

The Spectrum of a Compact Internal Space II: Effective Couplings and Mass Scales

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ABSTRACT: Minimal unbiased compositional change generates the Gaussian source of the compact internal construction. We derive joint constraints on effective couplings and mass scales from its gauge and matter content, scalar sector, spectral endpoint and kinetic normalizations. Compatibility of source and colour screening selects three matter copies. Within the response family indexed by integers, shell support and position retention select the level $R = 5$, whose transverse traceless endpoint equation has a unique positive root. Eliminating shared transfer parameters gives a relation among six geometric fermion masses. Bandwidth variation correlates the electron and electroweak scales, while two cosmological ratios are independent of the common neutrino mass normalization. Finite matching through one loop implements the generated Fermi, top and Higgs conditions in a common action. The absorptive top self energy gives the Born decay width as an additional output. The compact source covariance and positive RMS amplitude determine a local nonlinear flux that couples matter motion to source feedback. A model with an auxiliary radius admits autonomous source evolution, positive reduced propagation forms and a matter phase clock on a specified finite cosmological interval. Physical comparisons use the stated mass definitions, matching and thermal approximations. The fixed DESI DR2 distance comparison gives $\chi^2 = 29.5087$ for thirteen components and $p = 0.00554$, indicating tension in the adopted cosmological mapping. Spectral tail and quadrature bounds control the finite response calculation.

KEYWORDS: Three fermion generations; Fermion mass hierarchies; Fermion mass relations; Gauge coupling constants; Higgs boson mass; Neutrino masses; Cosmological parameters

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1 Introduction

A common internal origin for particle interactions must account for their different strengths and the large separation of mass scales. The scalar and Dirac spectra of a compact space specify mode positions and multiplicities. Source operations couple the modes, and kinetic normalization converts the resulting matrix elements into interaction strengths. We study

how these ingredients determine effective couplings and masses and what relations follow when different sectors share one source and one internal scale. Their shared endpoint constrains the sectors jointly. Dimensionless combinations eliminate common transfer parameters or mass normalizations and test the relative response laws with the matching appropriate to each observable.

In the Standard Model the Yukawa matrices of charged fermions are independent inputs. At tree level, electroweak symmetry breaking converts their singular values into masses [1, Electroweak and Higgs reviews]. Here the allowed operations of an internal response space determine mass relations. Each physical quantity is associated with a source variation and a normalized field or observable response. The same source data constrain particle parameters and the collective response of the occupied system.

Minimal unbiased compositional change is the common starting point of the mathematical and physical constructions. The source law is isotropic and centered. Weak continuity and independent convolution composition together with self similarity under square root scaling and finite normalized covariance determine its Gaussian form. The correlation window and reflected spectrum supply the common carrier [2, 3]. The mathematical continuation uses native mode operations to generate arithmetic. The physical construction in Ref. [4] generates the compact projective geometry. Relative frame localization fixes the faithfully acting gauge group. Recursive minimal realization of faithful matter preserving adjoints fixes the sixteen dimensional matter module. An elementary reversal real chiral variation fixes the single complex scalar doublet. The corresponding uniqueness statements hold within the generated class; the constructions introduce no fitted numerical parameter.

Mode content determines particle identity, normalized source matrix elements determine interaction strengths, and repeated amplitude transfer produces mass hierarchies. Effective fields represent these quantities at the scale of observation and are varied through their actual dependence on the generated variables. Clock readings compare changes within the same history. The macroscopic metric represents relations between matter motion and source response together with clock rates in the inherited Einstein–matter description. This interpretation determines which variables and variations enter each calculation.

Spectral geometry provides precedents for connecting internal data to particle parameters. A spectral triple encodes metric and differential structure [5, Chapter VI]. The spectral action constructs a bosonic action from a Dirac spectrum [6]. For a product of a spin manifold and a finite noncommutative geometry the Chamseddine–Connes–Marcolli construction [7, Sections 3.7 and 5.1–5.4] obtains high-scale relations among gauge and Higgs coefficients together with Yukawa couplings. These relations are evolved to low energy. Ref. [8] develops the construction further. The response evaluated here is a projected Gaussian spectral sum with discrete multiplicities and finite source-frequency bandwidth retained. Its endpoint depends on the resolved modes and is evaluated directly. Heat-kernel methods [9] supply the spectral-geometric background.

The scalar analysis inherited from Ref. [4, Section 8 and Appendix E] establishes a unique condensed branch of the relative spectral potential and a positive gap for its assigned scalar mass operator. It also gives uniform truncation bounds for the stationary amplitude and normalized mass. We use the same feedback, subtraction, weights, drive and kinetic

normalization. The resulting positive restoring response supplies the scalar branch used in the mass and electroweak calculations.

The first result is a joint determination of content and endpoint. Extensive source screening and the colour-screening condition select three matter copies. Their charge traces fix the torsion response while the shell-support and three-position conditions select the integer response level $R = 5$. At fixed content and bandwidth homogeneity reduces the tensor endpoint equation to $A_R/c = 4/27$. Here $A_R > 0$ is independent of $c = kL$ and hence gives one positive root. That endpoint retains three positive Dirac positions. The three-position plateau persists for every continuously extended bandwidth at least five. The count is stable under this continuation, whereas the dimensional scales retain a bandwidth dependence.

The second result is a set of cross-sector constraints. Eliminating common transfer parameters gives a relation among six geometric fermion masses. Correlated running-mass data [10] constrain its finite matching at a common scale. The common neutrino mass scale cancels from the vacuum-density fraction and a dimensionless temperature–Hubble ratio. Bandwidth variation also correlates the electron and electroweak scales. These relations test relative transfer laws even when a common dimensional normalization is eliminated. A canonically normalized action realizes the family transfer and relative mixing rotations. Finite one-loop matching implements the generated Fermi condition and the top and Higgs conditions in that action. Its absorptive top self-energy supplies the Born decay width as an additional output.

The macroscopic applications use the Einstein–matter response of Ref. [11]. The homogeneous branch is tested against the DESI distance vector with its full covariance. The variational construction of Bekenstein and Milgrom [12] supplies the nonrelativistic field equation for local nonlinear gravity. The endpoint covariance and RMS response fix its acceleration normalization and interpolation function here. The static problem has a unique weak solution for the stated boundary data. Coupling it to pressureless trajectories gives source feedback and an energy balance during smooth evolution. A covariant auxiliary-radius realization further admits an occupied FLRW background with autonomous source feedback and positive reduced propagation forms together with a matter-phase clock on a finite interval. Source work and total stress conservation follow from the common action. Its conserved phase charge makes the matter phase monotone on that interval. A separate independent-foliation realization of the type introduced by Blanchet and Marsat [13] is analysed by the constrained slow-motion reduction of Ref. [14]. Its kinetic signs are determined for the backgrounds specified in that analysis.

The accompanying implementation [15] evaluates the spectral response mode by mode. It solves the endpoint and propagates shared inputs through the coupling and mass relations. Newton’s constant sets the dimensional normalization. The calculation distinguishes gauge evolution with fixed geometric Yukawa data from joint five-coupling evolution and from finite one-loop matching. Functional-renormalization methods [16, 17] provide the setting for scale-dependent composite responses. The finite response and transfer equations used here are specified explicitly. Physical comparisons retain the mass definition and perturbative order and observational covariance appropriate to each quantity.

The archived records document numerical reproduction of the spectral, coupling and mass calculations and interval certificates for the endpoint and its specified propagated readouts. The accompanying Lean development [18] provides machine-checked proofs of the indexed source and basic physical-interface statements in their declared comparison classes. Appendix B identifies the formal correspondence and numerical coverage and distinguishes floating-point values from rigorous enclosures.

The appendices contain the detailed source-response and transfer calculations, error bounds and numerical records.

2 Spectral inputs and physical normalization

Spectral positions and multiplicities specify the channels in the internal response sums. Source matrix elements couple these channels; division by the kinetic norms gives the physical interaction strengths. Every sector uses the same endpoint, content data and dimensional normalization.

Three dimensionless quantities organize the calculation. The charged content trace gives the torsion response τ . Shell support and position retention select the integer response level R . The response balance determines the tensor endpoint kL . Newton’s constant G_N sets the dimensional unit. These common data determine the electroweak and fermion scales together with the cosmological scales.

2.1 Dimensional normalization

Newton’s constant sets the dimensional normalization. In the natural unit convention of the PDG physical-constants table [1] its value is

$$G_N = 6.70883 \times 10^{-39} \text{ GeV}^{-2}.$$

It defines the reduced Planck mass

$$M_P = \frac{1}{\sqrt{8\pi G_N}} = 2.43532 \times 10^{18} \text{ GeV}.$$

The particle mass scales are dimensionless multiples of M_P determined by the spectral data and the stated hierarchy relations. The unreduced Planck mass $M_{P1} = \sqrt{8\pi} M_P$ is used in the conventional radiation-era expansion formulas. Appendix C.16 specifies the additional effective inputs and thermal thresholds in the abundance calculation.

The torsion modulus τ and the jointly selected content and endpoint kL are evaluated from the response relations. The root equation for $c = kL$ depends only on dimensionless spectral and content data.

2.2 The spectral-sum representation

The compact carrier and its response algebra are constructed in Sections 3–4 of Ref. [4]. The same source generates the projective geometry and global gauge group together with the charged module and elementary scalar connector. Its curvature and volume enter the

internal response sums. The Gaussian factor represents the inherited unbiased composition law. Appendix C.1 states the scale convention used here and cites the uniqueness proof.

The scalar Laplacian on RP^3 of radius L has spectrum and multiplicities

$$\lambda_l = \frac{l(l+2)}{L^2}, \quad d_l = (l+1)^2, \quad l = 0, 2, 4, \dots$$

The spin structure τ_- has positive Dirac eigenvalues $(n+3/2)/L$ for even n and negative eigenvalues $-(n+3/2)/L$ for odd n with multiplicity $(n+1)(n+2)$. The numerical channels resolve these signed branches with their respective projectors. The physical radius enters through dimensionless scale ratios. Converting back to masses uses the common anchor.

In the inherited field realization, expansion in the internal mode basis gives macroscopic fields as coefficients. The internal geometry determines their mode labels, representations and allowed products. The coefficient fields describe the corresponding amplitudes in the macroscopic spacetime representation. A source or scalar connector acts between modes through an overlap matrix element. Dividing out the kinetic norms of the fields gives the canonically normalized interaction vertex. Mode multiplicities count the contributions; matrix elements and kinetic forms determine their strength. The generated source identifies the physical variation being measured. A charge insertion probes its current response whereas a deformation of the internal quadratic form probes its restoring response. The associated projection and canonical normalization determine how that response enters the observable.

The scalar sector inherits three further results from Section 8 and Appendix E of Ref. [4]. These are the renormalized conformal spectral response, a positive gap for the assigned scalar mass operator and uniform truncation bounds on a condensed scale interval. They determine the positive stationary amplitude and colour-singlet scalar mass m_δ . A common cutoff controls the mass error along a compact condensed scale path. Their use retains the spectral feedback and subtraction with the original weights and drive and the same kinetic normalization. Appendix C.11 states the complete condensate used here and its relation to the electroweak window readout. These results determine the scalar branch on which the mass and electroweak responses are evaluated.

2.3 Response definitions and scale-transfer relations

The response relations comprise the $R = 5$ TT endpoint with normalization $4/27$ and the content screening and position-retention rules. Their scale transfers determine the gauge normalizations, sector mass ladder, VEV proportional to the squared window and QCD catalogue. Cosmological readouts and the RMS-to-flux map use the same source data. Each generated source operation fixes the admissible variation and channel. Its projection and normalization determine the amplitude, coupling or density being read. Representation and charge moments fix the algebraic factors. The scale convention identifies the physical regime of the response. In the local spatial channel the occupation covariance fixes the acceleration variance. The quadratic entry represents squared field strength, and the positive RMS readout supplies the response amplitude. Appendix C derives these responses and transfer operators with their compatibility relations and error bounds.

Physical readouts are functions of this fixed generated source and its allowed operations with the inherited comparison maps and normalizations retained. The generating arithmetic and the closure of admissible spectral operations are developed in Refs. [2, 3]. Here the response and transfer relations identify which of those operations realize each physical quantity. A change of coordinates must transform the source and kinetic form together with the observable. Such transformations preserve the physical content of a realization. Composition determines the kernel. Joint source compatibility selects the content. The response balance fixes the endpoint. The corresponding uniqueness results retain the source operations and projectors and normalizations entering each construction.

For a generated history H denote a clock readout by $T_C[H]$ and another observable by $O[H]$. On an interval where the clock is monotone their relation defines O as a function of T_C . The order of construction is determined by the dependencies of the generating operations [3]. The Gaussian composition parameter measures accumulated change. The renormalization-group parameter labels the scale of a response. Physical rates use the selected clock normalization. In the macroscopic metric representation a nonzero matter amplitude with phase ϑ and $X = -(\nabla\vartheta)^2 > 0$ gives the proper-time readout $ds = d\vartheta/\sqrt{X}$ along its future-directed phase flow (Proposition G.4). The clock counts phase change and calibrates each increment by its instantaneous frequency. This material readout supplies the interval used to express propagation and reaction rates.

For effective fields $\Phi = \Gamma(q)$ the generated action is $S_{\text{gen}}[q] = S[\Gamma(q)]$ and its equations follow from variations of the admissible q . Its source exchange, constraints and quadratic response are calculated from this same action. An invariant-subsystem realization in an independently specified field theory additionally satisfies its embedding and evolution-preservation conditions.

The physical state space used here is the generated image

$$\mathcal{P}_{\text{phys}} = \Gamma(\mathcal{Q}_{\text{gen}}),$$

with its source and admissible variations retained. The relative-frame and least-matter continuations together with the connector continuation act on this image with the universal properties established in the companion construction. Their finite responses and sector readouts define the observables evaluated here. Enlarged field spaces serve to compare effective realizations. Physical variations follow the stated generating map.

2.4 Theoretical inputs and comparison conventions

Table 1 distinguishes the generated source structure and its physical continuation from the finite response readouts evaluated here. It also identifies the dimensional and effective inputs and the quantities derived from them. Observations enter the comparison stage.

The comparison tiers A–D classify approximation level and evidentiary status. Table 1 instead identifies the logical origin of the inputs and outputs. Appendix A assigns a comparison tier to each numerical entry.

The response conditions fix the dimensionless endpoint before Newton’s constant supplies the unit for dimensional reconstruction. Gauge, flavour, electroweak and cosmological

Table 1. Logical status of the quantities entering the calculation. The generated structural and physical-continuation results are inherited from Ref. [4]. The source operations and response rules define the calculation. Their joint conditions select the content and finite response level and endpoint.

Logical status	Objects	Role in the calculation
Generated source structure	Minimal unbiased composition and its Gaussian source. Projective carrier and signed spectra with their multiplicities and source algebra.	Defines the admissible modes and products and source operations inherited from the common generating construction.
Generated physical continuation	Relative-frame localization and recursive minimal matter realization with the elementary reversal-real connector. The resulting gauge group and matter module with the scalar connector and the condensed scalar branch for its stated spectral feedback and mass operator.	Defines the physical representation class and the scalar background inherited from the companion construction.
Response structure	Channel projectors with kinetic and response normalizations. Integer response-level family and endpoint normalization. Scale-transfer and readout maps.	Defines the finite response problem whose joint consequences are calculated throughout the paper.
Dimensional and effective inputs	Newton’s constant for dimensional reconstruction together with the separately stated thermal thresholds and effective inputs used in the abundance calculation.	Supplies the physical unit and the declared closures of the extended sectors.
Derived quantities	Torsion response and replicated content. Selected level $R = 5$ and endpoint. Couplings and masses with widths and densities and macroscopic responses.	Obtained after the generated structure and physical realization have been fixed.
Comparison data	Reference measurements and Standard-Model extrapolations with observational covariance matrices.	Enter residuals and likelihoods and other tests after the corresponding theoretical quantities have been evaluated.

quantities use these common data with their respective matching conventions. Reference measurements and Standard-Model extrapolations enter the comparison after the generated response laws have been evaluated. An explicitly calibrated effective extension identifies its calibration at the defining relation and tests additional quantities independently. Spectral truncation, quadrature error and physical sensitivity are treated separately in Appendix D.

2.5 Common endpoint data

The content and endpoint first satisfy the joint conditions of Appendix C.1. On the selected branch the unrounded endpoint and source responses are propagated through the coupling and mass relations. The dimensionless root equation is independent of the dimensional anchor. Appendix B specifies the spectral traces and numerical precision together with the electroweak corrections used in the tables.

2.6 Renormalization-group evolution

The spectral responses supply boundary couplings at their designated scales. The retained two-loop RG equations [19, 20] evolve the emergence-scale data from M_G to the internally determined Z -boson mass M_Z . The representation content fixes their group factors. The production gauge evolution holds the geometric Yukawa input $y_0 = 1$ fixed. Joint five-coupling evolution and finite one-loop matching are separate calculations. Appendix D.6 gives the transfer convention. The high-scale reference couplings come from a separate Standard-Model extrapolation with the boundary values listed in Appendix B. Figure 1 shows the dependence of the computed outputs.

3 Matter content and torsion response

The charged representations specify the response of the matter content to chiral and weak sources and to hypercharge. Their finite traces determine both the normalized chiral response used for torsion and the charge susceptibilities entering gauge matching. We use the minimal joint module, its charged response quotient, the primitive charge lattice and the pointwise vacuum stabilizer developed in the section on joint response module and charge in Ref. [4]. Anomaly cancellation and Yukawa covariance check the resulting assignments. The chiral derivative and charge Hessians refer to the same normalized source and give $\tau = 1/50$. Its charge moments also enter screening and hypercharge matching.

3.1 Complete matter and the charged response

The full matter carrier has charge decomposition $8_L + (7_R + 1_{\text{neutral}})$. Exchange preserves its total capacity. A common response on the two complete chiral copies has zero centre trace. The charge operator has a one-dimensional neutral right kernel. The charged quotient has $8_L + 7_R$ components. These quotient counts enter the torsion trace. The neutral line remains in the full matter carrier for interactions whose operators act on it.

The five representations per generation are $Q_L = (3, 2)$ and $L_L = (1, 2)$ together with the singlets $u_R = (3, 1)$ and $d_R = (3, 1)$ and $e_R = (1, 1)$. The left-handed doublet Q_L has $3 \times 2 = 6$ components and L_L has $1 \times 2 = 2$ components. Hence

$$N_L = 6 + 2 = 8.$$

The right-handed singlets give

$$N_R = 3 + 3 + 1 = 7.$$

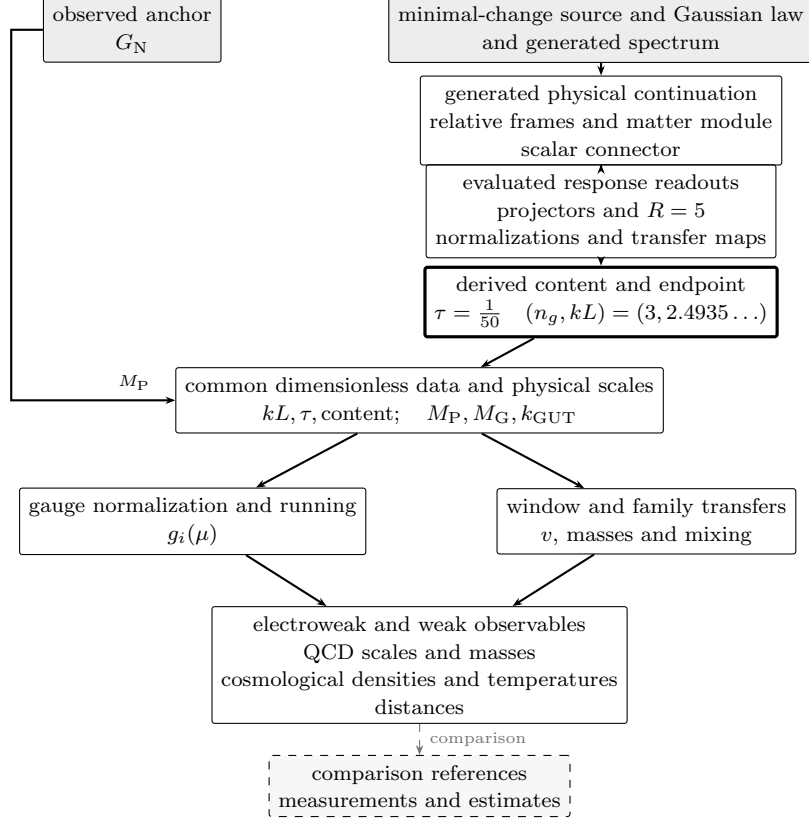


Figure 1. Dependence of the observables on the common generated structure. The upper branch shows the minimal-change source and spectrum followed by the generated physical continuation and the finite response readouts evaluated in this paper. Screening selects the content branch. The resulting response problem then gives the highlighted joint content and endpoint solution. Gauge normalization and mass transfer use the common source data in parallel. Their outputs combine in the lower-scale calculations. Arrows show the principal dependencies of these grouped calculations. The observed anchor supplies M_P for dimensional reconstruction. Comparison values enter at the last stage.

The number of Weyl components per generation in the charged response quotient is

$$N_f = N_L + N_R = 15,$$

and the total over three generations is 45. The left-handed count equals the number of colour generators

$$N_L = N_g = N_c^2 - 1 = 8,$$

and the right-handed count is one less. The chiral excess in the charged response is

$$N_L - N_R = 1.$$

The spin-structure convention on $\pi_1(RP^3) = \mathbb{Z}_2$ is specified by τ_- in the representative convention of Ref. [4].

Writing all fields as left-handed fields includes the conjugates of the right-handed species. In this convention the hypercharge assignment satisfies

$$\sum Y = 0,$$

the vanishing of the mixed gravitational anomaly. The quadratic hypercharge moment is

$$\sum Y^2 = \frac{10}{3},$$

the sum $6 \cdot (1/6)^2 + 3 \cdot (2/3)^2 + 3 \cdot (1/3)^2 + 2 \cdot (1/2)^2 + 1 \cdot (1)^2$. The fermion content also obeys the identity

$$N_f = 2 \sum Y^2 \Delta_f^2 = 15,$$

with fermion conformal weight $\Delta_f = 3/2$. These per-generation identities determine the normalized chiral-content ratio used for torsion.

3.2 The conformal-colour content identity

The full quadratic response algebra and local colour-frame propositions of Ref. [4] generate $SU(W) \cong SU(3)$ on $W = \text{Sym}^2 S$ under the frame-localization premise. Here S is its complex two-dimensional spinor carrier. The associated content identity uses the conformal coupling of a scalar in dimension d

$$\xi = \frac{d-2}{4(d-1)},$$

and with internal dimension $d = 3$ it is $\xi = 1/8$. The resulting identity is

$$N_g \xi = 1,$$

equivalently $N_g \Delta = 2(d-1)$ with scalar conformal weight $\Delta = (d-2)/2 = 1/2$. The generated value $N_g = 8$ and the conformal coefficient $\xi = 1/8$ satisfy the identity. Since $N_c^2 - 1 = 8$ has the single positive solution $N_c = 3$ the relation $d = N_c = 3$ compares the independently fixed internal dimension and colour number.

3.3 The torsion parameter

The torsion readout compares the net chiral response with the charge capacity of the same one-generation source. The first derivative with respect to the chiral source measures its left-right imbalance. The connected second derivative with respect to hypercharge measures the quadratic charge response. We evaluate both on the equal-weight charged carrier with the inherited normalization of the hypercharge operator. On the oriented three-dimensional carrier the isotropic and totally antisymmetric tensor channel is spanned by its volume form. The homogeneous torsion response therefore has a single amplitude parametrised by

$$T^a_{bc} = \frac{\tau}{L} \varepsilon^a_{bc},$$

where the dimensionless modulus τ is the normalised chiral-content invariant. Its numerator is the chiral excess per fermion component

$$\langle\chi\rangle = \frac{N_L - N_R}{N_f} = \frac{1}{15},$$

and its denominator is the quadratic hypercharge moment

$$\Pi_{\text{ren}} = \sum Y^2 = \frac{10}{3}.$$

Thus

$$\tau = \frac{\langle\chi\rangle}{\Pi_{\text{ren}}} = \frac{N_L - N_R}{N_f \sum Y^2} = \frac{1}{15 \cdot 10/3} = \frac{1}{50}.$$

Per-component normalization of the chiral response and per-generation normalization of the charge capacity give the source balance $\Pi_{\text{ren}}\tau = \langle\chi\rangle$. The chiral excess supplies the oriented bias. The positive charge susceptibility sets the restoring response against which that bias is measured. This balance fixes the constitutive normalization of the single isotropic torsion amplitude. The inherited normalization of Y fixes the charge unit. The per-component and per-generation traces fix the relative units of the two responses. Both are evaluated on the equal-weight finite content carrier before applying the spatial projectors $P_{\pm}$. The resulting τ then enters the spectral mass shifts of each assigned spatial branch. Replicating the one-generation carrier multiplies the extensive bias and capacity by the same factor and preserves their ratio. Relative chiral weights change the source itself and hence its normalized bias. The charge operator has the normalization of Ref. [4]. Appendix C.2 evaluates this source and gives the unequal-branch-weight expression (C.26).

3.4 Torsion and geometric response

The content parameter enters geometry through torsion and the $J = 2$ squash, which correct the tensor and gauge responses and the electroweak readouts. The $J = 2$ response uses the following dimensionless geometric coefficients. Their dimensional form carries an additional L^{-2} . The Einstein–Cartan coefficient is

$$\lambda_{\text{EC}} = N_g \left(1 + \frac{\tau}{2}\right)^2 + 6 = 14 + 8\tau + 2\tau^2,$$

and the effective tensor-response coefficient is

$$\lambda_{\text{TT}} = 14 = 2N_R.$$

Here TT labels the effective tensor response. The rough TT Laplacian eigenvalues are defined separately in Ref. [4]. The squash amplitude is

$$s_0 = n_{\text{broken}} \tau = 2\tau,$$

with $n_{\text{broken}} = 2$ the number of broken generators of $\mathfrak{su}(2)_R$. Its content ratio is

$$\frac{s_0}{N_R} = \frac{1}{175}.$$

The curvature relation of the Einstein-Cartan geometry is

$$R(\omega) = R_{\text{LC}} - \frac{3}{2} \left(\frac{\tau}{L} \right)^2,$$

with $R_{\text{LC}} = 6/L^2$. Torsion corrects the curvature at order $\tau^2 \sim 10^{-4}$. The gauge matching and electroweak order parameters use these same curvature and squash normalizations.

4 The internal scale and the window capacity

For fixed torsion τ the remaining dimensionless data are the response level R and the tensor endpoint kL . The ratio $c = kL$ compares the internal radius with the resolution length k^{-1} and enters both the generation window and the amplitude transfers that set the mass hierarchy. The common screening relation selects the three-copy content branch. At fixed content and bandwidth the tensor source response determines a unique endpoint, whose spectral window must retain the same number of family positions. These requirements give a unique endpoint on the selected branch and a finite bandwidth range with unchanged active shells. The same screening response enters the weak coupling.

4.1 The common scale parameter

The window capacity kL relates the emergence scale, gauge couplings, fermion masses and electroweak scale through a common internal resolution. The response problem is specified by the field content, species projectors, torsion and bandwidth. The Gaussian profile and transverse-traceless readout with endpoint normalisation $4/27$ determine the endpoint. The Planck anchor then supplies the mass unit. The resulting hierarchy is exponentially sensitive to kL . At fixed other inputs $v \propto (kL)^{-1} e^{-4\pi kL}$ and $m_e \propto e^{-20kL}$. The logarithmic derivatives and exact finite changes are given in Appendix D. The content ratio Ω_Λ is independent of this common scale.

