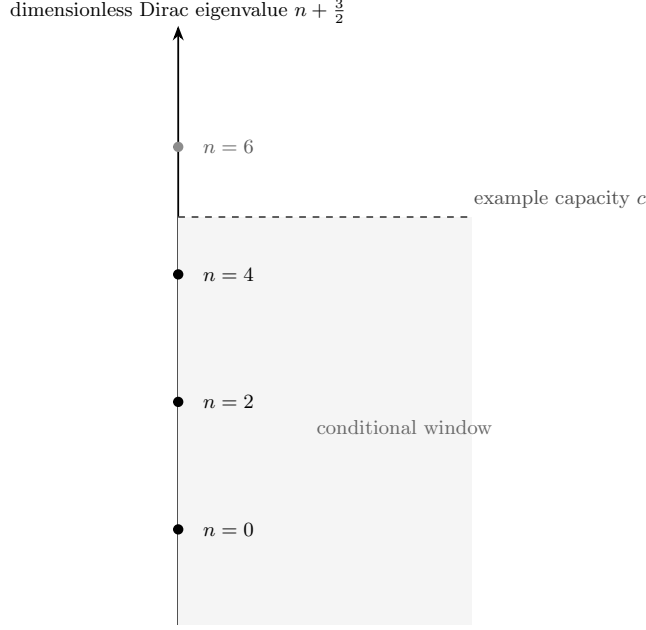


Figure 2. Three positive Dirac levels lie below the displayed capacity. Their multiplicities are not included in this count.

Haar normalization and multiplication. By Schur orthogonality for normalized Haar measure [15] the integral of $D_{mn}^j D_{ab}^k$ is $\delta_{jk} \delta_{ma} \delta_{nb} / (2j + 1)$. The matrix coefficients with integer j are unchanged under $U \mapsto -U$. So their inner product is equally the quotient Haar inner product. This proves the normalization in (5.1). The scalar eigenspace decomposition gives completeness.

The multiplicity-free Clebsch–Gordan decomposition [15] splits $V_j \otimes V_k$ into orthogonal summands V_J for $J = |j - k|, \dots, j + k$. Let $C_{jm,ka}^{JM}$ be the matrix entries of this unitary change of basis in the convention $D_{mn}^j D_{ab}^k = \sum_{JMN} C_{jm,ka}^{JM} \overline{C_{jn,kb}^{JN}} D_{MN}^J$. Multiplying the normalization factors proves (5.2). Changes of orthonormal bases transform the product coefficients and field coordinates covariantly.

The central-even matrix coefficients form the conjugation-closed unital Peter–Weyl algebra of the quotient. Spectral partial sums converge in each $H^\eta(X_3)$ for vectors in that space and multiplication is continuous for $\eta > 3/2$. With volume-normalized modes in place of the Haar-normalized ones the three-mode coefficients acquire the factor $(\pi^2 L^3)^{-1/2}$.

The highest-weight vector in $V_J \otimes V_J$ belongs entirely to V_{2J} . This proves the nonzero high-shell coefficient

$$(e_{JJ}^J)^2 = \frac{2J + 1}{\sqrt{4J + 1}} e_{2J,2J}^{2J}. \quad (\text{A.22})$$

The real function $e_{JJ}^J + \overline{e_{JJ}^J}$ has the same nonvanishing highest-weight square coefficient. The failure of finite closure therefore applies to real fields as well as to their complex coordinates.

Analytic completion. For $\alpha > 0$ write $f(U) = \sum_{j \geq 0} \text{tr}(B_j D^j(U))$ and set

$$\|f\|_{\mathcal{A}_\alpha} = \sum_{j \geq 0} e^{\alpha j} \|B_j\|_1,$$

where $\|\cdot\|_1$ is the matrix trace norm. This completion of the finite mode algebra is a Banach algebra contained in $C(X_3)$ with

$$\|fg\|_{\mathcal{A}_\alpha} \leq \|f\|_{\mathcal{A}_\alpha} \|g\|_{\mathcal{A}_\alpha}. \quad (\text{A.23})$$

For $t > 0$ the first bound below holds for $f \in L^2(X_3)$. The derivative and tail bounds hold for $f \in \mathcal{A}_\alpha$ and $0 < \beta < \alpha$

$$\|G_t f\|_{\mathcal{A}_\alpha} \leq \left[\sum_{j \geq 0} (2j+1)^2 e^{2\alpha j - 2t\lambda_j} \right]^{1/2} \|f\|_2, \quad (\text{A.24})$$

$$\|E_a f\|_{\mathcal{A}_\beta} \leq \frac{2}{eL(\alpha - \beta)} \|f\|_{\mathcal{A}_\alpha}, \quad (\text{A.25})$$

$$\|(I - P_{\leq J})f\|_\infty \leq e^{-\alpha J} \|f\|_{\mathcal{A}_\alpha}. \quad (\text{A.26})$$

Unitarity gives $|\text{tr}(B_j D^j)| \leq \|B_j\|_1$. In the Clebsch–Gordan decomposition of $V_j \otimes V_k$ let T_J be the diagonal block of $B_j \otimes C_k$ on V_J . Block pinching is an average of unitary conjugations. So $\sum_J \|T_J\|_1 \leq \|B_j\|_1 \|C_k\|_1$. Since $J \leq j+k$ the weighted sum proves (A.23). Completeness follows from the weighted summable matrix coefficients. The product extends by its bilinear bound. Schur orthogonality gives $\|f\|_2^2 = \sum_j \|B_j\|_F^2 / (2j+1)$. The inequality $\|B_j\|_1 \leq \sqrt{2j+1} \|B_j\|_F$ and Cauchy–Schwarz prove (A.24). Its series converges because $\lambda_j = 4j(j+1)/L^2$. The representation derivative has norm at most $2j/L$. $j e^{-(\alpha-\beta)j} \leq [e(\alpha-\beta)]^{-1}$ proves the derivative estimate. Summing the omitted coefficients gives the tail bound.

The support condition $J \leq j+k$ makes the exponential weight submultiplicative as in the spectral dynamics of Ref. [21]. Finite matrix-valued fields inherit the bounds by summing over their matrix entries. Thus curvature and covariant derivatives with their convergent mode approximations have common domains with explicit loss of analytic width under differentiation.

Proposition A.1 (Exact Gaussian product defect). Choose the left-invariant orthonormal frame with $[E_a, E_b] = (2/L)\epsilon_{abc}E_c$ and $\Delta_0 = -\sum_c E_c^2$. For $f, g \in H^\eta(X_3)$ with $\eta > 5/2$ and $G_t = e^{-t\Delta_0}$

$$G_t(fg) - (G_t f)(G_t g) = 2 \int_0^t G_{t-u} \sum_c (E_c G_u f)(E_c G_u g) du. \quad (\text{A.27})$$

The pairing is bilinear with no complex conjugation. The identity holds in $H^{\eta-1}$ and its norm is bounded by $C_{\eta,L} t \|f\|_{H^\eta} \|g\|_{H^\eta}$. In modal coordinates the coefficient of each pair is exactly

$$C_{\alpha\beta}^\gamma \left(e^{-t\lambda_\gamma} - e^{-t(\lambda_\alpha + \lambda_\beta)} \right). \quad (\text{A.28})$$

Infinite modal expressions mean Sobolev limits of finite spectral sums.

Proof. For finite sums differentiate $G_{t-u}[(G_u f)(G_u g)]$ with respect to u . The product rule $\Delta_0(fg) = (\Delta_0 f)g + f\Delta_0 g - 2\sum_c (E_c f)(E_c g)$ makes this derivative the negative of the integrand in (A.27). Integration between the two endpoints gives the sign and factor 2 displayed there. The round Casimir commutes with E_c and its heat maps are contractions on the spectral Sobolev spaces. Since $E_c : H^\eta \rightarrow H^{\eta-1}$ is bounded and $H^{\eta-1}$ is a multiplication algebra the integrand has norm at most $C_{\eta,L}\|f\|_{H^\eta}\|g\|_{H^\eta}$. This both proves the bound and allows finite-sum approximation in the identity. Applying the diagonal multiplier to the already proved modal product gives (A.28). $\square$

The defect is fixed by the multiplication and the heat map. For $t > 0$ the inverse G_t^{-1} is unbounded on a fixed Sobolev space. So $G_t[(G_t^{-1}f)(G_t^{-1}g)]$ is defined on the heat-image domain where the inverse images have the required Sobolev regularity.

Curvature coefficients and their domain. In a fixed trivialization let $A = \sum_a A_a \theta^a$ take values in the generated real internal Lie algebra with $A_a \in H^\eta$ and $[T_A, T_B] = f_{AB}^C T_C$. By the definition of exterior derivative

$$F_{ab} = E_a A_b - E_b A_a - \frac{2}{L} \epsilon_{abc} A_c + [A_a, A_b]. \quad (\text{A.29})$$

It lies in $H^{\eta-1}$. Expanding its nonlinear term by (5.2) gives exactly the coefficients in (8.2). The internal mode product couples to the already obtained matrix commutator.

Because G_t commutes with each E_a applying it componentwise cancels every linear term in $G_t F_{ab}(A) - F_{ab}(G_t A)$. Applying the preceding proposition to the brackets proves

$$G_t F_{ab}(A) - F_{ab}(G_t A) = 2 \int_0^t G_{t-u} \sum_c [E_c G_u A_a, E_c G_u A_b] du. \quad (\text{A.30})$$

The same $H^{\eta-1}$ estimate holds with an additional fixed finite-dimensional bracket norm. For $A_1 = fT_A$ and $A_2 = fT_B$ with $A_3 = 0$ and the real f above and $[T_A, T_B] \neq 0$ the highest-shell term cannot be cancelled by the linear derivative terms which stay in the initial shell. Moreover $\lambda_{2J} - 2\lambda_J = 8J^2/L^2 > 0$ proving a nonzero scale defect for every $t > 0$.

Here G_t acts on scalar coefficients in the fixed frame. A position-dependent change of frame need not commute with this heat map. For a different scalar operator $\tilde{\Delta}$ with heat map $\tilde{G}_t$ the exact algebraic identity on a common smooth core is instead

$$\begin{aligned} \tilde{G}_t F_{ab}(A) - F_{ab}(\tilde{G}_t A) &= [\tilde{G}_t, E_a] A_b - [\tilde{G}_t, E_b] A_a \\ &\quad + \tilde{G}_t [A_a, A_b] - [\tilde{G}_t A_a, \tilde{G}_t A_b]. \end{aligned} \quad (\text{A.31})$$

Its last two terms depend on the product rule of $\tilde{\Delta}$. For a squashed metric both the derivative commutators and the product defect must be computed for that operator.

Gaussian resolution damps the finer modes whereas nonlinear products mix modes. Resolving a response and evaluating the response of resolved fields therefore give different results. The following compensation law carries their exact difference with the unresolved response.

Proposition A.2 (Nonlinear response compensation under Gaussian resolution). Work in the fixed coefficient realization above with the full-module representation ρ and $\eta > 5/2$. Put $Q_a(A, \psi) = E_a\psi + \rho_*(A_a)\psi$ and write $C_t^F(A) = G_t F(A) - F(G_t A)$. The matter response has the exact defect

$$\begin{aligned} C_{a,t}^Q(A, \psi) &:= G_t Q_a(A, \psi) - Q_a(G_t A, G_t \psi) \\ &= 2 \int_0^t G_{t-u} \sum_c \rho_*(E_c G_u A_a)(E_c G_u \psi) du. \end{aligned} \quad (\text{A.32})$$

It belongs to $H^{\eta-1}$ and is bounded there by $C_{\eta,L,\rho} t \|A\|_{H^\eta} \|\psi\|_{H^\eta}$. For $U = (A, \psi)$ and $C_t(U) = (C_t^F(A), C_t^Q(A, \psi))$ both defects obey

$$C_{s+t}(U) = G_s C_t(U) + C_s(G_t U). \quad (\text{A.33})$$

Let $R = (R_F, R_Q)$ be unresolved responses in $H^{\eta-1}$ of the same coefficient types as (F, Q) . On the product Banach space of $A, \psi \in H^\eta$ and $R_F, R_Q \in H^{\eta-1}$ the prescribed Gaussian resolution determines

$$\mathcal{T}_t(U, R) = (G_t U, G_t R + C_t(U)), \quad \mathcal{T}_s \mathcal{T}_t = \mathcal{T}_{s+t}. \quad (\text{A.34})$$

This is a global strongly continuous locally Lipschitz semigroup. It preserves the joint response exactly in the sense

$$F(A_t) + R_{F,t} = G_t(F(A) + R_F), \quad Q(A_t, \psi_t) + R_{Q,t} = G_t(Q(A, \psi) + R_Q). \quad (\text{A.35})$$

The heat evolution of U and these identities uniquely determine R_t . Finite spectral approximations of the initial data converge to (U_t, R_t) uniformly on each finite resolution interval. Each finite modal product retains its full spectral support.

Proof. Commutation of G_t with E_a removes the linear part of Q . Proposition A.1 applied to each matrix–vector product gives (A.32) and its bound. For either response $B(U) = F(A)$ or $Q(A, \psi)$ insert and subtract $G_s B(G_t U)$ in $G_{s+t} B(U) - B(G_{s+t} U)$ to obtain (A.33). The continuous triangular change of coordinates

$$\Psi(U, R) = (U, (F(A), Q(A, \psi)) + R)$$

has a continuous inverse obtained by subtracting (F, Q) . Both maps are locally Lipschitz. The linear derivatives map H^η to $H^{\eta-1}$ and the remaining terms are continuous bilinear products. Consequently $\mathcal{T}_t = \Psi^{-1}(G_t \oplus G_t) \Psi$ with each heat map acting on all its coefficient components. This proves the stated semigroup continuity and preservation properties. On bounded sets and $0 \leq t \leq T$ the contraction of G_t and these bilinear bounds give a common Lipschitz constant. Spectral projections converge in the stated Sobolev spaces. So this estimate proves the asserted uniform approximation by finite spectral sums. $\square$

Equation (A.35) identifies the transported quantity as the field response together with its residual. The cocycle makes successive resolutions agree with their combined step. For

initially resolved data $R_F = R_Q = 0$. Subsequent residuals are fixed by this compensation law. For $A, \psi \in H^{\eta+2}$ and $R \in H^{\eta+1}$ differentiation gives the generator in the displayed state space

$$\begin{aligned}\dot{A} &= -\Delta_0 A, & \dot{\psi} &= -\Delta_0 \psi, \\ \dot{R}_{F,ab} &= -\Delta_0 R_{F,ab} + 2 \sum_c [E_c A_a, E_c A_b], \\ \dot{R}_{Q,a} &= -\Delta_0 R_{Q,a} + 2 \sum_c \rho_*(E_c A_a) E_c \psi.\end{aligned}\tag{A.36}$$

The interaction products supply the residual sources at each resolution. The reconstructed $F(A_t)$ and $Q(A_t, \psi_t)$ obey the connection identities while their sums with R_t obey the linear heat equations. The parameter t describes coefficient resolution at fixed L in the chosen trivialization.

A resolved spinor multiplication example. An explicit Clebsch–Gordan expansion determines whether scalar multiplication preserves a selected Dirac spectral subspace. In the left-trivialized spin frame the coupled modes are

$$\Psi_{m;JM}^j = \sum_{n,s} C_{jn, \frac{1}{2}s}^{JM} e_{mn}^j \chi_s, \quad j = 0, 1, \dots, \quad J = j \pm \frac{1}{2} \geq 0. \tag{A.37}$$

Here χ_s is an orthonormal spin-1/2 basis and Haar measure is normalized as above. In the orientation of (A.19) the round Dirac operator in this frame is $L^{-1}(4\mathbf{J} \cdot \mathbf{S} + 3/2)$ where $\mathbf{J}$ acts on the right scalar index n . Addition of angular momenta therefore gives eigenvalues $(2j + 3/2)/L$ at $J = j + 1/2$ and $-(2j + 1/2)/L$ at $J = j - 1/2$. Their multiplicities are $(2j + 1)(2j + 2)$ and $(2j + 1)(2j)$ respectively reproducing the complete selected signed tower.

Multiplication of the lowest positive mode by $e_{1,0}^1$ is explicit

$$e_{1,0}^1 \Psi_{0; \frac{1}{2}, \frac{1}{2}}^0 = \sqrt{\frac{2}{3}} \Psi_{1; \frac{3}{2}, \frac{1}{2}}^1 - \sqrt{\frac{1}{3}} \Psi_{1; \frac{1}{2}, \frac{1}{2}}^1. \tag{A.38}$$

This follows by resolving $e_{1,0}^1 \chi_{1/2}$ with the two Clebsch–Gordan coefficients in (A.37). On the entire lowest positive doublet the negative-sign multiplication block B_{-+} obeys $B_{-+}^\dagger B_{-+} = I_2/3$. Thus this particular multiplier does not preserve the positive Dirac subspace. A constant multiplier does. These signs label the geometric Dirac spectrum. A static source may be restricted to one sign whereas an interaction containing this multiplier couples the two spectral subspaces.

Finite-mode realizations. The finite-mode realizations of the normalized product tensors and invariant derivatives together with curvature and associated matter response used below retain the complete spectral support of each finite modal product. For the non-commuting configurations considered here direct evaluation of the resolution defects agrees with the Duhamel representations (A.30) and (A.32). These examples illustrate the joint composition law proved in Proposition A.2.

B Product-space realization on the generated event carrier

The product-space realization of Section 10 uses the generated Euclidean event carrier $\mathbb{E}_V$ and the compact internal response space. An action representation and regulator turn the native products into loop integrals containing event momenta and the internal spectrum. The estimates below give sufficient subtractions for these combined traces and a uniform heavy-mode expansion at fixed regulator and perturbative order.

B.1 Action and endpoint data

Definition B.1 (Action-level realization). Put $(M_4, \theta) = (\mathbb{E}_V, \vartheta_E)$ as in (10.9). An action-level realization consists of the generated G_{int} bundles over $M_7 = M_4 \times X_3$ with the endpoint-scaled internal metric and G_{int} as in (4.53). Its local real gauge- and spin-covariant action is

$$S_{\text{prod}} = S_{\text{YM}} + S_{\text{mat}} + S_{\text{resp}} + S_{\text{gf}} + S_{\text{gh}} \quad (\text{B.1})$$

with the native connection and matter normalization. The realization specifies boundary conditions and a chiral projection or twisting together with gauge fixing and ghosts. It also specifies the regulator and local renormalization conditions. The realization specifies a physical observable algebra and its Euclidean reflection. Its Schwinger functional is reflection positive on the positive-time physical algebra. The bosonic reflected form and the fermionic graded reflection obey their respective statistics. Gauge fixing and ghosts provide an auxiliary representation of the gauge identities [40, 41]. An interacting continuum realization supplies a limiting Schwinger hierarchy or an equivalent physical operator construction. Matter traces use $\mathcal{M}_{\text{full}}$ with descent governed by the [quotient and trace conditions](#) of Section 5.4. The fluctuation and ghost operators have the common closed form domains specified in Definition B.5. At a scale where its conditional flow exists the endpoint datum consists of the running couplings and background Hessian with ghost spectra together with the renormalization conditions and operator domains. Admissible spectral functionals are absolutely convergent traces or renormalized relative traces with their specified subtraction.

With the skew-adjoint connection convention of Section 8 choose on each gauge factor the positive invariant pairing

$$\mathcal{B}(X, Y) = -\text{tr}_\rho(XY). \quad (\text{B.2})$$

The faithful unitary representation ρ and its trace normalization are chosen to agree with the Lie-algebra inner product used above. An overall normalization is included in the coupling. The Yang–Mills term is then

$$S_{\text{YM}}[A] = \frac{1}{4g_7^2} \int_{M_7} d^7z \sqrt{g} \mathcal{B}(F_{MN}, F^{MN}). \quad (\text{B.3})$$

Here $[g_7^2] = \text{length}^3$. The action is understood with the regulator and renormalization conditions of Definition B.1. The long-root field and its fluctuation operator are specified in Section B.6.

Proposition B.2 (Physical positivity under a hierarchy limit). Let $\mathfrak{A}_+$ be a fixed algebra of positive-time physical observables with reflection Θ . Suppose normalized regulated Schwinger functionals ω_N satisfy $\omega_N(\Theta F F) \geq 0$ for $F \in \mathfrak{A}_+$ and all reflected polynomial pairings converge. Their limit is reflection positive. On a flat reflected base if its full hierarchy satisfies the covariance and statistics with the regularity conditions of reconstruction it determines a positive physical Hilbert-space representation. Clustering gives the corresponding uniqueness of the invariant cyclic vacuum [42, 43].

Proof. For any finite family F_i the matrices $G_{ij}^{(N)} = \omega_N(\Theta F_i F_j)$ are positive semidefinite. Entrywise convergence gives $c^* G c = \lim_N c^* G^{(N)} c \geq 0$. The limiting reflected form therefore has a Hilbert completion after quotienting its null space. The stated hierarchy assumptions permit Osterwalder–Schrader reconstruction. Polynomial observables are included through the assumed convergence of their pairings. $\square$

B.2 External trace and its subtraction

The Hessian spectrum and signed ghost weights determine the internal determinant and its coefficients (Theorem B.6 and Proposition B.7). A full one-loop action also integrates over external momenta as in (B.23). Absolute convergence must therefore hold for the combined sum and integral with local renormalization terms fixed by Definition B.1.

Proposition B.3 (A sufficient full-trace subtraction). Consider a finite number of separated fluctuation sectors on $\mathbb{R}^4 \times X_3$ with positive squared masses $a_l \asymp l^2$ with $d_l = O(l^2)$ and bounded real g_l and η_l . Define

$$R_3(x) = \log(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3}. \quad (\text{B.4})$$

On any compact y -interval satisfying $\sup_l |g_l y|/a_l \leq r < 1$ the relative four-dimensional density

$$\mathcal{V}_{4,\text{rel}}(y) = \frac{1}{2} \sum_l \eta_l d_l \int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4} R_3\left(\frac{g_l y}{p^2 + a_l}\right) \quad (\text{B.5})$$

converges absolutely and locally uniformly. The sum and integral with the Taylor series through all orders y^n for $n \geq 4$ may be interchanged. The local polynomial of degree at most three in y is fixed separately by the renormalization conditions.

Proof. The logarithmic remainder obeys $|R_3(x)| \leq |x|^4/[4(1-r)]$ for $|x| \leq r$. Direct radial integration gives for $n > 2$

$$\int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4} (p^2 + a)^{-n} = \frac{\Gamma(n-2)}{16\pi^2 \Gamma(n)} a^{2-n}. \quad (\text{B.6})$$

At $n = 4$ the absolute majorant is therefore a constant times $\sum_l d_l a_l^{-2} < \infty$. The geometric majorant for the higher Taylor terms then permits all stated interchanges by dominated convergence and Tonelli’s theorem applied to absolute values. Each coefficient is

$$[y^n] \mathcal{V}_{4,\text{rel}} = \frac{(-1)^{n+1}}{32\pi^2 n} \frac{\Gamma(n-2)}{\Gamma(n)} \sum_l \eta_l d_l g_l^n a_l^{2-n}, \quad n \geq 4. \quad (\text{B.7})$$

The internal asymptotics give $d_l a_l^{2-n} = O(l^{6-2n})$. So $n \geq 4$ is a sufficient absolute-convergence threshold for the stated class. In contrast the $n = 2$ external integral diverges for a single nonzero g_l . At $n = 3$ the external integral is finite but the internal sum can diverge. Thus one internal subtraction cannot justify the full trace in this class. $\square$

This estimate is restricted to the displayed constant-background Hessian. Derivative interactions and other regulator limits require separate bounds. The radial mass in this four-dimensional realization uses the complete V_4 with kinetic normalization and mixing conditions of Theorem B.8.

B.3 Zero modes and chirality

Zero-mode content and normalized projection. On round $X_3 = \mathbb{RP}^3$ the scalar Laplacian has one constant zero mode. Untwisted ordinary one-forms and spinors have none. The TT rough Laplacian also has no zero mode. Internally constant $A_\mu(x)$ are scalar zero modes valued in the supplied gauge algebra not Killing one-forms or isometry generators. Assume all bundles are pullbacks over the compact factor and retain the relevant constant scalar modes. Assume also that an explicit chiral mechanism realizes the module of Section 4.6. Restriction of (B.1) to these modes gives a four-dimensional action with the retained bundle representations. For the Yang–Mills term (B.3) canonical zero-mode normalization gives

$$g_4^2 = \frac{g_7^2}{\text{Vol}(X_3)} = \frac{g_7^2}{\pi^2 R_b^3}. \quad (\text{B.8})$$

This fixes the projected coupling g_4 for the supplied g_7 and radius. Integrating out the omitted modes can add threshold corrections. Connectedness gives the scalar zero mode and finite fundamental group gives $H_{\text{dR}}^1(X_3; \mathbb{R}) = 0$. The identity $D^2 = \nabla^* \nabla + R_{\text{scal}}/4$ excludes spinor zero modes while (A.6) starts at $6/R_b^2$. The internal tensor type of A_μ is scalar. Orthogonal low-mode projection retains the stated representations. Integration of the constant gauge kinetic term multiplies it by $\text{Vol}(X_3) = \pi^2 R_b^3$ giving (B.8).

Realizing the chiral module of Section 4.6 on this product space requires the stated chiral projection or twisting. The untwisted Dirac tower has the positive and negative levels listed in Appendix A.

For the heavy-mode expansion the needed resolvent bound is uniform in both momentum and internal level. Let $m_l \geq M > 0$ and $A_l(q) = q^2 + m_l^2$ with $\delta(q, P) = 2q \cdot P + P^2$. For every real Euclidean loop momentum q and omitted mode l with $|P| \leq \varepsilon M$ put $r = \varepsilon + \varepsilon^2 < 1$. Then

$$\frac{1}{(q+P)^2 + m_l^2} = \sum_{j=0}^N \frac{(-\delta(q, P))^j}{A_l(q)^{j+1}} + \mathcal{R}_{N,l}(q, P), \quad (\text{B.9})$$

$$|\mathcal{R}_{N,l}(q, P)| \leq \frac{r^{N+1}}{(1-r)A_l(q)}. \quad (\text{B.10})$$

The elementary inequalities $2|q|/(q^2 + m_l^2) \leq 1/m_l$ and $(q^2 + m_l^2)^{-1} \leq m_l^{-2}$ give $|\delta|/A_l \leq \varepsilon + \varepsilon^2 = r$. The finite geometric-series identity gives the stated expansion and remainder uniformly in both q and l .

Appelquist–Carazzone decoupling[44] motivates the local low-momentum expansion. For an infinite internal tower interchanging the mode sum and loop integral with the expansion also requires a common summable majorant.

Decoupling at fixed regulator and perturbative order. Consider the heavy-field contribution to a Wilson action or a heavy subgraph after its light-field contribution has been separated at one fixed perturbative order and with a fixed regulator. Suppose every integrated propagator in this contribution has the form in (B.9) and the regulator is independent of the external momenta for the chosen routing. For a graph with E heavy propagators and L loop momenta let $\mathbf{l}$ label its internal modes. Include all multiplicities and vertex coefficients in a nonnegative majorant $B_{\mathbf{l}}(\mathbf{q})$ for the absolute sum of its numerator monomials on the external momentum ball and require

$$C_{\Lambda} := \sum_{\mathbf{l}} \int \frac{d^{4L}\mathbf{q}}{(2\pi)^{4L}} |w_{\Lambda, \mathbf{l}}(\mathbf{q})| B_{\mathbf{l}}(\mathbf{q}) \prod_{e=1}^E \frac{1}{q_e^2 + m_{l_e}^2} < \infty. \quad (\text{B.11})$$

If each routed external shift obeys $|P_e| \leq \varepsilon M$ with $r = \varepsilon + \varepsilon^2 < 1$ replacing every heavy denominator by the degree- N sum in (B.9) produces a polynomial in the external momenta with amplitude error bounded by

$$|\mathcal{A}_{\Lambda} - \mathcal{A}_{\Lambda, N}| \leq C_{\Lambda} \frac{E r^{N+1}}{(1-r)^E}. \quad (\text{B.12})$$

Consequently the internal sum and loop integration may be interchanged with this convergent momentum expansion at the fixed regulator and order. Its coefficients give local operators with a controlled expansion in external momentum over M . Their sizes and relevant or marginal threshold corrections depend on the vertices and renormalization conditions. Both an exact denominator and its truncated sum are bounded by $[(1-r)A_l]^{-1}$. Telescope the difference of the products of E denominators and apply (B.10) to each of its E terms. The pointwise difference is bounded by $E r^{N+1} (1-r)^{-E} \prod_e A_{l_e}^{-1}$. Multiplication by the numerator majorant and integration gives (B.12). The same majorant bounds the absolutely summed geometric expansions. So Fubini’s theorem and dominated convergence justify the interchanges. Each finite term is polynomial in external momenta and therefore corresponds to a local derivative operator.

Finite mode and compact momentum cutoffs satisfy (B.11) for polynomial vertices and positive masses. For an infinite tower this bound controls the combined mode sum and loop integral. Removing the regulator further requires a uniform majorant and convergent local counterterms. Nonanalytic light-line terms remain in the light theory and are subtracted in matching. Symmetry-preserving regulators and counterterms give gauge-covariant local operators.

B.4 Gauge identities and matter representations

For the connection on the generated G_{int} bundle of Definition B.1

$$F_A = dA + A \wedge A. \quad (\text{B.13})$$

Together with the matter derivative of Proposition 8.1 this gives the component vertices. In native variables $D = d + A$ fixes the relative coefficients of the derivative and cubic terms together with the quartic and matter vertices. Writing $A = g\mathcal{A}$ after normalizing the quadratic term introduces the usual action coordinate g . It records the conversion from the generated trace normalization to the chosen canonical field coordinate.

Gauge vertices and component identities. Let an admissible realization contain

$$S_{\text{YM}}[A] = \frac{1}{2g^2} \int_{M_4 \times X_3} \mathcal{B}(F_A \wedge \star F_A) \quad (\text{B.14})$$

with the positive pairing (B.2) and the covariant Dirac form for the generated matter module. Expansion gives the quadratic propagator term and cubic and quartic gauge vertices together with minimal matter couplings. The overall value assigned to g fixes the action unit while all relative vertex coefficients are already fixed by $F_A = dA + A \wedge A$ and the generated representation. The Gaussian source state supplies the quadratic reference covariance. The interacting local quantization of Theorem C.4 is defined by relative S -operators and therefore requires no additional global path-integral measure. Write $A = A_M^a T_a dz^M$ with $[T_b, T_c] = f_{bc}^a T_a$ and define the matter current by $\delta S_{\text{mat}} = - \int \mathcal{B}(\delta A \wedge \star J)$. Then

$$F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + f_{bc}^a A_M^b A_N^c, \quad (\text{B.15})$$

$$D_A F_A = 0, \quad D_A \star F_A = g^2 \star J. \quad (\text{B.16})$$

Gauge covariance of the matter action implies $D_A \star J = 0$ on the matter equations of motion. The covariant derivative is $D_M \psi = (\partial_M + A_M^a T_a) \psi$ in a local frame. Expansion of $F_A = dA + A \wedge A$ gives the displayed components and the quadratic and cubic terms with the quartic terms of the action. The first identity in (B.16) follows from the graded Jacobi identity. Variation of (B.14) gives the second identity with the stated current convention. Gauge covariance and the matter equations give covariant current conservation.

Proposition B.4 (BRST symmetry and the Slavnov–Taylor identity). Let the action-level realization use a gauge-fixing fermion Ψ_{gf} and fields $c, \bar{c}, B$ with

$$sA = D_A c, \quad sc = -\frac{1}{2}[c, c], \quad s\bar{c} = B, \quad sB = 0, \quad S_{\text{gf}} + S_{\text{gh}} = s\Psi_{\text{gf}}. \quad (\text{B.17})$$

Classically $s^2 = 0$ [40, 41]. For a gauge-compatible finite internal resolution of the renormalizable four-dimensional gauge–scalar–spinor realization the classical BV master equation is preserved. In causal perturbation theory the anomaly identities (4.54) and the perturbative Adler–Bardeen theorem then permit time-ordered products to be normalized so that the quantum master identity and hence the Slavnov–Taylor identity holds to all orders [40, 45–47]. At a finite FRG scale the identity is generally modified by the regulator and must be restored as $k \rightarrow 0$ [48]. The anomaly check following Theorem 4.9 supplies the required local coefficients. The even number of weak doublets removes the global mod-two obstruction. The resolution retains complete representation multiplets and the original matrix Lie bracket. The classical closure and power-counting conditions are spelled out in Appendix C.8.

Fermion doubling. For a lattice discretization of the generated event base fermion doubling is governed by the locality and charge assumptions with the chiral-fermion hypotheses of the Nielsen–Ninomiya theorem [49]. Its application depends on the chosen external discretization. The internal expansion used here is in the eigenmodes of the compact continuum operator.

B.5 Regulated scale equations

Wetterich’s exact equation [50, equation (3)] expresses the flow through the full regulated propagator. For the generated gauge realization we use its background-field form developed for gauge theories in [51] with the field and ghost contributions in a supertrace

$$\partial_k \Gamma_k[\Phi; \bar{g}, \bar{A}] = \frac{1}{2} \text{STr} \left[(\Gamma_k^{(2)} + R_k)^{-1} \partial_k R_k \right]. \quad (\text{B.18})$$

The RG scale k parametrizes the regulator while b is the Gaussian width. The dimensional radius is $R_b = \sqrt{2\pi} b$. Its numerical value $L = R_b/\ell_*$ uses the fixed reference unit of (5.17). Thus the same radius convention is used in the relative drive and in the scale equations. Gauge consistency also requires the modified Ward or Slavnov–Taylor identity of Proposition B.4.

After a truncation $\Gamma_k = \sum_i g_i(k) \mathcal{O}_i$ and projectors $\mathcal{P}_i$ have been specified the beta functions are

$$\partial_k g_i(k) = \mathcal{P}_i \frac{1}{2} \text{STr} \left[(\Gamma_k^{(2)} + R_k)^{-1} \partial_k R_k \right]. \quad (\text{B.19})$$

The projected coefficients depend on the full supertrace including tensor projectors and ghosts together with endomorphism terms and regulator weights. For a locally Lipschitz finite projected system the standard initial-value theorem gives a unique local evolution while the solution remains in its coefficient domain. Preservation of a positivity domain requires the corresponding inward-pointing condition. A global endpoint requires bounds that keep the projected solution in its coefficient domain. The Gaussian resolution of Proposition A.2 acts directly on the mode coefficients at fixed radius.

For an interacting OS realization a regulated positive measure must also admit a limit with the hierarchy properties stated in Appendix C.1. Uniform moment bounds are needed to pass reflected polynomial inequalities to that limit. The free case is constructed in Appendix C.

B.6 The long-root response

We write χ_R for the right geometric response and $O_c = \text{Tr} F_c^2$ for the colour composite. The symbols E_c and ρ_H denote respectively the normalized collective colour scalar and radial fluctuation of the weak doublet H .

The long-root Hessian contains the Laplace contribution (A.12) together with the background gauge-fixing and endomorphism terms of the supplied action. Its sign must be computed from their sum.

Definition B.5 (Gauge-fixed long-root Hessian). Let $\chi_R = \sqrt{y} u_R$ where u_R is a normalized vector in the real long-root response line and $y \geq 0$. For an admissible action-level realization define

$$\mathcal{H}_{\text{gf}}(y) := \frac{\delta^2(S + S_{\text{gf}})}{\delta\Phi^2} \Big|_{\chi_R = \sqrt{y} u_R}, \quad (\text{B.20})$$

together with its ghost operator and a common closed quadratic-form domain. The chosen action specifies the field space and internal multiplicities. For matter contributions replacing $\mathcal{M}_{\text{full}}$ by $\mathcal{M}_{\text{ch}}$ is permitted only after verifying the [quotient and trace conditions](#) of Section 5.4 for the relevant operators. Otherwise the trace is taken over the full module including its neutral direction.

Conditional right-sector breaking. Suppose the renormalized action is $SU(2)_R$ invariant and its first physical negative Hessian eigenvalue lies in a real $j_R = 1$ component of the long-root sector and the nonlinear effective action stabilizes a nonzero minimum $\langle \chi_R \rangle$ in that component. Then

$$SU(2)_R \longrightarrow \text{Stab}(\langle \chi_R \rangle) \cong U(1)_R, \quad (\text{B.21})$$

up to conjugation. The two transverse right generators are broken while the generator along the selected axis is unbroken. The real spin-one representation of $SU(2)_R$ factors through the vector representation of $SO(3)$. The stabilizer of a nonzero vector is the group of rotations about its axis whose connected lift is a maximal torus of $SU(2)_R$. All maximal tori are conjugate.

B.7 Internal relative determinant and four-dimensional action

Theorem B.6 (Internal relative determinant and matching). Suppose the compact internal parts of $\mathcal{H}_{\text{gf}}(y)$ and the ghost operator form analytic families with common form domains and suppose that after sector decomposition their nonzero eigenvalues have the form $a_l + g_l y$ with $a_l > 0$ and real g_l . The trace in this theorem is over the internal modes only. Equivalently it can be taken at one fixed external momentum after that momentum has been included in a_l . With a specified linear subtraction define

$$\Phi_R^{\text{int}}(y) - \Phi_R^{\text{int}}(0) = c_{1,\text{int}} y + \frac{1}{2} \sum_l \eta_l d_l \left[\log \left(1 + \frac{g_l y}{a_l} \right) - \frac{g_l y}{a_l} \right], \quad (\text{B.22})$$

where η_l includes statistics and ghost signs with tensor weights. If $a_l \asymp l^2$ and $d_l = O(l^2)$ with $g_l = O(1)$ and $\eta_l = O(1)$ the subtracted series and each of its derivatives converge locally uniformly on the real domain $a_l + g_l y > 0$ for every l . The coefficient $c_{1,\text{int}}$ specifies the linear term of this internal functional. The corresponding four-dimensional mass also depends on the external trace and kinetic normalization.

Proof. At a finite internal cutoff the difference of logarithmic determinants gives the logarithm in (B.22). Its first-order Taylor term is removed by the stated subtraction before the cutoff is removed. On a compact subset of the positive domain the remaining summand and its first derivative are $O(d_l a_l^{-2}) = O(l^{-2})$. Derivatives of order $j \geq 2$ are $O(d_l |g_l|^j a_l^{-j})$

uniformly for large l . The finitely many remaining denominators stay bounded away from zero. The Weierstrass test therefore justifies the asserted differentiations as well as the sum. $\square$

The full four-dimensional potential $V_4(y)$ also traces over the external operator. In the separated flat-base realization its regulated one-loop part instead contains

$$\begin{aligned} V_{4,\Lambda}(y) - V_{4,\Lambda}(0) &= V_{\text{loc},\Lambda}(y) - V_{\text{loc},\Lambda}(0) \\ &+ \frac{1}{2} \sum_l \eta_l d_l \int \frac{d^4 p}{(2\pi)^4} w_{\Lambda,l}(p) \\ &\times \left[\log \left(1 + \frac{g_l y}{p^2 + a_l} \right) - \frac{g_l y}{p^2 + a_l} \right]. \end{aligned} \quad (\text{B.23})$$

The cutoff expression is well defined for an integrable external regulator. Its quadratic term has a logarithmically divergent external integral. Proposition B.3 gives the sufficient three-subtraction bound for the specified separated towers.

Renormalization dependence. Two admissible renormalization schemes may differ by finite local terms allowed by the symmetries. A critical radius or mass in this realization depends on those terms and on the kinetic normalization together with the internal Casimir contribution $16/L^2$.

Proposition B.7 (Internal quartic coefficient). Under the hypotheses of Theorem B.6 the coefficient of y^k for $k \geq 2$ for a single bosonic tower with constant coupling g is

$$c_k = \frac{(-1)^{k+1} g^k}{2k} \sum_l d_l a_l^{-k}. \quad (\text{B.24})$$

For the full signed Hessian the coefficient of y^2 is

$$\frac{\lambda_{R,\text{int}}}{4}, \quad \lambda_{R,\text{int}} = - \sum_l \eta_l d_l \frac{g_l^2}{a_l^2}. \quad (\text{B.25})$$

The sum converges in three internal dimensions. Its sign is determined by the signed Hessian and ghost weights. One positive bosonic tower gives a negative coefficient. The negative-logarithm spectral response has the opposite sign. The four-dimensional quartic coefficient further includes the external trace and local terms with field normalization.

Theorem B.8 (Normalized radial Hessian and conditional pole). Suppose the complete four-dimensional effective action after eliminating gauge directions and resolving any mixing has an isolated physical radial coordinate $r = \sqrt{y}$ with two-derivative terms

$$\Gamma_4[r] = \int d^4 x \left\{ \frac{1}{2} Z_R(r^2) (\partial r)^2 + V_4(r^2) \right\} + \Gamma_{\geq 4\partial}[r]. \quad (\text{B.26})$$

Assume $V_4'(y_*) = 0$ with $y_* > 0$ and $V_4''(y_*) > 0$ together with $Z_* = Z_R(y_*) > 0$. Then $\rho_R = \sqrt{Z_*}(r - \sqrt{y_*})$ has positive curvature mass

$$m_{R,\text{curv}}^2 = \frac{4y_* V_4''(y_*)}{Z_*}. \quad (\text{B.27})$$

This is the mass parameter of the two-derivative quadratic action. For an isolated pole additionally suppose the full normalized Euclidean inverse covariance is analytic through $z = p^2 \in [-2m_{R,\text{curv}}^2, 0]$ and has the form $K_R(z) = m_{R,\text{curv}}^2 + z + E_R(z)$ with $E_R(0) = E'_R(0) = 0$ and real values on that interval satisfying $|E'_R(z)| \leq \epsilon < 1/2$ there. The covariance $1/K_R$ then has a unique simple pole of positive residue at $z = -m_{R,\text{pole}}^2$ in that interval and

$$\frac{m_{R,\text{curv}}^2}{1 + \epsilon} \leq m_{R,\text{pole}}^2 \leq \frac{m_{R,\text{curv}}^2}{1 - \epsilon}. \quad (\text{B.28})$$

Proof. At the stationary point $\partial_r^2 V_4(r^2) = 4y_* V_4''(y_*)$. Division by Z_* gives (B.27). For the second assertion $K'_R(z) \geq 1 - \epsilon > 0$ and $K_R(0) > 0$ with $K_R(-2m_{R,\text{curv}}^2) < 0$. The intermediate value theorem and strict monotonicity give a unique zero with positive residue $1/K'_R$. The bound $|E_R(z)| \leq \epsilon|z|$ gives (B.28) at this zero. $\square$

The coefficient Z_R includes internal integration and wave-function normalization. Coupled channels use the generalized eigenvalue problem of Section 9.4 or a justified Schur reduction.

Controlled use of a Landau truncation. An even Landau normal form applies to a signed response such as a component of χ_R when its action is invariant under sign reversal. For the colour-singlet composite $O_c = \text{Tr } F_c^2$ an even expansion requires an additional symmetry that acts by $O_c \mapsto -O_c$.

C Field reconstruction on the generated Lorentzian carrier

The product covariance of Section 10 admits the standard Gaussian and Osterwalder–Schrader reconstruction. The Euclidean carrier $\mathbb{E}_V$ with its reflection and Lorentzian continuation were generated in Section 10.1. The internal spectral data are those of the preceding construction. The kinetic operator is the translation-invariant quadratic operator associated with the generated trace metric. A common Wick moment bound controls the full internal tower. On the condensed quadratic branch its averaged intensity has an isolated pole at the scalar gap and a positive two-particle continuum with an explicit truncation bound.

C.1 Reconstruction conditions

We use signature $(+ - - -)$ and the covering Poincaré group $\mathbb{M}_V \rtimes SL(V)$ generated in Theorem 10.1.

The Wightman conventions are those of Ref. [28]. Tempered fields and their adjoints act on a common invariant dense domain and transform covariantly under a strongly continuous unitary Poincaré representation and graded-commute at spacelike separation. Translations have spectrum in the closed forward cone and the invariant vacuum is cyclic and unique up to phase. For neutral scalar fields we use Osterwalder–Schrader reconstruction with Euclidean covariance and reflection positivity together with permutation symmetry and the regularity and growth conditions for tempered reconstruction. Clustering gives vacuum uniqueness [42, 43, 52].

These assumptions also fix the role of Haag's theorem. For Hermitian scalar Wightman fields admitting sharp-time restrictions a unitary intertwiner of the fields and vacua on one spacelike hyperplane with a positive-mass free field forces the other field to be free with the same mass [27–29]. The reconstruction below builds the free representation from its covariance. So its equivalence statement has precisely this fixed-covariance scope.

C.2 Covariance and reflection positivity

Identify the generated Euclidean event carrier $\mathbb{E}_V$ with $\mathbb{R}^4$ by the trace-orthonormal coordinates of (10.9). Its reflection is $\vartheta_E(\tau, \mathbf{x}) = (-\tau, \mathbf{x})$. On the compact internal space X_3 let $\Delta_3 \geq 0$ be a self-adjoint Laplace-type operator with eigenvalues ε_l and spectral projectors P_l of finite multiplicities $d_l = \text{rank } P_l$. For $m^2 > 0$ put

$$P_E = -\sum_{\mu=0}^3 \partial_\mu^2, \quad C = (P_E \otimes I + I \otimes \Delta_3 + m^2)^{-1}. \quad (\text{C.1})$$

On $\mathcal{S}(\mathbb{R}^4) \hat{\otimes} C^\infty(X_3)$ spectral calculus gives

$$C = \sum_l (P_E + M_l^2)^{-1} \otimes P_l, \quad M_l^2 = m^2 + \varepsilon_l, \quad (\text{C.2})$$

$$C((x, y), (x', y')) = \sum_l K_l(y, y') G_{M_l}(x - x'), \quad (\text{C.3})$$

where K_l is the kernel of P_l and G_{M_l} the massive Euclidean free propagator. The covariance is directly related to the internal restoring semigroup. For any strictly positive internal one-response operator $\mathcal{M}^2$ spectral calculus gives at fixed Euclidean momentum

$$(p_E^2 + \mathcal{M}^2)^{-1} = \int_0^\infty e^{-sp_E^2} e^{-s\mathcal{M}^2} ds. \quad (\text{C.4})$$

Here s has units of inverse mass squared. The integral retains every internal spectral projector and converts its restoring eigenvalue into a one-particle mass square. It applies to the scalar tower above and to the positive blocks of $\mathcal{M}_{\text{YM}}^2$. Physical time is the Lorentz translation parameter reconstructed from this covariance.

Reflection positivity can be checked before reconstruction. For a positive-time test function write $\hat{f}_l(t, \mathbf{p})$ for its spatial Fourier transform projected by P_l . Its reflected covariance form is

$$\sum_l \int \frac{d^3 \mathbf{p}}{2\omega_l(\mathbf{p})} \left\| \int_0^\infty e^{-\omega_l(\mathbf{p})t} \hat{f}_l(t, \mathbf{p}) dt \right\|^2 \geq 0, \quad \omega_l(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M_l^2},$$

with a unitary Fourier convention. The positive orthogonal sum converges on the test space. Gaussian Wick extension preserves this positivity. A deterministic translation commuting with reflection preserves the same quadratic forms for Gaussian cylinder observables. In particular a reflection-invariant constant mean preserves positivity.

C.3 Gaussian measure and reconstruction

Theorem C.1 (Free reconstruction of the internal tower). The functional

$$Z(f) = \exp[-\tfrac{1}{2}\langle f, Cf \rangle] \quad (\text{C.5})$$

defines a unique centred Gaussian measure on the distribution dual. Its Wick Schwinger functions are symmetric and Euclidean covariant. They are also tempered and reflection positive. OS reconstruction yields local bosonic free fields with the vacuum in Fock space over

$$\mathcal{H}_1 = \bigoplus_l L^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{2\sqrt{\mathbf{p}^2 + M_l^2}}\right) \otimes P_l L^2(X_3). \quad (\text{C.6})$$

The one-particle masses are M_l with multiplicities d_l . For this fixed covariance the spectral and quasi-free descriptions are unitarily equivalent preserving the vacuum and projectors. The full Hamiltonian is $d\Gamma(\omega)$ with the continuous momentum and multiparticle spectra obtained by second quantization. The associated Weyl algebras form a local isotonic and Poincaré-covariant net with positive-energy translations. Connected correlations of fixed local smeared polynomial observables decay exponentially at the least mass to which they couple up to polynomial factors. Every rate strictly below the positive mass lower bound is an exponential bound.

Proof. The positive covariance is continuous on the nuclear test space. So Bochner–Minlos gives the Gaussian measure [53]. Euclidean covariance follows from the mode resolvents. To control the full Schwinger hierarchy use $\|C\| \leq m^{-2}$ on the product L^2 space. Wick’s rule gives for $n \geq 1$

$$|S^{(2n)}(f_1, \dots, f_{2n})| \leq (2n-1)!! m^{-2n} \prod_{k=1}^{2n} \|f_k\|_2, \quad S^{(2n+1)} = 0.$$

The continuous test-space seminorm on the right and the Gaussian factorial bound give the required hierarchy regularity. Smooth internal smearing controls the mode sum. $m > 0$ controls the infrared limit and gives free clustering. The reflected form above supplies positivity. The OS theorem reconstructs the Hilbert space and vacuum with positive-energy fields [42, 43, 52]. Modewise reconstruction gives (C.6) and bosonic second quantization gives the full Hamiltonian. Uniqueness of the quasi-free reconstruction gives the asserted equivalence. The massive propagator bounds persist under smearing and the convergent internal sum. Wick contraction expresses connected polynomial correlations through cross propagators giving the stated clustering. The modewise Pauli–Jordan commutators have causal support. Hence the Weyl algebras are local and inclusion of supports gives isotony. $\square$

This realizes the state and operation construction of Section 5 on the generated event base. The generated Poincaré action commutes with compatible internal actions.

The relation between the internal mass and Lorentz translations is exact on the one-particle space. On its l th summand $P_1^0 = \sqrt{\mathbf{p}^2 + M_l^2}$ and $P_1^i = p_i$. Consequently

$$(P_1^0)^2 - \sum_i (P_1^i)^2 = \mathcal{M}^2, \quad \mathcal{M}^2 = \Delta_3 + m^2 \quad (\text{C.7})$$

under the direct-integral identification. Multiparticle translations add the four-momenta. For future-directed on-shell momenta this gives

$$P_{\text{tot}}^2 = \sum_r M_r^2 + 2 \sum_{r < s} (E_r E_s - \mathbf{p}_r \cdot \mathbf{p}_s) \geq \left(\sum_r M_r \right)^2. \quad (\text{C.8})$$

The inequality follows from the reverse Cauchy inequality for timelike future vectors. It gives the two-particle threshold $4M^2$ for two equal masses. In contrast the internal response completion adds the one-response eigenvalues through $d\Gamma(\mathcal{M}^2)$. The two generators agree on their internal one-response mass data and organize composite excitations according to their respective composition laws. On the Lorentz Fock space $H \geq m_{\min} \mathbf{N}$ with $m_{\min} = \sqrt{\inf \text{Spec } \mathcal{M}^2} > 0$. In infinite event volume its lower positive energy is a momentum threshold while the internal operator has the attained eigenvalues proved above.