4.2 Spectral data

The internal space is $RP^3 = S^3/\mathbb{Z}_2$ with round radius L . We use the spectral data and spin conventions collected in Appendix A of Ref. [4]. The scalar Laplacian has the spectrum

$$\lambda_l = \frac{l(l+2)}{L^2}, \quad d_l = (l+1)^2, \quad l = 0, 2, 4, \dots,$$

where the restriction to even l is the antipodal projection and d_l is the multiplicity. Fix the spin structure τ_- and the orientation and deck-lift convention supplied by the companion paper. Its signed Dirac spectrum is

$$\begin{aligned} \text{Spec } D_{\tau_-} &= \left\{ \frac{n+3/2}{L} : n \in 2\mathbb{N}_0 \right\} \cup \left\{ -\frac{n+3/2}{L} : n \in 2\mathbb{N}_0 + 1 \right\}, \\ d_n^D &= (n+1)(n+2). \end{aligned} \tag{4.1}$$

The other spin structure interchanges the two parity assignments. Thus $|D_{\tau_-}|$ has eigenvalues $(n+3/2)/L$ for all $n \in \mathbb{N}_0$ and $D_{\tau_-}^2$ has eigenvalues $(n+3/2)^2/L^2$ for all n with

multiplicity d_n^D in each case. Each n contributes one signed eigenspace with this multiplicity.

Write $P_{\pm} = \mathbf{1}_{\mathbb{R}_{\pm}}(D_{\tau_{\pm}})$. The three quantities entering the response and generation count are

$$\begin{aligned} \text{generation positions: } & \text{Spec}(LD_{\tau_{\pm}}|_{P_{\pm}}) = \{n + 3/2 : n \in 2\mathbb{N}_0\}, \\ \text{positive-branch squared tower: } & D_{\tau_{-}}^2|_{P_{+}}, \quad n \in 2\mathbb{N}_0, \\ \text{negative-branch squared tower: } & D_{\tau_{-}}^2|_{P_{-}}, \quad n \in 2\mathbb{N}_0 + 1. \end{aligned} \tag{4.2}$$

The position-retention rule counts distinct positive positions without weighting by d_n^D . The fermionic spectral trace on the charged response carrier $\mathcal{M}_{\text{ch}}$ uses squared eigenvalues and their multiplicities. The chiral assignment uses P_{-} for Q_L, L_L and P_{+} for u_R, d_R, e_R . Its uniform-even comparison uses P_{+} for every species. These are specified restrictions of $D_{\tau_{-}}^2$. The association of intrinsic sign projectors with the indicated chiral species is part of the retained matter assignment. Appendix B.4 specifies these traces and their numerical evaluation.

The volume is $\text{Vol}(RP^3) = \pi^2 L^3$ and is half the round-sphere volume. The six Killing vectors span the isometry algebra $\mathfrak{so}(4)$. The vanishing first cohomology $H^1(RP^3, \mathbb{R}) = 0$ excludes harmonic one-forms. Connectedness likewise makes the constants the only scalar Laplacian zero modes.

4.3 Internal radius and Kaluza–Klein masses

The remaining radius is fixed by the dimensionless endpoint together with the Planck anchor. We use the Kaluza–Klein mass readout of the inherited internal spectrum [4] with the usual Kaluza–Klein interpretation [21, 22]. The scale relation is

$$G_N \rightarrow M_P = \frac{1}{\sqrt{8\pi G_N}} \rightarrow M_G = \frac{M_P \sqrt{\pi}}{kL},$$

where M_G is the emergence scale. The Kaluza–Klein masses are

$$m_n^{\text{KK}} = \frac{n + 3/2}{kL} M_G,$$

and the window edge is

$$kL M_G = M_P \sqrt{\pi}.$$

Put $\varkappa := kL$. At the emergence scale the physical radius and dimensionless Dirac operator and generation cutoff are

$$\begin{aligned} L_{\text{phys}} &= \frac{\varkappa}{M_G}, & \widehat{D} &= L_{\text{phys}} D_{\tau_{-}}, \\ c_{\text{gen}} &= \varkappa^2, & \Lambda_{\text{gen}} &= \frac{c_{\text{gen}}}{L_{\text{phys}}} = \varkappa M_G = M_P \sqrt{\pi}. \end{aligned} \tag{4.3}$$

On the positive branch $\widehat{\lambda}_n = n + 3/2$ so the condition $\widehat{\lambda}_n < c_{\text{gen}}$ is equivalent to $m_n^{\text{KK}} < \Lambda_{\text{gen}}$. The generation cutoff selects the family positions. The Gaussian spectral trace evaluates

each assigned spatial tower with its own mode multiplicities and response bandwidth. The Kaluza–Klein scale measures the internal spectral cost of resolving each mode. The family masses measure the chirality-changing response between retained matter states. They are read from the mass vertex in the canonical kinetic normalization with the hierarchy set by accumulated transfer along the retained positions.

4.4 The fixed-branch spectral endpoint

The endpoint is defined by the transverse-traceless (TT) response at zero external momentum. Each internal mode contributes its tensor response with its field weight, multiplicity and Gaussian suppression. Summing these contributions fixes the compact-source scale used by the different sectors. The tensor-survival maximum fixes the normalization of the summed response. Before restriction to the source bandwidth the Gaussian sum on RP^3 is

$$\Pi_{\mu\nu}^2(L, \Lambda^2) = \frac{1}{V_3} \sum_{\text{fields}} \sum_{\text{modes}} d_n w \int_0^\infty \frac{d\omega}{\pi} \frac{(\omega^2 + \lambda_n)^2}{(\omega^2 + \lambda_n + m_{\text{eff}}^2)^2} e^{-(\omega^2 + \lambda_n)/\Lambda^2},$$

Here $V_3 = \pi^2 L^3$. The mode multiplicity is d_n and $w = 4$ for each gauge polarisation or Weyl fermion and $w = 1$ for each real scalar. The second argument is the squared Gaussian cutoff. The field content and τ are suppressed. With $q^2 = \omega^2 + \lambda_n$ the kernel satisfies $K_{\text{TT}}(q^2, 0) = 1$ for $q^2 > 0$. The Gaussian factor encodes the inherited composition law for unbiased change. The rational factor is the nonlinear response of each mode to its effective mass. The trace combines this response with the Gaussian weight on the same generated spectrum. The flat-space comparison uses the Ward-normalised subtracted amplitude. Appendix C.1 specifies the curvature and torsion shifts and the assigned spectral branches.

The endpoint normalization is the mass-response extremum

$$\frac{4}{27} = \max_{0 \leq y \leq 1} y(1-y)^2 \quad \text{at} \quad y = \frac{1}{3},$$

where $y = m^2/(q^2 + m^2)$ is the mass fraction of $K_{\text{TT}}(q^2, m^2) = (1-y)^2$. The variable q^2 is the internal spectral argument whereas k is the running scale. The single-leg survival fraction $1-y$ multiplies under tensor composition and gives the two-leg response $K_{\text{TT}} = (1-y)^2$. For $0 < y < 1$ its response to the logarithmic mass–momentum ratio $\zeta = \log[y/(1-y)] = \log(m^2/q^2)$ is normalized per tensor leg as

$$-\frac{1}{2} \frac{dK_{\text{TT}}}{d\zeta} = y(1-y)^2.$$

Thus the extremum measures the largest change of the surviving tensor response under a relative change of mass and momentum. The endpoint rule uses the maximum $4/27$ of this two-leg response derivative to normalize the summed compact-source response. The macroscopic propagation criterion uses the metric response described in Appendix C.1. The emergence endpoint is

$$\frac{V_3 \hat{\Pi}_{\mu\nu}^2(kL, k^2/M_{\text{P}}^2)}{32\pi^2} = \frac{4}{27} \quad \text{at} \quad k = M_{\text{G}}.$$

The endpoint calculation uses the dimensionless arguments $\ell_{\text{end}}(k) = \varkappa M_{\text{G}}/k = C/k$ and $\hat{\Lambda}_{\text{tr}}(k) = k/M_{\text{P}}$ where $C = M_{\text{P}}\sqrt{\pi}$. Its amplitude $\hat{\Pi}^2$ is the displayed spectral sum restricted to the specified response domain and evaluated with $L = \ell_{\text{end}}$ and squared cutoff $\hat{\Lambda}_{\text{tr}}^2$. The physical radius is instead L_{phys} in (4.3). At $k = M_{\text{G}} = C/\varkappa$ the arguments are $(\ell_{\text{end}}, \hat{\Lambda}_{\text{tr}}) = (\varkappa, \sqrt{\pi}/\varkappa)$ and leave one equation in one unknown. The numerical calculation uses bisection on $[0.3 M_{\text{G}}, 3 M_{\text{G}}]$ and self-consistency iteration. The frequency response in the numerical tables has support $\omega^2 + \lambda_n < R\Lambda^2$ with $R = 5$. In operator terms the common collective source is read through the spectral projection $P_R = \mathbf{1}_{[0, R\Lambda^2)}(H)$ with $H = \omega^2 + A$. Thus the retained Gaussian multiplier has relative weight greater than e^{-5} throughout the response domain. The inherited Gaussian law fixes the envelope while P_R specifies the bandwidth accepted by the collective source. The $R = 5$ response defines one common endpoint for the downstream sectors. Numerical quadrature evaluates the integral on this domain. The endpoint and mass relations test the physical consequences of the common bandwidth. Here TT denotes the tensor projection used to read the compact-source susceptibility and fix this internal endpoint. Macroscopic tensor propagation is described by the Einstein–matter realization of Section 8.6. The common Newton normalization links that description to the dimensional scale used here. The shell support and the three-position condition select the finite response level used in the numerical evaluation.

Definition 4.1 (Finite response level). The finite response domain is parametrized by a real cutoff R in $P_R = \mathbf{1}_{[0, R\Lambda^2)}(H)$. The physical level family consists of the projections P_N indexed by positive integers N with Λ^2 fixing the unit of the cutoff. A real cutoff $\mathcal{R}$ interpolates this family in the sensitivity analysis.

The shell-support condition preserves the species projectors entering the compact-source trace, while position retention fixes the family coefficient space. Both conditions must hold on the same endpoint branch. Their inequalities select one member of the integer response family.

Proposition 4.1 (Finite response level). For the three-copy branch with $\tau = 1/50$ and the stated positive shell support the unique integer response level is $R = 5$.

Proof. The endpoint $c_R = 27A_R/4$ is strictly increasing in R by Proposition C.1. The outward interval evaluation at $R = 4$ gives

$$c_4 \in \left[\frac{187}{80}, \frac{23939}{10240} \right] = [2.3375, 2.33779296875], \quad (4.4)$$

which lies below $\sqrt{11/2} = 2.3452078799$. The interval also separates the endpoint response from $4/27$ with one derivative sign on the bracket. It certifies a unique root at $R = 4$. Since $c_5 = 2.49353$ lies above $\sqrt{11/2}$ the crossing $c_{R_*} = \sqrt{11/2}$ satisfies $4 < R_* < 5$. The shell support of Eq. (C.11) gives $R \leq 16/\pi < 6$ and the three-position condition requires $R > R_*$. The only integer in this interval is $R = 5$. $\square$

For $R = 5$ the endpoint is

$$kL = 2.49353.$$

Proposition 4.2 (Unique endpoint at fixed content and bandwidth). The dimensionless endpoint has fixed-point function

$$F(\varkappa) = \frac{\pi^2 \varkappa^3 \widehat{\Pi}_{\mu\nu}^2(\varkappa, \pi/\varkappa^2)}{32\pi^2} - \frac{4}{27}.$$

Fix the field content and τ together with the species projectors and dimensionless integration domain $R > 0$. With $c = \varkappa = kL$ the coefficient A_R defined in (C.7) is independent of c and M_P and satisfies

$$F_R(c) = \frac{A_R}{c} - \frac{4}{27}, \quad F'_R(c) = -\frac{A_R}{c^2} < 0, \quad c_R = \frac{27}{4}A_R.$$

Thus each fixed content branch has exactly one positive endpoint. The reference level is $R = 5$. The value $R = \infty$ denotes the full Gaussian integral with the full specified towers. These roots have separate error bounds. This specializes the TT scaling law derived in Ref. [4]. Appendix C.1 checks the present inputs and the infinite-domain extension. The finite- and full-domain values are compared in Appendix D.1.

The domain comparison is quantitatively significant. Holding the content and projectors fixed together with the torsion modulus, Planck anchor and loop factor, Eq. (D.15) gives

$$v_\infty/v_5 \in [0.10, 0.13], \quad m_{e,\infty}/m_{e,5} \in [0.03, 0.04].$$

The position count is discrete whereas the endpoint coefficient integrates continuously varying mode weights over the response domain. The displayed ratios measure the dependence of absolute masses on that bandwidth. The integration error at a fixed bandwidth is controlled separately. Proposition C.1 also shows that the three-position plateau persists for every $R \geq 5$ whereas the two mass scales decrease together according to

$$\frac{v_R}{v_5} = \frac{c_5}{c_R} \left(\frac{m_{e,R}}{m_{e,5}} \right)^{\pi/5}.$$

The common bandwidth therefore produces a correlated displacement of the electroweak and electron scales, which can be tested across the two sectors.

The shell support provides a further constraint. For the positively weighted endpoint trace used here retaining the same active shells as at $R = 5$ and imposing three generation positions gives

$$R_* < R \leq \frac{16}{\pi}, \quad c_{R_*} = \sqrt{11/2}, \quad R_* \simeq 4.0187119842.$$

Equation (C.12) proves existence and uniqueness of this lower edge using monotonicity and outward finite-shell bounds. The ratio of the two mass scales then removes both the Planck anchor and their common squash factor

$$\frac{v}{m_e} = \frac{3}{16\pi^{5/2}} \frac{e^{(20-4\pi)c}}{c}.$$

With both masses in common units this ratio is strictly increasing on the three-position branch. An independently matched value therefore determines at most one compatible endpoint. At fixed source data it also determines at most one bandwidth. Equation (C.16) gives the inverse response. The charged-lepton invariant in (C.18) tests the transfer rule after eliminating the bandwidth as well.

Under the stated torsion mass shifts the three-copy capacity plateau is stable throughout $0 \leq \tau \leq 0.1$ with the other response data fixed. The finite endpoint bound is derived in Appendix C.2. The central calculation uses the content value $\tau = 1/50$.

Weak-coupling inversion. For fixed $N_c > 0$ and real τ the screened law $4\pi/g_2^2 = c^5/2 + 1/N_c - \pi\tau^2/2$ is strictly monotone in $c > 0$. A positive reference coupling therefore determines the unique capacity

$$c_{\text{ref}} = \left[2 \left(\frac{4\pi}{g_{2,\text{ref}}^2} - \frac{1}{N_c} + \frac{\pi\tau^2}{2} \right) \right]^{1/5}, \quad (4.5)$$

whenever the quantity in parentheses is positive.

The independently extrapolated Standard-Model reference coupling gives c_{ref} while the spectral coefficient A_R determines the endpoint of Proposition 4.2 before that comparison. Under the common content and torsion response and matching convention their unrounded central values satisfy

$$\left| \frac{c_{\text{ref}}}{c} - 1 \right| < 3 \times 10^{-6}. \quad (4.6)$$

Both give 2.49353 at the displayed precision with signed percentage residual $100(c_{\text{ref}}/c - 1)\% \simeq 0.00022\%$. This is a central-value consistency test of the spectral endpoint and screening law. The certified endpoint enclosure and reference-coupling rounding retain the separate accuracy stated in Appendix D.

The Gaussian width of the scale field is $L_{C_g} = \sqrt{\pi}$ which fixes the critical curvature

$$R_c = \frac{6}{\pi}.$$

The window capacity is then

$$(kL)^2 = 6.2177,$$

which sits below the Euclidean-period value $2\pi = 6.2832$ by 1.04%.

4.5 The generation count

The generation rule assigns a family position to each retained positive Dirac level with unit weight for each position. The retained-position count must agree with the copy number entering the same source trace. With the map (4.3) the criterion is

$$n + \frac{3}{2} < (kL)^2.$$

With $(kL)^2 = 6.2177$ the retained positions are

$$n = 0 \text{ (1.5)}, \quad n = 2 \text{ (3.5)}, \quad n = 4 \text{ (5.5)},$$

The position $n = 6$ gives 7.5 and lies above the edge. The selected window therefore contains exactly three positions and the generation count under this retention rule is

$$n_g = 3.$$

The family coefficients have a concrete representation on these shells. Write F_j^+ for the positive shell with $n = 2j$ and $S \simeq \mathbb{C}^2$ for the spin-1/2 carrier of the isotropy group $H_{\text{iso}} \simeq SU(2)$. Each of the three shells contains one copy of S . Its normalized radial intertwiner supplies one family coefficient. Their coefficient space is

$$K = \text{Hom}_{H_{\text{iso}}}(S, F_0^+ \oplus F_1^+ \oplus F_2^+), \quad \dim K = 3.$$

The products of these intertwiners act as matrix units on K and identify it with the minimal three-copy family carrier. The kinetic form and mass vertices transform together under this identification (Proposition C.3). The spin structure and species projectors fix the spatial modes in each copy. The family matrices act on their common coefficient space. The full $|D_{\tau_-}|$ window at the same edge contains five distinct positions $n = 0, 1, 2, 3, 4$ and the retained positive-branch eigenspaces have total dimension $2 + 12 + 30 = 44$. These are the spatial counts. The three-dimensional intertwiner space carries the family count.

The mass ladder assigns family amplitudes $m_n \propto e^{-\alpha_f n}$ to these positions with extrusion orders $n = \{0, 2, 4\}$ and sector-dependent rates α_f . The capacity check fixes the retained positions. The weighted traces remain those assigned to each species. The step structure of the criterion is shown in Figure 2.

The step boundaries give a direct interval certificate for this count. Every positive capacity satisfying $11/2 < c^2 \leq 15/2$ selects exactly $n = 0, 2, 4$. The third position is $4 + 3/2 = 11/2$ whereas the next is $6 + 3/2 = 15/2$. Hence an enclosure $[c_-, c_+]$ certifies the three-position plateau whenever $c_-^2 > 11/2$ and $c_+^2 \leq 15/2$. The content used in the trace must agree with the count read from its endpoint. For g copies of the inherited one-generation carrier the extensive adjoint-to-matter response ratio is $1/g$. Requiring the same inverse weak-source coefficient to obey the colour-normalised screening closure $N_c \delta_{\text{col}} = 1$ gives $1/g = 1/N_c$ and hence $g = N_c = 3$ (Proposition C.2). The endpoint of this three-copy source then passes the three-position plateau test. The screening conditions select the content branch and the window verifies that its retained family positions reproduce the same count. These selections are preserved by simultaneous unitary changes of mode basis. The source and kinetic form transform together with the mass vertices and leave the response traces and canonically normalized masses invariant. The endpoint branches are enumerated in Appendix C.1.

5 Gauge couplings

Ref. [4] fixes the common connection and its action on each matter representation. Canonical normalization of the quadratic response determines the interaction strengths. The endpoint of Section 4 fixes the common scale, and the representations of Section 3 determine the matter response. Channel normalization, screening and finite matching give the

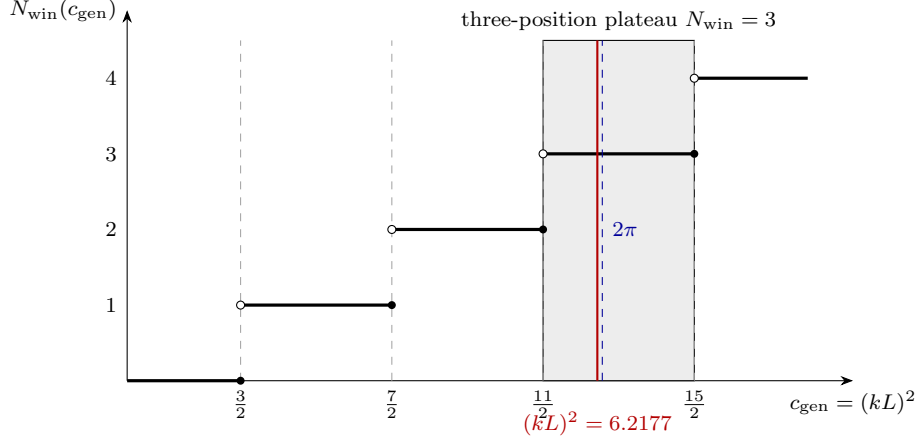


Figure 2. Window count for the selected positive Dirac positions. Each position has unit weight. The thresholds are $n + \frac{3}{2}$ for $n = 0, 2, 4, 6$ with the strict retention rule giving $N_{\text{win}} = 3$ on $\frac{11}{2} < c_{\text{gen}} \leq \frac{15}{2}$. Both the recorded endpoint $(kL)^2 = 6.2177$ and the Gaussian benchmark $2\pi = 6.2832$ lie inside this interval.

boundary couplings for scale evolution. Experimental values serve as comparison data. The isometry-breaking matching scale is $k_{\text{GUT}} = M_{\text{P}}\sqrt{\pi}\tau/\sqrt{3}$ set by the $J = 2$ isometry breaking at $L_{\text{GUT}} = \sqrt{3}/\tau$. Here k_{GUT} denotes this geometric matching scale. Each channel has its own generator normalization and finite matching coefficient. The three boundary couplings are normalized channel responses of one connection. Their evolution with the retained matter content correlates the weak, electromagnetic and strong interactions across scales.

5.1 Weak coupling and screening

For the $\mathfrak{su}(2)_L$ isometry modes on RP^3 the integrated Killing-mode norm fixes the quadratic response. Canonical normalization therefore makes the squared coupling inversely proportional to this norm. The Killing-response convention with scale factor $(M_{\text{G}}/M_{\text{P}})^2$ gives the bare coupling

$$g_2^2 = 8 \left(\frac{M_{\text{G}}}{M_{\text{P}}} \right)^2 (kL)^{-3}.$$

The window-edge identity $kLM_{\text{G}} = M_{\text{P}}\sqrt{\pi}$ of Section 4.3 gives

$$g_2^2 = \frac{8\pi}{(kL)^5}, \quad \alpha_W(M_{\text{G}}) = \frac{g_2^2}{4\pi} = \frac{2}{(kL)^5},$$

so the bare weak coupling at the emergence scale depends only on the window capacity in this response normalization. The common action fixes the strength by normalizing the gauge kinetic term together with the matter current. A constant rescaling of a mode changes both coefficients and leaves the canonical coupling invariant. Appendix C.4 gives this reduction and identifies the absolute response normalization. In the common connection convention screening changes the coefficient of the quadratic gauge term. The

matter and torsion contributions are organized additively in the inverse coupling through the relation

$$\frac{1}{\alpha_{\text{SM}}} = \frac{1}{\alpha_W} + \frac{1}{N_c} - \frac{\tau^2 \pi}{2}, \quad N_c \left(\frac{1}{\alpha_{\text{SM}}} - \frac{1}{\alpha_W} + \frac{\tau^2 \pi}{2} \right) = 1,$$

where $\alpha_W = 2/(kL)^5$. The colour term is the ratio of extensive adjoint and matter responses equal to $1/N_c$ on three generations. The torsion term combines the quadratic Einstein–Cartan coefficient with the Euclidean period and $N_g \xi = 1$ in this additive screening relation. Appendix C.4 evaluates these factors. Proposition C.2 selects the common copy number $g = 3$ under the two screening rules. The resulting value is

$$g_2(M_G) = \sqrt{\frac{4\pi}{1/\alpha_W + 1/N_c - \tau^2 \pi/2}} = 0.508848,$$

The unrounded value differs from the reference 0.508845 by +0.0005%. The two screening terms reduce the bare Killing value. Its difference from the same reference is +0.34%.

5.2 The hypercharge coupling

The $U(1)_Y$ matching rule accompanies the $\mathfrak{su}(2)_R \rightarrow U(1)_R$ breaking by the $J = 2$ squash. The surviving axial mode acquires a different curvature norm from its round-space value. With the axial Killing generator held fixed its metric-dual one-form changes with the squash. For an amplitude constant over the internal space, preservation of the curvature quadratic norm of the same Abelian connection rescales its amplitude by $\kappa(s)$. The inverse squared norm ratio is

$$\kappa^2(s) = \frac{1+s}{(1-2s)^{5/2}},$$

which at $s_0 = 2\tau$ gives $\kappa(2\tau) = 1.13183$. Appendix C.4 computes the norm ratio and the corresponding matter-mode profile. The matching is imposed at $k_{\text{GUT}} = M_P \sqrt{\pi} \tau / \sqrt{3}$ for which $k_{\text{GUT}}/M_G = kL\tau/\sqrt{3} \simeq 0.02879$. The coupling is then transported to M_G by the running equations. The multiplicative matching correction uses

$$\delta_{g_1} = -\tau r_{23} \sum Y^2 \Delta_f \xi,$$

where $r_{23} = 1/(\sqrt{3} \sum Y^2) = 3/(10\sqrt{3})$ is the hypercharge-trace hierarchy ratio. The content product is $\sum Y^2 \Delta_f \xi = 5/8$ with $\Delta_f = 3/2$ and $\xi = 1/8$ from Section 3. The complete boundary relation is

$$g_1(k_{\text{GUT}}) = g_2(k_{\text{GUT}}) \kappa(2\tau) (1 + \delta_{g_1}).$$

It combines the axial curvature factor with the finite content correction in the stated external-mode normalization before scale evolution. The curvature and derivative terms of the internal axial profile have distinct Gram factors; their calculation is given in Appendix C.4. Transport to the emergence scale gives

$$g_1(M_G) = 0.60499.$$

Here and in the running equations $g_1 = \sqrt{5/3} g_Y$. Together with g_2 it determines the fine-structure constant through $e^{-2} = g_Y^{-2} + g_2^{-2}$ and $\alpha_{\text{em}} = e^2/(4\pi)$. The common two-loop transport gives $\alpha_{\text{em}}^{-1}(M_Z) = 128.206$. The reference value 127.952 gives a deviation of +0.20%.

5.3 The colour coupling

The colour coupling is matched to the $\mathfrak{su}(2)$ Killing normalization at leading order. The long-root correction at the breaking scale is

$$g_3(k_{\text{GUT}}) = g_2(k_{\text{GUT}}) \left(1 + \frac{\alpha_{\text{GUT}}^2}{K} \right),$$

where $\alpha_{\text{GUT}} = g_2(k_{\text{GUT}})^2/(4\pi)$ and

$$K = \frac{J(J+2)}{d} = \frac{8}{3}$$

is the coefficient assigned by the long-root matching rule as the squash eigenvalue $J(J+2) = 8$ divided by $d = 3$. Its positive correction places g_3 above g_2 at the breaking scale. Two-loop transport with the geometric Yukawa input $y_0 = 1$ gives

$$g_3(M_G) = 0.49776,$$

and the strong coupling at the electroweak scale is

$$\alpha_s(M_Z) = 0.11804,$$

a signed residual of +0.120% against the reference value 0.1179.

5.4 The two-loop coefficients

The running uses the two-loop $\overline{\text{MS}}$ beta functions of Sections IV.1 and VI of Ref. [19] evaluated on the retained representations. The endpoint, torsion modulus and family carrier supply the generated data for this evolution. The running scale labels the external energy resolution at which the effective interaction coefficients are evaluated. The same matter charges control the relative running from the geometric matching and emergence scales to laboratory scales. Appendix C.5 specifies the scale derivative and loop factors together with generator normalisation. It computes the coefficients from the Casimirs and Dynkin indices. At one loop they are

$$b_1 = \frac{41}{10}, \quad b_2 = -\frac{19}{6}, \quad b_3 = -7,$$

and the two-loop matrix is

$$B = \begin{pmatrix} 199/50 & 27/10 & 44/5 \\ 9/10 & 35/6 & 12 \\ 11/10 & 9/2 & -26 \end{pmatrix}.$$

The top-Yukawa and Higgs-quartic beta functions use the same content. For example the one-loop Yukawa coefficient is $9/2 = 3/2 + N_c$ and each gauge term is $-3[C_2(Q_L) + C_2(u_R)]$. Appendix C.6 supplies the two-loop expressions and potential convention.

6 Electroweak breaking

The scalar sector supplies a stable nonzero condensate and an electroweak amplitude far below the emergence scale. The internal squash, colour-singlet condensate and electroweak vacuum expectation value have distinct field normalizations. The elementary connector couples retained modes, while the condensate is a collective coordinate of their spectral response. With canonical kinetic normalization the fermion mass measures the coupling of opposite chiralities and the radial scalar mass measures the restoring response of this coordinate. Squash stationarity fixes its quartic coupling. The complete spectral potential of Ref. [4] determines the condensate and the restoring mass of its radial fluctuations. Two successive window crossings determine the canonically normalized electroweak amplitude v . These constructions supply the Higgs-quartic matching condition in the common action. The same v normalizes the fermion mass vertices.