C.4 Positive spectral measure

For an internal test vector $g \in L^2(X_3)$ the field $\phi_g(x) = \phi(x \otimes g)$ has positive measure

$$\rho_g(d\mu^2) = \sum_l \|P_l g\|^2 \delta_{M_l^2}(d\mu^2). \quad (\text{C.9})$$

These one-particle weights differ from the multiplicities d_l of an internal trace. Composite correlations can also contain multiparticle continua. The masses use the supplied shift m^2 and the internal eigenvalues through $M_l^2 = m^2 + \varepsilon_l$. Theorem 7.2 controls the associated internal traces.

C.5 The intensity spectrum of the condensed quadratic tower

Fix a condensed branch of Corollary 8.3 with $y_L > 0$ and $Z_L > 0$ together with $m_\delta^2 = 4y_L F_L''(y_L)/Z_L$. Use the quadratic external realization (C.1) with $m^2 = m_\delta^2$ and the even scalar Laplacian on $\mathbb{RP}^3$ of radius L . Let e_{la} be a real orthonormal basis for normalized Haar measure where $l = 0, 2, \dots$ and $a = 0, \dots, (l+1)^2 - 1$ with $e_{00} = 1$. The canonically normalized independent fields φ_{la} have masses

$$M_l^2 = m_\delta^2 + \frac{l(l+2)}{L^2}.$$

The fluctuation of the amplitude is $E(x, \xi) = \sqrt{y_L} + Z_L^{-1/2} \sum_{la} \varphi_{la}(x) e_{la}(\xi)$. Its centred internally averaged Wick intensity is defined by spectral cutoffs and external test-function smearing

$$\mathcal{I}(x) = \int_{X_3} : (E(x, \xi)^2 - y_L) : d\mu_3(\xi) = \frac{2\sqrt{y_L}}{\sqrt{Z_L}} \varphi_{00}(x) + \frac{1}{Z_L} \sum_{la} : \varphi_{la}(x)^2 :. \quad (\text{C.10})$$

Here x belongs to the reconstructed Minkowski representation and normal ordering is with respect to its free vacuum. The mass and momentum variables use the same response units.

Proposition C.2 (Positive pole and continuum of the Wick intensity). With Fourier measure $d^4p/(2\pi)^4$ the connected two-point function of (C.10) has positive spectral measure

$$\rho_{\mathcal{I}}(ds) = \frac{4y_L}{Z_L} \delta_{m_\delta^2}(ds) + \frac{1}{8\pi^2 Z_L^2} \sum_{l \text{ even}} (l+1)^2 \sqrt{1 - \frac{4M_l^2}{s}} \mathbf{1}_{s \geq 4M_l^2} ds. \quad (\text{C.11})$$

The one-particle pole has positive weight $4y_L/Z_L$ and is isolated from the continuum which starts at $s = 4m_\delta^2$. The infinite sum defines a tempered positive two-point distribution. For $z = Q^2 \geq 0$ choose the continuum subtraction with value and first z derivative zero at $z = 0$. The corresponding response is

$$C_{\mathcal{I}}^{\text{R}}(z) = \frac{4y_L}{Z_L(z + m_\delta^2)} + \frac{2}{Z_L^2} \sum_{l \text{ even}} (l+1)^2 B_{\text{R}}(z; M_l^2), \quad (\text{C.12})$$

$$B_{\text{R}}(z; a) = \frac{1}{16\pi^2} \int_0^1 \left[\frac{u(1-u)z}{a} - \log \left(1 + \frac{u(1-u)z}{a} \right) \right] du. \quad (\text{C.13})$$

If the internal sum retains $l = 2j$ for $0 \leq j \leq J$ its omitted continuum contribution obeys

$$0 \leq \mathcal{R}_J(z) \leq \frac{81z^2 L^4}{61440\pi^2 Z_L^2 (2J+1)}. \quad (\text{C.14})$$

Proof. Orthonormality gives (C.10). Odd Gaussian moments make its linear-quadratic cross term vanish. Wick contraction of each squared field gives $2G_{M_l}^2$. The two-particle phase-space density of this contraction is $(8\pi^2)^{-1} \sqrt{1 - 4M_l^2/s} \mathbf{1}_{s \geq 4M_l^2}$. The linear term supplies the displayed atom. At every bounded s only finitely many internal shells contribute. Their total multiplicity is $O(s^{3/2})$ giving polynomial spectral growth. External Schwartz smearing therefore makes the cutoff two-point functions converge. Equivalently the cutoff observables applied to the vacuum converge in the free Hilbert space.

For each mass $a > 0$ two subtractions of the Euclidean bubble give (C.13). Its dispersion form is

$$B_{\text{R}}(z; a) = \frac{z^2}{16\pi^2} \int_{4a}^{\infty} \frac{\sqrt{1 - 4a/s}}{s^2(s+z)} ds.$$

The bound $0 \leq t - \log(1+t) \leq t^2/2$ and $\int_0^1 u^2(1-u)^2 du = 1/30$ imply $0 \leq B_{\text{R}}(z; a) \leq z^2/(960\pi^2 a^2)$. For the omitted shells $j \geq J+1 \geq 1$

$$M_{2j}^2 \geq \frac{8(2j+1)^2}{9L^2}, \quad \sum_{j>J} \frac{(2j+1)^2}{M_{2j}^4} \leq \frac{81L^4}{64} \sum_{j>J} \frac{1}{(2j+1)^2} \leq \frac{81L^4}{128(2J+1)}.$$

This proves (C.14) and convergence of the subtracted response. The bubble sum is locally normally convergent off its cut $(-\infty, -4m_\delta^2]$. So is analytic at $z = -m_\delta^2$. The isolated pole and its weight follow. Local contact polynomials affect neither the spectral measure nor this residue. $\square$

The intensity therefore measures the internal scalar gap in this quadratic realization. Its complete two-point response can be bounded using finite spectral sums. For $0 \leq z/a \leq 4$ the bubble integral before its factor $1/(16\pi^2)$ has the convergent alternating expansion

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \left(\frac{z}{a}\right)^n \frac{(n!)^2}{(2n+1)!}.$$

Adjacent even and odd partial sums give rational bounds for each retained shell. Together with (C.14) these propagate the existing intervals for y_L and m_δ^2 directly to the response. For a colour observable in a common source representation a nonzero cross weight at the same mass gives a positive lower bound on its mass weight (Appendix G.8.5).

C.6 The radial curvature polynomial

The Gaussian colour source selects the radial polynomial $P(N) = \lambda_R N + \kappa_R N^2$ of (G.54). In the condensed quadratic realization above evaluate this polynomial on the internally integrated scalar amplitude

$$\mathcal{R}(x) =: P\left(\int_{X_3} E(x, \xi)^2 d\mu_3(\xi)\right) : - P(y_L). \quad (\text{C.15})$$

At finite internal cutoff expand the polynomial and Wick order its elementary fields together. The limit is taken after external smearing. The internal integral precedes the polynomial evaluation.

Proposition C.3 (Spectrum of the radial curvature observable). Write $y = y_L$ and $Z = Z_L$ with $m = m_\delta$ and enumerate the real internal modes including multiplicities by i with $i = 0$ the constant mode. Set

$$\begin{aligned} a &= \frac{2\sqrt{y}}{\sqrt{Z}}, & b &= \frac{1}{Z}, & g_1 &= a(\lambda_R + 2\kappa_R y), \\ g_{2i} &= b(\lambda_R + 2\kappa_R y) + \kappa_R a^2 \delta_{i0}, & h &= 2\kappa_R a b, & k &= \kappa_R b^2. \end{aligned}$$

Let $\rho_r(s; m_1, \dots, m_r)$ be the positive spectral density of the product of r free two-point functions normalized by

$$\rho_2(s; m_1, m_2) = \frac{\sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}}{16\pi^2 s} \mathbf{1}_{s \geq (m_1 + m_2)^2}.$$

The connected measure of (C.15) is

$$\begin{aligned} \rho_{\mathcal{R}}(ds) &= g_1^2 \delta_{m^2}(ds) + \rho_{\text{cont}}(s) ds, \\ \rho_{\text{cont}}(s) &= 2 \sum_i g_{2i}^2 \rho_2(s; M_i, M_i) \\ &\quad + h^2 \left[2 \sum_i \rho_3(s; m, M_i, M_i) + 4 \rho_3(s; m, m, m) \right] \\ &\quad + k^2 \left[8 \sum_{ij} \rho_4(s; M_i, M_i, M_j, M_j) + 16 \sum_i \rho_4(s; M_i, M_i, M_i, M_i) \right]. \end{aligned} \quad (\text{C.16})$$

The atom has weight

$$g_1^2 = \frac{4y}{Z}(\lambda_R + 2\kappa_R y)^2 > 0.$$

The continuum begins at $4m^2$. The three- and four-particle contributions begin at $9m^2$ and $16m^2$. The full internal tower defines a tempered positive two-point distribution with $\rho_{\text{cont}}(s) = O((1+s)^5)$ in fixed response units. Six continuum subtractions give the convergent response

$$C_{\mathcal{R}}^{\text{R}}(z) = \frac{g_1^2}{z + m^2} + z^6 \int_{4m^2}^{\infty} \frac{\rho_{\text{cont}}(s)}{s^6(s+z)} ds, \quad z = Q^2 \geq 0. \quad (\text{C.17})$$

Proof. Orthonormality gives $\int E^2 = y + a\varphi_0 + b \sum_i \varphi_i^2$. Polynomial expansion and common Wick ordering yield

$$\mathcal{R} = g_1\varphi_0 + \sum_i g_{2i} : \varphi_i^2 : + h \sum_i : \varphi_0 \varphi_i^2 : + k \sum_{ij} : \varphi_i^2 \varphi_j^2 : .$$

Different Wick orders are orthogonal. With $D_i = \Delta_+(x; M_i^2)$ and $S_r = \sum_i D_i^r$ the connected covariance is

$$g_1^2 D_0 + 2 \sum_i g_{2i}^2 D_i^2 + h^2(2D_0 S_2 + 4D_0^3) + k^2(8S_2^2 + 16S_4).$$

The cubic zero-mode term has six contractions. Each mixed cubic term has two. The quartic equal-mode term has twenty-four contractions and each unordered unequal pair has sixteen. This proves the spectral coefficients. Successive phase-space convolution gives

$$\rho_{r+1}(s; m_1, \dots, m_{r+1}) = \int_{(\sum_{a=1}^r m_a)^2}^{(\sqrt{s} - m_{r+1})^2} \rho_r(u; m_1, \dots, m_r) \rho_2(s; \sqrt{u}, m_{r+1}) du,$$

with zero for an empty interval. In particular $0 \leq \rho_r(s) \leq s^{r-2}/[(r-2)!(16\pi^2)^{r-1}]$. Only modes with $M_i \leq \sqrt{s}$ contribute and their number is $O((1+s)^{3/2})$. The quadratic and cubic densities and the quartic density are therefore bounded respectively by $O((1+s)^{3/2})$ and $O((1+s)^{5/2})$ and $O((1+s)^5)$. External Schwartz smearing makes the cutoff observables applied to the vacuum a Cauchy sequence by positivity and these bounds. The six-subtracted integral is integrable at infinity as $O(s^{-2})$. Its continuum part is analytic off $(-\infty, -4m^2]$ proving isolation of the positive atom at $z = -m^2$. $\square$

The mixed response with the canonical constant mode is exactly $\langle \varphi_0(x) \mathcal{R}(0) \rangle_c = g_1 \Delta_+(x; m^2)$. Thus this radial observable couples to the scalar excitation through a nonzero one-particle vertex. The subtraction in (C.17) fixes the first six continuum Taylor coefficients at zero. Other local choices add a polynomial of degree at most five and preserve the spectral weights.

C.7 The free local net

For the bosonic tower of Theorem C.1 let F be a real test function on $\mathbb{M}_V \times X_3$ with finite internal spectral support. With $\Phi(F)$ the reconstructed field set

$$\mathfrak{A}(\mathcal{O}) = \{e^{i\Phi(F)} : \text{supp}_{\mathbb{M}_V} F \subset \mathcal{O}\}'' \quad (\text{C.18})$$

for bounded open event regions $\mathcal{O}$. The net is isotonic and Poincaré covariant. It is local with positive-energy translations and the reconstructed vacuum. The Reeh–Schlieder and CPT conclusions hold under the standard Wightman hypotheses [27, 28].

C.8 The interacting local net

Use the four-dimensional gauge–matter realization of the generated carriers. Its event connection A has structure group G_{int} . Retained internal connection components are adjoint scalar fields on $\mathbb{M}_V$. A finite internal resolution Π_N retains complete G_{int} multiplets of these scalar fields and the full matter module $\mathcal{M}_{\text{full}}$ with the elementary connector. The projector acts on mode labels and commutes with the fibre action. The notation Π_N distinguishes this resolution from the neutral projection P_N in Section 5. Locality refers to $x \in \mathbb{M}_V$. Internal overlap coefficients fix the vertices and leave the causal cone unchanged. Products are evaluated with their full internal support before integration over X_3 .

For this field realization gauge parameters depend on the event variable and take values in the generated Lie algebra $\mathfrak{g}_{\text{int}}$. Writing ϕ_N collectively for associated scalar and matter fields the classical differential is

$$s_N A = dc + [A, c], \quad s_N c = -\tfrac{1}{2}[c, c], \quad s_N \phi_N = -\rho_*(c)\phi_N. \quad (\text{C.19})$$

The internal projector commutes with $\rho_*(c)$. So these transformations close on the retained fields. The graded Jacobi identity and the representation identity give $s_N^2 = 0$. In particular the gauge bracket is the original matrix bracket rather than a projected product of internal functions.

For a compactly supported test function f on $\mathbb{M}_V$ write the full retained field multiplet as $\Phi_N = (A, \phi_N)$ and set

$$\mathcal{V}_N(f) = \int_{\mathbb{M}_V} f(x) \mathcal{L}_{\text{int},N}(\Phi_N, \partial\Phi_N)(x) d^4x, \quad (\text{C.20})$$

where $\mathcal{L}_{\text{int},N}$ is obtained by expanding the native curvature $F_A = dA + A \wedge A$ and common matter transport with the scalar connector of Section 8. Its matrix commutators and internal mode products together with the charges and source-natural relative structure are those fixed by the source response. Numerical Yukawa weights of distinct quark and lepton channels remain action-level dynamical data. The free quadratic part is the covariance of Theorem C.1 together with the gauge-fixed normally-hyperbolic and Dirac operators of the same generated modules. The local action used here consists of the curvature square and covariant scalar and Dirac terms together with Yukawa terms and an invariant scalar polynomial of degree at most four. With $[A] = [\phi] = 1$ and $[\psi] = 3/2$ together with $[\partial_\mu] = 1$ its vertices have engineering dimension at most four. Internal derivatives

contribute fixed mode coefficients. This is the renormalizable local field realization. The complete logarithmic functional of Section 8 determines the collective response and its equilibrium masses.

Theorem C.4 (Finite-resolution perturbative local net on the generated Lorentz carrier). At every gauge-compatible finite internal resolution described above the local functional (C.20) determines a finite-resolution Poincaré-covariant causal perturbative local net on $\mathbb{M}_V$ as a formal power series in the native interaction. If T_n are locally covariant time-ordered products and

$$S(F) = I + \sum_{n \geq 1} \frac{i^n}{n!} T_n(F^{\otimes n}), \quad S_V(F) = S(V)^{-1} S(V + F), \quad (\text{C.21})$$

then causal factorization gives

$$S_V(F_+ + F_0 + F_-) = S_V(F_+ + F_0) S_V(F_0)^{-1} S_V(F_0 + F_-) \quad (\text{C.22})$$

whenever $\text{supp } F_+$ is causally later than $\text{supp } F_-$. For a bounded causally convex region $\mathcal{O} \subset \mathbb{M}_V$ choose $f = 1$ on a neighbourhood of its causal hull. The algebraic adiabatic construction

$$\mathfrak{A}_{V,N}(\mathcal{O}) = \text{alg}\{S_{V_N(f)}(F) : F \text{ local, } \text{supp } F \subset \mathcal{O}\} \quad (\text{C.23})$$

is independent of the choice of f up to the canonical relative S -matrix isomorphism. These local equivalence classes form an isotonic local Poincaré-covariant net satisfying the time-slice property. For the gauge theory the physical net is

$$\mathfrak{A}_{\text{phys},N}(\mathcal{O}) = H^0(\mathfrak{A}_{V,N}(\mathcal{O}), s_q), \quad (\text{C.24})$$

where s_q is the renormalized BRST differential. The anomaly identities in (4.54) and the even number of weak doublets permit a BRST-preserving normalization to every perturbative order. Hence $s_q^2 = 0$ and (C.24) is independent of the gauge-fixing representative.

Proof. The generated Minkowski carrier is globally hyperbolic and at fixed N the free gauge-fixed system is a finite Green-hyperbolic multiplet. Its advanced and retarded propagators therefore have support in the cone (10.4). The Peierls bracket of functionals with spacelike separated supports vanishes. The microlocal extension theorems of causal perturbation theory construct the T_n while preserving causal factorization and covariance with the field equation [54, 55]. Equation (C.22) proves isotony and locality. The causal isomorphism between two switching functions that agree near the causal hull of $\mathcal{O}$ gives the algebraic adiabatic limit and the local equation of motion gives the time-slice property.

All internal overlap tensors are intertwiners of the fibre action. Since Π_N commutes with that action restriction to complete retained multiplets preserves invariance of the integrated action S_N so that $s_N S_N = 0$. Couple each field antifield to its variation in (C.19) and include the usual antighost–auxiliary-field pair. Invariance and $s_N^2 = 0$ give the classical BV master equation $(S_{\text{BV},N}, S_{\text{BV},N}) = 0$. Thus the finite field system has the same closed gauge algebra as its source carriers.

For gauge fields one applies the BV–BRST extension. Perturbative Yang–Mills observables are the degree-zero cohomology of the renormalized BRST differential and the Ward identities can be imposed whenever the local anomaly class vanishes [46]. The generated chiral module has vanishing cubic and mixed gauge-anomaly coefficients together with the gravitational gauge-anomaly coefficients by (4.54). The perturbative Adler–Bardeen theorem applies to this renormalizable gauge–scalar–spinor system. Here G_{int} is compact with one Abelian Lie-algebra factor. The classical master equation holds and complete chiral multiplets have the vanishing anomaly coefficients above. In a locally covariant normalization satisfying perturbative agreement the one-loop cancellation therefore gives all-order gauge-anomaly freedom [47]. Retained adjoint scalars add no chiral anomaly and each full matter copy has the same cancelling coefficients. The global mod-two obstruction is absent because the number of weak doublets is even. It follows that the quantum master identity can be normalized so that $s_q^2 = 0$. Standard BRST homological arguments give gauge-fixing independence of the physical cohomology and preserve causal factorization. $\square$

The source scale selects the gauge-compatible finite internal resolution at which an observation is made. At that resolution the theorem is exact in event support and causal order and perturbative in the native vertices. Its gauge symmetry is the local G_{int} action on the event carrier. Internal modes label complete associated multiplets. Joint resolution carries the internal products together with their unresolved product correction as established in Section 8.1. The complete internal spectral response is retained in the condensation and mass-gap theorem of Section 8.8. The local Lorentzian net is its event-space field representation.

D Statistical interpretation of the internal geometry

The level-set interpretation of Section 4 concerns Gaussian fluctuations on the generated Euclidean event carrier $\mathbb{E}_V$. On its real nuclear test space a continuous positive covariance C defines a centred Gaussian measure on the distribution dual through

$$Z(f) = \exp[-\tfrac{1}{2}C(f, f)]$$

by Bochner–Minlos [53]. Reflection positivity is a further condition proved for the product covariance in Theorem C.1. Here C specifies field fluctuations on the base. The tangent Gaussian law of Section 3 specifies the source increments.

For a translation-invariant tempered covariance on $\mathbb{E}_V \cong \mathbb{R}^4$ Bochner–Schwartz gives the distributional representation

$$C(x, y) = \int e^{ip \cdot (x-y)} d\nu(p), \quad \nu \geq 0.$$

Rotation invariance makes the kernel radial and when ν is absolutely continuous its density a function of p^2 . If the field additionally has scaling dimension one its covariance has away from contact terms

$$C(x, y) = \frac{c}{|x - y|^2}, \quad \widehat{C}(p) \propto p^{-2};$$

The degree -2 follows from homogeneity in four dimensions.

For a smooth mollification σ_ε of this field the regular-value theorem makes every regular level $\{x : \sigma_\varepsilon(x) = s\}$ a smooth hypersurface. The round/projective geometry can then serve as an ensemble-level model for these surfaces. This is a geometric interpretation of the fluctuations. The internal classification is proved in Theorem 4.3 and Proposition 4.4.

E Relative spectral estimates

E.1 Relative information and Hilbert–Schmidt convergence

The subtracted response $x - \log(1 + x) = O(x^2)$ makes Hilbert–Schmidt summability sufficient to control the potential and its first two derivatives. This control passes through the stationary equation to the normalized radial mass. The finite Gaussian relative-entropy identity fixes the sign and comparison direction of the prescribed response (Theorem 8.3). The bosonic fluctuation determinant has the sign convention of Theorem B.6.

Let $A > 0$ and $B \geq 0$ be real symmetric matrices on an n -dimensional real mode space. Set $A_y = A + yB$ for $y \geq 0$ and $T = A^{-1/2}BA^{-1/2}$. For the centred Gaussian laws $\nu_0 = N(0, A^{-1})$ and $\nu_y = N(0, A_y^{-1})$

$$2D_{\text{KL}}(\nu_0 \parallel \nu_y) = \text{Tr}\{yT - \log(I + yT)\}. \quad (\text{E.1})$$

The Gaussian relative-entropy formula [12] also applies when A and B do not commute. The log density ratio is $-\frac{1}{2} \log \det(A_y A^{-1}) + \frac{1}{2} x^\top (A_y - A)x$. Its ν_0 expectation is $\frac{1}{2} \{\text{Tr}(yBA^{-1}) - \log \det(I + yT)\}$ giving the result by cyclicity of the finite trace.

Reversing the two laws changes the eigenvalue function to $\log(1+x) + (1+x)^{-1} - 1$ whose second derivative changes sign at $x = 1$. Equation (E.1) uses real Gaussian coordinates. For the usual proper complex Gaussian convention the factor 2 on its left is absent. General fixed weights define the weighted response of Theorem 8.2.

Theorem E.1 (Hilbert–Schmidt convergence of the relative potential). Let $A \geq a_* I > 0$ be self-adjoint and let b be a nonnegative form on $\mathcal{D}(A^{1/2})$. Suppose its relative form is represented by a nonzero positive Hilbert–Schmidt operator T meaning

$$b[u, v] = \langle A^{1/2}u, TA^{1/2}v \rangle.$$

The forms $a_y = a + yb$ for $y \geq 0$ are closed on the same domain. The relative functional

$$I_T(y) = \text{Tr}\{yT - \log(I + yT)\} = -\log \det_2(I + yT) \quad (\text{E.2})$$

is finite and nonnegative. It is strictly convex with

$$\begin{aligned} I'_T(y) &= \text{Tr}\{yT^2(I + yT)^{-1}\}, \\ I''_T(y) &= \text{Tr}\{T^2(I + yT)^{-2}\} > 0. \end{aligned} \quad (\text{E.3})$$

For finite-rank orthogonal projections $P_N \rightarrow I$ strongly $T_N = P_N T P_N$ converges to T in Hilbert–Schmidt norm and $I_{T_N} \rightarrow I_T$ in $C^2([0, Y])$ for each finite Y .

Proof. Since T is bounded and positive $a[u] \leq a_y[u] \leq (1 + y\|T\|)a[u]$. So the form norms are equivalent. For the positive eigenvalues t_j of T one has $0 \leq yt_j - \log(1 + yt_j) \leq y^2 t_j^2 / 2$. The sum and its first two derivatives together with its higher derivatives on compact positive y intervals therefore converge normally. The regularized determinant in (E.2) is the convergent product $\prod_j (1 + yt_j) e^{-yt_j}$. These facts prove the formulas and positivity.

Finite-rank approximation of a Hilbert–Schmidt operator gives the asserted norm convergence. More explicitly for positive Hilbert–Schmidt S, T put $\delta = \|S - T\|_2$ and $c = \|S\|_2 + \|T\|_2$ with $q = \|T\|_2^2$. The resolvent identity and the trace-ideal product estimate give uniformly on $0 \leq y \leq Y$

$$\begin{aligned} |I_S(y) - I_T(y)| &\leq (Y^2 c / 2 + Y^3 q / 3) \delta, \\ |I'_S(y) - I'_T(y)| &\leq (Yc + Y^2 q) \delta, \\ |I''_S(y) - I''_T(y)| &\leq (c + 2Yq) \delta. \end{aligned} \tag{E.4}$$

Indeed $\|S^2 - T^2\|_1 \leq c\delta$ and $\|(I + yS)^{-1} - (I + yT)^{-1}\| \leq y\delta$. For the squared resolvents the latter bound is $2y\delta$. Substitution in (E.3) and integration prove the bounds. $\square$

If the truncation commutes with A expressing the compressed form relative to A gives $P_N T P_N$. Other truncations require a bound for their relative operators such as (E.4). For $B = gI$ the inverse three-dimensional conformal Laplacian is not trace class on L^2 . Proposition G.8 constructs the Gaussian law on a negative Sobolev space and computes its density and restoring moments there. Convergence of the compensated trace uses the Hilbert–Schmidt relative operator. A principal-symbol deformation can instead have relative order zero for which the displayed quadratic summability condition must be checked separately.