6.1 The squash quartic

The geometric order parameter is the $J = 2$ squash amplitude φ . Stationarity at its torsion-selected value $\varphi_0 = 2\tau$ fixes the quartic coefficient

$$\lambda = \frac{\xi(R_c - R_{\text{GUT}})}{(2\tau)^2} = 149.145,$$

where $\xi = 1/8$ and $R_c = 6/\pi$ and $R_{\text{GUT}} = 6/L_{\text{GUT}}^2$ with $L_{\text{GUT}} = \sqrt{3}/\tau$. Appendix C.11 specifies the torsion coefficients and the amplitude convention. This λ belongs to the squash coordinate. The Higgs quartic λ_H follows the separate pseudo-dilaton matching relation of Appendix C.12. That relation and the generated v determine the tree-level mass $m_H = \sqrt{2\lambda_H} v$. The finite matching of Proposition C.5 then determines the renormalized action coefficients with this generated mass as the Higgs physical condition.

6.2 The positive colour-singlet scalar mass

Below the critical internal curvature, the quadratic coefficient at zero condensate becomes negative. With the inherited feedback and subtraction the complete spectral potential of Ref. [4] then has a nonzero stationary amplitude with a positive normalized Hessian. This Hessian measures the restoring response of the condensate and gives its colour-singlet radial mass. Appendix C.11 places the internal-radius path used here in that branch. The long-root quadratic coefficient at zero amplitude is

$$m_{\text{long}}^2 = K(R - R_c) < 0 \quad \text{for} \quad R < R_c,$$

with $R_c = 6/\pi$ and $K = 8/3$. In the unit-normalized Landau limit the radial squared mass is

$$m_{\delta, \text{L}}^2 = \frac{16}{3}(R_c - R) > 0.$$

For the complete logarithmic potential this is a strict upper bound $0 < m_{\delta}^2 < m_{\delta, \text{L}}^2$. The exact value is the normalized Hessian in (C.86). Ref. [4] gives the conditions and error bound for the Landau approximation. In the dimensionless internal-radius convention

$L(M_G) = c$ and $L_c = \sqrt{\pi}$. The certified endpoint satisfies $c > 2.491 > \sqrt{\pi}$ and gives the negative quadratic coefficient $m_{\text{long}}^2(M_G) = 16(c^{-2} - \pi^{-1}) < -2.50$. Its numerical value is -2.52 in those units. Along $L(k) = cM_G/k$ every $0 < k \leq M_G$ remains in the condensed domain. This includes the infrared endpoint $k = \sigma_C^{-1}$ where the Hubble length is $\sigma_C = 6.95 \times 10^{41} \text{ GeV}^{-1}$. This establishes the condensed stationary branch over the stated scale range. For specified coefficient paths the same analysis bounds the gap uniformly on a finite scale interval as detailed in Appendix D.5.

For the radial curvature observable the proposition “Spectrum of the radial curvature observable” in Ref. [4] gives a positive mass atom at m_δ^2 together with the two-particle and three-particle and four-particle continua. This is the spectral response of the Wick-ordered polynomial of the internally integrated scalar intensity in the common quadratic external theory. Its mass atom occurs at the radial gap.

The Landau stationary amplitude is

$$\langle E \rangle_L^2 = \frac{8}{3} \frac{R_c - R}{\lambda_E},$$

and $m_{\delta, L}^2 = 2\lambda_E \langle E \rangle_L^2$. Here λ_E multiplies the quartic term in the condensate coordinate E . Appendix C.11 gives the coordinate and kinetic normalization needed to compare it with the squash coefficient λ . The inherited complete scalar mass operator is $\Delta_0 + m_\delta^2 I$ including the constant mode as its lowest positive-mass excitation. Its mass is therefore the lower edge of this scalar spectrum. Section 9.2 uses the separate $8\Lambda_{\text{QCD}}$ mass normalization. An identification with the lowest condensate mass requires a common scale and source normalization including the constant scalar mode.

6.3 The vacuum expectation value

The electroweak readout follows the creation and stabilization transfers of the mass amplitude. Each crossing contributes $e^{-2\pi kL}$. Their product carries the hierarchy between the emergence scale and the electroweak scale. With the loop factor and the channel and squash normalizations the resulting vacuum expectation value is

$$v = M_G \frac{3\alpha}{\pi} e^{-4\pi kL} (1 - s_0 \kappa) = 246.19 \text{ GeV},$$

where $\alpha = 1/(16\pi^2)$ is the loop factor and $s_0 = 2\tau$ and $\kappa = 1.13183$ are the squash data of Section 5.2. The value differs from the reference 246.22 GeV by -0.012% . The exponential $e^{-4\pi kL}$ supplies the hierarchy $v/M_G \sim 10^{-16}$. Appendix C.11 gives the two window crossings and their prefactor.

The total generator $4\pi kL$ and the electron cascade generator $20kL$ both depend on the same window capacity. Their prefactors and source-channel assignments in Table 2 therefore imply the correlated scale relation (C.10). Changing the bandwidth moves the electroweak and electron masses together. The complete condensate amplitude E_* and feedback coefficient g retain their independent stationary definitions in Appendix C.11.

7 Flavour structure

At fixed electroweak amplitude v , the flavour construction assigns hierarchical mass amplitudes to the three retained family positions. Repeated transfers of the same source amplitude accumulate suppression, so different family positions and sector rates produce large mass ratios from a common internal scale. The window capacity and torsion enter these rates. The electron scale supplies the second ladder normalization while v normalizes the top mass. The neutrino spectrum and angular texture use their own source and propagation relations (Appendices C.10 and E.1). The charged-fermion transfer operators give two mass combinations independent of the common transfer parameters. Their comparison with physical masses uses finite conversion to a common renormalization scheme.

7.1 The ladder

The three retained generation positions have extrusion orders $n = \{0, 2, 4\}$ as specified in Section 4.5. The transfer acts on the coefficient of the canonically normalized chirality-changing mass vertex. It composes source operations on the retained modes and thus determines an effective coupling between their left and right components. Successive operations multiply this amplitude and make their attenuation additive in its logarithm. Propagation and reaction rates follow from the common action containing these vertices with evolution read against a physical clock. On an unmodified rung, stationary composition of positive amplitudes gives $a_f(\ell_1 + \ell_2) = a_f(\ell_1)a_f(\ell_2)$ and $a_f(0) = 1$. Continuity then gives $a_f(\ell) = e^{-\alpha_f \ell}$. The sector rules determine its rate and the light-family insertions. In the up sector the rate is the window width reduced by the torsion deficit

$$\alpha_{\text{up}} = kL - 2\tau,$$

The same source response supplies the positive tilt decrement $1 - n_s = \tau w_W$ with $w_W = (1 + 2 + 1 + 3)/(d + 1) = 7/4$ for $d = 3$ and the local window $kL_{\text{CMB}} = kL(1 - \tau/4)$. These are determined by the common endpoint and torsion. The Weyl-window ratio of RP^3 enters their cosmological interpretation in Appendix C.14. The subscript CMB labels the corresponding cosmic-microwave pivot readout. Summing the mean response over the six internal isometry directions gives the sector decrement

$$\Delta = 6(1 - n_s)kL_{\text{CMB}},$$

We denote the positive decrement by $n_s^{\text{tilt}} \equiv 1 - n_s$. The down and lepton ladders split by the hypercharge ratio

$$\frac{9}{8} = \frac{1}{1 - (Y_d/Y_l)^2},$$

where $Y_d = |Y_{d_R}| = 1/3$ and $Y_l = |Y_{e_R}| = 1$ are the right-singlet hypercharge magnitudes. The rung exponents are

$$\alpha_{\text{dn}} = \alpha_{\text{up}} - \frac{18}{17}\Delta, \quad \alpha_{\text{lp}} = \alpha_{\text{up}} - 2\Delta,$$

Exponent	Source operation and transfer	Coefficient rule	Testable consequence
$2\alpha_{\text{up}}$	Two-order transfer on the family projector P_1 in B_u	Width $c = kL$ per order reduced by the two torsion directions gives $\alpha_{\text{up}} = c - 2\tau$	$\log(m_t/m_c) = 2(c - 2\tau)$
$2\alpha_{\text{dn}}$ and $2\alpha_{\text{lp}}$	Charge-weighted shifts of the same family transfer generators B_d, B_ℓ	Mean step $\Delta = 6(1 - n_s)c_C$. Consecutive decrements in ratio 9/8 with total 2Δ	The joint ratio relation (7.3)
Light-family insertions	Diagonal P_2 terms in (C.62)	Up-sector factor 4 and down-sector dilution $c_C/6$ with lepton period $\sqrt{2\pi}$ enter their stated amplitude channels	The first-generation mass ratios
$20c$	Four successive electron-amplitude transfers each with generator $5cI$	One-generation charge weight $\sum Y^2 \Delta_f = 5$. Equal attenuation per level and content unit	$m_e/M_P = e^{-20c}(1 - s_0\kappa)$
$4\pi c$	Creation and stabilization transfers each $e^{-2\pi c}$ in the electroweak amplitude	Two equal crossings with the squash normalization and loop and Fourier prefactor of Appendix C.11	The shared bandwidth response (C.10)

Table 2. Source data and transfer generators with the physical selection rules for the main hierarchies. Here $c_C = kL_{\text{CMB}}$ and P_1, P_2 are the middle- and light-family projectors. The rates specify mass amplitudes. Transition probabilities require the separate dynamical identification in Appendix C.7.

with the down first-generation step carrying the colour-dilution correction

$$\alpha_{\text{sd}} = \alpha_{\text{dn}} - \frac{kL_{\text{CMB}}}{6}.$$

Proposition C.4 realizes these ladder and colour-dilution rules by positive operators on the generated family module. The angular rotations orient these mass operators in a common left-handed kinetic space. If L_u and L_d diagonalize the canonical left mass-squared operators the charged-current overlap is $V_{\text{CKM}} = L_u^\dagger L_d$. Thus the masses measure the strengths along the family directions while mixing measures their relative orientation. A common change of mode basis preserves both the masses and these overlaps as expressed by the projector invariants in (C.66). Appendix C.7 specifies the additional source path and level-crossing data needed for a Landau–Zener interpretation.

The source algebra supplies the projectors, charges and allowed operations [4]. The positive transfers in Table 2 act on the mass amplitudes. Equal attenuation per content unit, the sector split and the number of crossings determine the coefficients and their joint mass relations.

The light-family factors resolve the last step of the transfer. In the up sector the insertion $I - 3P_2/4$ transmits one quarter of the light amplitude and leaves the heavy and middle amplitudes unchanged. Its logarithmic contribution is $(\log 4)P_2$ in B_u . The resulting geometric mass ratios obey $m_c/m_u = 4e^{2c_C}$ whereas $m_t/m_c = e^{2\alpha_{\text{up}}}$ tests the

common rung. Thus the light ratio probes the relative normalization of this insertion and the pivot width separately from the heavy-to-middle transfer. The down dilution and lepton period in Table 2 act on the same light-family projector in their respective sectors. Comparisons of these relations use the finite matching of the geometric amplitudes to masses at a common scheme and scale.

The masses m_i in the ladder formulas are the singular values of the canonically normalized generated mass vertex. At fixed generated background, they are the positive rest energies of the quadratic matter modes measured by the phase clock. The interacting readout uses the connected matter two point response with its mass and kinetic terms normalized together. For charged leptons we write the geometric masses explicitly as m_ℓ^{geom} . For comparison with pole masses in a specified realization, denote the corresponding poles by M_ℓ and their adopted full Standard Model running representation by $\overline{m}_\ell(\mu)$. Define

$$M_\ell = G_\ell m_\ell^{\text{geom}}, \quad \overline{m}_\ell(\mu) = D_\ell(\mu) M_\ell, \quad C_\ell(\mu) = D_\ell(\mu) G_\ell. \quad (7.1)$$

Here G_ℓ records the ratio of the pole readout to the generated mass. The specified readout fixes this ratio with the source normalization and scale retained. In the adopted running representation D_ℓ converts the pole to the stated convention and C_ℓ is their product. The pole to running calculation uses the self-energies of that effective realization.

Perturbative relations between running Yukawa couplings and pole masses apply to specified action parameters [23]. Preserving a relation among different masses under radiative corrections is a separate dynamical question, as illustrated by the Koide relation [24].

For a realization described by a perturbative effective two point function expand about a canonical reference mass matrix M_0 with kinetic corrections $\delta K_L, \delta K_R$ and mass correction δB . Restoring canonical fields gives

$$\delta M = \delta B - \frac{1}{2} \delta K_L M_0 - \frac{1}{2} M_0 \delta K_R. \quad (7.2)$$

Indeed $(I + \delta K)^{-1/2} = I - \delta K/2 + O(\delta^2)$ on the positive kinetic subspace. For a simple nonzero singular value m_i of M_0 , with normalized left and right singular vectors u_i, v_i , its variation is $\delta \log m_i = \text{Re}(u_i^\dagger \delta M v_i)/m_i$. Taking the generated diagonal mass vertex as M_0 in this representation gives the first order on shell readout ratio

$$(\log G_\ell)^{(1)} = \frac{\text{Re} \delta B_\ell}{m_\ell^{\text{geom}}} - \frac{1}{2} \text{Re}(\delta K_{L,\ell} + \delta K_{R,\ell}).$$

These coefficients refer to the same generated source and mass definition, with the low-energy loop contribution subtracted once when it is also included in the effective action. The condensate amplitude is already part of B in this convention. The weighted combination isolates the flavour dependence of this readout ratio. The separate pole-to-running calculation determines D_ℓ for specified pole conditions.

For the geometric masses defined by these transfers the hypercharge split eliminates the common window rate and sector decrement and gives

$$8 \log \frac{m_t}{m_c} + 9 \log \frac{m_\tau}{m_\mu} = 17 \log \frac{m_b}{m_s}. \quad (7.3)$$

This identity tests the relative attenuation of the up, down and charged-lepton sectors independently of their common window rate, decrement and absolute mass scales. An experimental test compares all six masses in a common convention. Running and finite matching convert the geometric amplitudes to the chosen mass definitions and scale. If $m_i^{\text{phys}}(\mu) = C_i(\mu)m_i$ the logarithmic combination formed from the physical masses must be reduced by $8\log(C_t/C_c) + 9\log(C_\tau/C_\mu) - 17\log(C_b/C_s)$ before comparison with zero. The uncertainties in the masses and matching factors are propagated together. The reference masses in Appendix A use their customary individual scales and schemes. In the species order (t, c, τ, μ, b, s) set $w = (8, -8, 9, -9, -17, 17)$ and let z be the column vector formed from the six $\log(m_i^{\text{phys}}/\text{GeV})$ followed by the six $\log C_i$. The matched residual and its variance are

$$r_m = (w, -w)z, \quad \sigma_{r_m}^2 = (w, -w)\Sigma_z(w, -w)^\top, \quad (7.4)$$

where Σ_z includes the mass-matching cross correlations. The numerical implementation [15] evaluates this combination from the six mass definitions and reference scales together with the matching coefficients and their joint logarithmic covariance. Common multiplicative mass normalizations cancel because $\sum_i w_i = 0$.

The lepton invariant and pole-mass comparison. The light-family insertion gives the geometric identity

$$\frac{(m_\mu^{\text{geom}})^2}{m_e^{\text{geom}}m_\tau^{\text{geom}}} = e^{\sqrt{2\pi}}$$

by cancellation of the common mass normalization and rung attenuation. The invariant isolates the light-family insertion. With $w_e = -1$ and $w_\mu = 2$ and $w_\tau = -1$ define

$$\Delta_L^{\text{pole}} = \sum_\ell w_\ell \log G_\ell, \quad \Delta_L^{\text{run}}(\mu) = \sum_\ell w_\ell \log D_\ell(\mu). \quad (7.5)$$

Taking the logarithm of Eq. (7.1) and summing with these weights gives

$$\begin{aligned} \log \frac{M_\mu^2}{M_e M_\tau} - \sqrt{2\pi} &= \Delta_L^{\text{pole}}, \\ \log \frac{\bar{m}_\mu(\mu)^2}{\bar{m}_e(\mu)\bar{m}_\tau(\mu)} - \sqrt{2\pi} &= \Delta_L^{\text{pole}} + \Delta_L^{\text{run}}(\mu). \end{aligned} \quad (7.6)$$

The pole inputs in Ref. [10, Eq. (5)] give

$$\frac{M_\mu^2}{M_e M_\tau} = 12.2951727421, \quad \Delta_{L,\text{ref}}^{\text{pole}} = 0.00257845 \pm 0.00006753. \quad (7.7)$$

The quoted reference uncertainty follows by propagating the input errors as independent Gaussian errors using $\sigma_{\text{ref}}^2 = \sum_\ell w_\ell^2 (\sigma_{M_\ell}/M_\ell)^2$. The tau mass dominates this uncertainty. Under these reference errors the exact pole identification $G_e = G_\mu = G_\tau = 1$ has standardized residual $\Delta_{L,\text{ref}}^{\text{pole}}/\sigma_{\text{ref}} = 38.18$. For a specified realization the charged-lepton readout determines G_ℓ . Its weighted logarithmic ratio Δ_L^{pole} and uncertainty are tested against Eq. (7.7). The pole and running comparisons share the same experimental inputs. Their correlations enter a combined test.

Correlated running-mass constraints. An experimental constraint on the required matching can be obtained without assigning the individual C_i . We use the full Standard-Model $\overline{\text{MS}}$ masses and their joint uncertainties at $\mu = M_Z = 91.1876 \text{ GeV}$ from Tables 2 and 3 and 8 of Ref. [10]. The common convention is $\overline{m}_i = y_i v_F / \sqrt{2}$ with $v_F = 246 \text{ GeV}$. This common normalization cancels from both matching combinations. The two geometric identities constrain the total matching factors C_i to this running convention. Define

$$\begin{aligned}\Delta_F &= 8 \log(C_t/C_c) + 9 \log(C_\tau/C_\mu) - 17 \log(C_b/C_s), \\ \Delta_L &= 2 \log C_\mu - \log C_e - \log C_\tau.\end{aligned}\tag{7.8}$$

Thus $\Delta_L = \Delta_L^{\text{pole}} + \Delta_L^{\text{run}}$ by Eq. (7.5). The published running masses then give

$$\begin{pmatrix} \Delta_F \\ \Delta_L \end{pmatrix} = \begin{pmatrix} 2.69537 \pm 1.48017 \\ 0.0112736 \pm 0.0005817 \end{pmatrix}, \quad \Sigma_{FL} = \begin{pmatrix} 2.1909009 & 8.68052 \times 10^{-7} \\ 8.68052 \times 10^{-7} & 3.38356 \times 10^{-7} \end{pmatrix}.\tag{7.9}$$

Here the delta method converts the published mass covariance to the logarithmic covariance before contraction by the two weight vectors. The reference error model combines propagated input uncertainties and perturbative truncation estimates. The correlation between the two resulting constraints is 0.0010082. These are constraints on the finite matching of the generated masses. A realization with $\Delta_F = \Delta_L = 0$ at M_Z has standardized residuals 1.82 and 19.38 respectively. The uncertainty in Eq. (7.9) sets the precision with which the lepton matching combination is tested.

The scale dependence supplies a separate consistency check. At one loop the common gauge terms and common Yukawa trace cancel within each equal-charge pair. In the diagonal-CKM limit

$$\begin{aligned}16\pi^2 \frac{dF}{d \log \mu} &= \frac{3}{2} \{25(y_t^2 - y_c^2 - y_b^2 + y_s^2) + 9(y_\tau^2 - y_\mu^2)\}, \\ 16\pi^2 \frac{dL}{d \log \mu} &= \frac{3}{2} (2y_\mu^2 - y_e^2 - y_\tau^2),\end{aligned}\tag{7.10}$$

where F is the six-mass logarithmic combination and $L = \log(\overline{m}_\mu^2/(\overline{m}_e \overline{m}_\tau)) - \sqrt{2}\pi$. At the reference point their derivatives are 0.22214 and -9.31×10^{-7} . The calculation [15] also retains a general unitary CKM matrix in the Yukawa terms. For the charged leptons the one-loop QED relation [25]

$$\log \frac{\overline{m}_\ell(\mu)}{M_\ell} = -\frac{\alpha}{\pi} \left[1 + \frac{3}{4} \log \frac{\mu^2}{M_\ell^2} \right] + O(\alpha^2)$$

gives a weighted correction 0.00874265 independent of the common scale at this order. This explicitly evaluates the electromagnetic part of the conversion. The common full Standard-Model reference in Eq. (7.9) includes the additional matching contributions. The generated action's flavour-dependent finite coefficients enter through Eq. (7.8).

Finite lepton transfer. In a fixed Standard-Model scheme with vanishing neutrino Yukawa couplings put $t = \log(\mu/\mu_0)$ and write the one-loop charged-lepton equations [20]

as

$$\frac{dy_i}{dt} = y_i[A(t) + by_i^2], \quad b = \frac{3}{32\pi^2}, \quad i = e, \mu, \tau.$$

Here $16\pi^2 A = \text{Tr}(3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d + Y_e^\dagger Y_e) - 9g_1^2/4 - 9g_2^2/4$ with GUT-normalized g_1 evaluated along the same running solution. Define $E(t) = \exp \int_0^t A(s) ds$ and $I(t) = 2b \int_0^t E(s)^2 ds$. As long as $d_i(t) = 1 - I(t)y_i(0)^2 > 0$ the finite solution and its lepton combination are

$$y_i(t) = \frac{E(t)y_i(0)}{\sqrt{d_i(t)}}, \quad (7.11)$$

$$R_L(t) := \log \frac{y_\mu(t)^2}{y_e(t)y_\tau(t)} - \log \frac{y_\mu(0)^2}{y_e(0)y_\tau(0)} = \frac{1}{2} \log \frac{d_e(t)d_\tau(t)}{d_\mu(t)^2}. \quad (7.12)$$

For a forward interval $0 \leq t \leq T$ uniform bounds $A_- \leq A(t) \leq A_+$ imply

$$2b \frac{e^{2A_-T} - 1}{2A_-} \leq I(T) \leq 2b \frac{e^{2A_+T} - 1}{2A_+}, \quad (7.13)$$

where each quotient is T at $A_\pm = 0$. Substitution into (7.12) by interval arithmetic encloses the running contribution provided all denominator lower bounds are positive.

For a fixed sequence of running and threshold conversions let Δ_L^{known} be the independently calculated contribution in that convention and let x_i be the logarithms of the remaining finite matching factors. Then

$$\max_i x_i - \min_i x_i \geq \frac{1}{2} |\Delta_L - \Delta_L^{\text{known}}|. \quad (7.14)$$

This bound is sharp and invariant under a common finite mass or vacuum-expectation-value normalization. The finite running solution follows by integrating $d(E^2/y_i^2)/dt = -2bE^2$ and taking the weighted logarithm. Upper and lower bounds on the integral of A give the stated enclosure. For the matching remainder $\Delta_L - \Delta_L^{\text{known}} = 2x_\mu - x_e - x_\tau$ lies between $-2(\max x - \min x)$ and $2(\max x - \min x)$. Equality is attained with the electron and tau logarithms at one endpoint and the muon logarithm at the other while a common additive shift cancels because the weights sum to zero.

The common integrating factor E cancels from the lepton ratio. Its dependence on the coupled running solution is retained through the single path integral I in Eq. (7.12). The remaining finite coefficients combine the flavour-dependent self-energies, vertex normalizations and external-field normalizations in the same scheme. Equation (7.14) constrains their required spread once the independently calculated conversions have been subtracted. The implementation [15] checks the resulting constraint in the registered comparison. At the reference central values subtracting the displayed leading QED term leaves $\Delta_L - \Delta_L^{\text{QED}} = 0.00253093$ and requires $\max x_i - \min x_i \geq 0.00126546$. This is a constraint on the combined remaining conversion from the geometric readout through the pole definition to the common running scheme. Numerical checks compare this transfer with direct integration of all three eigenvalue equations and verify composition and reversal of the scale interval. Outward interval evaluation uses uniform bounds on A throughout the interval.

Pole to running conversion in the specified electroweak realization. Consider the canonical electroweak realization with the pole identification $G_\ell = 1$. We extend the physical renormalization conditions of Proposition C.5 to the charged leptons. The geometric masses supply the pole conditions M_e, M_μ, M_τ of this action. The calculation determines D_ℓ at these conditions. At $Q_0 = M_Z^{\text{gen}} = 91.12484 \text{ GeV}$ use its tadpole-free Landau-gauge $\overline{\text{MS}}$ convention with the generated VEV and gauge coefficients and $y_\ell^2 = 2M_\ell^2/v^2$ at leading order. Neutrino Yukawa couplings vanish in this calculation. The photon self-energy and the W and Z and Higgs self-energies retain the charged-lepton masses in their loop functions. They follow from Ref. [26, Eq. (2.20)] with $Q_\ell = -1$ and $I_3^\ell = -1/2$ with a massless weak partner and zero colour coupling. The light-mass weak threshold agrees with Ref. [27, Sec. II.C].

Let $\delta s_\ell^{(1)}$ include the one-loop factor in the pole-squared-mass shift. Linear inversion gives

$$(\log D_\ell)^{(1)} = -\frac{\text{Re } \delta s_\ell^{(1)}}{2M_\ell^2}, \quad \Delta_L^{\text{run},(1)} = -\sum_{\ell=e,\mu,\tau} w_\ell \frac{\text{Re } \delta s_\ell^{(1)}}{2M_\ell^2}, \quad w = (-1, 2, -1). \quad (7.15)$$

The common VEV conversion to the reference convention $\bar{m}_\ell = y_\ell v_F/\sqrt{2}$ cancels since $\sum_\ell w_\ell = 0$. This direct pole-to-full-SM calculation includes the heavy electroweak thresholds in the finite self-energy. Its gauge coefficients are those of the same action. In particular $\alpha(Q_0) = 0.00779993272$ and

$$\begin{aligned} \Delta_{L,\gamma}^{\text{run},(1)} &= \frac{3\alpha(Q_0)}{2\pi} \log \frac{M_\mu^2}{M_e M_\tau} = 0.009335168839, \\ \Delta_{L,\text{weak}}^{\text{run},(1)} &= 2.348914726 \times 10^{-6}, \\ \Delta_L^{\text{run},(1)}(Q_0) &= 0.009337517754. \end{aligned} \quad (7.16)$$

All inputs to these coefficients are generated before the comparison. At the stated input values analytic interval evaluation of the nine two-point functions gives

$$\Delta_L^{\text{run},(1)} \in [0.009337517754041472, 0.009337517754041475].$$

Direct high-precision Feynman-parameter integrals check the finite components independently.

The weighted one-loop scale derivative is -9.77377×10^{-7} . Transferring the result to the comparison scale 91.1876 GeV therefore changes it by -6.73×10^{-10} at this order. Under $G_\ell = 1$ the running-mass constraint minus this one-loop conversion is 0.00193607 with reference uncertainty 0.0005817 . For a general geometric-to-pole conversion the total prediction is

$$\Delta_L^{\text{th}}(\mu) = \Delta_L^{\text{pole,th}} + \Delta_L^{\text{run},(1)}(\mu; \{G_\ell m_\ell^{\text{geom}}\}) + \mathcal{R}_{L,\text{run}}^{(2+)}. \quad (7.17)$$

The one-loop kernels are evaluated at the poles supplied by that conversion. The remainder $\mathcal{R}_{L,\text{run}}^{(2+)}$ contains higher-order electromagnetic and weak corrections to D_ℓ . The specified physical readout or pole identification supplies the geometric to pole term and its uncertainty. At fixed pole renormalization conditions, higher orders determine the corresponding

running parameters while preserving those poles. The displayed interval certifies the numerical one-loop conversion at the stated inputs. Equation (7.7) tests the pole identification directly. Eq. (7.9) tests the total conversion to the common running convention.