Theorem 8.2 classifies stability. For a finite positive matrix $S = \text{Tr } T$ and the condensed stability interval is $-2S < \mu < 0$. The same restriction persists with infinitely many modes if fixed Gaussian weights make S finite. With the unweighted positive Hilbert–Schmidt form of Theorem E.1 the only distinction is whether T is trace class. If weights vary with y or the scale parameter their derivatives contribute to the stationary equation. For a signed coordinate r uniqueness of $y_* = r^2 > 0$ gives two coordinate minima $r = \pm\sqrt{y_*}$. Whether they represent distinct vacua depends on the symmetry and observable algebra.

E.2 Gaussian-weighted scale dependence and stability

Theorem E.2 (Scale dependence of the Gaussian-weighted potential). For the same commuting conformal deformation $A_y = A + gyI$ let $L, g, \tau > 0$ and μ be C^1 functions of a parameter σ on an open interval. Put $c_j = (2j+1)^2 - 1/4$ and $d_j = (2j+1)^2$. Set $a_j = c_j/L^2$ and $f(x) = x - \log(1+x)$. Retain the unnormalized Gaussian weights independent of y in the complete response

$$F(\sigma, y) = \frac{\mu y}{2} + \sum_{j \geq 0} d_j e^{-\tau a_j} f(gy/a_j), \quad y \geq 0. \tag{E.5}$$

The dimensionless variables

$$q = \frac{\tau}{L^2}, \quad \kappa = gL^2, \quad z = \kappa y, \quad \chi = \frac{\mu}{2\kappa}, \quad J_q(z) = \sum_j d_j e^{-q c_j} f(z/c_j) \tag{E.6}$$

give $F = \chi z + J_q(z)$. Define

$$S(q) = \sum_j \frac{d_j e^{-q c_j}}{c_j}, \quad H(q, z) = \sum_j \frac{d_j e^{-q c_j}}{(c_j + z)^2}, \quad B(q, z) = \sum_j \frac{d_j e^{-q c_j}}{c_j + z}.$$

The response has a unique positive global minimizer precisely when $-S(q) < \chi < 0$. On this open stable set its stationary branch is C^1 in σ and with primes denoting parameter derivatives

$$z'_* = \frac{-\chi' + q' z_* B(q, z_*)}{H(q, z_*)}, \quad y'_* = \frac{z'_*}{\kappa} - \frac{\kappa'}{\kappa} y_*. \quad (\text{E.7})$$

Proof. On compact parameter intervals $q \geq q_0 > 0$. For bounded $z \geq 0$ every derivative used here is bounded by a polynomial in c_j times $e^{-q_0 c_j}$ after reducing q_0 if necessary. These bounds are summable since $c_j \asymp j^2$ and $d_j \asymp j^2$. Hence the series may be differentiated locally. Direct calculation gives

$$\partial_z J_q = \sum_j \frac{d_j e^{-q c_j} z}{c_j (c_j + z)}, \quad \partial_z^2 J_q = H > 0, \quad \partial_q \partial_z J_q = -z B.$$

The derivative $\chi + \partial_z J_q$ increases from χ to $\chi + S(q)$. Theorem 8.2 gives exactly the stated domain of positive minimizers. The implicit function theorem applied to $\chi + \partial_z J_q(z_*) = 0$ gives a C^1 branch and (E.7). $\square$

The branch derivative is along the specified path L, g, τ, μ . Constant τ/L^2 leaves the weights unchanged. Amplitude-dependent weights or running prefactors require additional derivatives.

For attenuation alone the boundary can be read directly from this branch. Fix L, g and $\mu < 0$ and increase $q = \tau/L^2$. There is a unique $q_c > 0$ with $S(q_c) = -\chi$. The positive minimizer exists exactly for $0 < q < q_c$. It increases strictly with q and tends to infinity as $q \uparrow q_c$. Its unnormalized radial curvature $4y_* F_{yy} = 4\kappa z_* H(q, z_*)$ tends to zero. Since $S'(q) = -\sum_j d_j e^{-q c_j} < 0$ with $S(0+) = \infty$ and $S(\infty) = 0$ there is exactly one q_c . The branch equation gives $dz_*/dq = z_* B/H > 0$. At q_c every finite z has $\partial_z J_{q_c}(z) < S(q_c)$. Hence $z_* \rightarrow \infty$. Finally

$$0 < 4\kappa z_* H(q, z_*) \leq \frac{4\kappa}{z_*} \sum_j d_j e^{-q c_j} \longrightarrow 0,$$

since the last sum is uniformly bounded near $q_c > 0$.

This attenuation boundary at fixed spectrum and drive differs from a path with running spectrum and drive. Compact subsets of the stable parameter domain instead have uniform positive lower bounds for the root and H .

E.3 Truncation, stationary amplitudes and mass errors

Proposition E.3 (Uniform truncation and response error control). For the scalar conformal operator on $\mathbb{RP}^3$ of radius L write

$$a_j = \frac{(2j+1)^2 - 1/4}{L^2}, \quad d_j = (2j+1)^2, \quad j = 0, 1, \dots$$

For the commuting deformation $A_y = A + gyI$ with $g > 0$ take $t_j = g/a_j$ and $\beta_j = d_j$. Then $Q_2 < \infty$ but $S = \infty$. So every $\mu < 0$ has the positive global minimizer described above. Retaining $0 \leq j \leq N$

$$T_{2,N} := \sum_{j>N} \frac{d_j}{a_j^2} \leq \frac{8L^4}{9(2N+1)}. \quad (\text{E.8})$$

On $0 \leq y \leq Y$ the errors in I, I', I'' are respectively bounded by $g^2 Y^2 T_{2,N}/2$ and $g^2 Y T_{2,N}$ and $g^2 T_{2,N}$ and all three errors are nonnegative.

Proof. For $u = 2j + 1 \geq 1$ one has $u^2 - 1/4 \geq 3u^2/4$. Consequently $d_j/a_j^2 \leq 16L^4/(9(2j+1)^2)$ and the decreasing-function integral bound gives $\sum_{j>N} (2j+1)^{-2} \leq [2(2N+1)]^{-1}$. Also $d_j/a_j \geq L^2$ proving divergence of S . The scalar bounds $0 \leq x - \log(1+x) \leq x^2/2$ and the derivative formulas prove the three remainder estimates. The constant conformal mode is included throughout with $a_0 = 3/(4L^2)$. $\square$

For fixed weights $w_j = e^{-\tau a_j}$ with $\tau > 0$ the restoring slope is finite and obeys

$$S_\tau = g \sum_j \frac{d_j e^{-\tau a_j}}{a_j} \leq \frac{4}{3} g L^2 e^{-3\tau/(4L^2)} \left(1 + \frac{\sqrt{\pi} L}{4\sqrt{\tau}} \right). \quad (\text{E.9})$$

This follows from $d_j/a_j \leq 4L^2/3$ and $a_j \geq (4j^2 + 3/4)/L^2$. Its weighted $T_{2,N}$ is at most $e^{-\tau a_{N+1}}$ times (E.8). The unweighted potential F_L uses the divergent restoring slope $S = \infty$ established above.

Stationary amplitudes and radial masses. Suppose the finite and completed commuting responses share g and their retained a_j, β_j with the drive μ . Suppose their positive roots y_N, y_* lie in one interval $[u, Y]$ on which the finite second derivative has a lower bound $m > 0$. With $T_{2,N} = \sum_{j>N} \beta_j/a_j^2$

$$0 \leq y_N - y_* \leq \frac{g^2 Y T_{2,N}}{m}. \quad (\text{E.10})$$

More generally if μ_N replaces μ and all other changes produce a C^1 derivative error at most ϵ_1 on this interval the absolute root error is at most

$$\delta_y := \frac{|\mu_N - \mu|/2 + g^2 Y T_{2,N} + \epsilon_1}{m}. \quad (\text{E.11})$$

The common bracket must contain both roots. The lower bound m must apply to the finite potential after all parameter and operator modifications. Let $Q_k = \sum_j \beta_j (g/a_j)^k$ for $k = 2, 3$ and assume $Q_3 < \infty$. For the conformal tower this follows from $a_j \geq a_0 > 0$ and $Q_3 \leq (g/a_0)Q_2$. With no additional second-derivative changes the radial curvatures $C_N = 4y_N I_N''(y_N)$ and $C_* = 4y_* I''(y_*)$ satisfy

$$|C_N - C_*| \leq 4\{(Q_2 + 2YQ_3)\delta_y + Yg^2 T_{2,N}\}. \quad (\text{E.12})$$

If a further second-derivative error is bounded by ϵ_2 add $4Y\epsilon_2$ to the right side. For kinetic coefficients $Z_N, Z_* \geq z_0 > 0$

$$\left| \frac{C_N}{Z_N} - \frac{C_*}{Z_*} \right| \leq \frac{|C_N - C_*|}{z_0} + \frac{C_* |Z_N - Z_*|}{z_0^2}. \quad (\text{E.13})$$

Proof. The full derivative exceeds the finite derivative. So its root is no larger. Apply the mean value theorem to the finite derivative and then the tail estimate. The same argument with the joint derivative defect gives (E.11). The bounds $I_N'' \leq Q_2$ and $|I_N'''| \leq 2Q_3$ give (E.12). Subtraction of the two normalized fractions gives (E.13). $\square$

Opposite derivative signs at u and Y including the error bounds give a common root bracket. One retained mode with $\beta_j > 0$ supplies $m = \beta_j g^2 / (a_j + gY)^2$. For fixed $\mu < 0$ sufficiently large unweighted conformal truncations inherit such a bracket. Uniformity near $\mu = 0$ requires additional bounds. Both the positive root and its radial curvature can approach zero.

A finite truncation has a finite limiting restoring slope. The low modes must therefore balance the largest negative drive on the scale interval before the cutoff can approximate the condensed branch. Those same modes give a common root bracket and a positive derivative bound. The remaining modes determine the truncation error.

Uniform control on a condensed scale interval. Along the path of Corollary 8.3 suppose the coefficient functions satisfy

$$0 < \nu_- \leq \nu_k \leq \nu_+, \quad 0 < \kappa_- \leq \kappa_k \leq \kappa_+, \quad 0 < Z_- \leq Z_k \leq Z_+.$$

Truncate only the unweighted potential at $j = N$ keeping its drive and normalization with the constant conformal mode. Put

$$c_j = (2j+1)^2 - \frac{1}{4}, \quad d_j = (2j+1)^2, \quad \mathcal{H}_N(z) = \sum_{j=0}^N \frac{d_j z}{c_j(c_j + z)}.$$

Choose an integer $M \geq 0$ with $M+1 > 2\nu_+$ and define

$$Y = c_M, \quad B_2 = \frac{8}{3}, \quad B_3 = \frac{32}{9}, \quad u = \frac{\nu_-}{2B_2}, \quad m = \mathcal{H}'_M(Y) > 0, \quad r_N = \frac{8}{9(2N+1)}.$$

For every $N \geq M$ both the complete and truncated potentials are coercive and have unique positive global minimizers with

$$u < z_k \leq z_{N,k} < Y, \quad 0 \leq z_{N,k} - z_k \leq \frac{Y r_N}{m}, \quad \mathcal{H}_N(z_{N,k}) = \nu_k. \quad (\text{E.14})$$

Set $m_{N,k}^2 = 4\kappa_k z_{N,k} \mathcal{H}'_N(z_{N,k}) / Z_k$. The full scalar mass towers constructed from either amplitude on their common H^2 domain satisfy

$$\inf_{k \in I} \min\{m_{N,k}^2, m_{\delta,k}^2\} \geq \Gamma_I := \frac{4\kappa_- u m}{Z_+} > 0, \quad (\text{E.15})$$

$$\sup_{k \in I} |m_{N,k}^2 - m_{\delta,k}^2| \leq K_I r_N, \quad K_I = \frac{4\kappa_+ Y}{Z_-} \left(1 + \frac{B_2 + 2Y B_3}{m}\right).$$

The last error is also the norm of the bounded difference between the two mass-squared operators uniformly in k . Thus a single finite $N \geq M$ with $8K_I/[9(2N+1)] \leq \varepsilon$ achieves any prescribed mass-squared accuracy $\varepsilon > 0$ on the whole interval.

Proof. Proposition E.3 at $L = 1$ gives $\sum_j d_j/c_j^2 \leq 16/9 + 8/9 = B_2$ and $\sum_j d_j/c_j^3 \leq (4/3)B_2 = B_3$. Hence $\mathcal{H}_N(u) \leq \mathcal{H}(u) \leq B_2 u < \nu_-$ whereas

$$\mathcal{H}_M(Y) \geq \frac{1}{2} \sum_{j=0}^M \frac{d_j}{c_j} > \frac{1}{2}(M+1) > \nu_+.$$

Monotonicity brackets every root as asserted. The finite limiting restoring slope exceeds ν_+ . So the stability classification of Theorem 8.2 also proves coercivity of each truncated potential. On $[u, Y]$ one has $\mathcal{H}'_N \geq \mathcal{H}'_M(Y) = m$ and $0 \leq \mathcal{H} - \mathcal{H}_N \leq Y r_N$. The mean value theorem gives the root error. Moreover $\mathcal{H}'_N \leq B_2$ and $|\mathcal{H}''_N| \leq 2B_3$. Applying these bounds to $z\mathcal{H}'_N(z)$ and using $0 \leq \mathcal{H}' - \mathcal{H}'_N \leq r_N$ proves the mass error. The common lower bounds $z > u$ and $\mathcal{H}' \geq m$ give Γ_I . Both mass operators retain the same complete Laplace tower and differ only by their constant shifts proving the operator assertion. $\square$

The coefficient bounds must hold on the entire path. The auxiliary constants use finite sums. So interval arithmetic with outward rounding can include their evaluation error. The displayed errors concern the exact truncated root $z_{N,k}$. A numerical root enclosure must also bound the root-solving error. The derivative lower bound m converts a residual bound into such an enclosure and Proposition E.3 propagates it to the radial mass. Choosing $\varepsilon = \Gamma_I/2$ bounds the spectral truncation error relative to the positive gap.

E.4 Landau approximation and the exact stationary equation

Proposition E.4 (Landau approximation and exact conformal stationarity). In Theorem 8.2 suppose $M := \sup_j t_j < \infty$ and put $Q_3 = \sum_j \beta_j t_j^3 \leq M Q_2$. Writing $\lambda_{\text{spec}} = 2Q_2$

$$F_\mu(y) = \frac{\mu}{2}y + \frac{\lambda_{\text{spec}}}{4}y^2 + \mathcal{R}(y),$$

one has for all $y \geq 0$

$$|\mathcal{R}(y)| \leq Q_3 y^3/3, \quad |\mathcal{R}'(y)| \leq Q_3 y^2, \quad |\mathcal{R}''(y)| \leq 2Q_3 y. \quad (\text{E.16})$$

If $\mu < 0$ let $y_{\text{quad}} = -\mu/\lambda_{\text{spec}}$ be the stationary amplitude square of the quartic approximation. If $\eta = M y_{\text{quad}} < 1$ the complete potential is stable even in the finite-restoring-slope case and

$$y_{\text{quad}} \leq y_* \leq \frac{y_{\text{quad}}}{1-\eta}, \quad (1-\eta)(-2\mu) \leq 4y_* I''(y_*) \leq -2\mu. \quad (\text{E.17})$$

With kinetic coefficient $Z_* > 0$ the radial mass is $m_{\text{curv}}^2 = 4y_* I''(y_*)/Z_*$. Unit kinetic normalization sets $Z_* = 1$.

Proof. For $x \geq 0$ one has $x^2/2 - [x - \log(1+x)] = \int_0^x s^2/(1+s) ds$. Differentiation gives the three remainder bounds. Furthermore

$$\frac{yQ_2}{1+yM} \leq I'(y) \leq yQ_2, \quad S \geq Q_2/M > Q_2 y_{\text{quad}} = -\mu/2.$$

The stationary equation $I'(y_*) = Q_2 y_{\text{quad}}$ yields the root bounds. Each summand in $y_* I''(y_*)$ is its corresponding summand in $I'(y_*)$ times $(1+y_* t_j)^{-1}$. This factor is between $(1+y_* M)^{-1} \geq 1-\eta$ and 1 giving the curvature bounds. $\square$

For $B = gI$ the quartic coefficient at $E = 0$ is $\lambda_{\text{spec}} = 2g^2 \text{Tr } A^{-2}$. The formula $m_{\text{curv}}^2 = -2\mu$ holds for a quartic potential with unit kinetic normalization. Under (8.14) it gives $\frac{16}{3}(\mathcal{R}_c - \mathcal{R})$. The complete logarithmic response has a strictly smaller curvature with the difference given by (E.20).

Exact stationary equation for the conformal tower. For the unweighted full conformal tower set $\kappa = gL^2$ and $c_j = (2j+1)^2 - 1/4$. Put $d_j = (2j+1)^2$ and $J(z) = \sum_j d_j [z/c_j - \log(1 + z/c_j)]$. Its derivative is

$$\mathcal{H}(z) := J'(z) = \begin{cases} \frac{\pi}{8} - \frac{\pi b}{4} \tan \frac{\pi b}{2}, & 0 \leq z < \frac{1}{4}, \quad b = \sqrt{\frac{1}{4} - z}, \\ \frac{\pi}{8}, & z = \frac{1}{4}, \\ \frac{\pi}{8} + \frac{\pi s}{4} \tanh \frac{\pi s}{2}, & z > \frac{1}{4}, \quad s = \sqrt{z - \frac{1}{4}}. \end{cases} \quad (\text{E.18})$$

The function is smooth and strictly increasing from zero to infinity with $\mathcal{H}'(0) = \pi(\pi+2)/8$ and $\mathcal{H}'(1/4) = \pi^2/8$. Consequently for every $\mu < 0$ the full potential $F_\mu(y) = \mu y/2 + J(\kappa y)$ has stationary amplitude and radial curvature given by

$$\mathcal{H}(z_*) = -\frac{\mu}{2\kappa}, \quad y_* = \frac{z_*}{\kappa}, \quad C_* = 4\kappa z_* \mathcal{H}'(z_*), \quad \lambda_{\text{spec}} = \frac{\pi(\pi+2)}{4} \kappa^2. \quad (\text{E.19})$$

Here C_* is the radial second derivative and the squared radial mass is C_*/Z_* in the stated normalization. The correction to the quartic curvature is

$$-2\mu - C_* = 4\kappa z_*^2 \sum_{j \geq 0} \frac{d_j}{c_j(c_j + z_*)^2} > 0. \quad (\text{E.20})$$

Proof. Logarithmic differentiation of $\cos(\pi b/2) = \prod_{j \geq 0} (1 - b^2/(2j+1)^2)$ gives

$$\sum_{j \geq 0} \frac{1}{(2j+1)^2 - b^2} = \frac{\pi}{4b} \tan \frac{\pi b}{2}.$$

The product follows by dividing the Euler products for $\sin(\pi b)$ and $\sin(\pi b/2)$. Take continuous limits at zero. For $b^2 = 1/4 - z$ termwise rearrangement of the convergent derivative yields

$$J'(z) = \frac{1}{4} \sum_j \frac{1}{(2j+1)^2 - 1/4} - b^2 \sum_j \frac{1}{(2j+1)^2 - b^2}.$$

Substitution and $b = is$ prove the real formulas on the two intervals. Normal convergence of the differentiated defining series proves smoothness also at $z = 1/4$ and gives $\mathcal{H}'(z) = \sum_j d_j/(c_j + z)^2 > 0$. Evaluation at zero gives the stated quartic coefficient. The hyperbolic formula gives the infinite limiting slope. The stationary equation then has exactly one positive root. Differentiating $F_\mu(E^2)$ gives C_* . Finally subtract $z_* \mathcal{H}'(z_*)$ from $\mathcal{H}(z_*)$ term by term and use the stationary equation to obtain the positive correction. $\square$

The stationary equation retains the complete logarithm including $j = 0$. The zero-mode realization instead omits that term from its eliminated tower. Gaussian weighting

replaces the defining series by the weighted sum. For the present unweighted response the tail and Landau estimates give controlled approximations to the exact equation.

The hyperbolic expression also fixes the large-amplitude growth. Since $s = \sqrt{z} + O(z^{-1/2})$ and $\tanh(\pi s/2) = 1 + O(e^{-\pi s})$ it gives

$$J'(z) = \frac{\pi}{4}\sqrt{z} + \frac{\pi}{8} + O(z^{-1/2}), \quad J(z) = \frac{\pi}{6}z^{3/2} + \frac{\pi}{8}z + O(\sqrt{z}).$$

The second formula follows by integration from any fixed positive z . With $I(y) = J(\kappa y)$ these formulas give the restoring asymptotics in Section 8. They describe growth only as the amplitude tends to infinity. Normal convergence controls the response and its first two derivatives at every finite amplitude.

F Auxiliary Yang–Baxter and RTT algebra

The exchange-algebra comparison in Section 10 uses the standard rational R -matrix[36, Chapter VI, Sections 1–2]. The generated phase-sensitive carrier fixes its equivariant class. The retained Dirac levels supply the monodromy inhomogeneities. Thus the exchange construction uses the existing mode labels to organize a finite auxiliary family of commuting operators. The classification is within rational difference families regular at the origin and nonconstant modulo scalar functions.

F.1 Carrier and spectral inhomogeneities

Let $S \cong \mathbb{C}^2$ be the minimal phase-sensitive spin module of Theorem 4.5. On the selected spin lift put

$$\widehat{D} := R_b D_{X_3}, \quad \lambda_n^+ = n + \frac{3}{2}, \quad n = 0, 2, 4, \dots \quad (\text{F.1})$$

Let $P_+ = \mathbf{1}_{(0,\infty)}(\widehat{D})$ and $\widehat{D}_+ = \widehat{D}|_{\text{ran } P_+}$. The resolvent $(\widehat{D}_+ + uI)^{-1}$ has poles at $-(n+3/2)$ for even n . For a cutoff c the retained absolute positions of these poles form

$$\Theta_c = \left\{ n + \frac{3}{2} : n \in 2\mathbb{N}_0, n + \frac{3}{2} < c \right\}. \quad (\text{F.2})$$

If $11/2 < c < 15/2$ then $\Theta_c = \{3/2, 7/2, 11/2\}$. Using these pole positions as spin-chain inhomogeneities defines a finite spin chain for each cutoff.

F.2 The rational equivariant family

Let $P(v \otimes w) = w \otimes v$ on $S \otimes S$.

The multiplicity-free decomposition $S \otimes S = \text{Sym}^2 S \oplus \Lambda^2 S$ and Schur's lemma give

$$\text{End}_{SU(2)}(S \otimes S) = \mathbb{C}\Pi_{\text{sym}} \oplus \mathbb{C}\Pi_{\text{alt}} = \text{span}_{\mathbb{C}}\{I, P\}. \quad (\text{F.3})$$

Here $\Pi_{\text{sym}} = (I + P)/2$ and $\Pi_{\text{alt}} = (I - P)/2$.

Rational difference solution. Consider a difference-type $SU(2)$ -equivariant family $R(u) = a(u)I + b(u)P$ with rational coefficients that are regular at the origin $a(0) = 0$ and $b(0) \neq 0$. Require the family to be nonconstant modulo scalar functions. The rational function $a(u)/b(u)$ is nonconstant. Every solution of the additive Yang–Baxter equation in this class is after multiplication by a nonzero rational scalar function and rescaling the spectral parameter equivalent to

$$R_\eta(u) = uI + \eta P, \quad \eta \neq 0. \quad (\text{F.4})$$

By (F.3) divide by $b(u)$ near zero and write $R(u)/b(u) = r(u)I + P$. Scalar factors cancel in YBE and RTT. The standard flip calculation [36, Chapter VI] gives

$$[r(u+v) - r(u) - r(v)](P_{12}P_{23} - P_{23}P_{12}) = 0. \quad (\text{F.5})$$

The second factor is nonzero. So r is locally additive. Regularity gives $r(u) = \alpha u$ by differentiation at zero. Rational continuation extends the identity. Nonconstancy of a/b fixes $\alpha \neq 0$ and division by α gives (F.4) with $\eta = 1/\alpha$. Scalar equivalence is understood away from isolated zeros and poles with the identities continued rationally across them.

The qualification modulo scalar functions is essential. For example $R(u) = (1 + u^2)P$ is nonconstant as a matrix-valued function and is regular at zero. It satisfies YBE but $a/b = 0$ and its normalized family is the constant flip P .

The regular family obeys

$$R_\eta(0) = \eta P, \quad R_\eta(u)R_\eta(-u) = (\eta^2 - u^2)I. \quad (\text{F.6})$$

Its eigenvalues are $u + \eta$ on the three-dimensional symmetric summand and $u - \eta$ on the antisymmetric line. The second identity is the algebraic unitarity relation with scalar factor $\eta^2 - u^2$.

F.3 Yang–Baxter, RTT, and commuting transfer matrices

The standard rational construction [36, Chapter VI, Sections 1–2] now applies to the generated carrier and the retained pole positions. The family (F.4) satisfies

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u). \quad (\text{F.7})$$

Fix $\eta \neq 0$ in the same dimensionless spectral units as u and the pole positions. For any finite list $\theta_i \in \Theta_c$ the monodromy

$$T_a(u) = R_{aN}(u - \theta_N) \cdots R_{a1}(u - \theta_1) \quad (\text{F.8})$$

satisfies

$$R_{ab}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u-v). \quad (\text{F.9})$$

Consequently

$$t(u) = \text{Tr}_a T_a(u), \quad [t(u), t(v)] = 0. \quad (\text{F.10})$$

The flip relations give (F.7). Iteration along the finite chain gives RTT. Generic invertibility of $R_{ab}(u-v)$ and cyclicity of the trace over the auxiliary spaces give the commuting transfer

matrices. Their identities extend through exceptional spectral parameters by polynomial continuation.

For the chosen rational normalization η and finite list of retained Dirac levels the carrier S supports this auxiliary exchange algebra and its commuting transfer matrices. A scattering interpretation additionally requires a particle sector in which the R -matrix obeys elastic factorization with Hilbert-space unitarity and analyticity together with crossing.

G Source response algebras and representations

The ladder matrix units generate the response algebras and compatible positive states used in Sections 4 and 5.2. Coherent observations distinguish their two-level operations whose diffusion limit has the round spectrum. The collective-amplitude and Gaussian-field constructions realize the spectral response of Section 8. The same ladder operations occur in the minimal-change construction of Ref. [8].

G.1 Finite response capacity and a sourced susceptibility

Write $\mathcal{H}_G = L^2(\mathbb{R})$ and $N = a_b^* a_b$ with $h_j = h_{b,j}$. The lowering partial isometry in the polar decomposition $a_b = S\sqrt{N}$ satisfies

$$Sh_0 = 0, \quad Sh_j = h_{j-1} \quad (j \geq 1).$$

Thus

$$P_0 = I - S^*S, \quad E_{ij} = (S^*)^i P_0 S^j = |h_i\rangle\langle h_j|, \quad P_n = \sum_{j=0}^{n-1} E_{jj}. \quad (\text{G.1})$$

The same matrix units construct every finite response corner. Their linear span is norm dense in the compact-operator ideal $\mathcal{K}(\mathcal{H}_G)$. A ladder phase change conjugates matrix coordinates and preserves the algebras and ranks for every n .

Full and connected source-response ranks. Let $H_n = P_n \mathcal{H}_G$ and $N_n = N|_{H_n}$. Fix $t > 0$ and $K \in \text{Herm}(H_n)$. Define the sourced finite-resolution Gaussian response by

$$R_n(K) = e^{-tN_n + K}, \quad Z_n(K) = \text{Tr } R_n(K), \quad W_n(K) = \log Z_n(K). \quad (\text{G.2})$$

At $K = 0$ the real-linear derivative of R_n has rank n^2 . The connected susceptibility $D^2 W_n(0)$ is nonnegative with kernel $\mathbb{R}I_n$ and therefore has rank $n^2 - 1$.

Proof. Put $R = e^{-tN_n}$ and $\rho = R / \text{Tr } R$. Differentiating the matrix exponential gives

$$DR_n(0)[K] = \int_0^1 R^{1-s} K R^s ds.$$

In the eigenbasis of R the ij matrix entry is multiplied by $\int_0^1 r_i^{1-s} r_j^s ds > 0$. The derivative is consequently invertible on Hermitian matrices. With $K_0 = K - \text{Tr}(\rho K)I_n$ the second derivative of the logarithm is

$$D^2 W_n(0)[K, K] = \int_0^1 \text{Tr}(\rho^{1-s} K_0 \rho^s K_0) ds = \sum_{i,j} |(K_0)_{ij}|^2 \int_0^1 p_i^{1-s} p_j^s ds. \quad (\text{G.3})$$

Every coefficient is strictly positive. The expression vanishes exactly when $K_0 = 0$ namely when K is scalar. This proves both ranks. $\square$

The rank calculation allows every Hermitian perturbation in the generated corner algebra. Restricting the probes can reduce that rank. Connected normalization removes the scalar direction which changes only the unnormalized response strength.

The Neumann window admits a separate matrix-algebra realization. Let P_n^N project onto $\text{span}\{\phi_{b,1}, \dots, \phi_{b,n}\}$ and let X be multiplication by $\cos[\pi(d - \ell_{UV})/\Delta_b]$ the first positive Neumann profile. The compressed $\hat{A}_b$ has distinct eigenvalues $1^2, \dots, n^2$. So its polynomial functional calculus contains all diagonal projections. The cosine product identity gives

$$\langle \phi_{b,j+1}, X \phi_{b,j} \rangle = \frac{1}{2}.$$

Compressing to the positive modes removes the constant term at $j = 1$. Hence

$$2E_{j+1,j+1} X E_{jj} = E_{j+1,j},$$

and adjunction and composition again generate $M_n(\mathbb{C})$. This realization requires the specified local multiplication probe to be an allowed response operation.

Corners and shells. The corners $P_n \mathcal{K}(\mathcal{H}_G) P_n$ form a nested sequence. Their Hilbert–Schmidt increments have dimensions

$$n^2 - (n-1)^2 = 2n - 1. \quad (\text{G.4})$$

Moreover $[N_n, E_{ij}] = (i-j)E_{ij}$ while the corresponding window commutator has eigenvalues $i^2 - j^2$. Each corner therefore contains several commutator energies. The degenerate spherical shells instead arise from diffusion on the operation group in Theorem G.2.

G.2 Compatible positive source states

The finite reference states have a common positive completion. The parameter t in this subsection is the resolution parameter of the ladder spectrum. Write $Q_n = I - P_n$. A finite matrix or density is extended by zero on $Q_n \mathcal{H}_G$ whenever a trace norm on the full source space is used.

Compatible source states and observable completion. Fix $t > 0$ and put $q = e^{-t}$. The normalized finite reference states and their completion are

$$\rho_{t,n} = \frac{1-q}{1-q^n} P_n q^N P_n, \quad \rho_t = (1-q)q^N, \quad \|\rho_{t,n} - \rho_t\|_1 = 2q^n. \quad (\text{G.5})$$

For $m > n$ their exact compatibility is conditional

$$\frac{P_n \rho_{t,m} P_n}{\text{Tr}(P_n \rho_{t,m})} = \rho_{t,n}. \quad (\text{G.6})$$

Consequently every bounded source observable A satisfies $|\text{Tr}((\rho_{t,n} - \rho_t)A)| \leq 2e^{-nt}\|A\|$. Let $\mathfrak{A}_n = M_n(\mathbb{C}) \oplus \mathbb{C}$ represented by $(X, c) \mapsto X \oplus cI_{Q_n \mathcal{H}_G}$ and for $m > n$ set

$$\iota_n^m(X, c) = (X \oplus cI_{m-n}, c), \quad \omega_{t,n}(X, c) = (1 - q^n) \text{Tr}(\rho_{t,n} X) + q^n c. \quad (\text{G.7})$$

These are unital injective $*$ -homomorphisms and compatible positive normalized states. Their norm-completed union is

$$\mathfrak{A}_{\text{src}} = \mathcal{K}(\mathcal{H}_G) + \mathbb{C}I, \quad \omega_t(A) = \text{Tr}(\rho_t A). \quad (\text{G.8})$$

Proof. The geometric weights q^j sum to $(1 - q)^{-1}$. The positive and negative parts of $\rho_{t,n} - \rho_t$ supported on the retained corner and its complement each have trace q^n . This gives the trace-norm identity and trace duality gives the observable bound. Compressing the weights and renormalizing proves conditional consistency.

An observable in $\mathfrak{A}_n$ is $cI + (X - cI_n)$ a scalar plus a finite-corner operator. The embeddings preserve its product and adjoint with the identity while its expectation in ρ_t is precisely (G.7). The states are therefore compatible. Norm density of the finite corners in the compact operators gives (G.8) with positivity and normalization preserved on completion. $\square$

The estimate is uniform for $t \geq t_0 > 0$. If $t = t_n$ also varies the displayed truncation error tends to zero exactly when $nt_n \rightarrow \infty$. The limits of the complete states depend on t . In particular $\rho_t \rightarrow |h_0\rangle\langle h_0|$ in trace norm as $t \rightarrow \infty$ with error $2e^{-t}$. As $t \downarrow 0$ every fixed finite-rank projection has expectation tending to zero. So there is no normal limiting state on $\mathcal{B}(\mathcal{H}_G)$.

The scalar c is the common value of an observable on the unresolved tail. Although ω_t extends normally to $\mathcal{B}(\mathcal{H}_G)$ the generated norm closure is the smaller algebra $\mathfrak{A}_{\text{src}}$. On that algebra the limit $t \downarrow 0$ is the quotient character $C + cI \mapsto c$.

G.3 A common carrier for the Gaussian and rotational centres

Proposition G.1 (Common source centre and vacuum phase transport). The first non-commutative corner is $V = P_2\mathcal{H}_G$. On this derived two-dimensional complex space put

$$A = P_2 a_b P_2, \quad J_+ = A^*, \quad J_- = A, \quad J_3 = A^* A - \frac{1}{2}I_V. \quad (\text{G.9})$$

These operators generate the defining action of $G_{\text{op}} = SU(V)$. Its nonidentity centre is the restriction of the Gaussian full-cycle operator

$$\mathbf{H}_b^G|_V = I_V + J_3, \quad e^{-i\theta\mathbf{H}_b^G}|_V = e^{-i\theta}e^{-i\theta J_3}, \quad e^{-2\pi i\mathbf{H}_b^G}|_V = -I_V \in G_{\text{op}}. \quad (\text{G.10})$$

On the real space $\mathfrak{e}_3 = \text{Herm}_0(V)$ with $\langle x, y \rangle = \frac{1}{2} \text{Tr}(xy)$ conjugation gives the double cover $G_{\text{op}} \rightarrow SO(\mathfrak{e}_3)$. The determinant-line transport of the source vacuum is unique up to phase. On the common carrier its transported number flow has the unique continuous lift $\Gamma(t) = (e^{-itJ_3}, e^{it/6})$ whose image is the pointwise vacuum stabilizer.

Proof. In the ordered basis h_0, h_1 one has $A = |h_0\rangle\langle h_1|$. Direct multiplication gives $[J_+, J_-] = 2J_3$ and $[J_3, J_{\pm}] = \pm J_{\pm}$ with $\sum_{\alpha=1}^3 J_{\alpha}^2 = 3I_V/4$ where $J_1 = (J_+ + J_-)/2$ and $J_2 = (J_+ - J_-)/(2i)$. Their skew-adjoint multiples span all traceless skew-adjoint matrices. Exponentiation therefore generates $SU(V)$. Restriction of $N + \frac{1}{2}I$ gives (G.10). The real vector space of traceless Hermitian two-by-two matrices has dimension three. Conjugation preserves its inner product. Its differential is an isomorphism onto $\mathfrak{so}(\mathfrak{e}_3)$ and its kernel consists of scalar determinant-one unitaries exactly $\{\pm I_V\}$. $\square$

The two central actions agree as operators on V . The group G_{op} acts by reversible operations. The operators $J_{\pm}$ change the level while the stabilizer of $N|_V$ in $SU(V)$ is the maximal torus $\{\text{diag}(e^{it}, e^{-it}) : t \in \mathbb{R}\} \cong U(1)$. Local frame comparisons retain these generated operations and their action on the carrier. The half-step is obtained on the full ladder before compression to V .

G.4 Coherent observations of source operations

A coherent reference level makes the central sign of a two-level operation observable. The required preparations and operations with their probes all belong to the matrix algebra generated in (G.1).

Coherent bypass and identifiable processes. Let $V = \text{span}\{h_0, h_1\}$ and $H_3 = V \oplus \mathbb{C}h_2$ with $U \in SU(V)$. Its controlled extension

$$C(U) = U \oplus I_{\mathbb{C}h_2}$$

is a generated reversible operation on H_3 extended by the identity on its orthogonal complement. Generated pure states and bounded observables on H_3 distinguish any two such processes including U and $-U$. If $\mathcal{A}_{\text{proc}}$ is the unital conjugation-closed algebra generated by their continuous response functions on $SU(V)$ then

$$\overline{\mathcal{A}_{\text{proc}}}^{\|\cdot\|_{\infty}} = C(SU(V)). \quad (\text{G.11})$$

Proof. A finite-dimensional unitary has a Hermitian logarithm. Extending a logarithm of U by zero gives a finite Hermitian response whose exponential is $C(U)$. Similarly finite unitaries prepare $\psi_k = (h_k + h_2)/\sqrt{2}$ for $k = 0, 1$ from the source vacuum h_0 . The observables

$$X_j = E_{j2} + E_{2j}, \quad Y_j = i(E_{j2} - E_{2j}), \quad j = 0, 1,$$

and their spectral projections belong to the generated corner algebra. They have norm one. For $\psi'_k = C(U)\psi_k$ direct multiplication gives

$$\langle \psi'_k, X_j \psi'_k \rangle = \text{Re } U_{jk}, \quad \langle \psi'_k, Y_j \psi'_k \rangle = \text{Im } U_{jk}. \quad (\text{G.12})$$

Thus the generated responses recover every matrix entry. They separate points of the compact group and the complex Stone–Weierstrass theorem proves (G.11). $\square$

The quotient of completed operation words. Let $\mathcal{W}$ consist of finite words in the generated $SU(V)$ operations. Declare $w \sim w'$ when every generated source preparation and bounded observation with arbitrary allowed operations before and after the completed words has the same response. Then

$$w \sim w' \iff U(w) = U(w').$$

Equality gives identical controlled operators. Unequal products are separated by (G.12). Since $C(U_2 U_1) = C(U_2)C(U_1)$ this is a two-sided composition congruence. Thus the quotient is $SU(V)$ with continuous response algebra (G.11). Allowing interventions inside a word or operations on higher corners changes the observation class. Likewise observing only the vacuum-density orbit gives a coarser quotient.

Local Gaussian cycle. Compress the ordered full Gaussian generator to V . The bounded Hermitian response $H_V = P_2 H_b^G P_2 = \frac{1}{2}(E_{00} + 3E_{11})$ is generated by the same matrix units. Its exponential on the full source space satisfies

$$\begin{aligned} R_V(\theta) &:= e^{-i\theta H_V} = I - P_2 + e^{-i\theta H_b^G} P_2, \\ R_V(2\pi) &= I - 2P_2 = C(-I_V), \quad R_V(4\pi) = I. \end{aligned} \tag{G.13}$$

Indeed P_2 reduces H_b^G . Its two retained eigenvalues are $1/2, 3/2$ and H_V vanishes on the complement. Thus the angular centre is the endpoint of a Gaussian cycle implemented on V . The intervening path lies in $U(V)$ with determinant $e^{-2i\theta}$. Its endpoint $C(-I_V)$ is central in the controlled group $\{C(U) : U \in SU(V)\}$ but not in the whole source algebra.

The full-source cycle remains $e^{-2\pi i H_b^G} = -I$. On H_3 the local cycle instead has diagonal entries $(-1, -1, +1)$. The bypass detects the relative sign between V and its complement whereas a global sign leaves every density-state expectation unchanged. Equation (G.13) thus realizes the common central action of (G.10) locally on V .

Averaging over this central operation gives the coarse observation

$$E_Z f(U) = \frac{1}{2} \{f(U) + f(-U)\}. \tag{G.14}$$

It is the positive unital conditional expectation onto $C(SU(V))^{\{\pm I\}}$ implemented by an equal random choice between the identity and the controlled angular centre. Its character space is $SU(V)/\{\pm I\}$ the quotient determined by centre-invariant observations. Centre-sensitive observations remain available and a spin structure must be specified separately.

G.5 Finite unbiased controls and second-order comparison

Finite unbiased controls and response comparison. On H_n let $\zeta = e^{2\pi i/n}$ and define generated unitaries

$$U h_j = h_{j+1 \bmod n}, \quad Z_n h_j = \zeta^j h_j.$$

The equal-control channel satisfies

$$E_n(X) = \frac{1}{n^2} \sum_{a,b=0}^{n-1} U^a Z_n^b X (Z_n^b)^* (U^a)^* = \frac{\text{Tr } X}{n} I_n. \tag{G.15}$$

In particular it prepares the trace state I_n/n from every density matrix. After that preparation the logarithmic source response satisfies

$$\overline{W}_n(K) = \log \frac{\text{Tr } e^K}{n}, \quad D^2 \overline{W}_n(0)[K, L] = \frac{\text{Tr}(KL)}{n} - \frac{\text{Tr } K \text{Tr } L}{n^2}. \tag{G.16}$$

This comparison is unitarily invariant and is positive definite on the traceless Hermitian responses. For comparison at a fixed source state return to the two-level carrier $V = \text{span}\{h_0, h_1\}$ and let B be a symmetric bilinear response on $\mathfrak{e}_3 = \text{Herm}_0(V)$ with inner product $\langle x, y \rangle = \text{Tr}(xy)/2$. Let $\mathcal{O} \subset SO(\mathfrak{e}_3)$ be the 24 orientation-preserving signed permutations of an orthonormal triple and choose generated lifts $u_R \in SU(V)$. Then

$$\frac{1}{24} \sum_{R \in \mathcal{O}} B(u_R x u_R^*, u_R y u_R^*) = \frac{\text{tr}_{\mathfrak{e}_3} B}{3} \langle x, y \rangle. \tag{G.17}$$

The result is independent of the lift signs and of the orthonormal triple. It applies in particular to the Gibbs susceptibility $D^2W_2(0)$.

Proof. Averaging $Z_n^b E_{jk} Z_n^{-b} = \zeta^{b(j-k)} E_{jk}$ removes the off-diagonal matrix units. Averaging shifts replaces each diagonal unit by I_n/n . This proves (G.15). The quadratic expansion of $\log \text{Tr } e^{sK+tL}$ gives (G.16) including its invariance and scalar kernel.

Averaging the diagonal sign rotations removes the off-diagonal entries of the symmetric matrix representing B . The remaining permutations make its three diagonal entries equal preserving their sum. All lifts exist by Proposition G.1. Changing a lift by the centre does not change conjugation. The resulting trace formula is independent of coordinates. Operationally the left side is the Hessian of the equal average of the response functions $W_2(u_R K u_R^*)$. $\square$

Equal weights are the unique probability law on the finite control labels that is invariant under label translations. With this probability rule the depolarizing channel is independent of orthonormal matrix coordinates. Equation (G.16) describes perturbations after preparation of the trace state. Equation (G.2) describes perturbations of the Gibbs reference state.

State depolarization and susceptibility averaging act on different objects. In Pauli coordinates the two-level Gibbs Hessian is

$$B_t = \text{diag} \left(\frac{2 \tanh(t/2)}{t}, \frac{2 \tanh(t/2)}{t}, \text{sech}^2(t/2) \right).$$

Pauli conjugations only change coordinate signs and leave this generally anisotropic matrix unchanged. Preparing the trace state before probing gives (G.16). Averaging the Gibbs susceptibility over the 24 rotations gives (G.17).

Theorem G.2 (Unbiased finite-variance operations and the round spectrum). Let $\xi \in \mathbb{R}^3$ have a fixed law independent of ϵ with $\mathbb{E}\xi_a = 0$ and $\mathbb{E}\xi_a \xi_b = c\delta_{ab}$ for $0 < c < \infty$. Compose independent copies of the generated step $\exp(i\epsilon \sum_a \xi_a J_a)$ retaining the complete process observation algebra $C(SU(V))$. After $\lfloor 8t/(c\epsilon^2) \rfloor$ steps their laws converge weakly to the unit round heat semigroup. Its positive generator has

$$\lambda_n = n^2 - 1, \quad m_n = n^2, \quad n = 1, 2, \dots \quad (\text{G.18})$$

The complete centre-invariant algebra has character space $\mathbb{RP}^3$ and retains odd n . Independently averaging each step over the 24 generated controls supplies the moment conditions from any fixed law with $0 < \mathbb{E}|\xi|^2 < \infty$.

Proof. Put $X_j = \sum_a \xi_a J_a^{(j)}$. For each fixed ξ the remainder in the quadratic expansion of $e^{i\epsilon X_j}$ is $o(\epsilon^2)$. Its norm divided by ϵ^2 is at most $\|X_j\|^2 \leq C_j |\xi|^2$. Dominated convergence and centering with the Casimir identity give the one-step Fourier matrix

$$M_j(\epsilon) = \mathbb{E} e^{i\epsilon X_j} = I - \frac{c\epsilon^2}{2} j(j+1)I + o_j(\epsilon^2).$$

Its indicated powers tend to $e^{-4tj(j+1)}I$. Compactness supplies subsequential weak limits and density of matrix coefficients identifies each limit. The limiting multipliers compose in t . For $t > 0$ their heat series converges absolutely. The compact-group decomposition [15, Chapter 12] gives

$$L^2(SU(V), dg) = \widehat{\bigoplus_{j=0,1/2,\dots}} V_j \otimes V_j^*.$$

The angular restriction of the full trace comparison is

$$g_U(UY, UZ) = -\frac{1}{2} \operatorname{ReTr}(YZ), \quad Y, Z \in \mathfrak{su}(V).$$

This metric is bi-invariant and is the unit round metric in quaternion coordinates. Define its left-invariant vector fields by

$$T_a = 2iJ_a, \quad (X_a f)(U) = \left. \frac{d}{ds} f(Ue^{sT_a}) \right|_{s=0}.$$

Since $\operatorname{Tr}(J_a J_b) = \delta_{ab}/2$ one has $g_e(T_a, T_b) = \delta_{ab}$. The flows of X_a preserve the metric and its volume whose normalization is Haar measure. Thus $\operatorname{div}_g X_a = 0$ and the nonnegative Laplace–Beltrami operator is

$$\Delta_g = - \sum_a X_a^2.$$

On a spin- j matrix coefficient X_a acts through $d\pi_j(T_a) = 2iJ_a^{(j)}$. Consequently

$$\Delta_g|_{V_j \otimes V_j^*} = 4j(j+1)I.$$

The step $\exp(i\epsilon \sum_a \xi_a J_a)$ equals $\exp((\epsilon/2) \sum_a \xi_a T_a)$. On each fixed matrix-coefficient block the corresponding averaging operator $P_\epsilon f(U) = \mathbb{E} f(U \exp(i\epsilon \sum_a \xi_a J_a))$ therefore satisfies

$$P_\epsilon = I - \frac{c\epsilon^2}{8} \Delta_g + o_j(\epsilon^2).$$

The positive generator of the limit and Δ_g agree on the common core of finite sums of matrix coefficients for their self-adjoint closures. Hence the limiting semigroup is $e^{-t\Delta_g}$ with the stated number of steps. Each block has energy $4j(j+1)$ and multiplicity $(2j+1)^2$. Putting $n = 2j+1$ proves (G.18). The centre acts on matrix coefficients by $(-1)^{2j} = (-1)^{n-1}$. Its complete invariant algebra therefore retains odd n and has the stated quotient as character space.

Finally conjugate each step by an independently redrawn uniform signed-permutation control R and its generated lift u_R . This replaces ξ by $R\xi$ with

$$\mathbb{E}_R R\xi = 0, \quad \mathbb{E}_R (R\xi)(R\xi)^T = |\xi|^2 I/3.$$

Averaging over ξ gives $c = \mathbb{E}|\xi|^2/3$. So the finite second moment controls the same expansion after this averaging. $\square$

The six-direction and Haar-conjugated walks are special cases. For the equal six directions $c = 1/3$ and

$$\frac{1}{6} \sum_{a=1}^3 \left(e^{i\epsilon J_a^{(j)}} + e^{-i\epsilon J_a^{(j)}} \right) = I - \frac{\epsilon^2}{6} j(j+1)I + O_j(\epsilon^4), \quad N_\epsilon = \lfloor 24t/\epsilon^2 \rfloor. \quad (\text{G.19})$$

For a Haar-conjugated axis $\exp(-i\epsilon g J_3 g^{-1})$ the same $c = 1/3$ follows from rotational invariance and unit length. Schur averaging gives the scalar coefficient $(2j+1)^{-1} \sum_{m=-j}^j e^{-i\epsilon m}$ with the same expansion. Three independent unit-variance Gaussian coefficients instead have $c = 1$. All three choices give the same round heat semigroup after the stated variance normalization. The corresponding composition limit for scalar Gaussian increments is developed in Ref. [8].

The full multiplicity $(2j+1)^2$ comes from the matrix coefficients of the complete operation algebra. Class functions retain one character per irreducible representation while the vacuum-density orbit is $SU(V)/U(1) \cong S^2$. The three random coefficients act on the generated $\text{Herm}_0(V)$ and their fixed law and covariance set the unit-radius diffusion parameter. The ordered identification $n = 2j+1$ matches the Neumann eigenvalue n^2 to $\lambda_n + 1$. The Neumann window and the group diffusion retain their respective Hilbert spaces and generators.

G.6 The real algebra of reversible operation amplitudes

Proposition G.3 (Generated quaternion amplitude and polar metric). The smallest real associative algebra containing the determinant-one reversible operations on V is

$$\mathbb{H}_V = \text{span}_{\mathbb{R}} SU(V) = \mathbb{R}I_V \oplus i \text{Herm}_0(V). \quad (\text{G.20})$$

Every nonzero $h \in \mathbb{H}_V$ has a unique factorization $h = RU$ with $R > 0$ and $U \in SU(V)$. The prepared trace state $\tau(X) = \text{Tr}(X)/2$ defines the positive full second-moment comparison $\tau(dh^*dh)$ whose real part gives

$$\frac{1}{2} \text{ReTr}(dh^*dh) = dR^2 + R^2 g_{\text{round}}. \quad (\text{G.21})$$

The radial and angular directions of this amplitude space are therefore orthogonal in the trace metric.

Proof. Write $x = \sum_a x_a \sigma_a$. The Pauli multiplication relations show that $\mathbb{R}I \oplus i \text{Herm}_0(V)$ is a real associative algebra. Its basis $I, i\sigma_1, i\sigma_2, i\sigma_3$ lies in $SU(V)$. Conversely every determinant-one two-dimensional unitary has the form $aI + ix$ with a, x_a real and $a^2 + \frac{1}{2} \text{Tr} x^2 = 1$. This identifies the smallest real algebra containing $SU(V)$. For any such $h = aI + ix$

$$h^*h = \left(a^2 + \frac{1}{2} \text{Tr} x^2 \right) I, \quad \det h = a^2 + \frac{1}{2} \text{Tr} x^2.$$

Hence $R = (\text{Tr} h^*h/2)^{1/2}$ and $U = h/R$ give the unique factorization. Substituting $dh = U dR + R dU$ into the trace metric the cross term vanishes because $\text{ReTr}(U^* dU) = 0$. The unit slice is the unit sphere in the real coordinates (a, x_1, x_2, x_3) proving the metric formula. $\square$

On Hermitian perturbations the full trace comparison is the second derivative of $\tau(e^K)$. Differentiating $\log \tau(e^K)$ instead subtracts the scalar expectation. The two forms agree on the angular directions of (G.16). The full second moment also measures the radial response strength.