7.2 The top-quark normalization

The generated top-quark mass fixes the normalization of the up-sector ladder. The elementary scalar connector of Ref. [4] has unit geometric matrix element

$$y_0 = 1,$$

which is invariant under a common unitary transformation of the connector and its paired modes. With the vacuum expectation value of Section 6 the tree-level mass is

$$m_t = \frac{y_0 v}{\sqrt{2}} = 174.1 \text{ GeV}.$$

The unit finite-carrier matrix element and signed spatial connector with the canonical action are constructed in Appendix C.8. The resulting top and Higgs masses and Fermi constant serve as physical renormalization conditions for the one-loop $\overline{\text{MS}}$ action of Proposition C.5.

7.3 The lower masses

The bottom-to-top Yukawa ratio includes the down-sector suppression and the evolution of the pivot window

$$\frac{y_b}{y_t} = e^{-(2\alpha_{\text{dn}} - n_s^{\text{tilt}}(kL_{\text{CMB}} + 2\tau))} = 0.024344,$$

where n_s^{tilt} is the positive spectral-tilt decrement. This gives the bottom mass

$$m_b = \frac{y_b}{y_t} m_t = 4.238 \text{ GeV}.$$

The within-sector mass ratios are

$$\frac{m_s}{m_d} = e^{2\alpha_{\text{sd}}} = 19.62, \quad \frac{m_b}{m_s} = e^{2\alpha_{\text{dn}}} = 44.87, \quad \frac{m_\mu}{m_e} = e^{2\alpha_{\text{lp}} + \sqrt{2}\pi} = 206.36,$$

and the charm and up ratios are

$$\frac{m_t}{m_c} = e^{2\alpha_{\text{up}}} = 135.2, \quad \frac{m_t}{m_u} = e^{2\alpha_{\text{up}}} e^{2kL_{\text{CMB}} + \ln 4} = 77304.$$

The electron cascade gives the closed form in the common Planck normalization

$$m_e = M_{\text{P}} e^{-20kL} (1 - s_0 \kappa) = 0.510 \text{ MeV},$$

where $s_0 \kappa$ is the squash correction of Section 3.4. The remaining lepton step is $m_\tau/m_\mu = e^{2\alpha_{\text{lp}}}$. These ratios and the two generated sector normalizations determine the lower masses. Both v and m_e are calculated relative to the same M_{P} . They provide intermediate normalizations within the mass calculation while v/m_e tests the relative strength of their source

transfers independently of the common dimensionful scale. Appendix C.9 gives the electron cascade normalisation.

The quark-mixing readouts use the same family attenuation. The adjacent-generation amplitude is $|V_{cb}| = \frac{1}{2}e^{-\alpha_{\text{up}}}(1 - s_0\kappa)$ while the next-generation amplitude is $|V_{ub}| = \frac{1}{2}e^{-2\alpha_{\text{up}}}(1 + \tau\kappa)$ with the source-channel factors given in Appendix C.13. For the geometric masses and angular readouts of this action eliminating the common exponential gives

$$\frac{|V_{cb}|^2}{m_c/m_t} = \frac{(1 - s_0\kappa)^2}{4}, \quad \frac{|V_{ub}|}{|V_{cb}|^2} = \frac{2(1 + \tau\kappa)}{(1 - s_0\kappa)^2}. \quad (7.18)$$

The first relation connects a mass hierarchy to a mixing amplitude. The second tests the relative channel factors after that hierarchy has cancelled. Their experimental comparison uses mass and flavour matching in the same common-action convention as the conversion above.

8 Cosmology

The electroweak scale enters cosmology through the dimension-five neutrino mass relation. The lightest mass then supplies the common energy scale in the vacuum-density and photon-temperature readouts and the Gaussian entropy difference connects that vacuum floor to the Hubble scale. Eliminating the common mass gives joint tests of the density, temperature and expansion readouts through the dark energy fraction and the temperature–Hubble ratio. The homogeneous evolution also includes baryons and a pressureless component together with photons and three thermal neutrinos.

The macroscopic calculation has three linked levels. The homogeneous background turns the endpoint density and clock readouts into an expansion history. Linear propagation carries this history to the distance observables. The local variational sector resolves how a spatially varying source and potential exchange energy and momentum. Their common normalization is Newton’s constant and their common matter data are the generated masses and source charges. Each level uses its background response map with these shared inputs. The density budget and acoustic distances test the common thermal scale. Local source–potential dynamics and the occupied auxiliary radius background resolve source feedback and a material phase clock.

8.1 Two Gaussian entropies

The density readout uses the lightest-neutrino rest mass as the infrared energy scale of the vacuum floor. Its fourth power has the dimensions of an energy density in natural units. The coefficient is fixed by the selected hypercharge weight and squash response. The endpoint entropy difference compares this floor with the Planck anchor through a Gaussian variance ratio and a content-dependent offset. The Planck Gaussian has variance M_{P}^2 and the vacuum-floor Gaussian has variance $\sqrt{\rho_{\Lambda}}$. On the branch $0 \leq x = s_0\kappa < 1/4$ the vacuum density is positive and equals

$$\rho_{\Lambda} = Y_u m_{\nu_1}^4 (1 - 4s_0\kappa) = \frac{2}{3} m_{\nu_1}^4 (1 - 4s_0\kappa).$$

The factor $(1 - 4s_0\kappa)$ is the dark-energy-weight correction of the $J = 2$ squash. The squash dependence of v^2 also enters the neutrino mass. At fixed remaining inputs its density factor is $(1 - x)^8(1 + x)^4(1 - 4x)$ as proved in Appendix C.14. Writing $m_{\nu_1} = M_P \hat{m}_{\nu_1}$ and $\rho_\Lambda = M_P^4 \hat{\rho}_\Lambda$ gives the dimensionless entropy span

$$S_{\text{end}} = \ln \frac{M_P^2 \sqrt{2\pi + r_{23}}}{\sqrt{\rho_\Lambda}} = \ln \frac{\sqrt{2\pi + r_{23}}}{\sqrt{\hat{\rho}_\Lambda}} = 139.2537,$$

with $r_{23} = 3/(10\sqrt{3})$ the hypercharge-trace hierarchy ratio. Appendix C.14 derives this expression from twice the Gaussian entropy difference plus the content offset and shows how intermediate scales cancel.

The entropy difference is invariant under a common change of energy unit. Along successive intervals of a Gaussian variance path the entropy differences add and every intermediate scale cancels. The content offset is attached once to the complete pair of endpoints. Thus $W = e^{S_{\text{end}}}$ measures the ultraviolet-to-infrared scale ratio including its content factor independently of that subdivision. The Hubble readout converts this span into an inverse length. Appendix C.14 also evaluates the distinct entropy of the normalized internal heat-trace state.

8.2 The Hubble rate and the dark-energy fraction

The expansion rate is $H = d \log a / ds$ where s is the proper-time readout of the homogeneous comoving flow. For a parameter λ with $ds = N_\lambda d\lambda$ it is $H = N_\lambda^{-1} d \log a / d\lambda$. In the homogeneous dust realization $T = s + \text{constant}$. The background equations use this clock so that their time coordinate is $t = s$.

The Hubble rate is the infrared endpoint of the emergence window

$$H_0 = M_P \sqrt{\pi} e^{-S_{\text{end}}} = 1.4388 \times 10^{-42} \text{ GeV},$$

which corresponds to $67.45 \text{ km s}^{-1} \text{ Mpc}^{-1}$. The dark-energy fraction is

$$\Omega_\Lambda = \frac{\rho_\Lambda}{3H_0^2 M_P^2} = \frac{2}{3} + \frac{r_{23}}{3\pi} = 0.68504.$$

Its signed residual against the adopted Planck reference in Table 2 of Ref. [28] is +0.05%. Appendix C.14 gives the cancellation of the common density.

The macroscopic Einstein–matter realization of Ref. [11] gives the gravitational effective action

$$S_{\text{eff}} = \frac{M_P^2}{2} \int d^4x \sqrt{-g} R - \int d^4x \sqrt{-g} \rho_\Lambda,$$

with $M_P^2 = 1/(8\pi G)$. The constant vacuum floor contributes a uniform background stress through the volume term in this action. The entropy relation and neutrino mass scale enter the homogeneous densities through

$$\rho_{\text{tot}} = \rho_\Lambda + \rho_b + \rho_\Sigma^{\text{budget}} = 3H_0^2 M_P^2, \quad \Omega_b + \Omega_\Sigma^{\text{budget}} + \Omega_\Lambda = 1.$$

This relation imposes the spatially flat homogeneous budget on the endpoint densities.

8.3 The microwave background temperature

The photon temperature readout uses the same lightest neutrino mass as the vacuum density. Its content and squash factors convert that rest-mass scale into the raw photon floor

$$T_{\text{CMB}}^{\text{raw}} = \frac{m_{\nu_1} r_{12}}{\pi} (1 - s_0 \kappa) (1 - \tau \Delta_s) = 2.7310 \text{ K},$$

where $r_{12} = 3/10$ is the hypercharge ratio and $\Delta_s = 1/2$ is the scalar conformal weight. The factor $(1 - s_0 \kappa)$ multiplies the already corrected rest mass. The total temperature response includes both factors. The formula uses common energy units before conversion to Kelvin. The endpoint photon zero-mode factor is

$$C_\gamma = 1 - \frac{\tau}{\pi^2}, \quad T_{\text{CMB}} = C_\gamma T_{\text{CMB}}^{\text{raw}} = 2.72547 \text{ K}.$$

The corrected value is used for the CMB-monopole comparison and for the linear propagation. At this temperature the baryon-to-photon ratio is

$$\eta_B = \eta_B^{\text{raw}} C_\gamma^{-3} = 6.12823 \times 10^{-10}.$$

Consequently $\eta_B T_{\text{CMB}}^3 = \eta_B^{\text{raw}} (T_{\text{CMB}}^{\text{raw}})^3$. The baryon density has the same value in either photon convention. The common mass cancels exactly from $T_{\text{CMB}}^2 / (M_{\text{P}} H_0)$ in energy units. The temperature is linear in that mass while the entropy relation makes H_0 quadratic in it. Together with Ω_Λ this ratio tests the joint density and temperature readouts and the expansion independently of their common mass normalization. Equation (C.117) gives this cross-scale relation and Appendix D.4 bounds its finite squash correction.

8.4 The homogeneous density budget

The baryon fraction is

$$\Omega_b = \frac{\eta_B^{\text{raw}} n_\gamma^{\text{raw}} m_p}{\rho_{\text{crit}}} = 0.04941,$$

where η_B^{raw} is given by Eq. (C.105) with its photon-temperature conversion described in Appendix E.2 and the corresponding photon density is

$$n_\gamma^{\text{raw}} = \frac{2\zeta(3)}{\pi^2} (T_{\text{CMB}}^{\text{raw}})^3,$$

with m_p the proton mass of Section 9.1 and $\rho_{\text{crit}} = 3H_0^2 M_{\text{P}}^2$ calculated from the Hubble rate of Section 8.2. With $\Omega_\Lambda = 0.68504$ the remaining fraction is fixed by the homogeneous budget

$$\Omega_\Sigma^{\text{budget}} = 1 - \Omega_\Lambda - \Omega_b = 0.26554751.$$

The sum

$$\Omega_b + \Omega_\Sigma^{\text{budget}} + \Omega_\Lambda = 1$$

satisfies the flatness constraint. The thermal components are evaluated at the corrected photon temperature. With the common neutrino temperature

$$T_\nu = T_{\text{CMB}} \left(\frac{4}{11} \right)^{1/3} \left(\frac{N_{\text{eff}}}{3} \right)^{1/4} = 1.95242401 \text{ K},$$

their energy densities including both helicities are

$$\rho_{\nu_i} = \frac{T_\nu^4}{\pi^2} \int_0^\infty \frac{q^2 \sqrt{q^2 + (m_{\nu_i}/T_\nu)^2}}{e^q + 1} dq, \quad \rho_\gamma = \frac{\pi^2 T_{\text{CMB}}^4}{15}. \quad (8.1)$$

The common budget therefore fixes

$$\Omega_{\text{cold}} = \Omega_\Sigma^{\text{budget}} - \Omega_\gamma - \sum_i \Omega_{\nu_i} = 0.26403880, \quad (8.2)$$

where $\Omega_\gamma = 0.0000543504$ and $\sum_i \Omega_{\nu_i} = 0.0014543578$. The three neutrino masses are those of the rest spectrum in Appendix E. The same vacuum fraction $\Omega_\Lambda = 0.68504430$ enters the endpoint and the propagation.

8.5 Cold-matter evolution and linear cosmological perturbations

Linear cosmological perturbations are evolved with the two-derivative Einstein–matter action of Section 8.2 [11]. We take the vacuum density to be constant and the visible stress tensor to be separately conserved. The pressureless component carries a density transported with the matter flow and allows an inhomogeneous cold source while its spatial and mixed rest-frame stress projections vanish. In this subsection ρ_Σ denotes the cold density with present mean $\rho_{\text{crit}}\Omega_{\text{cold}}$. Photons and thermal neutrinos belong to $T_{\mu\nu}^{\text{vis}}$.

A dust action fixes the transported density and velocity together with their stress. Its field equations determine both the residual Einstein source and the rest-frame pressureless form.

Cold source and comoving transport. On a smooth spacetime with signature $(-, +, +, +)$ and Levi–Civita connection take $M_{\text{P}} > 0$ and constant ρ_Λ with a separately conserved visible stress tensor. Add to the Einstein and visible-matter actions

$$S_\Sigma = -\frac{1}{2} \int d^4x \sqrt{-g} \varrho (g^{\mu\nu} \partial_\mu T \partial_\nu T + 1), \quad u_\mu = -\partial_\mu T, \quad (8.3)$$

with ϱ varied independently and future-directed u . Define

$$h_\mu{}^\alpha = \delta_\mu{}^\alpha + u_\mu u^\alpha, \quad (8.4)$$

$$E_{\mu\nu} = M_{\text{P}}^2 G_{\mu\nu} + \rho_\Lambda g_{\mu\nu} - T_{\mu\nu}^{\text{vis}},$$

Every smooth solution has $u^\mu u_\mu = -1$ and $T_{\mu\nu}^\Sigma = \varrho u_\mu u_\nu$. The Einstein equation therefore sets $E_{\mu\nu} = T_{\mu\nu}^\Sigma$. The spatial and mixed rest-frame projections vanish and with $\rho_\Sigma = \varrho$

$$E_{\mu\nu} = \rho_\Sigma u_\mu u_\nu, \quad (8.5)$$

$$\nabla_\mu (\rho_\Sigma u^\mu) = 0, \quad \rho_\Sigma u^\mu \nabla_\mu u_\nu = 0.$$

In the nonrelativistic cosmological limit let $x(t, q)$ be an injective comoving flow map with $x(t_i, q) = q$ and $J(t, q) = \det(\partial x / \partial q) > 0$ and peculiar velocity $v(t, x(t, q)) = a\dot{x}(t, q)$. Then its physical cold density is

$$\rho_\Sigma^{\text{free}}(t, x(t, q)) = \rho_{\Sigma, i}(q) \frac{a_i^3}{a^3 J(t, q)}. \quad (8.6)$$

On a periodic comoving cell a degree-one flow preserves the cell volume. If $\langle \rho_{\Sigma,i} \rangle = \bar{\rho}_{\Sigma,i}$ then $\langle \rho_{\Sigma}^{\text{free}}(t) \rangle = \bar{\rho}_{\Sigma,i}(a_i/a)^3$. Nonnegative initial ϱ remains nonnegative throughout the smooth flow domain. This is the irrotational dust action of Ref. [29] in the present convention. Variation gives unit norm and conservation with the stated stress. The contracted Bianchi identity gives the same continuity and geodesic equations. Along the comoving flow Jacobi's determinant formula gives $\dot{J} = J \nabla_x \cdot (v/a)$ and hence $d(a^3 \rho J)/dt = 0$. This proves (8.6). Change of variables over the cell gives the mean relation.

The endpoint density fixes the mean at the selected reference epoch. The action and initial perturbations determine its subsequent cold transport. Before characteristic crossing this also supplies local free-source data. Rest-frame pressure and anisotropic stress vanish within the dust realization. Adiabatic initial data are selected separately. The linear-propagation model consequently uses

$$T_{\mu\nu}^{\Sigma} = \rho_{\Sigma} u_{\mu} u_{\nu}, \quad \nabla_{\mu}(\rho_{\Sigma} u^{\mu}) = 0, \quad p_{\Sigma} = 0, \quad c_{s,\Sigma}^2 = 0, \quad \pi_{\Sigma} = 0.$$

The homogeneous expansion and thermal history with the acoustic scales are calculated with CAMB [30]. Table 3 reports the numerical results [15] for $\Omega_{\text{cdm}} := \Omega_{\text{cold}}$ and the corrected photon monopole with the three rest masses and $Y_p = 0.246082$ and $N_{\text{eff}} = 3.043873$. Recombination uses the standard atomic constants of the thermal solver. The background and distance calculation is independent of the primordial perturbation amplitude and requires no matter or angular power spectrum. The thermal realization uses the common temperature in (8.1) for each mass eigenstate. The propagated matter fraction is

$$\Omega_m = \Omega_b + \Omega_{\text{cold}} + \sum_i \Omega_{\nu_i} = 0.31490135.$$

Together with photons and the endpoint vacuum it closes the same flat budget. The neutrino energy integral retains its relativistic-to-massive transition throughout the background evolution.

In the Einstein–radial–cold subsystem with constant vacuum density a stationary condensate with $U(\varphi_*) = U'(\varphi_*) = 0$ and $U''(\varphi_*) = m_{\delta}^2 > 0$ has a positive physical scalar kinetic form on the branch defined by $\rho_{\Sigma} > 0$ and $Z_* > 0$ and $M_*^2 > 0$ with $H \neq 0$ and nonzero spatial wave number. The result “Physical scalar kinetics at constant condensate” in Ref. [11] gives the cold coefficient $D^{-1} > 0$ where $D = \rho_{\Sigma} \kappa_{\text{phys}}^2 + 3\rho_{\Sigma}^2/(2M_*^2)$ with $M_* = M_P$ in the constant Einstein normalization used here. This establishes the kinetic sign in the Einstein–radial–cold subsystem. The thermal distance calculation also retains photons and visible matter with the inputs listed above.

The endpoint densities determine the expansion history while the thermal and recombination evolution determines the drag sound horizon. Their combination fixes the distance ratios tested by baryon acoustic oscillations. The BAO comparison evaluates a Gaussian compressed-distance likelihood using the thirteen published DESI DR2 mean distance ratios and their covariance [31]. These quantities are inferred from fits to the acoustic feature with the broadband clustering parameters marginalized. We apply this likelihood to the Einstein–matter thermal background and sound-horizon calculation specified above and retain the DESI acoustic templates and covariance. This tests the combined

Table 3. Background and acoustic quantities calculated with CAMB 2.0.4 from the endpoint densities and thermal model [15]. The matter fraction includes the massive-neutrino contribution.

Quantity	Calculated value
Ω_m	0.314901
r_{drag}	146.946 Mpc
$100\theta_*$	1.041246
z_{eq}	3408.92

Table 4. DESI DR2 distance ratios [31] and the fixed endpoint prediction of this paper. Here $d_i = (t_i - y_i)/\sqrt{C_{ii}}$ with t_i the prediction and y_i the measurement and C the published covariance. The d_i are correlated diagnostics. The total statistic uses C^{-1} . The drag scale is $r_d = r_{\text{drag}}$.

z	Ratio	Measurement y_i	Prediction t_i	d_i
0.295	D_V/r_d	7.941676	8.054941	+1.489
0.510	D_M/r_d	13.587584	13.497310	−0.536
0.510	D_H/r_d	21.862947	22.737024	+2.038
0.706	D_M/r_d	17.350691	17.697134	+1.925
0.706	D_H/r_d	19.455349	20.167880	+2.134
0.934	D_M/r_d	21.575640	21.991042	+2.568
0.934	D_H/r_d	17.641495	17.568638	−0.362
1.321	D_M/r_d	27.600856	28.078062	+1.470
1.321	D_H/r_d	14.176022	14.065368	−0.493
1.484	D_M/r_d	30.511901	30.272103	−0.314
1.484	D_H/r_d	12.817000	12.880688	+0.123
2.330	D_H/r_d	8.631546	8.616842	−0.145
2.330	D_M/r_d	38.988974	39.179741	+0.359

map from endpoint densities through thermal evolution to the sound horizon and distance vector within the compressed likelihood. The local nonlinear source–potential response is evaluated separately. The prediction uses $H_0 = 67.45284 \text{ km s}^{-1} \text{ Mpc}^{-1}$ and r_{drag} fixed by the same calculation. The distances include photons, thermal neutrinos, cold matter and the vacuum. The distances are $D_M(z) = c_{\text{light}} \int_0^z dz'/H(z')$ and $D_H = c_{\text{light}}/H$ and $D_V = (zD_M^2 D_H)^{1/3}$. For the thirteen-component vector formed from D_M/r_{drag} and D_H/r_{drag} and D_V/r_{drag} the statistic is $\chi^2 = 29.5087$ for thirteen degrees of freedom with fixed-model tail probability 0.00554011 [15]. Table 4 resolves this test into the measured distance ratios. The diagonal standardized residuals locate the discrepancies in redshift and observable. Their correlations enter the joint statistic through the full covariance. In the supplied covariance the two-component blocks at $z = 0.706$ and $z = 0.934$ contribute 13.6414 and 6.9114 respectively to the total χ^2 . The transverse and radial residuals at $z = 0.706$ have the same sign whereas their measurement covariance is negative. This increases their joint discrepancy relative to the sum of their squared diagonal standardized residuals.

A covariance-weighted projection separates a common distance scale from the remain-

Table 5. Logarithmic responses of the acoustic scale and two distance ratios at the fixed background [15]. Each row varies one physical coordinate while the others are held fixed. Central differences use logarithmic step 10^{-3} . The temperature and matter density rows include a half-step check.

x	$E_{r_d,x}$	$E_{(D_M/r_d)0.706,x}$	$E_{(D_H/r_d)0.706,x}$
Y_p	-0.00504	+0.00504	+0.00504
ω_b	-0.129	+0.129	+0.129
ω_m	-0.257	+0.114	-0.0210
T_{CMB}	+0.157	-0.157	-0.158
N_{eff}	-0.102	+0.102	+0.102
H_0	$+2.77 \times 10^{-8}$	-0.715	-0.445
Common neutrino mass	+0.00112	-0.00111	-0.00111

ing redshift and transverse-radial dependence. With $W = C^{-1}$ define

$$\alpha_* = \frac{t^\top W y}{t^\top W t}, \quad \sigma_\alpha = (t^\top W t)^{-1/2}, \quad r_\perp = \alpha_* t - y. \quad (8.7)$$

Minimizing $(\alpha t - y)^\top W (\alpha t - y)$ gives $t^\top W r_\perp = 0$. Hence the statistic of the fixed prediction has the exact decomposition

$$\chi^2 = \underbrace{\frac{(1 - \alpha_*)^2}{\sigma_\alpha^2}}_{\chi_{\text{scale}}^2} + \underbrace{r_\perp^\top W r_\perp}_{\chi_\perp^2} = 15.2965 + 14.2122. \quad (8.8)$$

The data give $\alpha_* = 0.989282$ and $\sigma_\alpha = 0.002740$. Thus a common 1.0718% decrease in the distance ratios accounts for the scale component of the discrepancy. The orthogonal component has twelve residual degrees of freedom and a Gaussian tail probability 0.2874 when a common scale is fitted for this diagnostic. The physical prediction retains its fixed normalization $\alpha = 1$ and the thirteen-degree-of-freedom probability quoted above.

To resolve the thermal and density dependence of the statistic, define

$$E_{X,x} = \left. \frac{\partial \log X}{\partial \log x} \right|_{\text{other inputs}}, \quad \omega_b = \Omega_b h^2, \quad \omega_m = \Omega_m h^2, \quad h = H_0 / (100 \text{ km s}^{-1} \text{ Mpc}^{-1}).$$

Table 5 gives these partial responses at the fixed prediction. Each row holds the other coordinates fixed. Flatness sets the vacuum density and varying ω_b at fixed ω_m compensates the cold density. The helium response is weak and acts through the drag scale. Matter-density changes also modify the redshift dependence of the distances and can give opposite radial and transverse responses. These derivatives distinguish the physical directions to which the data are sensitive.

The seven central differences require fourteen thermal background solutions. The two half step checks require four more. The central background is reused. For r_d the differences between the two step sizes are 6.93×10^{-5} and 9.78×10^{-5} in the temperature and matter-density elasticities. They provide numerical consistency estimates for the displayed

derivatives. A variation of the underlying generated inputs must additionally compose these partial derivatives with their common dependence. In particular the shared mass relation between T_{CMB} and H_0 supplies a joint direction.

The common mass scale and the temperature constraint. Let s be a logarithmic displacement of the common cosmological neutrino mass normalization. Hold the mass ratios and τ and M_{P} fixed together with η_b and the proton mass and weak-sector inputs. The relations of Eq. (C.117) give

$$H_0(s) = H_0 e^{2s}, \quad T_\gamma(s) = T_\gamma e^s, \quad m_{\nu_i}(s) = m_{\nu_i} e^s, \quad \Omega_m(s) = \Omega_m, \quad \Omega_b(s) = \Omega_b e^{-s}. \quad (8.9)$$

The neutrino energy density scales as e^{4s} because m_{ν_i}/T_ν is fixed. The critical density has the same scaling and preserves the photon and neutrino fractions and the vacuum fraction. Baryon density instead scales as $\eta_b m_p T_\gamma^3$ and gives the final relation in (8.9). In the coordinates of Table 5 this direction is $(\delta \log H_0, \delta \log T_\gamma, \delta \log m_\nu, \delta \log \omega_m, \delta \log \omega_b) = (2, 1, 1, 4, 3)s$. It preserves the cosmological scale relations while testing their common normalization. The production endpoint remains fixed.

Proposition 8.1 (Finite common-scale response of the thermal background). Consider the flat background with constant vacuum density and thermal Fermi–Dirac neutrinos of fixed distribution shape together with pressureless baryons and cold matter. Apply (8.9) at fixed η_b and m_p and N_{eff} with the remaining content factors held fixed. Compensate the cold density so that the total baryon–cold density scales as e^{4s} . On the interval where this cold density remains positive

$$E(z; s) = E(z; 0), \quad \frac{(D_X/r_d)(z; s)}{(D_X/r_d)(z; 0)} = \alpha(s) := e^{-2s} \frac{r_d(0)}{r_d(s)}, \quad X = M, H, V. \quad (8.10)$$

Let $a_d(s)$ be the drag scale factor and write the sound-horizon integral as

$$\begin{aligned} r_d(s) &= e^{-2s} I_s(a_d(s)), \\ I_s(a_d) &= \int_0^{a_d} \frac{c da}{\sqrt{3} a^2 H(a; 0) \sqrt{1 + e^{-s} \mathcal{R}_b(a)}}, \\ \mathcal{R}_b(a) &= \frac{3\rho_b(a; 0)}{4\rho_\gamma(a; 0)}. \end{aligned} \quad (8.11)$$

For fixed $a_d = a_d(0)$ the common multiplier obeys

$$-\frac{1}{2} \max(s, 0) \leq \log \alpha(s) \leq -\frac{1}{2} \min(s, 0). \quad (8.12)$$

Proof. At each redshift m_{ν_i}/T_ν is unchanged. The integral in (8.1) and the photon density scale as e^{4s} . The vacuum density has the same scaling. The baryon density scales as e^{3s} whereas the compensating cold contribution makes their sum scale as e^{4s} . Both components evolve as a^{-3} in this background approximation. The Friedmann equation therefore gives $H(a; s) = e^{2s} H(a; 0)$. Radial and transverse distances scale as e^{-2s} as does their volume-distance combination. This proves (8.10). The baryon loading is $e^{-s} \mathcal{R}_b(a)$ and gives (8.11).