This real algebra also equals the complex-linear endomorphisms commuting with the pseudoreal reversal $J = i\sigma_2 K$ independently of an overall phase choice for J . It therefore agrees with the connector space of Proposition G.4. Real coefficients give the reversal-compatible amplitude space $\mathbb{H}_V$. Complex coefficients give its complexification $\mathbb{H}_V + i\mathbb{H}_V = M_2(\mathbb{C})$. The central source phase is retained in $U(V) = (U(1) \times SU(V))/\{(1, I), (-1, -I)\}$. Its general element maps $\mathbb{H}_V$ into the complexified space. The polar metric above is the positive trace metric on the real operation amplitudes selected by the reversal rule.

G.7 Vacuum transport, Clifford connection, and local frames

The generated group acts on V by its defining representation and on G_{op} by left multiplication. Its third action is conjugation on $\mathfrak{e}_3$. On $\mathfrak{e}_4 = \mathbb{R}\nu \oplus \mathfrak{e}_3$ conjugation extends as $\mathbf{1} \oplus \text{Ad}$ whose spin lift is diagonal (g, g) . Its central element $(-I_V, -I_V)$ is trivial on $\mathfrak{e}_4$. Left multiplication on the group instead makes $-I_V$ antipodal and corresponds to $(-I_V, I_V)$ in the ambient $\text{Spin}(4)$ action.

Vacuum-line transport. Let $L_{\text{src}} = \Lambda^2 V$ and $\omega = h_0 \wedge h_1$. The map

$$F_\omega : V \longrightarrow L_{\text{src}} \otimes V, \quad v \longmapsto \omega \otimes v \quad (\text{G.22})$$

is a unitary $SU(V)$ intertwiner. It is unique up to phase among such unitary intertwiners. It sends the source vacuum to the explicitly constructed line $\mathbb{C}(\omega \otimes h_0)$. No nonzero $SU(V)$ -invariant vector lies on that line.

Proof. The determinant representation L_{src} is trivial on $SU(V)$ and ω is normalized proving unitarity and covariance. Schur's lemma gives the one-dimensional intertwiner space. The defining representation has no invariant vector. So $\text{Hom}_{SU(V)}(\mathbf{1}, L_{\text{src}} \otimes V) = 0$. $\square$

A ladder phase changes F_ω by an overall phase leaving its vacuum line unchanged. On $U(V)$ the determinant twist is nontrivial. The central weights of V and $L_{\text{src}} \otimes V$ are respectively one and three. More precisely every $U \in U(V)$ satisfies

$$(\Lambda^2 U \otimes U)F_\omega = (\det U)F_\omega U. \quad (\text{G.23})$$

Preserving the source operation on the transported carrier therefore fixes its relative action to be

$$R_{\text{rel}}(U) = (\det U)^{-1}(\Lambda^2 U \otimes U), \quad R_{\text{rel}}(U)F_\omega = F_\omega U. \quad (\text{G.24})$$

This correction is unique among scalar characters multiplying the natural tensor action. The intertwining identity forces that character to equal $(\det U)^{-1}$ for every U . It also preserves composition since the determinant is multiplicative. If the scale-covariant metric property is realized by the common-carrier Clifford model and the associated bundles share its diagonal spin action (G.22) gives an explicit transport to $L \otimes S_R$. For an independent right-frame bundle this identification is additional data. On the common carrier the source phase is consequently transported by R_{rel} .

Source phase lift and vacuum fixation. Use the common-carrier identification with $L \otimes S_R$ and transport the already generated source flow by $\tilde{U}_t = F_\omega e^{-itN_G}|_V F_\omega^*$. For the existing relative character B_0 the right scalar action $\rho(g, z) = z^{-3}g$ has a unique continuous lift of this real-parameter flow starting at the identity

$$\Gamma(t) = (e^{-itJ_3}, e^{it/6}), \quad \rho(\Gamma(t)) = \tilde{U}_t. \quad (\text{G.25})$$

Its image is the full pointwise stabilizer of $v_0 = F_\omega h_0$. On the full matter module its phase generator is $Y = T_R + B_0/6$ with $T_R = -J_3$ on S_R and zero on S_L . The lifted action has least period 12π although the transported two-dimensional source flow has least period 2π .

Proof. The conjugated flow is independent of the phase of F_ω and fixes v_0 because $N_G h_0 = 0$. In the source frame $N_G|_V = J_3 + I/2 = \text{diag}(0, 1)$. For $U_t = e^{-itN_G}|_V$ the natural tensor action has weights e^{-it} and e^{-2it} . The determinant compensation in (G.24) multiplies it by e^{it} and recovers the original eigenvalues 1 and e^{-it} . Thus the transported vacuum keeps the zero number eigenvalue of the source and supplies the same phase reference. Since $\det \tilde{U}_t = e^{-it}$ the equation $z_t^{-3}g_t = \tilde{U}_t$ forces $z_t^6 = e^{it}$. Continuity and $z_0 = 1$ uniquely give $z_t = e^{it/6}$ and $g_t = e^{-itJ_3}$. Conversely these formulas solve the equation. Every pointwise stabilizer element has $g = \text{diag}(z^3, z^{-3})$ by Theorem 4.9 and is attained by Γ . The action on the full module is therefore e^{itY} . It is generated on a finite source corner by its Hermitian matrix and functional calculus. On $W \otimes S_L$ its weight is $1/6$ forcing any period to be a multiple of 12π . All remaining weights are integer multiples of $1/6$. So 12π is a period. $\square$

Here ρ is a sixfold cover with kernel $\{(z^3 I, z) : z^6 = 1\}$. In particular $\Gamma(2\pi) = (-I, e^{i\pi/3})$ is invisible on the right scalar pair but not on the full matter module. The full module therefore carries the lifted circle of period 12π rather than a single-valued action of the source circle of period 2π . On the common carrier the least matter realization keeps this transported vacuum as its phase reference. The unique lifted source flow then gives the residual phase action and its hypercharge generator. Write $\theta = t/6$ for the primitive integer-charge circle parameter. For the colour-natural connector the B_0 factors cancel and

$$h \mapsto g_L h e^{-i6\theta T_R}.$$

Its first and second columns have integer charges $-3, +3$ respectively in agreement with $h = (\tilde{H}, H)$ and $Y_H = 1/2$ in the main text.

Proposition G.4 (Quaternion amplitude and elementary Clifford connector). In the common-carrier realization $S_L = S_R = V$ choose the pseudoreal operator $J = i\sigma_2 K$. For $x \in \text{Herm}_0(V)$ put

$$h(r, x) = rI_V + ix, \quad c(r, x) = \begin{pmatrix} 0 & h(r, x) \\ h(r, x)^* & 0 \end{pmatrix}. \quad (\text{G.26})$$

Then $hJ = Jh$ and $c(r, x)^2 = (r^2 + \frac{1}{2} \text{Tr } x^2)I$. Moreover c is the Euclidean Clifford multiplication for ϵ_4 . Its non-diagonal linear blocks exhaust the four-real-dimensional space of complex-linear maps commuting with J . In particular the distinguished normal $\nu = (1, 0)$

gives the nonzero connector $h = I_V$. The same blocks and their colour-natural lifts are finite-corner elements of the generated source algebra.

Proof. The pseudoreal structure satisfies $JxJ^{-1} = -x$ and is antilinear. So $J(ix)J^{-1} = ix$. A traceless Hermitian two-by-two matrix obeys $x^2 = \frac{1}{2} \text{Tr}(x^2)I_V$. Multiplication proves the square identity and polarization gives the Clifford anticommutator. The four matrices $I_V, i\sigma_1, i\sigma_2, i\sigma_3$ form a real basis of the complex-linear maps commuting with J . $\square$

Source-corner realization of the elementary connector. The Clifford blocks and their colour-natural lift belong to the generated source algebra. For $d = 1, 4$ identify $\mathbb{C}^d \otimes (S_L \oplus S_R)$ with H_{4d} by sending the ordered left and right basis vectors to $h_{4\alpha+i}$ and $h_{4\alpha+2+i}$ respectively where $0 \leq \alpha < d$ and $i = 0, 1$. For $h \in \mathbb{H}_V$ the generated self-adjoint odd operator is

$$\hat{c}_d(h) = \sum_{\alpha=0}^{d-1} \sum_{i,j=0}^1 (h_{ij} E_{4\alpha+i, 4\alpha+2+j} + \overline{h_{ij}} E_{4\alpha+2+j, 4\alpha+i}). \quad (\text{G.27})$$

It represents $I_d \otimes c(h)$ and its exponentials are generated reversible finite-corner operations. For $d = 4$ the first factor is the already selected pre-exchange carrier C . So the upper odd block is precisely $I_C \otimes h$ on the full modules. Independent generated left and right transformations $u_L, u_R \in SU(V)$ act by $h \mapsto u_L h u_R^*$ and preserve $\mathbb{H}_V$ and its trace comparison and have kernel $\{(I, I), (-I, -I)\}$ on this real amplitude space.

Proof. The matrix-unit relations (G.1) give the displayed tensor action. Oddness and its scalar square follow from Proposition G.4. The generated finite functional calculus gives the unitary exponential as in the finite-corner construction of Theorem G.2. Tensoring with I_C proves pre-exchange naturality. Block-diagonal source controls give $c(h) \mapsto c(u_L h u_R^*)$ preserving the real algebra and comparison of Proposition G.3. A pair fixing all h has $u_L = u_R$ by $h = I$ then lies in the scalar commutant of $\mathbb{H}_V$ giving the stated kernel. $\square$

The block-diagonal representation is faithful while its amplitude action has the stated diagonal kernel. A single central sign sends h to $-h$. The Gaussian global sign restricts to $(-I, -I)$ on the selected corner. The controlled operation acts as the identity on the complement. The connector construction names this generated odd response with its source naturality and reversal reality as the elementary chiral variation. Independent copies require a further copy space.

G.8 Zero-mode amplitude and colour-singlet response

Retaining the scalar coordinate $E_{\text{int}} = \int \Phi$ changes which modes are traced and adds a direct amplitude term to the response. The resulting Hessian and Schur complement differ from the full-diagonal prescription F_L . For the long-root connection invariant mode sums determine the mean comparison operator and the static overlap of colour strength with radial intensity. These calculations fix the normalization factors and observable responses used in Section 9.2.

G.8.1 Invariant strength of a complete mode family

Let a compact group act transitively and isometrically on X of volume V and unitarily on a Hermitian bundle over X . Let $\mathcal{U}$ be a d -dimensional invariant space of smooth sections with an L^2 -orthonormal basis u_a . Then

$$\sum_{a=1}^d |u_a(x)|^2 = \frac{d}{V}. \quad (\text{G.28})$$

For $\chi = \sum_a c_a u_a$ a positive state with second moments $\omega(c_a^* c_b) = q \delta_{ab}/d$ therefore gives

$$\omega(|\chi(x)|^2) = \frac{q}{V}, \quad \omega(\|\chi\|_{L^2}^2) = q. \quad (\text{G.29})$$

If the complex representation on $\mathcal{U}$ is irreducible invariance of its second-moment matrix and its trace q imply the stated matrix. The identities sum over the full invariant mode family. Indeed the diagonal contraction of the equivariant projection kernel is constant by transitivity and its integral is d . The second-moment matrix is scalar by Schur's lemma in the irreducible case. Its trace then fixes the weight q/d .

For the long-root family the addition formula makes the mean squared one-form norm spatially uniform. Its second-moment weight q specifies intensity independently of the mean field. The connection Laplacian determines its effect on scalar modes.

Proposition G.5 (Connection strength and retained collective amplitude). Let u_a for $a = 1, \dots, d_R$ be a real orthonormal basis of a complete isometry-invariant coexact one-form space on round RP^3 and let T_A for $A = 1, \dots, h$ be a basis orthonormal for the fixed Ad-invariant inner product on the generated compact gauge algebra in a finite-dimensional unitary representation ρ . On the corresponding trivial associated bundle form the connection $\nabla_c = d + B_c$ using real square-integrable coefficient variables with

$$B_c = \sum_{a,A} c_{aA} u_a \rho_*(T_A), \quad \mathbb{E} c_{aA} = 0, \quad \mathbb{E}(c_{aA} c_{bB}) = \frac{q}{d_R h} \delta_{ab} \delta_{AB}. \quad (\text{G.30})$$

Its mean quadratic comparison on H^1 and its operator on H^2 are

$$\mathbb{E}(\nabla_c^* \nabla_c) = \Delta_0 \otimes I + \frac{q}{Vh} \mathcal{C}_\rho, \quad \mathcal{C}_\rho = - \sum_A \rho_*(T_A)^2. \quad (\text{G.31})$$

On an irreducible representation the added term is $qC_2(\rho)/(Vh)$ times the identity on every scalar shell. For the long-root space (2,1) in Appendix A $d_R = 15$. The same normalized source decomposition gives the exact tower Hessian and completed negative-log response for the retained zero coordinate.

Proof. The associated connection was constructed in Proposition 8.1. Its squared comparison norm expands as $\|df\|^2 + 2 \operatorname{Re} \langle df, B_c f \rangle + \|B_c f\|^2$. The linear term has zero mean. Contracting the stated second moments in the last term using skew-adjointness of $\rho_*(T_A)$ and (G.28) gives $q \langle f, \mathcal{C}_\rho f \rangle / (Vh)$. Smoothness of the finite mode family bounds its coefficients and derivatives. Finite second moments justify averaging the forms and the operator on their stated domains. The mean form is closed with domain H^1 since its added coefficient is a bounded nonnegative matrix. The Casimir commutes with the representation and is scalar on each irreducible block. $\square$

The mean shift is uniform over scalar shells but depends on the representation. For $\mathfrak{g} = \mathfrak{su}(3)$ it vanishes on a colour singlet. The nonlinear colour-strength observable has a separate response (Proposition G.6). Averaging is taken at the level of the quadratic form. Averaging the resolvent $\mathbb{E}(\nabla_c^* \nabla_c + zI)^{-1}$ with $z > 0$ or eliminating source fluctuations gives a different operator.

The Gaussian law of the mean comparison. The averaged precision has an exact information-theoretic characterization. At a finite scalar cutoff include the positive conformal term in the connection precision K_c and let $\nu_c = N(0, K_c^{-1})$. Write $\bar{K}_q = \mathbb{E}_q K_c$. Among probability laws with finite mean relative entropy the unique minimizer of $\mathbb{E}_q D(\nu \| \nu_c)$ is

$$\begin{aligned} \nu_q^{\text{comp}} &= N(0, \bar{K}_q^{-1}), & \mathbb{E}_q D(\nu \| \nu_c) &= D(\nu \| \nu_q^{\text{comp}}) + C(q), \\ C(q) &= \frac{1}{2} (\log \det \bar{K}_q - \mathbb{E}_q \log \det K_c) \geq 0. \end{aligned} \tag{G.32}$$

Indeed the expectation of $\log(d\nu_c/dx)$ is a quadratic function with precision $\bar{K}_q$. Normalizing its exponential gives ν_q^{comp} . Subtraction of the two relative entropies gives the displayed identity and concavity of $\log \det$ gives $C(q) \geq 0$. The fixed positive conformal lower bound and the finite Gaussian moments make all determinant expectations finite. Equality in the minimization requires $\nu = \nu_q^{\text{comp}}$.

On an irreducible comparison block $\bar{K}_q = A + g_\rho q I$. Relative comparison with $q = 0$ therefore gives

$$2D(\nu_0^{\text{comp}} \| \nu_q^{\text{comp}}) = \text{Tr}\{qT - \log(I + qT)\}, \quad T = g_\rho A^{-1}. \tag{G.33}$$

The Hilbert–Schmidt completion in Theorem E.1 makes this identity independent of the finite cutoff. It controls the response and its first two derivatives uniformly on bounded intensity intervals. Thus the connection law γ_q determines a comparison law through the averaged log likelihood. The completed potential compares these laws. The minimum averaged cost also contains $C(q)$ which depends on the intensity. A mixture of the conditional laws instead averages their covariances. These are different statistical operations with explicitly different responses.

The quadratic coefficient can also be computed before averaging. For a realization B_1 of the source in (G.30) put $B(r) = rB_1$ and $L(r) = (d + rB_1)^*(d + rB_1)$. Let P_0 project onto constant sections. Because each u_a is coclosed $L'(0)P_0 = d^*B_1P_0 = 0$. Eliminating the orthogonal complement on which Δ_0 is strictly positive therefore gives

$$L_{\text{coll}}(r) = r^2 P_0 B_1^* B_1 P_0 + O(r^3). \tag{G.34}$$

Indeed the first off-diagonal block is zero. So its Schur term has no quadratic contribution. Averaging this coefficient gives the same Casimir term as (G.31). On nonconstant shells the off-diagonal first variation contributes its Schur term to the quadratic coefficient.

G.8.2 Normalized zero coordinate and scalar feedback

Retained amplitude and exact tower Hessian. For $g > 0$ on the real scalar space $L^2(X, dv)$ put $u_0 = V^{-1/2}$ with $P_0 = |u_0\rangle\langle u_0|$ and $Q_0 = I - P_0$. Every field has the unique orthogonal decomposition

$$\Phi = E_0 u_0 + \eta, \quad \eta \in Q_0 L^2, \quad E_0 = V^{-1/2} \int_X \Phi dv, \quad E_{\text{int}} = \sqrt{V} E_0. \quad (\text{G.35})$$

Consider the scalar feedback functional

$$F_g(\Phi) = \frac{g}{2} \langle 1, \Phi \rangle^2 \|\Phi\|^2 = \frac{\gamma}{2} E_0^4 + \frac{\gamma}{2} E_0^2 \|\eta\|^2, \quad \gamma = gV. \quad (\text{G.36})$$

At $\Phi_* = E_0 u_0$ its full Hessian and its fixed-amplitude tower Hessian are respectively

$$D^2 F_g(\Phi_*) = \gamma E_0^2 (Q_0 + 6P_0), \quad D_\eta^2 F_g(E_0, 0) = \gamma E_0^2 I_{Q_0 L^2}. \quad (\text{G.37})$$

The uniform shift $gE_{\text{int}}^2 I$ therefore acts on the eliminated mean-zero tower while the retained coordinate has a different Hessian entry.

Proof. Orthogonal projection gives the decomposition and $\|\Phi\|^2 = E_0^2 + \|\eta\|^2$. Substitution proves the polynomial identity. Its second derivatives at $\eta = 0$ are $6\gamma E_0^2$ on the retained line and $\gamma E_0^2 I$ on the tower. The mixed derivative is zero there. Continuity of the polynomial on L^2 gives these Fréchet derivatives. Away from $\eta = 0$ the mixed Hessian is $2\gamma E_0 \eta$. $\square$

For the positive conformal operator A already used in the spectral construction P_0 is reducing. On the tower its domain is $Q_0 H^2(X)$ and the bounded perturbation is

$$A_\perp(E_0) = Q_0 A Q_0 + \gamma E_0^2 I_\perp. \quad (\text{G.38})$$

On round RP^3 its base eigenvalues are $a_l = [l(l+2) + 3/4]/L^2$ with $l = 2, 4, \dots$ and multiplicities $(l+1)^2$. The trace runs over the mean-zero scalar tower at fixed $l = 0$ amplitude. This retained coordinate is a scalar mode. The long-root calculation above varies a coexact connection.

The full relative-information theorem applies unchanged to this restricted carrier. With $y = E_0^2$ and $t_l = \gamma/a_l$ it gives

$$I_\perp(y) = \sum_{l=2,4,\dots} (l+1)^2 [y t_l - \log(1 + y t_l)]. \quad (\text{G.39})$$

This information is realized by the scalar feedback itself. On a finite tower the same comparison $\mathcal{S}(\Phi) = \langle \Phi, A\Phi \rangle / 2 + F_g(\Phi)$ at fixed $E_0 = e$ has tower precision $D_e = A_\perp + \gamma e^2 I$. Its normalized Gaussian comparison is $\nu_e = N(0, D_e^{-1})$. Writing p_e for its density in common real coordinates Gaussian normalization and the relative-information identity in Appendix E.1 give with both traces and $I_\perp$ restricted to this same finite tower

$$-2 \log \frac{p_e(0)}{p_0(0)} = -\text{Tr} \log(I + \gamma e^2 A_\perp^{-1}), \quad 2D_{\text{KL}}(\nu_0 \| \nu_e) = I_\perp(e^2). \quad (\text{G.40})$$

The common coordinate Jacobian cancels in the density ratio. The negative logarithm and its compensated relative information follow from this fixed-amplitude density ratio. The unweighted infinite-mode limit exists for the compensated expression with the same comparison direction and retained direct term.

For I_{all} on the same scalar family including $l = 0$ one has $I_{\perp} = I_{\text{all}} - [yt_0 - \log(1 + yt_0)]$. Fixed physical weights multiply the corresponding term as well. Removing one mode preserves the Hilbert–Schmidt and tail estimates but changes the stationary equation. The retained direct term $\gamma y^2/2$ in (G.36) remains in the response together with the tower contribution and the specified drive. Thus the retained coordinate and the modes in the tower trace form an orthogonal decomposition of the scalar field.

Completed negative-log response of the retained coupling. Retain the coupling (G.36) together with the base quadratic term $a_0 y/2$ and the negative-logarithmic tower response with fixed weights $\beta_j \geq 0$. Let $a_0, \gamma, a_j > 0$ and set $t_j = \gamma/a_j$. Assume $\sup_j t_j < \infty$ and $S_{\perp} = \sum_{j \in \perp} \beta_j t_j < \infty$. Its complete response relative to zero amplitude is

$$\begin{aligned} W(y) &= \frac{a_0}{2}y + \frac{\gamma}{2}y^2 - \sum_{j \in \perp} \beta_j \log(1 + yt_j) \\ &= \frac{\mu_{\text{ret}}}{2}y + \frac{\gamma}{2}y^2 + I_{\perp}(y), \quad \mu_{\text{ret}} = a_0 - 2S_{\perp}, \end{aligned} \quad (\text{G.41})$$

where $I_{\perp}$ uses these same weights. It is strictly convex and coercive on $y \geq 0$. Its unique minimizer is positive exactly when $2S_{\perp} > a_0$. Otherwise the unique minimizer is $y = 0$. In the positive case the stationary equation and radial curvature are

$$\frac{a_0}{2} + \gamma y_* = \sum_{j \in \perp} \frac{\beta_j t_j}{1 + y_* t_j}, \quad \left. \frac{d^2}{dr^2} W(r^2) \right|_{r=\sqrt{y_*}} = 4y_* \left[\gamma + \sum_{j \in \perp} \frac{\beta_j t_j^2}{(1 + y_* t_j)^2} \right]. \quad (\text{G.42})$$

Proof. The stated bounds imply that all series and their first two derivatives converge normally on compact positive intervals. Differentiating gives $W' = a_0/2 + \gamma y - \sum \beta_j t_j/(1 + yt_j)$ and $W'' = \gamma + \sum \beta_j t_j^2/(1 + yt_j)^2 \geq \gamma > 0$. Moreover $W(y) \geq \gamma y^2/2 + (a_0/2 - S_{\perp})y$ since $\log(1 + x) \leq x$. The sign of $W'(0)$ proves the minimum statement. Twice differentiating $W(r^2)$ at its positive stationary point proves the curvature formula. $\square$

The terms of W must be expressed in the same response units. In a dimensionful potential β_j includes the prefactor converting the dimensionless logarithm to the chosen response units in addition to the multiplicity. In the conformal realization the retained base eigenvalue is $a_0 = 3/(4L^2)$ before this common normalization.

For a fixed spectrum with fixed weights and response units the quadratic and quartic coefficients of W obey an algebraic relation. Put $S_1 = \sum_{\perp} \beta_j/a_j > 0$ and $S_2 = \sum_{\perp} \beta_j/a_j^2$. A pair (μ_*, λ_*) is the quadratic and quartic coefficient of this same W precisely when

$$\mu_* < a_0, \quad \lambda_* = \frac{a_0 - \mu_*}{S_1} + \frac{(a_0 - \mu_*)^2 S_2}{2S_1^2}, \quad \gamma = \frac{a_0 - \mu_*}{2S_1}. \quad (\text{G.43})$$

Indeed $\mu_* = a_0 - 2\gamma S_1$ fixes the unique positive γ and substitution in $\lambda_* = 2\gamma + 2\gamma^2 S_2$ proves both directions. This criterion applies to the finite- S_1 realization. In the renormalized full-diagonal potential F_L the linear coefficient is fixed separately by the reference slope.

Both μ_{ret} and the direct quartic term follow from this retained realization. Its restoring slope is unbounded unlike a finite-restoring-slope F_μ . The result applies to finite cutoffs and fixed weighted towers with $S_\perp < \infty$. The unweighted three-dimensional tower has divergent $S_\perp$ and needs its own linear subtraction. The polynomial functional F_g is defined on L^2 . The compensated series is defined through its Hilbert–Schmidt relative operator as in Theorem E.1.

G.8.3 Amplitude and scale normalization

At fixed volume the zero-mode kinetic norm is $(\partial E_{\text{int}})^2/(2V)$. It equals $(\partial E_0)^2/2$. For a homothetic family identify points by their Haar coordinates and write $\Psi = \sqrt{V}\Phi$ in the fixed space $L^2(X, dh)$ with $dh = dv/V$. Then $\sqrt{V}\partial_\sigma\Phi = \Psi' - (\log V)'\Psi/2$. For the zero component this gives

$$\int_X |\partial_\sigma(E_0 u_0)|^2 dv = \left| E_0' - \frac{1}{2}(\log V)'E_0 \right|^2, \quad V = \pi^2 L^3. \quad (\text{G.44})$$

This is the ordinary derivative in the unnormalized field identification. Equivalently the derivative compatible with this unitary identification is $\nabla_\sigma\Phi = \partial_\sigma\Phi + (\log V)'\Phi/2$. The connection term is the derivative of the varying normalization.

More generally take an invertible real amplitude coordinate $E = a\varphi$ with $a > 0$. At a fixed scale the kinetic and potential terms together with the source pairing transform as

$$\begin{aligned} Z_E &= Z_\varphi/a^2, & \mu_E &= \mu_\varphi/a^2, & \lambda_E &= \lambda_\varphi/a^4, \\ j_E &= j_\varphi/a, & g_E &= g_\varphi/a^2 & \text{when } g_\varphi\varphi^2 &= g_E E^2. \end{aligned} \quad (\text{G.45})$$

These identities follow by substitution with the quadratic and quartic potential written as $\mu\varphi^2/2 + \lambda\varphi^4/4$. For a scale-dependent a the corresponding derivative is

$$\varphi' = a^{-1}[E' - (\log a)'E]. \quad (\text{G.46})$$

At a stationary point the potential curvature and kinetic coefficient transform by the same factor. So their ratio is invariant. An overall rescaling of the functional also rescales its kinetic and source terms.

G.8.4 Collective response by Schur elimination

For a strictly positive finite-mode comparison kernel K and a surjective linear amplitude map C the constrained quadratic response at amplitude e is

$$\inf_{Cx=e} \frac{1}{2}\langle x, Kx \rangle = \frac{1}{2}\langle e, K_{\text{coll}}e \rangle, \quad K_{\text{coll}} = (CK^{-1}C^*)^{-1}. \quad (\text{G.47})$$

The minimizing lift is $x_* = K^{-1}C^*K_{\text{coll}}e$. For a coisometry $CC^* = I$ identify C with the first block projection and write

$$K = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \quad K_{\text{coll}} = A - BD^{-1}B^*. \quad (\text{G.48})$$

For $t > 0$ shifting and reducing obey

$$(C(K + tI)^{-1}C^*)^{-1} = K_{\text{coll}} + tI + B[D^{-1} - (D + tI)^{-1}]B^*. \quad (\text{G.49})$$

The extra term is nonnegative and vanishes exactly when $B = 0$. The minimizing lift is K -orthogonal to $\ker C$. Completing the square therefore proves the constrained minimum. Block elimination gives $A - BD^{-1}B^*$ and applying it to $K + tI$ gives the shift correction. The strict positivity of $D^{-1} - (D + tI)^{-1}$ proves the equality condition.

The correction in (G.49) occurs in a two-dimensional example. The choices $K = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $C = (1, 0)$ with $t = 1$ give $K_{\text{coll}} = 3/2$ and $(C(K + I)^{-1}C^*)^{-1} = 8/3$ not $5/2$. In an infinite-mode application the same minimization proof works for a strictly positive self-adjoint K with bounded inverse and a bounded surjective map to finitely many retained coordinates. The minimization is taken on the closed form domain $D(K^{1/2})$ with quadratic cost $\|K^{1/2}x\|^2/2$. Passing compressed kernels or derivatives to a limit requires uniform bounds on those kernels and derivatives.

The retained scalar kernel. For $\mathcal{S}$ above and $\Phi = eu_0 + \eta$ in its form domain use the projection $C_0\Phi = \langle u_0, \Phi \rangle$. Differentiating the polynomial (G.36) and adding the base quadratic form gives the full Hessian

$$K(e, \eta) = \begin{pmatrix} a_0 + 6\gamma e^2 + \gamma\|\eta\|^2 & 2\gamma e\langle \eta, \cdot \rangle \\ 2\gamma e\eta & A_\perp + \gamma e^2 I \end{pmatrix}. \quad (\text{G.50})$$

The form domain is $\mathbb{R} \oplus Q_0 H^1(X)$ and the self-adjoint operator domain is $\mathbb{R} \oplus Q_0 H^2(X)$. The additional diagonal and mixed terms are bounded. The mixed terms have finite rank. Since $D_e = A_\perp + \gamma e^2 I > 0$ minimizing its quadratic Hessian response over tower variations at a fixed amplitude variation gives exactly

$$S(e, \eta) = a_0 + 6\gamma e^2 + \gamma\|\eta\|^2 - 4\gamma^2 e^2 \langle \eta, D_e^{-1} \eta \rangle. \quad (\text{G.51})$$

Indeed the minimizing tower variation for amplitude variation v is $-2\gamma e D_e^{-1} \eta v$. Substitution leaves $S(e, \eta)v^2/2$. The minimization is justified by $D_e > 0$. Where the full Hessian is strictly positive it also equals $(C_0 K^{-1} C_0^*)^{-1}$. At $\eta = 0$ the mixed term vanishes exactly. So the retained coefficient is $a_0 + 6\gamma e^2$ while the tower shift is $\gamma e^2 I$. At nonzero η the direct and negative Schur terms both contribute. This gives the complete reduced Hessian of the retained scalar realization. Its comparison with the full-diagonal prescription is governed by (G.43).

G.8.5 Colour-singlet observables and collective excitations

For the generated connection Wick contraction of the real Gaussian source determines the fourth moments entering the mean colour strength and its covariance with radial intensity.

Proposition G.6 (Colour-singlet response of the Gaussian long-root source). Use the connection family of Proposition G.5 with the full real $(2, 1)$ coexact mode space $d_R = 15$ and the colour algebra of dimension $h = 8$. Set $M = d_R h$ and $c_{aA} = \sqrt{q/M} \xi_{aA}$ where

the ξ_{aA} are independent standard real Gaussians and $q > 0$. Here $B_c = \sum_{a,A} c_{aA} u_a T_A$ is the principal Lie-algebra-valued connection before applying an associated representation ρ_* . With the same Ad-invariant inner product and orthonormal T_A put

$$C_{\text{ad}} = \frac{1}{h} \sum_{A,B} \|[T_A, T_B]\|_{\mathfrak{g}}^2, \quad \alpha_R = \frac{C_{\text{ad}}}{3Vh}, \quad \lambda_R = \frac{16}{L^2}.$$

Let $\mathcal{N}(c) = \|B_c\|_{L^2}^2 = \sum_{a,A} c_{aA}^2$ and $\mathcal{Q}(c) = \|dB_c + B_c \wedge B_c\|_{L^2}^2$ using $\|F\|^2 = \sum_{i < j} \|F_{ij}\|_{\mathfrak{g}}^2$ for two-forms. Then

$$\mathbb{E}_q \mathcal{N} = q, \quad \mathcal{O}(q) := \mathbb{E}_q \mathcal{Q} = \lambda_R q + \alpha_R q^2, \quad \alpha_R > 0. \quad (\text{G.52})$$

In the real L^2 space of this source the orthogonal projection onto $\text{span}\{\mathcal{N} - q\}$ obeys

$$\begin{aligned} \text{Cov}_q(\mathcal{Q}, \mathcal{N}) &= \frac{2q^2}{M} (\lambda_R + 2\alpha_R q) > 0, \\ \Pi_{\text{rad}}(\mathcal{Q} - \mathcal{O}(q)) &= (\lambda_R + 2\alpha_R q)(\mathcal{N} - q). \end{aligned} \quad (\text{G.53})$$

Conditioning on the intensity gives the radial polynomial

$$P(N) := \mathbb{E}[\mathcal{Q} \mid \mathcal{N} = N] = \lambda_R N + \kappa_R N^2, \quad \kappa_R = \frac{M}{M+2} \alpha_R = \frac{60}{61} \alpha_R. \quad (\text{G.54})$$

Proof. The mode space is invariant under the connected orientation-preserving isometry group. Its diagonal tensor kernel is isotropic on each tangent fibre. Hence $\sum_a u_{a,i} u_{a,j} = d_R \delta_{ij} / (3V)$ by the addition formula. Thus $\mathbb{E} B_i^A B_j^B = q \delta_{ij} \delta_{AB} / (3Vh)$. Coexactness and the Hodge eigenvalue λ_R give $\mathbb{E} \|dB\|^2 = \lambda_R q$. The cubic cross term has zero Gaussian expectation. For each $i < j$ Wick contraction gives $\mathbb{E} \| [B_i, B_j] \|^2 = (q/(3Vh))^2 h C_{\text{ad}}$. There are three such pairs. Integration over X proves the first formula. The finite smooth mode expansion has all Gaussian polynomial moments. So the observables lie in L^2 and differentiation under the integral is legitimate. Its Gaussian density has logarithmic derivative $M(\mathcal{N} - q)/(2q^2)$ with respect to q . Thus $\mathcal{O}'(q) = M \text{Cov}(\mathcal{Q}, \mathcal{N}) / (2q^2)$. Finally $\text{Var}(\mathcal{N}) = 2q^2/M$ gives the projection. For the conditional mean write $c = \sqrt{N} \theta$ with θ uniform on S^{M-1} . The quadratic term is $\lambda_R N$. The cubic term averages to zero and the quartic mean is proportional to N^2 . Its Gaussian mean $\alpha_R q^2$ and $\mathbb{E}_q N^2 = q^2(M+2)/M$ fix κ_R . $\square$

Here B already includes the connection coupling. If $B = g\mathcal{B}$ and q denotes the strength of $\mathcal{B}$ instead the two terms in (G.52) acquire g^2 and g^4 . The positive norm fixes the trace convention. Using $\frac{1}{2}\|F\|^2$ halves both the mean and the covariance. The observable $\mathcal{Q}$ is gauge invariant and its expectation here is taken in the fixed coexact Gaussian source law.

The source polynomial representation and its radial grading. Write γ_q for the Gaussian law of the M real coefficients and put $\xi_i = \sqrt{M/q} c_i$. The source algebra is the polynomial $*$ -algebra $\mathcal{P}_M = \mathbb{C}[c_1, \dots, c_M]$ with $\omega_q(A) = \int A d\gamma_q$. Its cyclic Hilbert space is $L^2(\gamma_q)$ with multiplication defined on $\mathcal{P}_M$. For a multi-index $\nu \in \mathbb{N}^M$ set

$$h_\nu(\xi) = \prod_i \frac{\text{He}_{\nu_i}(\xi_i)}{\sqrt{\nu_i!}}, \quad U_q h_\nu = |\nu\rangle \in \Gamma_s(\mathbb{C}^M),$$

where He_n denotes the probabilists' Hermite polynomial. Then U_q is unitary with $U_q 1 = \Omega$ and on the polynomial core

$$\begin{aligned} \langle U_q A, U_q B \rangle &= \omega_q(A^* B), \\ U_q(AB) &= \pi_q(A) U_q B, \quad \pi_q(c_i) = \sqrt{q/M} (a_i + a_i^*). \end{aligned} \quad (\text{G.55})$$

These identities retain ordinary source multiplication including every Gaussian contraction. To verify them the generating function $\exp(t\xi - t^2/2)$ gives $\mathbb{E} \text{He}_n(\xi) \text{He}_m(\xi) = n! \delta_{nm}$ and $\xi \text{He}_n = \text{He}_{n+1} + n \text{He}_{n-1}$. Tensor products give the displayed representation. The Hermite polynomials are complete. A function orthogonal to them has identically zero Gaussian exponential transform and hence zero Fourier transform. All finite products act on the same dense polynomial domain. Density extends U_q to every square-integrable source observation. Unbounded products retain their common polynomial domain.

The intensity is a quadratic source observable. Put $\alpha_G = M/2$ and $X = \alpha_G \mathcal{N}/q$ with $s = (X - \alpha_G)/\sqrt{\alpha_G}$. Radial integration gives the density $x^{\alpha_G-1} e^{-x}/\Gamma(\alpha_G)$ for X . Consequently

$$\mathbb{E} s^2 = 1, \quad \mathbb{E} s^3 = \frac{2}{\sqrt{\alpha_G}}, \quad \mathbb{E} s^4 = 3 + \frac{6}{\alpha_G}. \quad (\text{G.56})$$

The first three orthonormal radial polynomials are $p_0 = 1$ and $p_1 = s$ together with

$$p_2 = \frac{X^2 - 2(\alpha_G + 1)X + \alpha_G(\alpha_G + 1)}{\sqrt{2\alpha_G(\alpha_G + 1)}}.$$

The Gamma moments $\mathbb{E} X^j = \Gamma(\alpha_G + j)/\Gamma(\alpha_G)$ give by direct substitution

$$s^2 = 1 + \frac{2}{\sqrt{\alpha_G}} p_1 + \sqrt{\frac{2(\alpha_G + 1)}{\alpha_G}} p_2, \quad U_q s = \frac{1}{\sqrt{2M}} \sum_i (a_i^*)^2 \Omega. \quad (\text{G.57})$$

Thus the normalized intensity occupies the second Hermite sector of the original coefficients. Its radial multiplication retains the nonzero third moment in (G.56).

For the full curvature observation conditional expectation gives the orthogonal decomposition

$$\mathcal{Q} = P(\mathcal{N}) + R_{\text{ang}}, \quad \mathbb{E}[R_{\text{ang}} | \mathcal{N}] = 0. \quad (\text{G.58})$$

Both terms are finite source polynomials and are retained by U_q . The first has the Gamma radial expansion above. The second retains the angular and colour correlations of the connection. The decomposition preserves $\|\mathcal{Q}\|_{L^2(\gamma_q)}^2$ and every mixed polynomial moment through (G.55).

The complete curvature spectrum in the source state. The same representation resolves the curvature observation into its orthogonal Hermite components. Flatten the mode and colour indices to $i = 1, \dots, M$ and write its homogeneous parts as

$$\mathcal{Q}(c) = \lambda_R \mathcal{N}(c) + \sum_{ijk} T_{ijk}^{(3)} c_i c_j c_k + \sum_{ijkl} T_{ijkl}^{(4)} c_i c_j c_k c_l.$$

The real symmetric tensors are obtained by polarizing $2\langle dB, B \wedge B \rangle$ and $\|B \wedge B\|^2$. Their coefficients are fixed by mode multiplication and the colour bracket. With Wick ordering in the original law γ_q one has

$$\begin{aligned}\mathcal{Q} - \mathcal{O}(q) &= \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4, \\ \mathcal{Q}_2 &= (\lambda_R + 2\alpha_R q)(\mathcal{N} - q), \\ \mathcal{Q}_3 &= \sum T_{ijk}^{(3)} : c_i c_j c_k :_q, \quad \mathcal{Q}_4 = \sum T_{ijkl}^{(4)} : c_i c_j c_k c_l :_q.\end{aligned}\tag{G.59}$$

Their squared norms are

$$\begin{aligned}W_{\mathcal{Q},2}(q) &= \frac{2q^2}{M}(\lambda_R + 2\alpha_R q)^2 > 0, \\ W_{\mathcal{Q},3}(q) &= 6(q/M)^3 \|T^{(3)}\|_F^2, \quad W_{\mathcal{Q},4}(q) = 24(q/M)^4 \|T^{(4)}\|_F^2 \geq \frac{8\alpha_R^2 q^4}{M(M+2)} > 0.\end{aligned}\tag{G.60}$$

Here $\|\cdot\|_F$ is the Euclidean tensor norm in the fixed orthonormal mode basis. For the equilibrium update (G.75) the exact connected correlation and its response spectral measure are consequently

$$\begin{aligned}C_{\mathcal{Q}}(\tau) &= \omega_q [(\mathcal{Q} - \mathcal{O}(q))T_{\tau}^{\text{coeff}}(\mathcal{Q} - \mathcal{O}(q))] = \sum_{n=2}^4 W_{\mathcal{Q},n}(q) e^{-n\omega_R \tau}, \\ \rho_{\mathcal{Q}}^{\text{src}} &= \sum_{n=2}^4 W_{\mathcal{Q},n}(q) \delta_{n\omega_R}, \quad \omega_R = \lambda_R + 2\alpha_R q.\end{aligned}\tag{G.61}$$

To prove the decomposition contract the invariant tensors with the coefficient covariance $(q/M)I$. A contraction of $T^{(3)}$ is an invariant vector and hence vanishes. A single contraction of $T^{(4)}$ is an invariant symmetric form. Absolute irreducibility of the real $(2, 1)$ carrier times the adjoint colour carrier gives

$$\sum_i T_{iik}^{(3)} = 0, \quad \sum_i T_{iikl}^{(4)} = \frac{\alpha_R M}{3} \delta_{kl}.$$

The constant in the second identity follows from $\mathbb{E}_q \mathcal{Q}^{(4)} = \alpha_R q^2$ and the three Gaussian pairings. Wick expansion now gives (G.59). Orthogonality of Hermite orders gives their norms. The fourth-order radial projection is $\kappa_R : \mathcal{N}^2 :_q$ with

$$:\mathcal{N}^2 :_q = \mathcal{N}^2 - \frac{2q(M+2)}{M} \mathcal{N} + \frac{q^2(M+2)}{M}, \quad \|:\mathcal{N}^2 :_q\|^2 = \frac{8q^4(M+2)}{M^3}.$$

Its squared norm gives the lower bound in (G.60). The Mehler update multiplies degree n by $e^{-n\omega_R \tau}$ which proves (G.61). In particular the original curvature has a nonzero component at $2\omega_R$ and retains its higher-order source correlations. This spectrum measures restoration within the fixed connection law.

Joint curvature and intensity sources. For $h_N < M/(2q)$ and $J \geq 0$ define

$$W(h_N, J) = \log \mathbb{E}_q \exp(h_N \mathcal{N} - J \mathcal{Q}), \quad n = \partial_{h_N} W, \quad \Gamma(n, J) = h_N n - W(h_N, J).$$

The intensity Legendre transform exists locally for $n > 0$ near $J = 0$. At $J = 0$ with right derivatives in J

$$\begin{aligned} \Gamma(n, 0) &= \frac{M}{2} \left(\frac{n}{q} - 1 - \log \frac{n}{q} \right), & \Gamma_J(n, 0) &= \lambda_R n + \alpha_R n^2, \\ \Gamma_{nJ}(n, 0) &= \lambda_R + 2\alpha_R n, & \Gamma_{nn}(n, 0) &= \frac{M}{2n^2}, \\ \Gamma_{JJ}(n, 0) &= -\text{Var}_n \mathcal{Q} + \frac{\text{Cov}_n(\mathcal{Q}, \mathcal{N})^2}{\text{Var}_n \mathcal{N}} < 0. \end{aligned} \tag{G.62}$$

For the mean-amplitude coordinate $e = \sqrt{n}$ at $n = q$ one has $\Gamma_{ee} = 2M/q$ and $\Gamma_{eJ} = 2\sqrt{q}(\lambda_R + 2\alpha_R q) > 0$.

Proof. The remaining Gaussian precision is positive. Since $\mathcal{Q} \geq 0$ all polynomial source derivatives are dominated locally as $J \downarrow 0$. At $J = 0$ one has $n = q/(1 - 2qh_N/M)$. Inversion and substitution give $\Gamma(n, 0)$. Strictly positive intensity variance gives the local inversion for $J \geq 0$. Differentiating at fixed n gives $\Gamma_J = \mathbb{E} \mathcal{Q}$ and the covariance Schur complement in (G.62). The latter is the negative variance after projecting onto $\mathcal{N} - \mathbb{E} \mathcal{N}$. The nonzero quartic part of $\mathcal{Q}$ is not affine in $\mathcal{N}$. So this residual variance is strictly positive. The mean and mixed derivative follow from (G.52). The chain rule gives the amplitude derivatives. $\square$

The same tilted law also determines the mean connection comparison. The real $(2, 1)$ carrier and the adjoint colour carrier are absolutely irreducible under their independent connected symmetry groups. Invariance of $\mathcal{N}$ and $\mathcal{Q}$ therefore gives $\mathbb{E} c = 0$ and $\mathbb{E} c c^\top = (n/M)I$. For the associated representation ρ the preceding connection calculation consequently gives

$$\overline{K}_\rho(n, J) = \Delta_0 \otimes I + \frac{n}{Vh} C_\rho, \quad C_\rho = - \sum_A \rho_*(T_A)^2.$$

At fixed $h_N = 0$ its curvature insertion is

$$- \partial_J \overline{K}_\rho|_{J=0+} = \frac{2q^2(\lambda_R + 2\alpha_R q)}{MVh} C_\rho.$$

At fixed n the mean operator is fixed and the source response is retained by Γ_J and Γ_{JJ} . These formulas specify both the connection comparison and the direct curvature-source term.

Self-consistent curvature susceptibility. A curvature source changes both the observation at fixed mean intensity and the intensity selected by the complete potential. These responses can be combined without counting the intensity cost twice. Take the preceding

reference law at $q = q_L$ and write $F = F_L^{\text{YM}}$. The joint continuation which preserves the equilibrium potential at $J = 0$ and the fixed-intensity source derivative Γ_J is uniquely

$$\Phi_L(n, J) = F(n) + \Gamma(n, J) - \Gamma(n, 0). \quad (\text{G.63})$$

Indeed $\Phi_L(n, 0) = F(n)$ and $\partial_J \Phi_L = \partial_J \Gamma$ determine it by integration in J . The subtraction removes the bare intensity cost already replaced by the complete restoring potential. All source insertions still use the original connection polynomial $\mathcal{Q}$.

Put

$$V_\perp(q) = \text{Var}_q \mathcal{Q} - \frac{\text{Cov}_q(\mathcal{Q}, \mathcal{N})^2}{\text{Var}_q \mathcal{N}} = W_{\mathcal{Q},3}(q) + W_{\mathcal{Q},4}(q) > 0. \quad (\text{G.64})$$

This is the variance orthogonal to the intensity score. It includes the fourth-order radial component as well as angular fluctuations. For $J \geq 0$ sufficiently small there is a unique local stationary branch $n(J)$ through $n(0) = q_L$ with positive $\Phi_{L,nn}$. The curvature expectation along this branch is $\mathcal{O}_{\text{eq}}(J) = \Gamma_J(n(J), J)$. Its static susceptibility is

$$\begin{aligned} n'(0+) &= -\frac{\mathcal{O}'(q_L)}{F''(q_L)}, \\ \chi_{\mathcal{Q}}^{\text{stat}} &:= -\mathcal{O}'_{\text{eq}}(0+) = V_\perp(q_L) + \frac{\mathcal{O}'(q_L)^2}{F''(q_L)} = V_\perp(q_L) + \frac{b_{\mathcal{Q}}^2}{m_{\text{YM,rad}}^2} > 0, \\ b_{\mathcal{Q}} &= \sqrt{2q_L^2/M} \mathcal{O}'(q_L), \quad m_{\text{YM,rad}}^2 = (2q_L^2/M) F''(q_L). \end{aligned} \quad (\text{G.65})$$

The finite Gaussian source has all polynomial moments. For h_N in a compact subinterval of its stated domain these dominate the source derivatives as $J \downarrow 0$. The positive intensity variance gives a smooth local Legendre inversion on this one-sided domain. At $(q_L, 0)$ one has $\Phi_{L,n} = 0$ and $\Phi_{L,nn} = F''(q_L) > 0$. The implicit-function argument therefore gives the stationary branch. Differentiating its equation and then $\Gamma_J(n(J), J)$ using (G.62) proves (G.65).

The first term is the direct curvature response at fixed mean intensity. The second is the additional response when that mean relaxes in the complete potential. Both are measured by the same curvature insertion. The coefficient $b_{\mathcal{Q}}$ is therefore a source-derived coupling to the collective restoring mode. In its linear equilibrium dynamics this contribution has the resolvent $b_{\mathcal{Q}}^2/(m_{\text{YM,rad}}^2 - z)$. The fixed-law correlation (G.61) and the susceptibility above describe respectively an equilibrium fluctuation and a self-consistent change of the source intensity.

The colour composite thus has nonzero overlap with the radial intensity of the same source. For $q = r^2$ its mean derivative is $2r(\lambda_R + 2\alpha_R r^2) > 0$ at $r > 0$ and zero at $r = 0$. The source is centred with $\mathbb{E}B = 0$. The positive intensity describes fluctuations around zero mean. This composite observation remains distinct from the scalar precision $A + gE^2 I$.