At fixed upper limit the logarithmic derivative of its integrand is $e^{-s}\mathcal{R}_b/[2(1 + e^{-s}\mathcal{R}_b)] \in [0, 1/2]$. Positivity makes $\partial_s \log I_s$ a weighted average in the same interval. Integrating in s proves (8.12). $\square$

Composing the calculated derivatives gives $\partial_s \log(D/r_d) \simeq -0.74496$ for all thirteen entries. For $d_i = \partial_s t_i$ the covariance-weighted tangent optimum is $s_* = -d^\top C^{-1}(t - y)/(d^\top C^{-1}d) = 0.01439$ with tangent standard deviation 0.00368. The independent monopole measurement $T_\gamma = 2.72548 \pm 0.00057 \text{ K}$ [32] constrains the same variable to $s = (2.21 \times 10^{-6}) \pm 2.0914 \times 10^{-4}$. Within its three-standard-deviation interval the linearized distance change is at most 4.69×10^{-4} and the smallest BAO statistic is 28.20. Thus the allowed common-normalization displacement is insufficient to remove the distance tension. The extrapolated s_* locates the required displacement within this tangent approximation. Combining the monopole and BAO covariance blocks at the fixed prediction gives $\chi^2 = 29.50881$ for fourteen components and $p = 0.00891$ assuming independent measurement errors. The thirteen-component statistic uses BAO alone. The fourteen-component statistic adds the CMB-monopole block. The one-parameter BAO scale projection in Eq. (8.8) assigns 15.2965 to that direction and 14.2122 to its orthogonal residual. The physical background and all other generated parameters remain fixed.

The finite scale relation tests the required thermal response. The exact image of the monopole three-standard-deviation interval is

$$s \in \left[\log \frac{T_{\text{obs}} - 3\sigma_T}{T_\gamma}, \log \frac{T_{\text{obs}} + 3\sigma_T}{T_\gamma} \right] = [-0.000625400, 0.000629425]. \quad (8.13)$$

For any common multiplier the original covariance gives exactly

$$\chi^2(\alpha) = \chi_\perp^2 + \frac{(\alpha - \alpha_*)^2}{\sigma_\alpha^2}. \quad (8.14)$$

The level $\chi^2 \leq \chi_{13,0.95}^2 = 22.3620$ therefore requires $\alpha \in [0.981459, 0.997105]$. This interval identifies the common multipliers that reach the stated reference level of the fixed-model statistic. At fixed drag epoch (8.12) instead gives $\alpha \in [0.999685, 1.000313]$. Using the unrounded enclosure gives the conservative bound $\chi^2 \geq 28.6237$.

For an evolving drag epoch define $J_s = I_s(a_d(s))/I_s(a_d(0))$. Equations (8.10)–(8.12) give

$$\log \alpha(s) = -\log \frac{I_s(a_d(0))}{I_0(a_d(0))} - \log J_s.$$

Consequently reaching the above 95% reference level within (8.13) requires $\log J_s \geq 0.00258406$ at the selected displacement. Reaching α_* requires $\log J_s \geq 0.0104607$. Equivalently any differentiable response reaching that reference level has $\sup |\partial_s \log \alpha| \geq 4.60543$ between the fixed branch and the selected point. Relative to the calculated central slope -0.744959 its finite logarithmic remainder must be at least 0.00242988 in magnitude. These necessary response sizes follow from the finite temperature interval and measurement covariance. The actual change of a_d is determined by the recombination and drag evolution. Table 5 checks the central slope by finite differences. The finite remainder is the integrated departure from that slope over the allowed interval.

Finite recombination and drag response. At fixed expansion and baryon loading with helium abundance held fixed the remaining thermal dependence can be studied directly through the drag optical depth. Let $R_b(z) = 3\rho_b/(4\rho_\gamma) = R_{b0}/(1+z)$ and $n_e(z) = n_{H0}(1+z)^3 x_e(z)$ and define

$$\mathcal{T}_d(z) = \int_0^z \frac{c_{\text{light}} \sigma_T n_{H0} (1+z')^3 x_e(z')}{R_{b0} H(z')} dz', \quad \mathcal{T}_d(z_d) = 1, \quad r_d = r_s(z_d). \quad (8.15)$$

Here $r_s(z) = \int_z^\infty c_s(z')/H(z') dz'$ is the comoving sound horizon. The reference calculation uses the recombination electron history in this integral with astrophysical reionization excluded from the definition of its drag epoch.

Finite optical-depth propagation. Let $\mathcal{T}_0$ be continuous and strictly increasing across the drag-root interval and let r_s be positive and decreasing there. Suppose a second positive-opacity history obeys

$$|\mathcal{T}_d(z) - \mathcal{T}_0(z)| \leq \epsilon \mathcal{T}_0(z) + A, \quad 0 \leq \epsilon < 1, \quad 0 \leq A < 1.$$

Then its drag root lies between the reference inverse-depth levels

$$z_- = \mathcal{T}_0^{-1}\left(\frac{1-A}{1+\epsilon}\right), \quad z_+ = \mathcal{T}_0^{-1}\left(\frac{1+A}{1-\epsilon}\right), \quad r_s(z_+) \leq r_d \leq r_s(z_-). \quad (8.16)$$

At fixed $H(z)$ the thirteen distance ratios therefore have the common multiplier $\alpha = r_{d,0}/r_d$. Their statistic over this finite set is exactly (8.14). Its minimum is attained at the projection of α_* onto the allowed interval and its maximum at an endpoint. At the new root the assumed inequality gives $(1-\epsilon)\mathcal{T}_0 - A \leq 1 \leq (1+\epsilon)\mathcal{T}_0 + A$. Monotonicity gives (8.16). The distance numerators are fixed and the final statement follows from the positive quadratic coefficient in (8.14).

There is an independent electron-number bound for the low-redshift part of this problem. Keep the electron history fixed above z_ℓ and allow either history below it to satisfy $0 \leq n_e \leq n_{H0}(1+2f_{\text{He}})(1+z)^3$. Nonnegative radiation and neutrino densities together with nonnegative vacuum density imply $H(z) \geq H_0\sqrt{\Omega_b + \Omega_{\text{cold}}}(1+z)^{3/2}$. Integration then gives the finite absolute bound

$$|\Delta \mathcal{T}_d| \leq A_\ell := \frac{2c_{\text{light}} \sigma_T n_{H0} (1+2f_{\text{He}})}{5R_{b0} H_0 \sqrt{\Omega_b + \Omega_{\text{cold}}}} [(1+z_\ell)^{5/2} - 1]. \quad (8.17)$$

For $z_\ell = 30$ outward interval evaluation of this analytic expression at the fixed background gives $A_{30} \leq 0.01132773059844881$. Numerical inversion of the reference depth then gives $r_d \in [146.86995, 147.02327]$ Mpc and $\chi^2 \in [28.0490, 31.0265]$. This result bounds changes between all electron histories satisfying the stated electron-capacity bound in that low-redshift interval with the background and higher-redshift history held fixed.

The higher-redshift recombination response is evaluated with the full radiative-transfer solver supplied with HYREC-2 [33, 34]. The fixed background and thermal responses are preserved in the archived calculation [35]. Both integrations use the same tabulated $H(z)$ and hydrogen and helium number densities together with the photon temperature and

$r_s(z)$ of the reference background. One uses standard atomic constants. The other uses the generated $\alpha(0)$ and electron pole mass $m_e := M_e = G_e m_e^{\text{geom}}$. The displayed calculation uses $G_e = 1$ in Eq. (7.1) and gives the ratios

$$\frac{\alpha(0)}{\alpha_{\text{at}}} = 0.9999247538, \quad \frac{m_e}{m_{e,\text{at}}} = 0.9987383562,$$

$$\frac{\sigma_T}{\sigma_{T,\text{at}}} = \left(\frac{\alpha(0)/\alpha_{\text{at}}}{m_e/m_{e,\text{at}}} \right)^2 = 1.0023772036.$$

The reference atomic normalization is the declared standard-constant convention of the thermal calculation. The ratios enter the atomic rates and matter-temperature evolution together with the drag opacity. The generated values are fixed before the distance comparison.

Thermal calculation	z_d	r_d (Mpc)	χ^2
Reference RECFAST background	1060.219424	146.946194	29.5088
HYREC with standard atomic constants	1060.217303	146.946386	29.5051
HYREC with generated $\alpha(0)$ and m_e	1058.799882	147.074547	27.1191

Table 6. Finite recombination responses at the archived thermal-study background [35], with the expansion history, helium abundance and baryon loading held fixed. Every statistic uses the original thirteen-component observational covariance with no adjusted distance parameter.

The generated-atomic calculation has common distance multiplier 0.9991272939 and reference tail probability $p = 0.0119820$. Applying (8.17) also to this history with its corresponding Thomson cross section gives $r_d \in [146.99810, 147.15186]$ Mpc and $\chi^2 \in [25.7794, 28.5175]$. Thus the directly calculated atomic displacement together with the entire low-redshift electron capacity leaves the statistic above the 95% reference level. This is a finite test of these thermal effects with the upstream background held fixed.

The analytic capacity bound controls the allowed low redshift histories. Numerical integration and atomic model accuracy are assessed separately. The depth quadrature error estimates in the two direct integrations are below 1.6×10^{-11} . These estimates concern evaluation of the selected recombination equations. The physical remainder comprises the omitted atomic and radiative-transfer terms above $z = 30$ and joint changes of the upstream baryon and helium inputs and the thermal inputs. The displayed bounds quantify the low-redshift electron-history response within the retained atomic model at fixed upstream inputs. The difference between recombination implementations is a numerical comparison while a theory covariance requires a probability model for the physical remainder. The retained calculation reuses the background distance integrals and contains only the background and acoustic quantities needed for this test [15].

At fixed $E(z) = H(z)/H_0$ all three distance ratios scale as $(H_0 r_d)^{-1}$. The scale projection therefore tests this common combination. Changes in the redshift dependence of $E(z)$ also enter $r_\perp$ while the thermal history determines both r_d and the neutrino contribution to $E(z)$. Attribution to the Hubble scale or the sound horizon or the thermal mapping requires their joint physical variation. The present discrepancy constrains the

complete chain from the endpoint densities and rest masses through the photon and neutrino thermal history, cold transport and the flat Einstein background to the drag scale and observed distances. Its empirical assessment uses the published measurement covariance. Any additional theory covariance requires a separately derived input uncertainty or approximation error for that chain. For independent measurement and theoretical errors the joint covariance is $C_{\text{tot}} = C_{\text{obs}} + C_{\text{th}}$. The comparison implementation [15] accepts such an independently specified C_{th} with its source and approximation order. The associated χ^2 probability uses a zero-mean Gaussian likelihood for the combined errors. The values quoted here use C_{obs} .

8.6 The gravitational sector

The dimensional anchor fixes $G_N = 1/(8\pi M_P^2)$ in the macroscopic metric action [11]. The endpoint source variance and positive RMS amplitude response determine the instantaneous potential for specified source and boundary data. The joint matter–potential action then determines its feedback on the transported source. The auxiliary-radius model follows internal compression, occupied matter and external expansion, with the matter phase supplying a clock on its timelike branch. The homogeneous Einstein–matter background supplies the thermal evolution and distances calculated above. The finite responses of the five source channels use spectral variables expressed in Planck units.

At emergence $L_{\text{phys}} = kL/M_G$. The long-root normalization uses $C_2(2, 1) = 2[2(2 + 1) + 1(1 + 1)] = 16$ and gives

$$\lambda_{\text{long}} = \frac{16}{L_{\text{phys}}^2} = \frac{16M_G^2}{(kL)^2}.$$

Express the channel inputs in Planck units

$$\hat{L} = M_P L_{\text{phys}}, \quad \hat{k} = M_G/M_P, \quad \hat{V}_3 = \pi^2 \hat{L}^3, \quad \hat{\lambda}_{\text{long}} = \lambda_{\text{long}}/M_P^2 = \frac{16}{\hat{L}^2}.$$

Let $\hat{\Pi}_a^2(\hat{L}, \hat{k}^2, \tau)$ denote the dimensionless channel functional with its eigenvalues and curvature shifts together with frequency and Gaussian cutoff expressed in these same units. The five channels are the spin-2 and spin-0 responses of $T_{\mu\nu}$ together with F^2 and G^2 and J^μ . The dimensionally consistent normalization of their combined response is

$$\hat{\mathcal{S}} = \left| \sum_a \hat{V}_3 \hat{\Pi}_a^2(\hat{L}, \hat{k}^2, \tau) \right|, \quad \hat{\sigma} = \hat{\mathcal{S}} \hat{k}^2, \quad \sigma = M_P^2 \hat{\sigma}.$$

The associated dimensionless ratio and effective coupling are

$$r_{\text{read}} = \frac{\hat{\sigma}}{\hat{\lambda}_{\text{long}}} = \frac{(kL)^2}{16} \hat{\mathcal{S}}, \quad Z_{\text{read}} = \frac{1}{1 + r_{\text{read}}}, \quad G_{\text{read}} = \frac{G_N}{Z_{\text{read}}}.$$

For a finite channel sum $r_{\text{read}} \geq 0$ and $0 < Z_{\text{read}} \leq 1$. Evaluating the five channels in this common convention gives $r_{\text{read}} = 85.9303$ and $Z_{\text{read}} = 0.0115035$ [15]. The resulting G_{read} is the effective coupling associated with this channel sum. The metric action uses G_N fixed

by the dimensional anchor. Appendix B specifies the numerical evaluation. The sectoral quantities $p_{\text{TT}}^2 = 8/L_{\text{phys}}^2$ and $\lambda_{\text{TT}} = 14/L_{\text{phys}}^2$ retain their respective definitions.

On the matched Einstein–matter branch the positive tensor residue is normalized by the same Newton constant that fixes the dimensional scale. This macroscopic normalization also enters the local source response and its dynamical realizations.

Local source and potential. The invariant covariance on the decomposable pure-occupation orbit gives the angular factor $4/3$ in the acceleration variance. Direction averaging cancels the mean source and retains its mean-square amplitude. The normalized Gaussian susceptibility Z_{IR} compares squared response amplitudes; its positive RMS amplitude $\sqrt{Z_{\text{IR}}}$ gives the spatial flux. The reference endpoint fixes a_0 together with H_0 , and this normalization remains fixed during the subsequent local evolution. For the positive acceleration scale a_0 with $y = |\nabla\Phi|/a_0$ and $\mu(y) = y/\sqrt{1+y^2}$ this law determines a Bekenstein–Milgrom potential [12]. On a bounded Lipschitz domain the static weak solution is unique for Dirichlet data in H^1 and a source in H^{-1} . The same potential accelerates pressureless matter. Its evolving density feeds back into the nonlinear Gauss law. The joint source–potential action gives this feedback together with continuity. On a fixed smooth domain its energy balance accounts for matter flux and boundary work throughout smooth evolution before caustic formation. Impermeable matter boundaries and time-independent Dirichlet data conserve the total energy. These results are derived in Eq. (G.2) and Propositions G.1 and G.2.

The finite plane-symmetric realization evolves the same nonlinear field and material source with mass conservation. Halving the time step reduces the energy-balance residual by approximately a factor of four consistent with second-order time accuracy. At each time transport supplies the free density and the potential determined by that source and its boundary data supplies the polarization density. The latter contributes to the regional mean through its boundary flux. This density decomposition complements the cold-source transport and homogeneous averaging of Section 8.5. Its explicit relation is Eq. (G.8).

Matter response and the auxiliary collective radius. The canonical matter action determines matter evolution under a time-dependent internal conformal source. Internal geometry fixes the mode frequencies and kinetic norms that enter the macroscopic matter fields. The action’s positive kinetic form reproduces the static spectral frequencies and occupied source charge. The phase-space second moment obeys an exact source-work identity. The corresponding causal response retains the direct variation and both frequency branches. It supplies the matter contribution to a common constrained gravitational calculation. Covariant variation gives the complete matter stress tensor and, on the matter equations, the exchange $Q_a \nabla_\nu s^a$ where s^a are the retained sources and Q_a their matter responses. This exchange enters the collective source equation and the lapse and shift constraints together. Uniform compression has an exact lapse representation with a time-dependent dilation coupling. Its mixed velocity form retains the canonical source constraint. The derivation is given in Proposition G.3 and Eqs. (G.22)–(G.23) and Eqs. (G.30)–(G.31).

Allowing the inherited complete unweighted relative potential to vary in both amplitude and radius gives the unique joint vacuum radius $L_* = \sqrt{2}L_c$ where L_c is the conden-

sation threshold and a positive joint Hessian. This supplies a collective static restoring response for the common source balance. Retaining the radius as an auxiliary collective variable gives a closed covariant amplitude–matter model with occupied and expanding spatially flat FLRW solutions. These initial data satisfy the source balance and Einstein constraints. Their nonzero matter phase charge distinguishes them from the joint static vacuum. The source constraint balances the restoring force against the occupied kinetic and potential densities. As expansion changes this balance, the radius adjusts together with the matter fields. The internal radius sets the spectral scales while the external scale factor sets the physical volume occupied by the homogeneous matter. Their coupled evolution follows from this common balance and the Einstein constraints. Positivity of the algebraic source-constraint derivative $\mathfrak{h} > 0$ fixes this adjustment as a locally unique smooth function of the retained fields and their first derivatives. For the constructed occupied data the initial compression rate is strictly positive. After the common constraints are eliminated the three physical scalar kinetic and gradient forms are positive on a nonzero evolution interval of this occupied background alongside the two Einstein tensor polarizations. The global rotation charge of the retained real pair is conserved and its matter phase reads proper time on this same background. The static result and the occupied construction are proved in Propositions C.6 and G.5. The finite-interval bounds are given in Eq. (G.51).

For a retained matter mode of nonzero amplitude with phase ϑ and $X = -(\nabla\vartheta)^2 > 0$ choose the future-directed phase congruence. The matter equations determine its compression force and its radial derivative corrections. The homogeneous and static limits give geodesic flow and acceleration by the source mass gradient respectively with explicit error estimates. Along each flow curve $ds = d\vartheta/\sqrt{X}$ relates proper time to the matter phase. Phase increments count the matter oscillation and the instantaneous frequency $\sqrt{X}$ calibrates each increment into a time interval. Monotone changes of the evolution parameter assign different clock scales to this same physical trajectory and preserve its canonical evolution and source-work identity. A unit dust clock exists on a contractible spacetime patch when the one-form $X^{-1/2}d\vartheta$ is closed or equivalently when the phase congruence is geodesic. These relations follow from Proposition G.4. As in Section 2 generated dynamics follow by varying the action on its admissible variables and reducing their common constraints. Finite-window kinetic tests use the complete physical form and its error estimate.

Independent-foliation realization. For comparison, an effective action with an independent foliation field gives a realization of the local constitutive law and the Einstein–matter cosmological branch. A smooth geodesic foliation recovers the Einstein–matter equations while the static foliation with vanishing shift gives the RMS law at leading weak-field nonrelativistic order. On a spatially flat dust–vacuum background with positive dust density and nonzero Hubble rate together with constant vacuum density and a stationary radial condensate of positive mass squared the scalar constraint reduction has positive kinetic coefficients for nonzero Fourier modes. A conserved relative-foliation charge contributes to the total cold density without changing its linear metric response (Proposition G.6).

On accelerated static backgrounds the leading slow-motion reduction instead gives the

velocity operator $2M^2(DL^{-1}D - D)$ where M is the Einstein normalization and $D = -\Delta$ and $L = -\nabla \cdot [D\mathcal{J}(\nabla\bar{\Phi})\nabla]$ with $\mathcal{J}(g) = \mu(|g|/a_0)g$ and fixed Dirichlet perturbations. For a constant background and a nonzero longitudinal wavevector the velocity form is negative when $|\nabla\bar{\Phi}|/a_0 > 0.786151377\dots$. Its restriction to supplied foliation directions and their spherical support are described in Proposition G.7 and the radial specialization (G.65). These results describe the independent foliation realization on the two stated backgrounds. The cosmological distance calculation of Section 8.5 uses the homogeneous thermal background and linear Einstein–matter propagation specified there.

9 QCD scale and confinement

The colour coupling of Section 5.3 sets the QCD scale used for the proton and glueball channels, string tension and deconfinement temperature. Their absolute values test the conversion of the running coupling into a hadronic scale. Ratios in which this scale cancels test the content and channel assignments.

The weak-process chain brings these hadronic quantities together with the flavour and electroweak sectors. The light-quark mass difference and electromagnetic subtraction determine Δm_{np} in the native isovector readout with $g_S^{\text{eff}} = 1$. This mass difference sets the decay phase space, while G_F and V_{ud} with the axial response set the weak amplitude. The same gap and weak rate enter freeze-out. Neutron survival to the adopted capture temperature then determines the reduced helium estimate. The lifetime and abundance probe the common source through the decay phase space and thermal neutron population, respectively. Their joint response distinguishes changes in the hadronic mass difference from changes in the weak amplitude. Appendix F evaluates this colour-to-weak chain and retains the common source and thermal assumptions.

9.1 The QCD scale and the proton mass

The QCD scale follows by propagating the colour coupling $g_3(M_G)$ of Section 5.3 to the internal extraction scale and applying the five-flavour two-loop scale formula. The Yukawa-difference correction rescales the extracted coupling α_s by $(1 - s_0\kappa/N_g)$ before the nonlinear conversion to Λ_{QCD} with $s_0 = 2\tau$ and $N_g = 8$. The geometric seed $y_0 = 1$ is held fixed in the running. Appendix C.15 specifies the extraction point and the nonlinear scale map. Appendix D.3 gives their error propagation. The result is

$$\Lambda_{\text{QCD}} = 0.2080 \text{ GeV},$$

a deviation of -0.94% from the reference 0.210 GeV. The adopted constituent-content relation for the proton mass is

$$m_p = N_c \Delta_f \left(1 - \frac{1}{N_g^2 \Delta_s}\right) \Lambda_{\text{QCD}} \left(1 + \tau\kappa \sum Y^2 \Delta_s\right) = \frac{279}{64} \Lambda_{\text{QCD}} \left(1 + \frac{5\tau\kappa}{3}\right) = 0.9411 \text{ GeV},$$

where $N_c = 3$ and $\Delta_f = 3/2$ and $\Delta_s = 1/2$. The conformal-content correction is $1 - 1/(N_g^2 \Delta_s) = 31/32$. The final bracket is the effective hadronic response factor. Appendix C.15 evaluates both factors. The value differs from the observed proton mass by $+0.30\%$.

The neutron–proton splitting probes the response to an isovector source. The difference $m_d - m_u$ varies at fixed mean light-quark mass. The scalar response measures the resulting difference between the two baryon energies. The native normalization uses $g_S^{\text{eff}} = 1$ with the electromagnetic proton-minus-neutron contribution subtracted in the same source convention. The colour singlet and its allowed source insertions are explicit in Proposition C.7. Eliminating intermediate channels retains their contribution to the baryon source response and the full-state normalization in (C.140).

9.2 Glueball mass channels

Composite sources are products of retained source operators resolved into definite symmetry channels. The mass readout assigns a scale to each channel using the same mode algebra as the elementary couplings and collective restoring responses. The composite-channel readout pairs two internal transverse-vector sources. Their lowest vector modes belong to the Killing carrier of Ref. [4]. The two-leg contribution is common to the channel family while projection onto total rotation representations resolves its scalar and tensor responses. The spectral assignment is

$$\hat{\lambda} = 2\hat{\lambda}_{\text{gluon}} + C_2(J) + n N_g \xi,$$

where $\hat{\lambda}_{\text{gluon}} = 4$ and $N_g \xi = 1$ are dimensionless. The reported scalar reference and mass-ratio rule are

$$m_G = 8\Lambda_{\text{QCD}} = 1.664 \text{ GeV}, \quad m_a = m_G \sqrt{\hat{\lambda}_a/8}.$$

The conversion from dimensionless spectral labels to squared masses is $m_a^2 = 8\hat{\lambda}_a\Lambda_{\text{QCD}}^2$. Here the scalar reference $\hat{\lambda}_* = 8$ fixes the common mass normalization through $m_G/\Lambda_{\text{QCD}} = \hat{\lambda}_*$. The absolute scalar mass therefore tests the reference-to-mass conversion together with the extracted QCD scale. Ratios of channel masses cancel that normalization and test the relative spectral assignments. Appendix C.15 gives this operator mapping and checks the two-Killing-leg origin of the normal reference levels. The scalar reference and the selected tensor projection have labels 8 and 16 in the symmetric square of that carrier. The pseudoscalar and excited scalar use the additional conformal-channel assignments 17 and 18. Each mass therefore combines a resolved channel label with the same QCD scale. The selected channel ratios are

$$2^{++}/0^{++} = \sqrt{2}, \quad 0^{++*}/0^{++} = 1.50, \quad 0^{-+}/0^{++} = 1.46.$$

The tensor mass is 2.354 GeV. The first conformal-excitation mass is 2.496 GeV and the pseudoscalar mass is 2.426 GeV. For the pseudoscalar Chen et al. obtain 2560(35)(120) MeV in quenched lattice QCD [36, Table XXI]. The BESIII partial-wave analysis determined $J^{PC} = 0^{-+}$ for $X(2370)$ in 2024 [37]. The combined analysis of three radiative- J/ψ channels gives 2359_{-14}^{+13} MeV [38]. Our value lies 5.2% below the former central value and 2.8% above the latter.

The BESIII study of suppressed $K^*\bar{K}$ decays argues for a dominant pseudoscalar-glueball component in $X(2370)$ [39]. This motivates comparison with the channel mass

above. Quenched Yang–Mills calculations provide the unmixed reference spectrum [36, 40]. Comparison with a physical resonance also involves meson–glueball mixing [41]. The four channels considered here specify masses before such mixing. Their overall scale is fixed by Λ_{QCD} with tensor-to-scalar ratio $\sqrt{2}$. The complete condensate operator of Section 6.2 has its own kinetic and source normalization.

9.3 String tension and deconfinement

The string tension and deconfinement temperature are assigned to the effective tensor and lowest vector channels respectively. They give

$$\sigma = \frac{\lambda_{\text{TT}}}{\pi} \Lambda_{\text{QCD}}^2 = \frac{14}{\pi} \Lambda_{\text{QCD}}^2 = 0.1929 \text{ GeV}^2,$$

where $\lambda_{\text{TT}} = 14$ is the effective coefficient of the three-branch tensor catalogue in this string-tension relation. The rough-TT spectrum in Ref. [4] has minimum $6/L^2$ on its two-branch carrier. The deconfinement temperature is

$$T_d = \frac{\lambda_{\text{vector}}}{N_c} \Lambda_{\text{QCD}} (1 - \tau\kappa) = \frac{4}{3} \Lambda_{\text{QCD}} (1 - \tau\kappa) = 271.1 \text{ MeV},$$

where $\lambda_{\text{vector}} = 4$ is the lowest dimensionless transverse-vector label. The factor $(1 - \tau\kappa)$ assigns the chiral-squash correction of Section 3.4 to the deconfinement scale. The ratio

$$\frac{\sigma}{T_d^2} = \frac{14}{\pi} \cdot \frac{9}{16} (1 - \tau\kappa)^{-2} = \frac{126}{16\pi(1 - \tau\kappa)^2} = 2.6242$$

is independent of the common QCD scale. The content ratio $126/(16\pi) = 2.5068$ is enhanced by the inverse chiral-squash factor $(1 - \tau\kappa)^{-2}$. Appendix C.15 evaluates these confinement-scale relations.

10 Numerical results and comparison

Gauge couplings test channel normalization, boundary matching and scale evolution; mass combinations test relative amplitude transfer and finite matching. The neutron lifetime and helium abundance combine the hadronic and flavour responses with weak reaction rates. Cosmological distances test the thermal history, sound horizon and expansion mapping jointly. The benchmarks in Appendix A include measurements, extrapolated couplings and estimates of channel masses, each with its stated comparison convention.

The comparisons distinguish dimensionless structural relations from dimensional values normalized by G_N and joint tests of outputs sharing common inputs. Each benchmark retains its measurement definition and approximation level. Numerical errors, perturbative remainders and reference uncertainties have separate budgets.

The signed percentage residual is $100(X_{\text{calc}}/X_{\text{ref}} - 1)$ evaluated before display rounding. Outputs sharing the same endpoint or mass scale form correlated comparison channels. Joint statistical tests use the covariance and likelihood appropriate to the experimental data and the stated theoretical approximation.

For charged leptons the conversion (7.1) fixes the object tested at each stage. The pole invariant (7.7) constrains the factors G_ℓ whereas the common-scale constraints (7.9) involve their products $C_\ell = D_\ell G_\ell$.

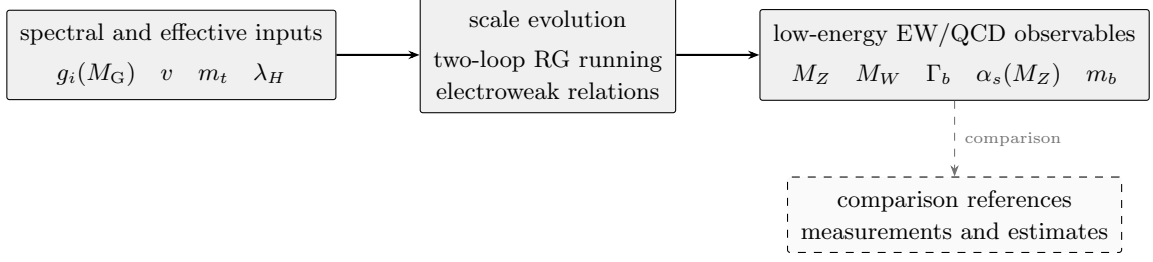


Figure 3. Spectral boundary data and two-loop running together with low-energy observables in the fixed-Yukawa calculation of Section 10.1. The electroweak relations and their perturbative orders are given in Section 10.6. The dashed branch introduces the reference values.

Table 7. Inputs and outputs of the three calculation procedures. The gauge normalization is $g_1 = \sqrt{5/3} g_Y$.

Procedure	Boundary or physical conditions	Result
Fixed-Yukawa evolution	Spectral $g_i(M_G)$ with fixed $y_0 = 1$ and geometric quartic and hierarchy VEV	Two-loop gauge evolution and the sector readouts in Tables 14 and 10.
Five-variable evolution	A common-scale tuple $(g_1, g_2, g_3, y_t, \lambda_H)$ and threshold maps	Joint scale dependence of all five couplings for that boundary.
Finite one-loop matching	Generated G_F, M_t, M_H and leading gauge coefficients at $Q = M_Z$	Tadpole-free $\overline{\text{MS}}$ $v_L, \bar{y}_t, \bar{\lambda}, \bar{m}^2$ with the real physical conditions preserved through one loop.

10.1 Scale evolution and finite matching

The spectral and effective relations supply $g_i(M_G)$ (Section 5) and $y_0 = 1$ (Section 7.2) together with λ_H (Appendix C.12) and v (Section 6.3). The geometric Yukawa and quartic are retained as fixed readouts distinct from the running variables in the adopted two-loop transfer of Appendix D.6. The flow ends at the self-consistent internal Z mass of Section 10.6. Figure 3 shows how these inputs enter the low-energy comparison.

The common action realizes the generated boundary relations in the mass terms, interaction vertices and reaction amplitudes of the same normalized fields. Finite matching expresses these responses in the adopted mass and renormalization scheme. The top decay tests the same generated parameters in an additional reaction amplitude. Table 7 summarizes the three uses of this action. In the fixed-Yukawa calculation the gauge couplings evolve while the geometric Yukawa and quartic determine the tabulated hierarchy. The five-variable system instead evolves all five couplings from common boundary data. Finite matching fixes the renormalized coefficients from the generated Fermi and mass conditions at one scale.

Table 8. The sectors of the computation and one representative value per sector with short comparison descriptors. Logical roles and comparison tiers are distinguished in Section 2.4. The full table is Appendix A.

Sector	Representative quantity	Computed value	Comparison status
Internal scale	generation count n_g	3	joint screening
Gauge couplings	weak coupling $g_2(M_G)$	0.50885	calculated
Flavour	top mass m_t	174.1 GeV	calculated
Electroweak	vacuum expectation v	246.19 GeV	calculated
Electroweak precision	Z mass M_Z	91.125 GeV	calculated
Cosmology	Hubble rate H_0	67.45 km s ⁻¹ Mpc ⁻¹	exploratory
QCD	proton mass m_p	0.9411 GeV	spectral est.

An additional decay observable. The imaginary part of the same top self-energy determines an output beyond the real conditions used in the matching. With $m_b = 0$ and $|V_{tb}| = 1$ Eq. (C.68) gives

$$\Gamma_t^{(0)} = \frac{G_F M_t^3}{8\pi\sqrt{2}}(1 - x_W)^2(1 + 2x_W) = 1.53445 \text{ GeV}, \quad x_W = \frac{(g_2 v_F/2)^2}{M_t^2}. \quad (10.1)$$

The charged-current interaction and all masses in this expression belong to the leading action used in the one-loop pole kernel. Its W mass is $g_2 v_F/2 = 79.79153 \text{ GeV}$. This is the Born decay width obtained from the one-loop pole’s absorptive part [1, 26]. The experimental compilation gives $1.42^{+0.19}_{-0.15} \text{ GeV}$ [1]. The width tests a decay channel whose value was not imposed in the real-part matching. A precision comparison includes the relative QCD and electroweak corrections of orders α_s/π and α/π together with finite bottom mass and physical charged-current mixing. These corrections set the accuracy beyond the Born approximation.

10.2 Couplings, masses and comparison conventions

Appendix A lists each computed value with its comparison reference and signed residual and defining relation. Table 8 gives one representative per sector. The tiers in Table 9 identify the approximation and interpretation of each comparison. Their meaning is independent of the numerical error or statistical confidence.

10.3 Shared constraints and numerical residuals

The relations between sectors provide tests that survive elimination of a common dimensional scale. A change of one source response propagates to every quantity that uses it. Joint observations constrain both the response and its relative contribution to different sectors. Write $D = 2\pi + r_{23}$ and $A_\gamma = C_\gamma(r_{12}/\pi)(1 - \tau\Delta_s)$ with $x = s_0\kappa(s_0)$ and $Y_u = 2/3$ in terms of the neutrino ratios and thermal factors with the hypercharge used in Section 8. The cosmological pair obeys

$$\Omega_\Lambda = \frac{D}{3\pi}, \quad \frac{(k_B T_{\text{CMB}})^2}{M_P H_0} = A_\gamma^2 \sqrt{\frac{D}{\pi Y_u}} \frac{(1-x)^2}{\sqrt{1-4x}}. \quad (10.2)$$

Table 9. Comparison tiers. Tier A identifies the anchor and selected structure. Tier B covers couplings and masses. Tier C records scheme-sensitive quantities. Tier D contains exploratory extensions.

Tier	Content	Reading
A	anchor and structure	Newton’s constant and inherited structural constants with the generation count selected by the joint rules. Fixed before the low-energy evaluation
B	couplings and masses	Gauge couplings and mass ratios with v and the electroweak and charged-fermion masses. Neutrino mass scale and rest-mass splittings
C	scheme-sensitive	m_b and Λ_{QCD} with Γ_b and $\sin^2 \theta_{\text{eff}}$. Consistency checks labelled by their perturbative and renormalisation status
D	exploratory	Unit-amplitude solar scalar source. PMNS and CP assignments with baryon asymmetry. Cosmological fractions and T_{CMB} with hadronic scales and other phenomenological extensions

Both relations eliminate the common neutrino mass scale. In particular a change of that scale moves T_{CMB} and H_0 together while preserving their displayed ratio at fixed content and correction factors. The hadronic channel ratios similarly eliminate Λ_{QCD} . These combinations test the relative readout rules while absolute masses and densities also test the hierarchy normalization. Their remaining dependence and finite error bounds are given in Appendix D.

The three gauge-coupling residuals are below 0.01% in magnitude. The fine-structure-constant residual is +0.20% and the electroweak-scale residual is −0.012%. The charged-fermion mass ratios and atmospheric neutrino splitting and listed homogeneous cosmological quantities lie within one percent of their adopted references. Figure 4 displays the residuals in Appendix A including separate rest-mass and unit-source solar entries. Its charged-lepton bars show the diagnostic realization with unit geometric-to-pole conversion.

The cosmological distance data provide a joint test with a measured covariance. The DESI DR2 comparison gives $\chi^2 = 29.5087$ for thirteen fixed predictions with $p = 0.00554011$. Its exact scale projection in Eq. (8.8) separates 15.2965 along a common distance-scale direction and 14.2122 in the orthogonal residual. Table 4 locates the discrepancies in redshift. This tension enters the empirical assessment of the complete endpoint

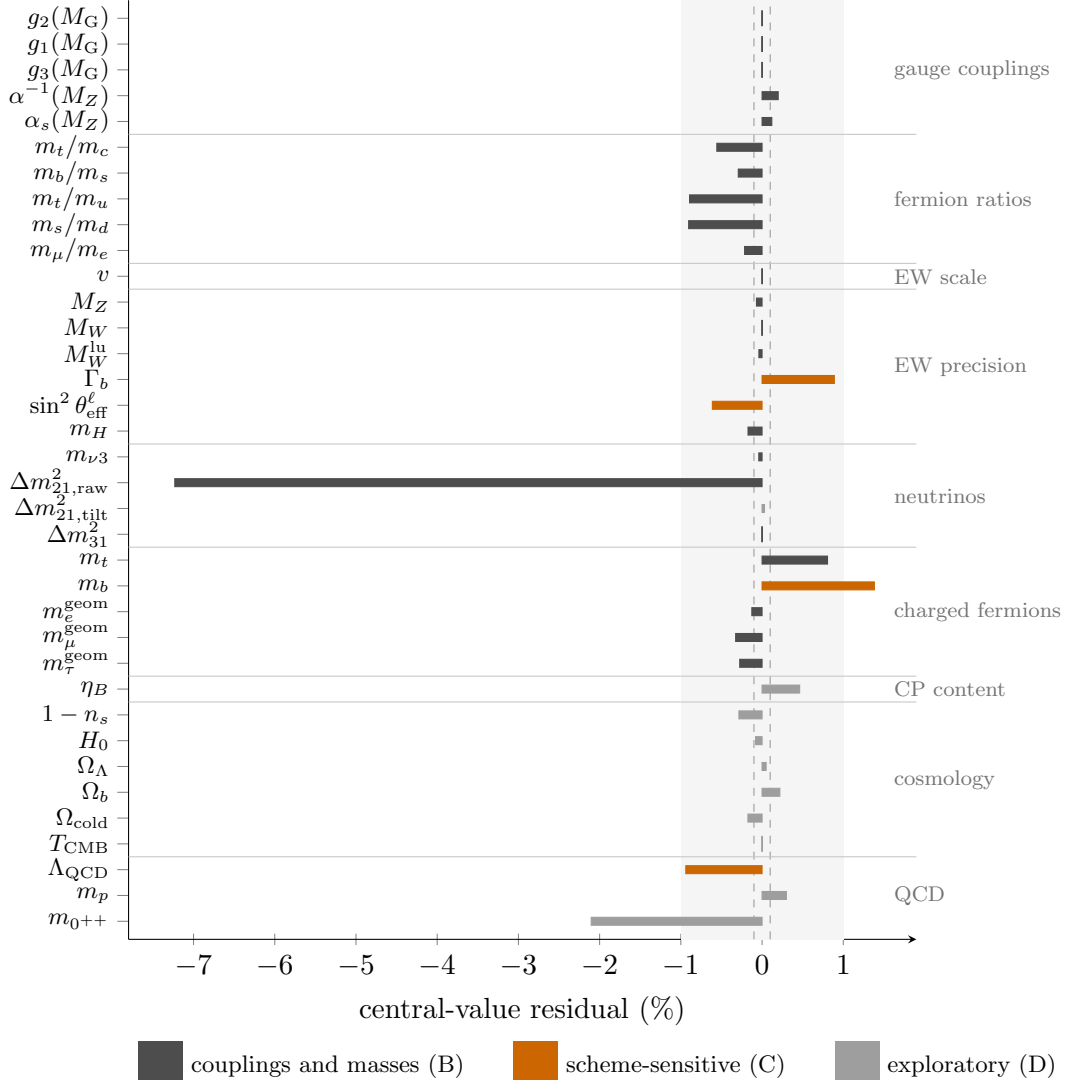


Figure 4. Selected signed central-value comparisons for 36 entries grouped by sector and coloured by comparison tier. The charged-lepton bars show the realization with unit geometric-to-pole conversion tested in the pole-invariant comparison of Section 7. The other entries use the residual definitions in Appendix A. The rest-mass difference and the unit-amplitude scalar-source readout are compared with the same reference value. The latter is Tier D. The shaded band marks $\pm 1\%$ and the dashed lines $\pm 0.1\%$. Residuals with $|\delta| < 0.02\%$ appear as origin ticks. The three plotted entries outside $\pm 1\%$ are discussed in Section 10.4. The electroweak and weak-process comparisons are given in Tables 10 and 22 respectively.

and thermal-background mapping through to the distances. The projection identifies a shared scale direction. Attributing it to a particular physical input requires varying that input through the full mapping.

The production calculation [15] evaluates the physical quantities before applying the comparison data. Additional tests hold the response and transfer rules fixed.

10.4 Leading comparison offsets and their physical origin

The following central-value offsets localize distinct stages of the calculation. Their statistical significance is governed by the uncertainty and matching convention together with the approximation level of each observable.

- The vacuum rest-spectrum value $\Delta m_{21}^2 = 6.874155 \times 10^{-5} \text{ eV}^2$ lies 7.231% below the adopted solar-oscillation benchmark. The unit-amplitude scalar-source propagation readout instead has offset +0.030% in Tier D. Its correlated splitting shifts are given in Eq. (C.81). The physical solar test uses the source-dependent probabilities in Eq. (E.1).
- The geometric bottom-mass readout of Section 7.3 with the fixed geometric anchor $y_0 = 1$ lies 1.38% above the displayed $\overline{\text{MS}}$ benchmark. This Tier-C offset compares the stated geometric and renormalized mass conventions.
- The lightest scalar-channel readout $8\Lambda_{\text{QCD}}$ lies 2.10% below the catalogue benchmark 1.7 GeV. It tests the common QCD extraction and the specified channel-to-mass assignment. The relevant lattice and resonance comparisons carry their own spectral and mixing uncertainties.
- The neutron lifetime is 915.54 s and lies 4.23% above its reference value. Its phase space depends strongly on the same neutron–proton mass difference that enters weak freeze-out and the helium calculation in Appendix F and makes the lifetime a correlated test of the hadronic–weak chain.

The QCD-scale residual is -0.94% for the specified two-loop extraction. The PMNS mixing angles and CKM phase together with primordial abundances use the mixing and thermal assumptions in Appendices E and F. The same reduced freeze-out and decay calculation gives $\Delta m_{np} = 1.28497 \text{ MeV}$ and $\tau_n = 915.54 \text{ s}$ and $Y_p = 0.246082$. The corresponding central-value residuals are -0.647% and $+4.23\%$ and $+0.44\%$. Table 22 compares these linked observables and Appendix F gives their common-input response. The common electroweak amplitude gives an R_b residual of $+0.384\%$.

The running and electroweak relations correlate m_b and Λ_{QCD} with m_t and M_W and Γ_b through their shared inputs across the comparison tiers.

The lifetime elasticity to the mass difference is -6.53633 . At fixed other inputs a smaller energy release reduces the available phase space and lengthens the lifetime. This local response maps the mass-gap deficit to a lifetime shift with the direction and approximate size of the reported discrepancy. Replacing that mass difference alone by its experimental value gives $\tau_n = 877.59 \text{ s}$ and $Y_p = 0.240716$ in the same weak approximation (Table 23). This externally conditioned diagnostic locates a strong phase-space contribution to the lifetime discrepancy while displaying its coupled effect on helium. The common input response propagates hadronic matching, the weak amplitude and radiative factors through both observables. Their joint comparison tests how the same hadronic variation changes decay phase space and neutron survival during the thermal history.

10.5 Numerical bounds and physical sensitivity

The tabulated central values are obtained from floating point evaluations before display rounding. The interval enclosures certify the specified spectral tails and endpoint ranges with retained-position counts at their stated accuracy. The precision of each tabulated observable also depends on its input propagation and physical approximation. Appendix B records the evaluation precision and rounding with verification coverage while Appendix D supplies the enclosures. For the production response the certified endpoint is $c_{3,5} \in [2.4912094, 2.4958593]$. At fixed remaining inputs its exact propagation gives relative ranges $[0.9702, 1.0306]$ for v and $[0.9544, 1.0476]$ for m_e and $[0.7853, 1.2722]$ for ρ_Λ each normalized at $c_o = 2.49353$ (Table 19). These guaranteed ranges quantify the coverage of the conservative quadrature certificate. The tabulated percentage residuals describe the floating-point central comparisons. The shared endpoint correlates all three outputs.

Appendix D gives the dependence of the observables on their common spectral and content parameters. At fixed other inputs the hierarchy obeys $v \propto (kL)^{-1}e^{-4\pi kL}$ and $m_e \propto e^{-20kL}$. Its local tangent estimates for a one-percent change of kL are -32% for v and -50% for m_e and -259% for ρ_Λ . Finite changes require the exact propagation formulas of that appendix. The dark-energy fraction Ω_Λ is a pure content ratio and has zero elasticity. The upstream definitions are the specified TT kernel $K_{\text{TT}} = (1 - y)^2$ and the endpoint normalisation $\max_{0 \leq y \leq 1} y(1 - y)^2 = 4/27$ and the torsion content ratio $\tau = (N_L - N_R)/(N_f \sum Y^2) = 1/50$. For the adopted bandwidth and content these relations fix the endpoint. The elasticity matrix quantifies how its uncertainty is amplified in the mass hierarchy. Figure 5 summarises the local response.

For shared inputs θ with a justified probability model the logarithmic Jacobian $J_{ai} = \partial \log O_a / \partial \theta_i$ propagates their covariance as $J \Sigma_\theta J^\top$. Bounds on truncation and quadrature instead propagate as enclosures. Omitted perturbative terms retain their stated order. Variation of the response domain R or of a physical transfer rule compares different models. These three operations have distinct statistical meanings and are kept separate when assessing the residuals.

10.6 Electroweak precision observables

The electroweak calculation uses the common inputs of Section 10.1. These comprise v and $G_F = 1/(\sqrt{2}v^2)$ together with $\alpha(M_Z)$ of Section 5 and $\alpha(0)$ obtained from the one-loop QED vacuum-polarisation descent of Appendix C.16 with the geometric couplings at M_G and the fermion masses of Section 7 and the fixed Higgs-quartic readout λ_H of Appendix C.12.

The internal Z mass is the self-consistent fixed point of the tree-level mass formula evaluated on the two-loop geometric running

$$M_Z = \frac{v}{2} \sqrt{g_2^2 + \frac{3}{5}g_1^2} \Big|_{\mu=M_Z},$$

which gives $M_Z = 91.125$ GeV against the observed 91.1876 GeV (-0.069%) with

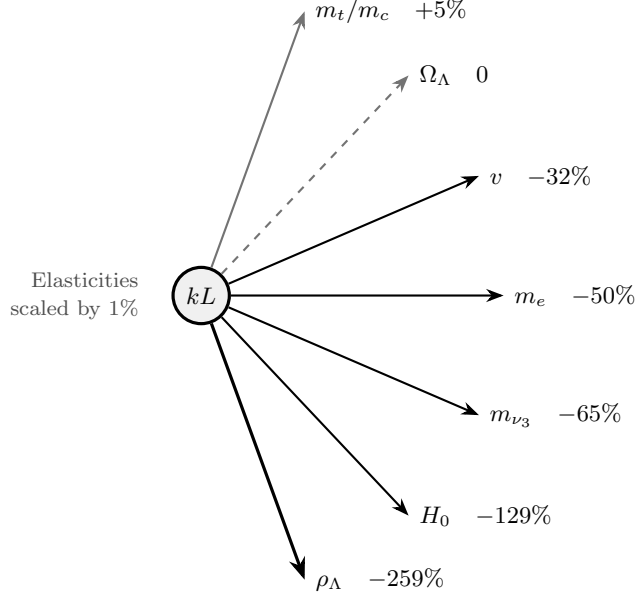


Figure 5. Local sensitivity to $c = kL$ at fixed other inputs. The labels give one percent of the logarithmic derivatives. The factors $c^{-1}e^{-4\pi c}$ and e^{-20c} and their powers produce the large elasticities of dimensionful quantities. The content ratio Ω_Λ has zero elasticity. Appendix D gives the matrix and the formulas for finite changes.

$\alpha^{-1}(M_Z) = 128.206$ (+0.20%). The W mass is the on-shell relation

$$s^2 c^2 = \frac{\pi\alpha(0)}{\sqrt{2}G_F M_Z^2} \frac{1}{1 - \Delta r}, \quad \Delta r = \Delta\alpha - \frac{c^2}{s^2} \Delta\rho,$$

where $s^2 = 1 - M_W^2/M_Z^2$ and $c^2 = M_W^2/M_Z^2$ with $\Delta\alpha = 1 - \alpha(0)/\alpha(M_Z)$ and $\Delta\rho$ is the one-loop t - b doublet contribution with its full mass dependence. The result is $M_W = 80.372$ GeV against the retained 80.369 GeV benchmark (+0.003%). The on-shell mixing angle is $\sin^2 \theta_W = 0.22208$ with a -0.50% residual against the value formed from the reference masses. The leading-universal approximation uses the leading-top baseline $\Delta\rho_{\text{top}}^{(1)} = 3x_t$ with $x_t = G_F m_t^2 / (8\sqrt{2}\pi^2)$ as in Eq. (8.22) of Ref. [42]

$$\Delta\bar{\rho} = 3x_t [1 + x_t(19 - 2\pi^2)] = 0.00948.$$

The retained two-loop heavy-top coefficient is its small- M_H/m_t limit. At the common electroweak reference scale write the retained running coupling as α_Z . The effective parameters satisfy

$$\bar{s}^2(1 - \bar{s}^2) = \frac{\pi\alpha_Z}{\sqrt{2}G_F M_Z^2} (1 - \Delta\bar{\rho}), \quad \rho_{\text{NC}} = \frac{1}{1 - \Delta\bar{\rho}}, \quad \bar{s}^2 = s_W^2 + c_W^2 \Delta\bar{\rho}.$$

These are the leading-universal relations of Ref. [42, Eqs. (8.21)–(8.25)]. The weak branch gives $M_W = 80.337$ GeV (-0.039%) and $\bar{s}^2 = 0.23011354$ and $\rho_{\text{NC}} = 1.00956729$. The identity $\pi\alpha_Z/(\bar{s}^2 \bar{c}^2) = \sqrt{2}G_F M_Z^2 \rho_{\text{NC}}$ fixes the normalization of the same neutral-current

amplitude. The coupling fraction from the geometric running is $\sin^2 \theta(M_Z) = 0.23327$. Its conversion to the effective angle is specified by the matching relations above. Table 11 states the perturbative approximation for each comparison.

The displayed $\alpha^{-1}(M_Z) = 128.206$ is evaluated at the self-consistent Z mass. The value 128.207 in Appendix A comes from two-loop evolution to the preliminary one-loop estimate of that scale. The scale-transfer calculation [15] uses the latter coupling in the W -mass relation and the width radiators. The two coupling readouts differ by about 0.0006%.

All Z partial widths use $v_f = T_3^f - 2Q_f \bar{s}^2$ and $a_f = T_3^f$ and the common normalization ρ_{NC} . The nonuniversal heavy-top contribution to the bottom amplitude is $\delta v_b = \delta a_b = x_t$ at first order [1, Electroweak review, Eqs. (10.53), (10.55)]. Keeping its interference with the leading amplitude gives

$$\Gamma_f = N_c^f \frac{G_F M_Z^3 \rho_{\text{NC}}}{6\pi\sqrt{2}} [v_f^2 + a_f^2 + 2\delta_{fb} x_t (v_b + a_b)] R_{\text{QCD}}^f R_{\text{QED}}^f. \quad (10.3)$$

Here $R_{\text{QCD}}^q = 1 + \alpha_s/\pi + 1.409(\alpha_s/\pi)^2$ for quarks and $R_{\text{QCD}}^\ell = 1$ and $R_{\text{QED}}^f = 1 + 3Q_f^2 \alpha_Z/(4\pi)$. The factorized radiators multiply the retained leading-universal amplitude and first-order bottom interference. The latter corresponds to

$$\delta_b = \frac{2x_t(v_b + a_b)}{v_b^2 + a_b^2} = -0.01448455, \quad \sin^2 \theta_{\text{eff}}^b = \bar{s}^2(1 + 2x_t) = 0.23157075$$

through first order in the nonuniversal term. Final-state masses enter at $O(m_f^2/M_Z^2)$. The table retains their massless limit.

The partial widths are summed before forming the ratios and peak cross section. The results are $\Gamma_Z = 2.50636 \text{ GeV}$ (+0.447%) and $\Gamma_b = 0.380407 \text{ GeV}$ (+0.890%) with $R_b = 0.2171203$ and $R_l = 20.79153$ and $\sigma_{\text{had}}^0 = 41.54848 \text{ nb}$ (+0.163%). The definition of the peak is $12\pi\Gamma_e\Gamma_{\text{had}}/(M_Z^2\Gamma_Z^2)$. In Table 10 Γ_l sums the three charged-lepton widths and R_l uses the single-flavour denominator Γ_e .

The strong-sector matching and additive generation contribution of Appendix C.12 fix $\lambda_H = 0.129$. Combined with the generated v they give the tree-level relation $m_H = \sqrt{2\lambda_H} v = 124.98 \text{ GeV}$ with central-value residual -0.17% . This value then supplies the Higgs mass condition in the one-loop matching of Section 10.1 and determines the corresponding renormalized parameters. The mass comparison tests the tree-level relation and the finite matching implements that same physical mass in the quantum action through the calculated order. The squash coefficient $\lambda = 149$ in Section 6.1 belongs to the internal scalar potential. The ladder's geometric muon and tau masses are 105.31 MeV and 1.772 GeV with residuals -0.33% and -0.28% .

10.7 Observable tests

Table 12 gives tests of the numerical branches. Neutrino ordering and the summed mass test the mass readout while the octant and phase test the separate mixing ansatz. The tensor reference is the BK18 limit $r_{0.05} < 0.036$ at 95% confidence [43] with pivot $k_* = 0.05 \text{ Mpc}^{-1}$ and $n_t = 0$. The local acceleration entry concerns the nonrelativistic static response of

Table 10. Electroweak comparisons for the common inputs of Section 10.1 with deviations evaluated before display rounding. The Z mass is tree-level on the two-loop running. The two W -mass entries use the one-loop- ρ and leading-universal approximations. All effective Z widths and ratios with the peak cross section follow Eq. (10.3) with a common effective angle and neutral-current normalization and radiators. The Higgs mass is at tree level.

Quantity	Computed	Reference	Deviation
M_Z (GeV)	91.125	91.1876	−0.069%
$\alpha^{-1}(M_Z)$ (self-consistent)	128.206	127.952	+0.20%
$\sin^2 \theta(M_Z)$	0.23327	0.23122	+0.89%
M_W (GeV)	80.372	80.369	+0.003%
M_W leading-univ. (GeV)	80.337	80.369	−0.039%
$\sin^2 \theta_W$ (on-shell)	0.22208	0.22321	−0.50%
ρ_{NC}	1.009567	—	—
$\sin^2 \theta_{\text{eff}}^l$	0.23011	0.23153	−0.61%
Γ_Z (GeV)	2.50636	2.4952	+0.447%
Γ_{had} (GeV)	1.75206	1.7444	+0.439%
Γ_b (GeV)	0.380407	0.37705	+0.890%
Γ_l (GeV)	0.252804	0.251898	+0.360%
Γ_{inv} (GeV)	0.501494	0.4990	+0.500%
σ_{had}^0 (nb)	41.5485	41.481	+0.163%
R_l	20.7915	20.767	+0.118%
R_b	0.217120	0.21629	+0.384%
m_H (GeV)	124.98	125.20	−0.17%

Appendix G. A lensing prediction requires the light-propagation response and source data in the same macroscopic realization as the matter trajectories and clock readouts. In a metric representation these determine the effective metric along the ray. Proposition G.6 examines one realization with an independent foliation field.