Proposition G.7 (Interaction-preserving resolution of the same long-root source). Keep the preceding Gaussian source at fixed L in the left-invariant coframe used in Section 8.1. Apply scalar coefficient heat $G_t = e^{-t\Delta_0}$ in this frame. Put $\rho_R = 24/L^2$ and $a_t = e^{-\rho_R t}$

with $H = B \wedge B$. Let P_J project each two-form coefficient onto scalar spin J and write $H_J = P_J H$. Then $H = \sum_{J=0}^3 H_J$ and

$$\begin{aligned} B_t &= a_t B, & q_t &= a_t^2 q, \\ F_{B_t} &= a_t dB + a_t^2 H, & G_t F_B &= a_t dB + \sum_{J=0}^3 e^{-t\lambda_J} H_J, & \lambda_J &= 4J(J+1)/L^2. \end{aligned} \quad (\text{G.66})$$

The connection–field-strength pair has the composition law

$$T_t(B, F) = (a_t B, a_t F + (a_t^2 - a_t) B \wedge B), \quad T_s T_t = T_{s+t}, \quad F = F_B. \quad (\text{G.67})$$

For $w_J = \mathbb{E}_q \|H_J\|^2$ one has

$$w_0 = \frac{C_{\text{ad}} q^2}{60Vh} = \frac{\alpha_R q^2}{20} > 0, \quad \sum_{J=0}^3 w_J = \alpha_R q^2. \quad (\text{G.68})$$

Consequently the nonlinear response lost by interchanging the two operations satisfies

$$\mathbb{E}_q \|G_t F_B - F_{B_t}\|^2 = \sum_{J=0}^2 (e^{-t\lambda_J} - a_t^2)^2 w_J \geq (1 - a_t^2)^2 \frac{\alpha_R q^2}{20} > 0 \quad (t > 0). \quad (\text{G.69})$$

Proof. The left coframe is fixed by left translations. A $(2, 1)$ one-form therefore has scalar spin 2 coefficients with scalar eigenvalue $24/L^2$. The Hodge eigenvalue governing $\|dB\|^2$ is $16/L^2$. Coefficient heat commutes with the invariant derivatives. This proves the first line and the two field-strength formulas initially allowing $0 \leq J \leq 4$. To remove $J = 4$ write V_j for the spin- j module. The spatial wedge factors through

$$\Lambda^2(V_2 \otimes V_1) = (V_1 \oplus V_3) \otimes (V_0 \oplus V_2) \oplus (V_0 \oplus V_2 \oplus V_4) \otimes V_1.$$

Its only left-spin-4 term is $(4, 1)$ whereas scalar-spin-4 two-form coefficients have right spins 3, 4, 5. Equivariance thus annihilates that term. Substitution proves the composition law.

For real orthonormal modes put $K_k^{ab} = \int_X \epsilon_{kij} u_{a,i} u_{b,j} dV$. These are the compressions of the three frame-spin generators. In the right coupling $2 \otimes 1 \rightarrow 1$ their coefficient relative to the total spin generators is

$$\frac{1(1+1) + 1(1+1) - 2(2+1)}{2 \cdot 1(1+1)} = -\frac{1}{2}.$$

Indeed a vector operator on V_1 has multiplicity one and contraction with total spin gives $(J^2 + S^2 - L_{\text{orb}}^2)/2$. The left multiplicity is 5. Hence $\sum_k \|K_k\|_{\text{HS}}^2 = 15/2$. With $*$ the three-dimensional Hodge identification the constant coefficients are

$$(P_0 * H)_k = \frac{1}{2V} \sum_{a,b,A,B} c_{aA} c_{bB} K_k^{ab} [T_A, T_B].$$

The internal Wick pairing vanishes. The two cross pairings agree because both K_k^{ab} and the bracket are antisymmetric. Thus

$$w_0 = \left(\frac{q}{15h}\right)^2 \frac{h C_{\text{ad}}}{2V} \frac{15}{2} = \frac{C_{\text{ad}} q^2}{60Vh}.$$

Parseval and the preceding quartic calculation give the sum of the w_J . Orthogonality gives the remainder formula. Its $J = 3$ term vanishes since $\lambda_3 = 2\rho_R$. All fields are finite smooth mode polynomials. So every stated Gaussian moment is finite. $\square$

Reconstructing the field strength from the resolved connection gives $\mathbb{E}\|F_{B_t}\|^2 = \mathcal{O}(q_t)$. Resolving the initial field strength gives $a_t^2 \lambda_R q + \sum_J e^{-2t\lambda_J} w_J$. Thus $G_t F_B \rightarrow H_0$ while $F_{B_t} \rightarrow 0$ in mean square. The surviving term lies in the kernel of scalar coefficient heat in the fixed left frame. The Hodge Laplacian acts on the entire two-form including the coframe. This accounts for the nonzero product correction under coefficient resolution. The decay rates in these formulas are scalar coefficient heat rates with respect to the resolution parameter t . The whole source algebra obeys the same transport. For a polynomial A on the resolved source put $(\mathcal{D}_t A)(c) = A(a_t c)$. Since $a_t c$ has law γ_{q_t}

$$\omega_q(\mathcal{D}_t A) = \omega_{q_t}(A), \quad \mathcal{D}_t(AB) = (\mathcal{D}_t A)(\mathcal{D}_t B), \quad U_q \mathcal{D}_t A = U_{q_t} A. \quad (\text{G.70})$$

The last identity follows because the standardized coefficients $\sqrt{M/q_t} a_t c$ equal $\sqrt{M/q} c$. For finite t this is an isometry between the two source laws including their complete curvature polynomials and angular correlations.

Full additive colour-probe response. For the same Gaussian principal connection B let a be a deterministic smooth $\mathfrak{su}(3)$ -valued one-form in the fixed trivialization. Use the response $\mathcal{Y}(A) = \frac{1}{2}\|F_A\|^2$ and put $\delta_q = 2\alpha_R q = 2qC_{\text{ad}}/(3Vh)$. Its exact translated response is

$$\mathcal{R}_q(a) := \mathbb{E}_q \mathcal{Y}(B + a) - \mathbb{E}_q \mathcal{Y}(B) = \frac{1}{2}\|da + a \wedge a\|^2 + \frac{\delta_q}{2}\|a\|^2. \quad (\text{G.71})$$

The noncommutative interaction is retained in $da + a \wedge a$. The second variation on these internal probe directions is

$$D^2 \mathcal{R}_q(0)[a, a] = \|da\|^2 + \delta_q \|a\|^2. \quad (\text{G.72})$$

Its quadratic-form domain is $H(d) = \{a \in L^2 : da \in L^2\}$ and the associated self-adjoint operator is $d^*d + \delta_q I \geq \delta_q I$. On the fixed coclosed probe space this becomes $\Delta_1^\perp + \delta_q I$ with lowest eigenvalue $4/L^2 + \delta_q$ and long-root eigenvalue $16/L^2 + \delta_q$.

Proof. Write $v = q/(3Vh)$ and $[B \wedge a]_{ij} = [B_i, a_j] - [B_j, a_i]$. Then $F_{B+a} = F_B + F_a + [B \wedge a]$. The diagonal colour covariance and centred Gaussian law give $\mathbb{E}F_B = 0$ and $\mathbb{E}[B \wedge a] = 0$. The cubic part of $\mathbb{E}\langle F_B, [B \wedge a] \rangle$ vanishes. Its quadratic part also vanishes. Derivative covariances remain diagonal in the colour labels and Ad-invariance gives $\langle T_A, [T_A, T_D] \rangle = 0$. Finally simplicity of $\mathfrak{su}(3)$ gives $\sum_A \|[T_A, X]\|^2 = C_{\text{ad}}\|X\|^2$. So

$$\mathbb{E}\|[B \wedge a]\|^2 = vC_{\text{ad}} \int_X \sum_{i < j} (|a_i|^2 + |a_j|^2) dV = 2vC_{\text{ad}}\|a\|^2.$$

Expanding the squared norm proves the exact identity and differentiating at zero proves the Hessian. The differential d is closed and smooth one-forms are a core. So the latter form closes on $H(d)$. On coclosed forms $d^*d = \Delta_1$. The spectrum in Appendix A gives the stated eigenvalues. The nonlinear expression is defined on smooth probes and extends to $H(d) \cap L^4$ where $a \wedge a \in L^2$. Its quadratic part extends to $H(d)$. $\square$

The colour interaction fixes the positive shift δ_q . Under fixed- L resolution $\delta_{q_t} = a_t^2 \delta_q \rightarrow 0$ giving a uniform lower bound on each bounded resolution interval. The variation is with respect to the independent additive probe a at fixed Gaussian source intensity q . Its unique minimum at $a = 0$ describes stability of the averaged connection response. F_L describes the separate collective coordinate E .

Common frame invariance of the source response. The Gaussian connection is expressed relative to the generated source frame. A change of this common frame transports the connection and source frame together with the probability law and observations on the allowed response space. It preserves their relative configuration. Varying the connection at a fixed source frame changes that configuration and can have a restoring response. Let $U : X_3 \rightarrow SU(3)$ be a smooth presentation of that frame and set

$$A_0(U) = -(dU)U^{-1}, \quad B_U = UBU^{-1}, \quad a_U = UaU^{-1}, \quad \gamma_{q,U} = (B \mapsto B_U)_* \gamma_q.$$

Here U labels a common frame presentation. The variables carried by the source remain its Gaussian coefficients and its generated response types. Writing $D_0 = d + [A_0(U), \cdot]$ direct differentiation gives $D_0 a_U = U(da)U^{-1}$ and

$$F_{A_0+B_U+a_U} = UF_{B+a}U^{-1}. \quad (\text{G.73})$$

Under a passive local change g the frame becomes $g^{-1}U$. The reference connection A_0 becomes $g^{-1}A_0g + g^{-1}dg$ and B_U, a_U transform by conjugation. Trace invariance and the displayed pushforward of the law therefore prove the exact common-frame identity

$$\begin{aligned} \mathcal{R}_{q,U}(a_U) &:= \mathbb{E}_{q,U} [\mathcal{Y}(A_0 + B_U + a_U) - \mathcal{Y}(A_0 + B_U)] \\ &= \frac{1}{2} \|D_0 a_U + a_U \wedge a_U\|^2 + \frac{\delta_q}{2} \|a_U\|^2 = \mathcal{R}_q(a). \end{aligned} \quad (\text{G.74})$$

In particular for every source polynomial A its transported observation $A_U(B_U) = A(U^{-1}B_UU)$ satisfies $\omega_{q,U}(A_U) = \omega_q(A)$. The same equality applies to A^*B . It transports the source Hilbert norm and multiplication together.

The generated connection responses are coexact in the source frame. If $P_\perp$ is their Hodge projector their presentation in the frame U is the range of

$$P_{\perp,U} = \text{Ad}_U P_\perp \text{Ad}_{U^{-1}}.$$

Indeed $D_0^* = \text{Ad}_U d^* \text{Ad}_{U^{-1}}$ since conjugation is unitary on the generated trace norm. This specifies both the form domain and its frame transport. Consider the common-frame orbit of pairs (I, a) with $a \in \text{ran } P_\perp$. Every orbit has the unique source-frame representative $a = U^{-1}a_UU$. The quotient norm and response are consequently those of the original coexact space.

The corresponding infinitesimal statement identifies the zero directions of the joint response. At $A_0 = 0, U = I$ put $\eta = \delta U$ and $b = \delta A + d\eta$. A pure frame change has $(\delta A, \eta) = (d\varepsilon, -\varepsilon)$. Hence $b = 0$. The joint quadratic form is

$$\|db\|^2 + \delta_q \|b\|^2, \quad b \in \text{ran } P_\perp.$$

For $q > 0$ its kernel consists precisely of pure frame directions in this generated orbit. Its quotient operator is $\Delta_1^\perp + \delta_q I$. In a fixed source frame a nonzero exact probe instead changes the relative connection and has response $\delta_q \|d\varepsilon\|^2$. The two variations have different source-frame data.

Equilibrium composition of the original coefficient observables. On the long-root space the linear restoring operator just derived is $\omega_R I$ where $\omega_R = \lambda_R + \delta_q$. Independent Gaussian composition at the same equilibrium intensity determines its update

$$(T_\tau^{\text{coeff}} A)(c) = \mathbb{E}_\eta A \left(e^{-\omega_R \tau} c + \sqrt{\frac{q}{M}(1 - e^{-2\omega_R \tau})} \eta \right), \quad \eta \sim N(0, I_M). \quad (\text{G.75})$$

The noise covariance follows uniquely by subtracting the covariance of the conditional mean from $(q/M)I_M$. This establishes uniqueness among updates with that restoring mean and an independent centred Gaussian increment. Gaussian addition proves the semigroup law and preservation of γ_q . Exchange symmetry of the joint Gaussian pair before and after the update proves self-adjointness on $L^2(\gamma_q)$. The Hermite generating function gives for every polynomial

$$U_q T_\tau^{\text{coeff}} A = e^{-\tau \omega_R \mathbf{N}} U_q A, \quad T_\tau^{\text{coeff}} s = e^{-2\omega_R \tau} s, \quad (\text{G.76})$$

where $\mathbf{N}$ is the Hermite number operator. All contractions remain present. For example with $a = e^{-\omega_R \tau}$

$$T_\tau^{\text{coeff}}(c_i c_j) = a^2 c_i c_j + (q/M)(1 - a^2) \delta_{ij}.$$

Equations (G.55) and (G.76) therefore preserve the state and multiplication representation together with the equilibrium update for the entire original polynomial algebra including R_{ang} .

This parameter τ measures equilibrium restoration. The earlier coefficient heat resolution sends B to $e^{-\rho_R t} B$ and hence q to $e^{-2\rho_R t} q$. It compares different source laws. Equation (G.75) compares fluctuations within one law at fixed q . The collective mean response in Section 8.8 varies that law through its intensity and uses the normalized curvature of F_L^{YM} .

To test whether an observation detects a collective excitation let x be the fluctuation coordinates at a background and $e = Cx$ the collective coordinate. If a colour-singlet observable has linear differential ℓ_c its response along the relaxed collective variation is

$$b_c = \ell_c K^{-1} C^* K_{\text{coll}}. \quad (\text{G.77})$$

Differentiating the minimizing lift gives the colour observable's linear response to the collective coordinate. More generally if a source j and state x satisfy a C^1 closure $\mathcal{F}(x, j) = 0$ on specified Banach spaces with boundedly invertible $D_x \mathcal{F}$ at the background and a C^1 observation $\mathcal{O}$ then

$$\frac{d\mathcal{O}(x(j), j)}{dj} = \partial_j \mathcal{O} - D_x \mathcal{O} (D_x \mathcal{F})^{-1} \partial_j \mathcal{F}. \quad (\text{G.78})$$

The implicit-function derivative proves the formula including the direct source term. The closure and the observation must describe the same source-dependent system.

If the generated kinetically normalized mass-squared operator $\mathcal{M}^2$ is self-adjoint and has an isolated eigenvalue m^2 with projection P_m a linear observation represented by a vector b in its Hilbert space has resolvent response

$$\langle b, (\mathcal{M}^2 - z)^{-1} b \rangle = \frac{\|P_m b\|^2}{m^2 - z} + \langle (I - P_m) b, (\mathcal{M}^2 - z)^{-1} (I - P_m) b \rangle. \quad (\text{G.79})$$

The spectral theorem proves the formula. The second term is analytic near the isolated eigenvalue. The mode is visible exactly when $P_m b \neq 0$. Analytic direct source terms leave its residue unchanged. Thus the dynamical counterpart of the static overlap (G.53) is the spectral weight $\|P_m b\|^2$ in the colour response. The lowest point in the support of that response determines its observed gap.

A mass atom from a common cross response. Let Q and ϕ be Hermitian scalar observables in the same positive state and translation representation with connected spectral matrix $d\rho_{ab}$ in a common normalization. If $Z_\phi = \rho_{\phi\phi}(\{m^2\}) > 0$ then

$$\rho_{QQ}(\{m^2\}) \geq \frac{|\rho_{Q\phi}(\{m^2\})|^2}{Z_\phi}. \quad (\text{G.80})$$

Thus a nonzero common cross weight detects the mass atom in Q . If the remaining spectrum is separated from m^2 this contribution is an isolated pole. Positivity of every linear combination $aQ + b\phi$ makes the two-by-two spectral matrix at $\{m^2\}$ positive semidefinite. Its determinant gives the bound. Contact polynomials leave this matrix unchanged while separation of the remaining support makes its resolvent analytic near the mass atom.

G.9 Gaussian spectral fields and regulator covariance

On the generated round $X_3 = \mathbb{RP}_L^3$ fix $L > 0$ and the normalized Haar measure $d\bar{v} = dv/(\pi^2 L^3)$. The positive conformal scalar operator used in the spectral response acts on real $L^2(X_3, d\bar{v})$ as

$$A = \Delta_0 + \frac{\mathcal{R}}{8} I, \quad \mathcal{R} = \frac{6}{L^2}, \quad D(A) = H^2(X_3).$$

The scalar spectrum (5.13) gives

$$a_j = \frac{(2j)(2j+2) + 3/4}{L^2}, \quad d_j = (2j+1)^2, \quad j = 0, 1, \dots \quad (\text{G.81})$$

It is positive self-adjoint with compact resolvent and lower bound $a_0 = 3/(4L^2)$. Its closed quadratic-form domain is $H^1(X_3)$. Choose real orthonormal eigenvectors e_i with i enumerating every shell and all its real multiplicities. In particular

$$\frac{3(2j+1)^2}{4L^2} \leq a_j = \frac{(2j+1)^2 - 1/4}{L^2} \leq \frac{(2j+1)^2}{L^2}.$$

The Gaussian law with covariance A^{-1} is distribution valued in three dimensions. Its support and compensated relative entropy are controlled by two different spectral sums. The field uses $\text{Tr}(1 + A)^{-q} A^{-1}$ for $q > 1/2$ and the response uses $\text{Tr} A^{-2}$ for the response of Section 8.

Theorem G.8 (Distribution-valued Gaussian spectral completion). For $q > 1/2$ use the real space H_A^{-q} with norm $\|x\|_{-q}^2 = \sum_i (1 + a_i)^{-q} |x_i|^2$. If Z_i are independent standard real Gaussians the series

$$X_\tau = \sum_i \sqrt{\frac{1 - e^{-\tau a_i}}{a_i}} Z_i e_i, \quad C_\tau = A^{-1}(I - e^{-\tau A}), \quad \tau > 0 \quad (\text{G.82})$$

defines a centred H_A^{-q} -valued Gaussian with covariance C_τ on spectral test coordinates. Replacing the coefficient by $a_i^{-1/2}$ gives the law $\mu_A = N(0, A^{-1})$. Both series converge in mean square in H_A^{-q} but have infinite L^2 norm almost surely. On this common completion the shifted covariances $(A + gyI)^{-1}$ realize the compensated relative response and its restoring moments described below.

Proof. The finite-coordinate product laws are compatible and the mean squared tail is

$$\sum_{i>N} (1 + a_i)^{-q} \frac{1 - e^{-\tau a_i}}{a_i} \leq \sum_{i>N} (1 + a_i)^{-q} a_i^{-1} \rightarrow 0.$$

Indeed for a cutoff after shell J the upper tail is bounded by

$$\sum_{j>J} d_j (1 + a_j)^{-q} a_j^{-1} \leq \left(\frac{4L^2}{3}\right)^{q+1} \sum_{j>J} (2j+1)^{-2q} \rightarrow 0.$$

The same estimate proves the A^{-1} claim. Characteristic functions of finite linear combinations identify the covariance of the limit. For either law the coordinate variances c_i are asymptotic to a_i^{-1} at large eigenvalue. Thus $d_j/a_j \rightarrow L^2$ whereas $d_j/a_j^2 = O_L((1+j)^{-2})$ proving $\sum_i c_i = \infty$ and $\sum_i c_i^2 < \infty$. The centred series $\sum_i c_i (Z_i^2 - 1)$ converges almost surely by independence and summable variances whereas the means diverge. Thus $\sum_i c_i Z_i^2 = \infty$ almost surely. $\square$

Using the same Z_i at different parameter values defines a joint Gaussian coupling of these laws on H_A^{-q} with cross covariances fixed by the shared spectral coordinates.

Relative response on the common field space. Fix $L, g > 0$ and $q > 1/2$ and let $\nu_y = N(0, (A + gyI)^{-1})$ for $y \geq 0$ on the same real H_A^{-q} with covariance understood on spectral test coordinates. For projections P_N through successive complete shells the reference-subtracted intensity

$$W_N(x) = \|P_N x\|_{L^2(d\nu)}^2 - \text{Tr}(P_N A^{-1}), \quad W_A = \lim_{N \rightarrow \infty} W_N \quad (\text{G.83})$$

exists almost surely and in $L^2(\nu_y)$ for every fixed $y \geq 0$. It defines the same Borel observable for these laws and is invariant under real orthogonal changes of basis inside every shell. For the unweighted response with i counting all real multiplicities

$$I(y) = \sum_i \left[\frac{gy}{a_i} - \log \left(1 + \frac{gy}{a_i} \right) \right], \quad (\text{G.84})$$

$$\frac{d\nu_y}{d\nu_0} = \exp \left[-\frac{gy}{2} W_A - \frac{1}{2} I(y) \right], \quad 2D_{\text{KL}}(\nu_0 \| \nu_y) = I(y).$$

The laws are mutually absolutely continuous and their restoring moments are

$$\mathbb{E}_{\nu_y} W_A = -\frac{I'(y)}{g}, \quad \text{Var}_{\nu_y}(W_A) = 2 \sum_i (a_i + gy)^{-2} = \frac{2I''(y)}{g^2}. \quad (\text{G.85})$$

Proof. The preceding completion applies since $(a_i + gy)^{-1} \leq a_i^{-1}$. Under ν_0 the centred independent terms of W_N have total variance $2 \sum_i a_i^{-2} < \infty$. So they converge almost surely and in L^2 . Take their pointwise limit where it exists and zero otherwise as a Borel version of W_A . The finite Gaussian density ratio is $R_{N,y} = \exp[-gyW_N/2 - I_N(y)/2]$. These ratios form a mean-one martingale under ν_0 . For any $p > 1$ direct Gaussian integration gives

$$\mathbb{E}_{\nu_0} R_{N,y}^p = \exp\{[I_N(py) - pI_N(y)]/2\}.$$

Local uniform convergence of the relative sum bounds this expression uniformly in N also for y in a compact interval. Thus $R_{N,y}$ converges in L^1 to the positive density in (G.84). Its finite-coordinate marginals are those of ν_y proving the density and equivalence. The negative logarithm has ν_0 -mean $I(y)/2$ proving the information identity. Under ν_y the means of W_N sum to $-\sum_i gy/[a_i(a_i + gy)]$ absolutely convergently and the centred terms have summable variances $2 \sum_i (a_i + gy)^{-2}$. This proves almost-sure and $L^2(\nu_y)$ convergence. Equivalence identifies the limit with the same W_A . Taking the means and variances and using the already convergent derivatives of I gives (G.85). Complete-shell projections make the basis invariance explicit. $\square$

W_A is a real reference-subtracted intensity and may be negative. The normalized Haar convention introduces no volume factor. Keeping the same truncated field but using dv and its corresponding reference subtraction gives $W_{dv} = \pi^2 L^3 W_A$ and changes its density coefficient to $gy/(\pi^2 L^3)$. This fixes the normalization of the field realization of the spectral feedback. The reference drive is prescribed in Theorem 8.3. Colour-composite observables and their pole residues use the response maps of Section 9.2.

The exponential regulator as accumulated covariance. For $k > 0$ and $z \geq 0$ let

$$R_k(z) = \frac{z}{e^{z/k^2} - 1}, \quad R_k(0) = k^2, \quad \tau = k^{-2}. \quad (\text{G.86})$$

Then the exact identity

$$[z + R_k(z)]^{-1} = \int_0^{k^{-2}} e^{-uz} du \quad (\text{G.87})$$

also holds at $z = 0$ where both sides equal k^{-2} . Applied to the fixed positive operator A the identity gives C_τ . The finite relative Gaussian normalization

$$\mathcal{F}(k, k_0) = \frac{1}{2} \text{Tr} \log\{[A + R_k(A)][A + R_{k_0}(A)]^{-1}\}$$

has scale derivative

$$\partial_{\log k} \mathcal{F}(k, k_0) = \frac{1}{2} \text{Tr} \frac{\partial_{\log k} R_k(A)}{A + R_k(A)} = \text{Tr} \frac{A/k^2}{e^{A/k^2} - 1}. \quad (\text{G.88})$$

The scale derivative is thus weighted by $y/(e^y - 1)$ with $y = A/k^2$. Adding z to (G.86) gives $z/(1 - e^{-z/k^2})$. Integration and the continuous zero limit prove (G.87). At fixed z differentiation gives $\partial_{\log k} \log[z + R_k(z)] = 2y/(e^y - 1)$ $y = z/k^2$. For k in a compact positive interval the relative logarithm and its derivatives decay exponentially at large z times polynomial factors. The shell multiplicities therefore justify both the trace and termwise differentiation. The factor $1/2$ is the real Gaussian normalization convention. $\mathcal{F}$ is the difference of logarithmic Gaussian normalization constants.

If a separate $m^2 \geq 0$ is added outside this fixed regulator its Gaussian covariance instead is

$$[A + R_k(A) + m^2 I]^{-1} = C_\tau(I + m^2 C_\tau)^{-1}.$$

Placing the mass inside the regulator instead gives $\int_0^\tau e^{-u(A+m^2 I)} du$. Equation (G.88) uses A fixed. A varying radius or source-dependent operator or moving unit convention adds the corresponding derivative and transport terms. The relation $\tau = k^{-2}$ specifies the scale conversion for this fixed-operator covariance.

Acknowledgments

AI-assisted tools were used for language editing and for drafting and checking the numerical verification and formalization code. They also assisted selected numerical consistency checks. The author is responsible for the scientific content and conclusions.

Declarations

Conflict of interest. The author declares no competing interests.

Data availability. The finite-dimensional matter responses and complete mode products together with joint resolution corrections and signed-spinor multiplication examples are described in the computational supplement distributed with the accompanying software [10]. The software is an independent check of finite examples. The analytic proofs are given in the paper. An archived version of this article is available at <https://doi.org/10.5281/zenodo.22067118>.

The physical branch of Lean Formalization of Arithmetic and Basic Physical Interfaces [9] formalizes the common source construction and its finite physical interfaces. It provides an independent check of the structural layer. The indexed clauses cover source exchange and capacity preservation with common-response centre-trace cancellation. They also cover the charge lattice and global-group kernel together with the common-frame response action and the composition and transport laws for finite coefficient resolution. The analytic mass-gap and source-state/GNS completion arguments are proved in the paper together with continuum reconstruction and physical calibration. These analytic arguments lie outside the stated formalization coverage. The library is archived at <https://doi.org/10.5281/zenodo.23254521>.

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