Accuracy of the comparisons. The quoted uncertainties describe propagation within the present truncation. Spectral accuracy is controlled by the finite sums and endpoint enclosure together with the mode ceiling. Perturbative accuracy is set by the loop order and threshold conversions. The Born top width has QCD and electroweak corrections together with finite bottom-mass and charged-current effects. The covariance budget for the cosmological comparison comprises the full DESI covariance and CMB-monopole errors together with the BBN-rate uncertainties. These contributions determine the accuracy within the stated definitions.

The charged-lepton conversion is different in character. The realization $G_\ell = 1$ is excluded by the pole invariant at 38σ . With the pole masses held as renormalization conditions higher-order D_ℓ determines the corresponding running parameters and preserves the specified poles. The pole prediction depends on G_ℓ calculated from the generated vertex and on the light-family transfer rule.

Table 11. The perturbative level of each electroweak precision observable in Table 10 together with the leading correction at the next level.

Observable	Evaluated level	Leading next correction
M_Z	tree level on the two-loop running	one-loop electroweak self-energy
M_W	one-loop ρ and leading-universal heavy-top terms with resummed α	finite Δr_{rem} including bosonic contributions with vertex and box contributions
$\sin^2 \theta(M_Z)$	geometric couplings at M_Z	full one-loop $\sin^2 \theta_{\text{eff}}$
$\sin^2 \theta_{\text{eff}}^l$	improved Born with the retained heavy-top $\Delta \bar{\rho}$	finite process-dependent κ_l and higher orders
Γ_f	Common leading-universal amplitude with heavy-top bottom interference and factorized QCD/QED radiators	finite-mass phase space and finite process-dependent EW terms and mixed radiative corrections
m_H	tree-level generated mass relation	radiative correction to the mass relation with independently fixed boundary parameters

The numerical evaluation uses the Coarse Graining Genesis Framework [15]. It evaluates the stated finite sums and comparisons at the precision and approximation orders specified above.

Table 12. Observable comparisons of the stated branches. The mass and flavour entries together with the cosmological and local-response entries retain their respective derivational and comparison status.

Quantity	Value fixed here	Discriminating measurement
Tensor-to-scalar ratio	$r = 0.0253$	a robust upper bound well below this value
Neutrino mass ordering	normal	inverted ordering
Atmospheric octant	$\sin^2 \theta_{23} = 0.551$	lower octant
Summed neutrino mass	$\sum m_\nu = 0.0615 \text{ eV}$	a robust upper bound below this value
PMNS phase (mixing ansatz)	$\delta_{\text{PMNS}} \simeq (8/7)\pi$	value far from this branch
Pseudoscalar glueball target	$m_{0-+} = 2.426 \text{ GeV}$	a determined glueball mass incompatible with the stated unmixed channel assignment
Low-acceleration branch	$a_0 = 1.204 \times 10^{-10} \text{ m s}^{-2}$ and $\mu(y) = y/\sqrt{1+y^2}$	rotation curves incompatible with the response for the stated baryonic source and boundary conditions

11 Discussion and conclusion

The effective couplings and mass scales share one source, whose Gaussian law and reflected spectrum follow from minimal unbiased compositional change. The physical construction of Ref. [4] supplies the compact response geometry, gauge group, matter module and elementary scalar sector within its realization class. Common kinetic normalization converts source matrix elements into interaction strengths; successive transmission of mass amplitudes generates hierarchies. Joint screening selects three matter copies. Shell support and position retention then select $R = 5$ and the tensor balance has one positive endpoint on that branch. The retained family positions reproduce the same count.

The relations among sectors test more than individual central values. The relation among six masses eliminates the common transfer parameters. In the adopted running representation its comparison uses masses in a common scheme and scale. The lepton identity belongs to the geometric mass eigenvalues. Its pole readout is recorded by G_ℓ , while the adopted running representation uses the additional factors D_ℓ . The geometric identities therefore constrain relative mass transmission independently of the common endpoint and dimensional normalization.

Two cosmological combinations eliminate the common neutrino mass normalization. These are the vacuum-density fraction and $T_{\text{CMB}}^2/(M_{\text{P}}H_0)$ in energy units. They test the relative density, temperature and expansion laws. The electron and electroweak scales provide another correlated test through their bandwidth response. The three-position plateau persists for every continuous bandwidth $\mathcal{R} \geq 5$ while the same active shells and three positions require $R_* < \mathcal{R} \leq 16/\pi$ with $R_* \simeq 4.01871$. The only integer level in this interval is five. The monotone dependence of v/m_e on the endpoint supplies a matched-ratio test of that level. Thus spectral retention and the sensitivity of absolute masses are measured by distinct relations.

The family transfer and mixing rotations are realized in one canonical action. Its replicated quadratic response retains the direct source and domain variations. Strong-sector and generation-count matching determine the Higgs quartic. Together with the generated v it gives the tree-level Higgs mass. The generated Fermi condition and the top and Higgs conditions then determine locally unique renormalized coefficients in the tadpole-free $\overline{\text{MS}}$ scheme. The absorptive top self-energy gives the additional Born output $\Gamma_t^{(0)} = 1.53445 \text{ GeV}$. The decay corrections and charged-current assumptions determine the accuracy beyond that order.

The macroscopic constructions resolve complementary parts of the matter–source response. On the decomposable occupation orbit, the source covariance sets the acceleration variance and the positive Gaussian RMS amplitude determines the spatial response. The resulting source–potential system has a unique static weak solution and a boundary energy balance during smooth matter evolution before trajectory crossing. The auxiliary-radius model admits an occupied expanding background with autonomous source feedback and positive constrained propagation and a phase clock on a finite interval. On that solution, source exchange balances in the total stress, and phase-charge conservation fixes the clock rate through the evolving volume and matter amplitude. For comparison, the action with

an independent foliation varies the congruence as an additional field. Its reduction gives positive scalar kinetic energy on the specified cosmological branch and a longitudinal negative direction on the accelerated-static branch. These signs characterize the stated effective action and background.

The empirical comparisons probe the physical transfer laws as well as the spectral data. The endpoint enclosure and its propagation give guaranteed ranges at fixed remaining inputs. The tabulated residuals describe the central comparisons within the stated mass and coupling conventions. The emergence-scale gauge couplings agree closely with the adopted Standard-Model extrapolation.

The correlated running-mass data give $\Delta_F = 2.69537 \pm 1.48017$ and $\Delta_L = 0.0112736 \pm 0.0005817$ at M_Z . The pole inputs require the weighted geometric-to-pole contribution $\Delta_{L,\text{ref}}^{\text{pole}} = 0.00257845 \pm 0.00006753$. With the quoted input errors the pole invariant excludes the exact identification $G_\ell = 1$ at $38.18\sigma_{\text{ref}}$. For this specified realization the independent one-loop pole-to-running calculation gives $\Delta_L^{\text{run},(1)} = 0.00933752$ including the finite-mass weak contribution 2.35×10^{-6} . Its running-mass comparison has central offset 0.00193607. Higher-order pole-to-running corrections enter that comparison with the specified poles held fixed. The physical mass response determines the geometric to pole ratio for each realization. Equation (7.2) expresses its first order value in the perturbative effective representation through the mass vertex and both kinetic responses. The finite lepton transfer and flavour-spread bound organize the running and remaining matching contributions.

The solar rest-mass splitting is 7.23% below its reference. The separate unit-source entry describes the scalar propagation model. Its solar test uses the environmental probabilities of Eq. (E.1). The geometric bottom mass lies 1.38% above the displayed $\overline{\text{MS}}$ benchmark and the scalar-channel glueball scale lies 2.10% below the catalogue reference. Their interpretation depends on the mass matching, channel assignment and reference uncertainties.

For the fixed cosmological mapping the DESI comparison gives $\chi^2 = 29.5087$ for thirteen components and $p = 0.00554011$. The covariance-weighted decomposition separates common-scale and shape residuals. The CMB monopole constrains the same normalization direction. Finite sound-horizon bounds specify the additional drag response needed along that direction. Recombination with generated atomic constants and the full allowed electron capacity below $z = 30$ at fixed upstream background leave the statistic above the stated 95% reference level. The atomic and radiative-transfer remainder above $z = 30$ and joint changes of upstream inputs enter separately. This distance tension therefore tests the adopted chain from endpoint densities and thermal history to the acoustic scale and expansion distances.

The reduced weak-freeze-out and decay calculation gives $\Delta m_{np} = 1.28497 \text{ MeV}$ and $\tau_n = 915.54 \text{ s}$ and $Y_p = 0.24608$ with central residuals -0.647% and $+4.23\%$ and $+0.44\%$. The lifetime and helium abundance probe the isovector response and electromagnetic subtraction and weak rate in different combinations. Replacing the total mass difference by its experimental value sharply reduces the lifetime offset and lowers the helium abundance. The source channel construction, constrained baryon resolvent and independent propagation of lattice inputs identify the hadronic matching quantities entering this joint test.

The rotation-curve comparison tests the pointwise approximation to the local static equation. The neutrino, tensor and hadronic targets retain the response laws fixed in these calculations.

The common source connects particle amplitudes to the restoring response of occupied matter. Material phase changes supply elapsed-time readings, and the effective metric expresses relations among motion, stress and clock rates. Transforming the source, kinetic form and observables together preserves these relations under a change of mode basis or clock parameter. The independent mass combinations and joint response tests give quantitative probes of this generated realization.

A Numerical values and reference conventions

Table 13 collects the inherited content identities and the endpoint quantities used throughout the calculation. Table 14 gives the resulting values and comparison references together with signed central-value residuals and the tiers defined in Section 10. Newton’s constant supplies the dimensional anchor. The remaining entries follow the spectral and effective relations specified in Section 2 and Appendix C.

The charged fermion entries are the geometric mass and ladder readouts of Section 7. The Higgs entry uses the tree relation $m_H^2 = 2\lambda_H v^2$. The reference mass definitions are as follows. The value 172.69 GeV is the PDG 2022 direct-reconstruction top-mass reference whereas 4.18 GeV denotes $\overline{m}_b(\overline{m}_b)$ in the $\overline{\text{MS}}$ scheme [44]. The reference charged-lepton masses are pole masses and the Z and Higgs values use the PDG 2024 resonance conventions [1]. For charged leptons Eq. (7.1) separates the geometric vertex from the physical pole and running definitions. The table labels the charged lepton entries as geometric masses m_ℓ^{geom} and compares them with pole mass references. Their pole residuals depend on the conversion G_ℓ . The specified realization $G_\ell = 1$ is excluded by the pole invariant at 38σ by Eq. (7.7). The general ratio G_ℓ follows from the physical readout of the specified realization. The finite one-loop calculation gives the pole-to-running factors D_ℓ for the specified pole conditions. The quark ratio values 136, 45, 78000 and 19.802 serve as individual comparison quotients. The joint geometric relation (7.3) is compared with physical masses through the six matching factors specified in Section 7. The W and strong-coupling reference values are 80.369 GeV and $\alpha_s(M_Z) = 0.1179$. All percentage entries are central-value comparisons in these stated conventions. The finite one-loop matching in Proposition C.5 uses the generated (G_F, M_t, M_H) as physical conditions and reports the corresponding running coefficients.

The two neutrino-splitting references are the normal-ordering values without Super-Kamiokande atmospheric data in NuFIT 5.2 [45, 46] and equal 7.41×10^{-5} and 2.511×10^{-3} eV². The rest-mass difference (Tier B) and the unit-amplitude scalar-source readout (Tier D) are compared with the same reference value. Equation (E.1) specifies the environmental propagation needed for a solar interpretation. The 0.0502 eV entry is an atmospheric mass-scale benchmark. The cosmological density and tilt references use the base- Λ CDM column labelled TT+TE+EE+lowE+lensing in Table 2 of Planck 2018 [28]. The rounded fractions use its baryon and cold-dark-matter densities and Hubble param-

Table 13. Content identities and endpoint quantities. Together with the dimensional anchor and the stated transfer relations they determine the entries of Table 14.

Quantity	Value	Origin
window capacity kL	2.49353	spectral endpoint in Sec. 4.4
emergence scale M_G	1.731×10^{18} GeV	$M_G = M_P \sqrt{\pi}/kL$ in Sec. 4.3
torsion modulus τ	1/50	content response in Sec. 3.3
left-handed count N_L	8	charged module in Sec. 3.1
right-handed count N_R	7	charged module in Sec. 3.1
chiral asymmetry $N_L - N_R$	1	difference of the two counts in Sec. 3.1
hypercharge moment $\sum Y^2$	10/3	generated charge weights in Sec. 3.1

eter. The H_0 benchmark is expressed in GeV. The microwave-temperature reference is Fixsen’s 2.72548 ± 0.00057 K [32].

The gauge references at the emergence scale are two-loop extrapolations from the fixed boundary values in Appendix B. Percentage residuals describe offsets from the displayed benchmarks. Statistical tests use experimental covariance and the theoretical uncertainties appropriate to each benchmark. Section 10 gives the electroweak comparison levels; Section 9.2 identifies the separate lattice and resonance references.

Table 14: Computed quantities and fixed comparison benchmarks grouped by sector. The computed values use the physical relations and numerical scheme of Section 2. The text of this appendix specifies the reference sources and versions together with the comparison limits. The derivation column points to the relations used in each evaluation.

Quantity	Computed	Reference	Residual	Tier	Relation
Internal scale					
generation count n_g	3	3	+0.000%	A	joint screening compatibility and retained-position check (App. C.1)
Gauge couplings					
weak coupling $g_2(M_G)$	0.508848	0.508845	+0.001%	B	geometric weak coupling after colour and torsion screening (App. C.4)

Quantity	Computed	Reference	Residual	Tier	Relation
hypercharge coupling $g_1(M_G)$	0.60499	0.605001	-0.002%	B	squashed-isometry normalisation transported from its scale (App. C.4)
colour coupling $g_3(M_G)$	0.49776	0.497765	-0.001%	B	common-origin colour coupling with conformal correction (App. C.4)
$\alpha^{-1}(M_Z)$	128.207	127.952	+0.199%	B	two-loop running at the preliminary internal Z scale (Sec. 10.6)
$\alpha_s(M_Z)$	0.118041	0.1179	+0.120%	B	two-loop gauge flow with derived content coefficients (App. C.5)
Fermion mass ratios					
top to charm m_t/m_c	135.242	136	-0.557%	B	assigned up-sector ladder ratio (App. C.7)
bottom to strange m_b/m_s	44.8679	45	-0.293%	B	assigned down-sector ladder ratio (App. C.7)
top to up m_t/m_u	7.730364e+04	7.800000e+04	-0.893%	B	up-sector ladder with first-generation window transfer (App. C.7)
strange to down m_s/m_d	19.623	19.802	-0.904%	B	down-sector first-generation ladder ratio (App. C.7)
muon to electron $\frac{m_\mu^{\text{geom}}}{m_e^{\text{geom}}}$	206.356	206.8	-0.215%	B	charged-lepton ladder with the Euclidean-period shift (App. C.9)
Electroweak scale					
Higgs vacuum expectation v (GeV)	246.19	246.22	-0.012%	B	Gaussian window hierarchy at the spectral fixed point (App. C.11)

Quantity	Computed	Reference	Residual	Tier	Relation
Electroweak precision					
Z mass M_Z (GeV)	91.1248	91.1876	-0.069%	B	self-consistent neutral electroweak mass relation (Sec. 10.6)
W mass M_W (GeV)	80.3717	80.369	+0.003%	B	on-shell relation with the one-loop rho correction (Sec. 10.6)
leading-universal W mass M_W^{lu} (GeV)	80.3374	80.369	-0.039%	B	leading-universal on-shell electroweak relation (Sec. 10.6)
Γ_b (GeV)	0.380407	0.37705	+0.890%	C	common effective neutral-current amplitude with heavy-top interference (Sec. 10.6)
$\sin^2 \theta_{\text{eff}}^l$	0.230114	0.23153	-0.612%	C	improved-Born leptonic effective angle (Sec. 10.6)
Higgs mass m_H (GeV)	124.983	125.2	-0.173%	B	tree-level Higgs mass from the spectral quartic and scale (Sec. 10.6)
Neutrinos					
heaviest neutrino scale $m_{\nu 3}$ (eV)	0.0501797	0.0502	-0.040%	B	Weinberg two-period texture. Atmospheric reference scale (App. C.10)
solar rest split $\Delta m_{21,\text{raw}}^2$ (eV ²)	6.874155e-05	7.410000e-05	-7.231%	B	light-pair rest-mass difference (App. C.10)
unit-source split $\Delta m_{21,\text{tilt}}^2$ (eV ²)	7.412190e-05	7.410000e-05	+0.030%	D	scalar-source read-out at $\zeta = 1$ (App. C.10)

Quantity	Computed	Reference	Residual	Tier	Relation
atmospheric splitting Δm_{31}^2 (eV ²)	0.00251121	0.002511	+0.008%	B	atmospheric split from the absolute mass eigenvalues (App. C.10)
lightest neutrino $m_{\nu 1}$ (eV)	0.00260742	–	–	B	absolute vacuum-floor mass from the hypercharge texture (App. C.10)
Charged-fermion masses					
top mass m_t (GeV)	174.082	172.69	+0.806%	B	geometric unit overlap at the electroweak scale (App. C.8)
bottom mass m_b (GeV)	4.23779	4.18	+1.383%	C	absolute bottom mass from the down-sector ratio (App. C.9)
electron mass m_e^{geom} (MeV)	0.510354	0.511	–	–	four-level electron cascade over one-generation content (App. C.9)
muon mass m_μ^{geom} (GeV)	0.105314	0.105658	–	–	electron mass propagated through the lepton ladder (App. C.9)
tau mass m_τ^{geom} (GeV)	1.77211	1.777	–	–	muon mass propagated through the lepton ladder (App. C.9)
CP content					
baryon asymmetry η_B	6.128230e-10	6.100000e-10	+0.463%	D	content asymmetry at the physical photon temperature (Sec. 8.3)
Cosmology					
scalar tilt $1 - n_s$	0.035	0.0351	-0.285%	D	torsion modulus times the seven-over-four content ratio (App. C.14)

Quantity	Computed	Reference	Residual	Tier	Relation
Hubble rate H_0 (GeV)	1.438850e-42	1.440000e-42	-0.080%	D	Gaussian endpoint logarithm and content offset (App. C.14)
dark-energy fraction Ω_Λ	0.685044	0.6847	+0.050%	D	dark-energy density divided by the critical density (App. C.14)
baryon fraction Ω_b	0.0494082	0.0493	+0.219%	D	raw photon floor and Sakharov ratio combined with the proton density (App. C.14)
cold fraction Ω_{cold}	0.264039	0.2645	-0.174%	D	common budget after photons and thermal neutrinos (Sec. 8.4)
CMB temperature T_{CMB} (K)	2.72547	2.72548	-0.000%	D	endpoint-corrected microwave monopole (App. C.14)
QCD					
QCD scale Λ_{QCD} (GeV)	0.208028	0.21	-0.939%	C	specified two-loop QCD extraction (App. C.15)
proton mass m_p (GeV)	0.941086	0.938272	+0.300%	D	conformal proton relation with constituent dressing (App. C.15)
lightest glueball m_G (GeV)	1.66422	1.7	-2.104%	D	scalar reference of the specified QCD channel readout (App. C.15)
Weak decay and primordial helium					
neutron–proton difference (MeV)	1.28497	1.29333236	-0.647%	D	common scalar-isovector and electromagnetic mass response (App. F)

Quantity	Computed	Reference	Residual	Tier	Relation
neutron life-time (s)	915.54	878.4 ± 0.5	+4.23%	D	Coulomb-weighted decay with the stated radiative factor (App. F)
helium baryon fraction Y_p	0.246082	0.245	+0.44%	D	instantaneous freeze-out followed by conserved decay and complete capture (App. F)

B Reproducibility

The numerical tables use the spectral data and response relations of Appendix C with Newton’s constant fixing the dimensional normalization. The calculation record specifies the inputs, numerical precision, consistency checks and rigorous enclosures [15].

Generated quantities and external reference data are stored separately and accessed through a common parameter interface. The reference values enter at the comparison stage. Source identities and input fingerprints associate each result with the frozen calculation.

B.1 Numerical precision and calculation records

Precision and displayed values. Unrounded floating-point records determine the reported central values. Mathematical interval bounds establish the three-position plateau. The endpoint comparison from Eq. (4.5) uses the relative residual $c_{\text{ref}}/c - 1$ where c_{ref} is obtained by inverting the separately extrapolated Standard-Model reference $g_2(M_G)$. This central-value comparison uses the unrounded calculation. The conservative endpoint enclosure certifies the retained-position count at the accuracy stated in Appendix D. Changes to the response domain or transfer rules measure model sensitivity. Numerical precision concerns evaluation of the fixed response.

The endpoint $kL = 2.493534\dots$ is displayed as 2.49353 and used without display rounding in the exponential hierarchy relations. For $X \propto e^{-akL}$ its leading effect is $\delta X/X = -a\delta kL$. Appendix D includes the algebraic prefactors. The reference-coupling inversion uses this fixed endpoint for comparison.

Calculation and verification. Table 15 summarizes the calculations and their numerical and formal checks. Table 16 identifies the software release. The response domain, channel weights, transfer relations and unit conversions determine the calculation independently of previously evaluated intermediate values.

The tables use the joint three-copy branch and the $R = 5$ endpoint. Its root and finite-source responses determine the coupling boundaries and hierarchy relations before the low energy ratios are formed. The mode bound, quadrature accuracy and root resid-

Table 15. Computational support and its scope [15].

Calculation	Evaluation or check
Spectral sums and end-point	Mode sums and endpoint enclosures. Finite-source responses and a uniform three-position certificate for $R \geq 5$.
Cosmological and external-data comparisons	CAMB propagation for the stated inputs. DESI DR2 covariance statistic and common-scale projection. SPARC statistics for the stated mass models.
Discrete algebraic relations	Formal checks of matter counts and rational two-loop coefficients together with the rest-frame cold-source decomposition.
Interacting responses	Neutral-mode elimination and mode products. Source Hessians and canonical vertices. Causal matter response and covariant source exchange.
Propagation and local fields	Labelled flavour evolution and neutron–proton weak rates. Non-spherical Dirichlet problems and finite nonlinear source–potential evolution.
Particle comparisons	Widths and ratios at a common electroweak order. Separate static-mass and solar-propagation comparisons.

ual control numerical error. Loop order and threshold matching specify the perturbative approximation.

The production run has identifier 20261024T084548Z-d52375b5 and is dated 24 October 2026. Table 17 identifies its source set, scientific records, comparison inputs and CAMB calculation by full SHA-256 checksums. The source-set checksum uses the sorted relative-source checksum mapping in compact UTF-8 JSON. The scientific and reference records use their complete byte contents. The remaining checksum conventions are specified in Ref. [15]. The archived fixed-background thermal study [35] in Section 8.5 has its own CAMB execution and HYREC-resource identities. Its thermal-response checksum identifies the atomic-rate comparison at that fixed background. The corresponding DESI projection uses its distance predictions and the published covariance.

B.2 Numerical evaluation and consistency checks

Electroweak widths and ratios. The numerical calculation [15] evaluates the partial widths from the common effective amplitude of Eq. (10.3). The same effective angle and neutral-current normalization enter every channel. The bottom vertex contributes its first-order interference and all channels retain their factorized radiators. The unrounded values are $\Gamma_b = 0.38040730308009635$ with $\Gamma_{\text{had}} = 1.7520580283558593$ and $\Gamma_Z = 2.5063559621245686$ GeV. They determine $R_b = 0.2171202648105626$ and $R_l =$

Table 16. Calculation records used in this manuscript. The production calculation and the fixed-background thermal evaluation are identified separately by the checksums below [15].

Record	Version or date	Relation to this manuscript
Software [15]	Release used for the frozen record	Spectral and finite-source calculations. Joint mass and bandwidth responses. One-loop matching and propagation.
Particle and mass tables	24 October 2026	Fixed- y_0 evolution and widths at a common correction order.
Finite matching and decay	Same calculation	Generated Fermi and top conditions together with the Higgs condition. Specified but disfavoured charged-lepton pole realization and its running conversion. Top width from the retained absorptive part.
External comparisons	Identified solver records	CAMB background at the stated inputs. Fixed-background recombination and DESI scale response. SPARC comparison.
Source dynamics	Same calculation	Nonlinear material flow and canonical phase-space evolution. Phase-clock readout and covariant source exchange. Joint static amplitude-radius response.
Verification	24 October 2026	Numerical reproduction and independent identity and convergence checks. Interval certification and input isolation. The finite structural checks described above.

20.791529232604695. The sensitivity table uses the analytic logarithmic derivatives of Appendix D evaluated at the same endpoint and couplings.

Numerical accuracy and interacting responses. Verification distinguishes algebraic and input consistency checks from independent quadrature, interval bounds and refinement calculations. Repeated checks of the same numerical example share its evaluated data. Independent formulas, alternative discretizations and altered inputs are evaluated separately. Stored responses are reused when their source dependencies and input fingerprints agree with the current calculation.

The fixed-Yukawa gauge evolution uses a fourth-order integrator with step-doubling control. The colour boundary is checked by forward evolution to the breaking scale with relative residual below 10^{-12} . In the declared normalization the connected finite-source Hessian and torsion coefficient are

$$\text{diag}(224/225, 2/9, 2/15, 2/15, 2/15), \quad \tau = 1/50.$$

Relations among quantities derived from the same inputs provide numerical consistency checks including the geometric mass combination (7.3) and its common-input derivatives.

Table 17. Full SHA-256 identities of the frozen calculation and its external inputs. The two lines in each entry concatenate to one digest.

Object	SHA-256
Calculation source set	057074ac9761f3d5d27a3bcc5d3b3b88 43ecb395c97da15e37347263ebcf19a4
Endpoint certificate	9b712f9770f80dd427a865b2c10543c6 5ff46887a62116716017742d62970324
Scientific records	a70c81546d8e7635c1ddc5e646fae9d4 17289460b20a1f534834c766fc3e9643
Reference records	749968311efddfc5b767403e7fcd6294 4cb11c434815e004214059a030ef447d
CAMB production	ee13f231337d3fc797ef5fd2f4b4c508 33fa6483643c07efe96c77bd0b82302c
Thermal-study CAMB	45c2439ee34a9a38f0b6cfd80572d86f 78b39267f9352fc403d8314919ba65df
HYREC resources	e212aab017f0d118a26192dcf3195ecd d4c1eada5f20fb9494c3d4ba692b447a
Thermal response	1074d4403d31f8b3d1ff8be9393e05b8 7609e6c3718f1f58cf3b6a548c8efae6
DESI DR2 input	2d7c70e3d9cb4fa27ade312869b76b32 b2bc6da9cafb492a8fa5eddfa0d53103
SPARC input	92158efa833151be9807e6fef83b390c 4b32297c877c88b0481e1b276e8c8fee

Further calculations evaluate the complete condensate for specified (g, Z, L) with spectral tail bounds on its mass, the complex mixing texture, the nonspherical static Dirichlet problem and the finite material flow realization of Section G.5. For inputs with an assigned covariance Σ the analytic logarithmic Jacobian gives the first-order output covariance $J\Sigma J^T$. Numerical residuals and interval bounds retain their respective meanings when no probability model is assigned to the inputs.

The joint vacuum of Proposition C.6 is certified in the same response units at $g = 1$ and $Z = 2$. One additional stationary enclosure at the exact radius $L_* = \sqrt{2\pi}$ with 128 shells and 55 bisections per bound gives $u_* \in [0.1135162098, 0.1137652465]$ and the relaxed compression curvature in $[0.1840255478, 0.1844292709]$. The joint Hessian bounds reuse this enclosure and its spectral tail. Independent checks evaluate the original potential and its square completion and its two source derivatives. The covariant matter calculation independently differentiates the Hilbert stress including the metric connection and compares its divergence with the matter equation and source exchange. The same source matrices enter the lapse, shift and canonical dilation checks. Finite matrix tests use known constrained quadratic actions to check the mixed field Schur reduction, basis covariance and sign intervals under a supplied error bound. The local material flow and independent foliation calculations have separate consistency checks. Each record identifies the action, background and varied fields.

The occupied common background of Proposition G.5 uses the same joint vacuum

normalization with $\mu = M_G$ and $g = 1$ with $Z_E = 2$ and $m_0^2/\mu^2 = 3/(8\pi)$ and $L_0^2 = 3\pi$. The dimensional inputs give $M_P/\mu = 1.40682610$ and $\rho_\Lambda/\mu^4 = 2.81008 \times 10^{-120}$. At each background evaluation the radius equation is solved at fixed canonical data. The full potential is evaluated only for the initial energy and the sampled Friedmann checks. Its analytic derivatives supply the evolution. The numerical trajectory covers $\mu\Delta s = 1/(4\sqrt{3/(8\pi)}) = 0.72360125$. The phase integral is checked against the polar angle of the evolved matter pair. The maximum relative Friedmann and phase-charge residuals are 2.9×10^{-12} and 6.4×10^{-13} respectively. The phase-reconstruction residual is below 2.7×10^{-14} . The trajectory uses 62 right-hand-side evaluations and 101 algebraic source solves including the output and phase-integral evaluations. Eighteen potential integrals supply the vacuum and initial energy together with the remaining output checks. These floating-point checks are separate from the continuous interval certificate on $0 \leq \mu s \leq 0.01$ in Appendix G. That certificate evaluates eight distinct spectral endpoints totaling 520 shell-pair evaluations and uses vector-field bounds to exclude exit from the phase-space box. It includes the infinite spectral tail and requires no interval time integration or potential quadrature [15].

For the elementary matter connector of Ref. [4] elimination of the neutral coordinate gives the charged compression and an effective source. The readout and direct term complete the reduced response, and all four terms are retained. Mode products are evaluated before applying the selected projection. The noncommuting finite-source Hessian is the Duhamel response of the matrix exponential. Spatial vertices and kinetic Gram matrices are computed separately and their canonically normalized combinations are compared under changes of mode basis. The five-variable gauge–Yukawa–quartic evolution uses specified matching boundaries and threshold conditions for each direction of evolution. The observational comparison values do not enter these boundary conditions. The finite electroweak calculation imposes the generated Fermi and top conditions together with the Higgs condition in Proposition C.5. Each distinct one-loop integral is evaluated once at the leading coefficients and reused in the finite inverse relations. Independent checks include the pure-QCD pole limit and Born top width together with the explicit one-loop scale cancellation and substitution into the forward conditions. The charged-lepton calculation retains the finite masses in the photon and weak kernels. Nine analytic two-point functions are evaluated by interval arithmetic and checked against independent parameter integrals and the small-mass weak limit together with the weighted scale derivative. The weak-decay and helium Jacobian reuses the accepted central phase-space partition for its derivatives and differentiates the freeze-out and conserved-decay relations analytically. The signed spatial kernel and generated family rotations are checked against independently evaluated Gaunt products on their complete finite mode spaces [15].

Canonical normalization combines the finite interaction coefficients, spatial vertices and kinetic Gram matrices. The scalar response uses the modes coupled to its source, the canonical two-derivative external action and the internal mass spectrum. The source and action parameters are included in Ref. [15].

An additive torsion probe h at fixed matter projection and response domain tests the

derivative of the composite endpoint and mass map

$$E(c, h) = \frac{A_R(\tau + h)}{c} - \frac{4}{27}, \quad O(c, h) = \log \frac{v(c, \tau + h)}{M_P}.$$

Here A_R is the coefficient in Proposition 4.2. Implicit differentiation gives

$$\frac{dO}{dh} = O_h - O_c(E_c)^{-1}E_h = -2.501835\dots \quad (R = 5, \tau = 1/50).$$

Independent re-solving of the original scaled spectral endpoint on both sides of the probe verifies this derivative with relative residual below 10^{-8} for the stated probe increments. The derivative includes both the direct torsion term and the change in the endpoint.

The fixed-connection scalar energy source of Ref. [11] is evaluated on the compact modes. For the positive quadratic form $A_B = D_B^\dagger D_B + 3/(4L^2)$ and stationary occupation N the occupied energy is

$$E_N(\eta) = \text{Tr} \left[N \sqrt{e^\eta A_B e^\eta} \right].$$

For the nonzero connection considered here the uniform source charge equals the total occupied energy. The three-source Hessian includes the second derivative of the source map and the full square-root feedback. Independent finite-source differences give a relative residual below 3×10^{-8} . The comparison between two retained mode spaces includes the full second-order intermediate support of the occupied states. The resulting occupied-matter susceptibility uses the inherited source convention. An external collective tensor propagator requires its own action and source map [15].

The same operator and occupied states also enter the canonical dynamics of Proposition G.3. Three finite source segments use amplitudes $(0, 1/32, 0)$ and $(0, 0, -1/32)$ followed by $(0, 0, 0)$ with durations $(1/4, 1/2, 1/4)/\omega_{\min}$ where $\omega_{\min} = 0.6837072629$ in the declared reciprocal-radius unit. The phase-space second moment is transported by the exact constant-source propagator. The relative canonical-form residual is below 2×10^{-14} and the absolute source-work balance residual is below 2×10^{-14} in the declared energy unit. These are floating-point consistency checks on the finite source history. Expressing the same segments with positive clock rates $(1/2, 3/2, 2)$ checks the invariance of the transported second moment and source work under Eq. (G.19). The second parametrization reuses all three source matrices and requires three additional exact propagators. The phase-clock rate and integrability condition of Proposition G.4 are checked on the existing homogeneous and static accelerated cases together with the radial-pressure case using their already evaluated local field derivatives. The causal response is evaluated at $\zeta/\omega_{\min} = i/2, i, 1 + i/2$. An independent one-oscillator formula and a direct phase-space resolvent test its frequency branches and source contact term. The mode-basis calculation uses entrywise frequency denominators and reuses the reference spectrum to avoid a dense phase-covariance superoperator factorization [15].

B.3 External comparisons

Reference-coupling extrapolation. The Standard-Model comparison starts at the reference $M_Z = 91.1876$ GeV with

$$(g_1, g_2, g_3, y_t, \lambda) = (0.461425, 0.64779, 1.217200, 0.940, 0.129).$$

It uses $g_1 = \sqrt{5/3} g_Y$ and the potential convention $V(H) = -m^2|H|^2 + \lambda|H|^4$ and evolves the five-coupling two-loop subsystem to M_G [15]. These boundary couplings define the separate Standard-Model extrapolation of Ref. [15] independently of the mass and α_s comparison benchmarks in Table 14.

Cosmological and external-data comparisons. The CAMB 2.0.4 background calculation uses the density components and thermal inputs specified in Section 8.5. It evaluates the homogeneous expansion and recombination together with the acoustic scale and distances. The seven-coordinate response uses the same central solution and eighteen thermal-background probes including the two half-step checks of Table 5. The calculation comprises one density initialization and nineteen thermal-background solutions in total. The independent distance quadratures reuse these solutions. The DESI DR2 combined-sample mean and covariance correspond to Table 4 of Ref. [31]. Its thirteen-residual statistic uses the full positive-definite covariance and calculated drag scale together with the full thermal background stated in Section 8.5. An independent quadrature and Cholesky evaluation agrees. The finite thermal-history calculation adds three RECFAST background evaluations and two full radiative-transfer recombination histories at the same expansion background. The latter propagate the standard and generated atomic constants separately [33, 34]. Their drag integrals and the low-redshift electron-capacity bound feed the same thirteen distance ratios and covariance projection. The solver resources and generated atomic inputs together with the execution identities are retained with the thermal result [15]. Ordinary verification reuses that result after checking its inputs. This budget uses thermal background solutions without additional power-spectrum calculations. The SPARC comparison uses the mass-model data of Ref. [47] with the pointwise acceleration map and cuts stated in Appendix G. The three-dimensional calculation tests the static field equation on specified source geometries.

B.4 Spectral traces and projection conventions

The calculation [15] uses the signed spectrum (4.1) in three distinct ways.

1. The generation count compares the positions $n + 3/2$ for $n = 0, 2, 4, \dots$ with $(kL)^2$. Distinct positions of $\hat{D}|_{P_+}$ have unit weight.
2. The spectral trace uses squared Dirac eigenvalues $(n + 3/2)^2/L^2$ with multiplicities $(n + 1)(n + 2)$. The even-index sum evaluates the trace of $P_+ f(D_{\tau_-}^2)$. The complete $f(D_{\tau_-}^2)$ trace includes both parities.
3. The chiral assignment uses odd indices for Q_L, L_L and even indices for u_R, d_R, e_R . The uniform-even comparison uses even indices for every species. Each response combines the squared eigenvalues and species-weighted multiplicities with the channel weight of Eq. (C.3).

In the spin-structure convention of Ref. [4] the species assignment is $D_{\tau_-}^2|_{P_-}$ for left species and $D_{\tau_-}^2|_{P_+}$ for right species. These projectors belong to one complete geometric

spin structure. Reversing the spin convention reverses the sign assignment while retaining the indexed sums.

The generation edge is $c_{\text{gen}} = (kL)^2$ whereas the endpoint trace uses the dimensionless radius $\ell_{\text{end}} = kLM_G/k$ and the squared Gaussian cutoff $\Lambda^2 = (k/M_P)^2$. Accordingly the second argument of $\Pi(L, \Lambda^2)$ denotes a squared cutoff. The physical radius is $L_{\text{phys}} = kL/M_G$ as in (4.3). Mode and quadrature bounds control the numerical evaluation of the smooth trace independently of the generation edge.

The calculation [15] evaluates all five gravitational channels in the common Planck-unit convention of Section 8. It gives $r_{\text{read}} = 85.9303$ with $Z_{\text{read}} = 0.0115035$ and $G_{\text{read}} = 5.8320 \times 10^{-37} \text{ GeV}^{-2}$. These values characterize the finite combined-channel readout in its stated convention. Newton’s constant sets the macroscopic metric normalization and remains the dimensional input throughout the particle and cosmological chains.

Spectral responses are compared at the same operators, domains, species projectors, multiplicities, channel weights and units. In particular identifying the positive rough TT Laplacian of Ref. [4] with a Casimir-plus-curvature operator requires an identification of their mode spaces and action on each mode.

B.5 Scope of the formal verification

The archived Lean proof library [18] verifies the indexed clauses connecting the source construction to the basic physical interface within the comparison classes recorded in its theorem index. A separate finite algebraic check verifies matter counts and representation-content identities together with normalisation factors. It also checks denominator-cleared rational two-loop coefficients and the rest-frame cold-source tensor decomposition from their declared premises. Continuum spectral and evolution arguments are given analytically in the manuscript. The formal records provide an independent check of the indexed constructions and their stated premises. The numerical evaluation described in this appendix also checks $M_\nu U_{\cdot i} = m_i U_{\cdot i}$ for every labelled column of the real CP-conserving mass texture. This identity tests the mass assigned to each mixing column. The mixing ansatz and the additional solar propagation relation are specified separately in Appendix C.

C Key derivations

The endpoint reduction and finite source responses determine the internal scale and coupling boundary of Sections 4–5. The remaining calculations give the renormalization group coefficients, mass and electroweak relations, and cosmological and abundance relations used in the numerical tables. The geometric and spectral data are those of Ref. [4]. Each readout fixes its source projection and normalization. Source derivatives measure changes in the mode response; kinetic normalization converts the resulting matrix elements into couplings or mass amplitudes. This common normalization permits the parameter eliminations and joint responses. Newton’s constant sets the dimensional scale.

C.1 The window-capacity endpoint

Inherited Gaussian rule. We use the “Gaussian uniqueness under composition” theorem of Ref. [4]. On its minimal change carrier, weak continuity, centred unbiased composition, square root scaling and fixed nonzero covariance determine the Gaussian semigroup. For the nonnegative spectral generator supplied by that construction we use the trace convention e^{-tH} with $t = \Lambda^{-2}$. The frequency resolved sum $H = \omega^2 + A$ is restricted to the spatial branch assigned to each species.

The window capacity kL is fixed by the spin-2 endpoint condition of Section 4.4. Its full-domain zero-external-momentum TT spectral readout has the form specified for the compact branch in Section 2.2 of Ref. [11]:

$$\Pi_{\mu\nu}^2(L, \Lambda^2) = \frac{1}{V_3} \sum_{\text{fields}} \sum_{\text{modes}} d_n w(\text{field}) \int_0^\infty \frac{d\omega}{\pi} K_{\text{TT}}(\omega^2 + \lambda_n, m_{\text{eff}}^2) e^{-(\omega^2 + \lambda_n)/\Lambda^2}, \quad (\text{C.1})$$

Here $V_3 = \pi^2 L^3$ and d_n is the mode multiplicity and $w(\text{field})$ is the channel weight. The second argument is the squared cutoff Λ^2 . The fixed content and torsion are suppressed. Set $q^2 = \omega^2 + \lambda_n$ for the internal spectral variable. The running scale is denoted by k . The kernel is

$$K_{\text{TT}}(q^2, m^2) = \frac{q^4}{(q^2 + m^2)^2}. \quad (\text{C.2})$$

For $q^2, m^2 \geq 0$ with nonzero denominator $0 \leq K_{\text{TT}} \leq 1$ and $K_{\text{TT}}(q^2, 0) = 1$ for $q^2 > 0$. The flat-space reference is the Ward-normalised subtracted amplitude. The weights are $w = 4$ for each gauge polarisation or Weyl fermion and $w = 1$ for each real scalar. The shifts are $m_{\text{sc}}^2 = 3/(4L^2)$ and $m_V^2 = C_2/(2L^2) + \tau^2/(6L^2)$ and $m_F^2 = 3\tau^2/(8L^2)$ where $C_2 = 3, 2, 0$ for $\text{SU}(3), \text{SU}(2), \text{U}(1)$. The scalar value is ξR with $\xi = 1/8$ and $R = 6/L^2$. The fermionic curvature is already included through $\not{D}^2 = \nabla^* \nabla + R/4$. The auxiliary TT vertex uses the Lichnerowicz shift $6/L^2$; the trace sums the matter rows listed above. The common scale path is $L = kLM_G/k$.

For the fermion contribution the mode sum in this formula has the explicit meaning

$$\sum_f \sum_{\substack{n \geq 0 \\ n \equiv \epsilon_f \pmod{2}}} N_{\text{int}}(f)(n+1)(n+2) w(f) \mathcal{I}\left(\frac{(n+3/2)^2}{L^2}, m_f^2; \Lambda^2\right), \quad (\text{C.3})$$

where $N_{\text{int}}(f)$ is the number of internal components of species f , $\mathcal{I}$ is the frequency integral above, and the default assignment is $\epsilon_{Q_L} = \epsilon_{L_L} = 1$ and $\epsilon_{u_R} = \epsilon_{d_R} = \epsilon_{e_R} = 0$. This is a species-weighted trace of the restrictions $D_{\tau_-}^2|_{P_-}$ and $D_{\tau_-}^2|_{P_+}$ defined in (4.2). An unrestricted trace of $D_{\tau_-}^2$ would sum over both parities for every species. The uniform-even comparison sets all $\epsilon_f = 0$. Field multiplicities and channel weights together with spatial branch projections are separate factors in the trace.

The Gaussian envelope is fixed by the inherited rule above. Appendix D compares the Gaussian response with a hard spectral window and quantifies its domain sensitivity.

The endpoint condition on the volume-weighted spectral readout is

$$\frac{V_3 \hat{\Pi}_{\mu\nu}^2(kL, k^2/M_P^2)}{32\pi^2} = \frac{4}{27} \quad \text{at} \quad k = M_G. \quad (\text{C.4})$$

The tensor mass response maximum fixes the endpoint normalization $4/27$:

$$\frac{4}{27} = \max_{0 \leq y \leq 1} y(1-y)^2 \quad \text{at} \quad y = \frac{1}{3}, \quad (\text{C.5})$$

where $y = m^2/(q^2 + m^2)$ and $K_{\text{TT}} = (1-y)^2$. On $L(k) = C/k$ with $C = M_{\text{P}}\sqrt{\pi}$ the volume and cutoff reduce the endpoint condition to one equation in kL . Numerical bisection on $[0.3 M_{\text{G}}, 3 M_{\text{G}}]$ and self-consistency iteration between $M_{\text{G}} = C/kL$ and the crossing scale $k_{\star}$ give

$$kL = 2.49353. \quad (\text{C.6})$$

In the endpoint formula the radius symbol denotes the dimensionless argument $\ell_{\text{end}}(k) = kLM_{\text{G}}/k = C/k$ and the Gaussian scale is $\widehat{\Lambda}_{\text{tr}} = k/M_{\text{P}}$ with its square as the second readout argument following Section 4.4. The physical KK radius and the generation edge are instead fixed by (4.3). The quoted fixed point is evaluated with this endpoint definition and the projected fermion trace (C.3).

The screened coupling law of Section C.4 gives a consistency test through the inverse in Eq. (4.5). It relates the screened reference coupling to the bare boundary $\alpha_W = 2/(kL)^5$. The endpoint geometry $L_{\text{Cg}} = \sqrt{\pi}$ fixes $R_c = 6/\pi$ and the resulting capacity is $(kL)^2 = 6.2177$.

Endpoint normalization and propagation. The modewise extremum above sets the endpoint rule. The two-leg response and its logarithmic derivative are given in Section 4.4. Here $4/27$ normalizes the modewise survival response. The spectral sum enters the separate endpoint equation above. The macroscopic gravitational realization in Ref. [11] uses one metric in the Einstein and matter actions. Its quadratic tensor response about flat spacetime has two physical helicities and positive residue g^2/Z . The “Matched critical branch” proposition identifies $(1 - x(\chi))/\Pi(\chi)$ with $\widehat{Z}Q^2/M_{\text{P}}^2$ and derives $\chi(0) = \chi_c$ and $-\text{d}\chi/\text{d}(Q^2/M_{\text{P}}^2) = \widehat{Z}\Pi(\chi_c)/x'(\chi_c) > 0$. The effective normalization $g = 1$ with $Z = M_{\text{P}}^2$ gives $G_N = (8\pi M_{\text{P}}^2)^{-1}$. The radius equation fixes the compact endpoint and the common metric action and inverse-response matching fix the macroscopic propagation and its coupling to conserved matter.

Homogeneous endpoint reduction. Fix the matter content and species projectors together with τ . Set $a_j = L^2\lambda_j$ and $b_j = L^2m_j^2$ and $D_j = d_j w_j \geq 0$ including field multiplicities in D_j . For a dimensionless frequency domain $R > 0$ define

$$A_R = \frac{1}{32\pi^3} \sum_j D_j \int_0^{\sqrt{(R\pi - a_j)_+}} \left(\frac{u^2 + a_j}{u^2 + a_j + b_j} \right)^2 e^{-(u^2 + a_j)/\pi} du, \quad (\text{C.7})$$

$$(x)_+ = \max(x, 0).$$

For $R = \infty$ integrate to infinity over the full assigned towers. Their polynomial multiplicities and quadratically growing eigenvalues make the Gaussian sum convergent. Finite R defines a restricted response with the same Gaussian weight.

Proof of Proposition 4.2. Write $c = kL$ and $C = M_P\sqrt{\pi}$ with $M_G = C/c$ and $t = k/M_G$. Apply the “Scale law for a TT spectral density” of Ref. [4] with $k_0 = C$ and $L_0 = 1$ together with $\Lambda_0 = \sqrt{\pi}$ and $r = k/C = t/c$. Use frequency parameter $b = R$ and reference eigenvalues and mass squares a_j, b_j . These give $L = c/t$ and $\Lambda^2 = \pi t^2/c^2$. Its reference readout is A_R in (C.7). Thus the inherited scale law yields

$$\frac{V_3 \hat{\Pi}_R^2}{32\pi^2} = \frac{tA_R}{c}. \quad (\text{C.8})$$

At $t = 1$ this gives $F_R(c) = A_R/c - 4/27$. A positively weighted mode with nonempty integration interval makes $A_R > 0$. The scalar constant mode already does so. The unique root is $c_R = 27A_R/4$ and no dimensional anchor occurs in A_R . The cited proposition applies directly to finite selections and finite R . For the full towers and $R = \infty$ pass through increasing finite modal and frequency domains. Positivity and the convergent Gaussian majorant preserve the same identity in the limit. $\square$

For fixed c (C.8) increases linearly with t and its crossing is $t_\star = 4c/(27A_R)$. Provided this lies in the chosen bracket the exact self-consistency update is $c_{\text{new}} = C/k_\star = c/t_\star = 27A_R/4$. The numerical iteration implements this fixed branch identity. Quadrature error concerns the fixed domain integral. The Gaussian tail measures its difference from the full-domain response.

At $R = 5$ the nonzero scalar degrees are $l = 0, 2$ and the vector degree is $l = 1$. The spinor degrees are $n = 0, 2$ on P_+ and $n = 1$ on P_- . The scalar values of a_j are 0, 8. The vector value is 4. The spinor values are 9/4, 49/4, 25/4 in the stated order. The numerical mode ceiling 60 contains all these modes. Along $c, t \in [0.3, 3]$ we have $L \in [0.1, 10]$ and $\Lambda^2 \geq \pi/100$ and $q^2 + m_j^2 \geq 3/400$ for every positively weighted mode. These lower bounds keep the numerical regularization of the denominator and cutoff inactive. The full Gaussian tail is bounded in Appendix D.1.

Proposition C.1 (Bandwidth dependence of the spectral response). Keep the full assigned towers and content together with the source projectors and mass shifts in (C.7) fixed. Then A_R is continuous and strictly increasing for $R > 0$ and $A_R \uparrow A_\infty$. Away from shell thresholds $R\pi = a_j$ its derivative is

$$\frac{dA_R}{dR} = \frac{e^{-R}}{64\pi^2} \sum_{a_j < R\pi} \frac{D_j}{\sqrt{R\pi - a_j}} \left(\frac{R\pi}{R\pi + b_j} \right)^2 > 0. \quad (\text{C.9})$$

For the three-copy trace at $\tau = 1/50$ every $R \geq 5$ satisfies $11/2 < c_R^2 < 15/2$ and hence retains the same three positive positions. At fixed Planck anchor with fixed loop factor and squash correction its mass readouts obey

$$\begin{aligned} \frac{v_R}{v_5} &= \frac{c_5}{c_R} e^{-4\pi(c_R - c_5)}, & \frac{m_{e,R}}{m_{e,5}} &= e^{-20(c_R - c_5)}, \\ \frac{v_R}{v_5} &= \frac{c_5}{c_R} \left(\frac{m_{e,R}}{m_{e,5}} \right)^{\pi/5}. \end{aligned} \quad (\text{C.10})$$

Proof. Each finite domain contains finitely many spectral shells. Their integrals are continuous as a shell enters the domain. Differentiating the upper limit gives (C.9). The positively weighted scalar constant mode makes the increase strict. Monotone convergence and the Gaussian majorant give the full-domain limit. Since $c_R = 27A_R/4$ the certified bounds in Appendix D.1 and (D.11) imply $2.490 \leq c_5 \leq c_R \leq c_\infty \leq 2.6631205$. The squares of the two outer bounds lie strictly between $11/2$ and $15/2$. Finally substitute $M_G = M_P\sqrt{\pi}/c_R$ into the electroweak readout and eliminate $c_R - c_5$ using the electron readout. $\square$

Fixed shell support and the generation interval. For the three-copy endpoint trace at $\tau = 1/50$ the reference shell set consists of the positively weighted shells with $a_j \leq 49/4$. It includes the $n = 2$ spinor at $a = 49/4$ and excludes the next $l = 3$ vector at $a = 16$. A domain R contains exactly this set of nonzero frequency integrals precisely when

$$\frac{49}{4\pi} < R \leq \frac{16}{\pi}. \quad (\text{C.11})$$

There is a unique R_* in this interval with $c_{R_*} = \sqrt{11/2}$. The simultaneous conditions of this shell support and three retained generation positions are exactly

$$R_* < R \leq \frac{16}{\pi}, \quad R_* \simeq 4.0187119842, \quad \frac{16}{\pi} \simeq 5.0929581789. \quad (\text{C.12})$$

The value displayed for R_* is the numerical solution of its defining equation. The existence and ordering assertions follow from the enclosures below and strict monotonicity. The largest active spectral value is $a = 49/4$ from the $n = 2$ spinor branch. The next positively weighted shell is the $l = 3$ vector with $a = 16$. A shell has a nonzero frequency integral exactly when $a < R\pi$. This proves (C.11). The new shell contributes zero at the upper endpoint.

The finite integrals at these two edges admit the outward enclosures

$$\begin{aligned} c_{49/(4\pi)} &\in [2.1897287252, 2.3186156701], \\ c_{16/\pi} &\in [2.4131554931, 2.5852484689]. \end{aligned} \quad (\text{C.13})$$

At $R = 4$ the same outward cell scheme separates the endpoint response from $4/27$ at the bracket boundaries and fixes its derivative sign, giving the enclosure

$$c_4 \in \left[\frac{187}{80}, \frac{23939}{10240} \right] = [2.3375, 2.33779296875]. \quad (\text{C.14})$$

Its upper endpoint is below $\sqrt{11/2} = 2.3452078799$. This certifies the separation used in Proposition 4.1 directly at $R = 4$.

Write $K_j(u) = [(u^2 + a_j)/(u^2 + a_j + b_j)]^2$. On a cell $[u_i, u_{i+1}]$ of width h_j its monotonic increase and the monotonic decrease of the Gaussian give

$$\begin{aligned} h_j K_j(u_i) e^{-(u_{i+1}^2 + a_j)/\pi} &\leq \int_{u_i}^{u_{i+1}} K_j(u) e^{-(u^2 + a_j)/\pi} du \\ &\leq h_j K_j(u_{i+1}) e^{-(u_i^2 + a_j)/\pi}. \end{aligned} \quad (\text{C.15})$$

Apply these bounds with 32 equal cells per nonzero branch using the coefficients of (C.7) and the outward arithmetic of Appendix D.1. The first edge has seven nonzero branches and the second has eight. The calculation therefore uses 480 branch cells. Sharing Gaussian nodes between the vector branches requires 363 exponential enclosures [15]. It gives (C.13). The squared bounds satisfy

$$c_{49/(4\pi)}^2 < \frac{11}{2} < c_{16/\pi}^2 < \frac{15}{2}.$$

Continuity and strict monotonicity from Proposition C.1 give the unique crossing R_* . The strict retention rule requires $11/2 < c_R^2 \leq 15/2$. This yields (C.12). The coarse enclosures certify these separations. The finer mass intervals use the quadrature bounds stated in Appendix D.

Scale-ratio identification of the endpoint. For the positive electroweak and electron readouts expressed in the same mass units put $\beta = 20 - 4\pi$. Then

$$\mathcal{Q}(c) := \frac{v}{m_e} = \frac{3}{16\pi^{5/2}} \frac{e^{\beta c}}{c}, \quad \frac{d \log \mathcal{Q}}{dc} = \beta - \frac{1}{c} > 0 \quad \left(\frac{11}{2} < c^2 \leq \frac{15}{2} \right). \quad (\text{C.16})$$

Thus a value of $\mathcal{Q}$ in the image of this branch determines a unique c . At fixed source content and τ it also determines a unique bandwidth whenever it lies in the corresponding scale-ratio range of (C.12). Bounds on $\mathcal{Q}$ propagate through these monotone inverses. On an open shell interval

$$\frac{dc}{d \log \mathcal{Q}} = \frac{1}{\beta - 1/c}, \quad \frac{dR}{d \log \mathcal{Q}} = \frac{1}{(\beta - 1/c_R) dc_R/dR}. \quad (\text{C.17})$$

With $q(\tau) = 1 - s_0 \kappa$ the two readouts are

$$v = \frac{3M_P q(\tau)}{16\pi^{5/2} c} e^{-4\pi c}, \quad m_e = M_P q(\tau) e^{-20c}.$$

Their ratio cancels both the dimensional anchor and the common squash factor. Differentiating its logarithm gives (C.16). On the stated branch $\beta > 7$ and $1/c < 1$. Strict monotonicity of c_R gives the bandwidth inverse and the chain rule gives (C.17).

An empirical use of this inverse first matches v and m_e to the common mass definitions of the readouts. At $R = 5$ the central values are

$$\mathcal{Q} = 4.8238982197 \times 10^5, \quad \frac{dc}{d \log \mathcal{Q}} = 0.14219508, \quad \frac{dR}{d \log \mathcal{Q}} = 2.29569558.$$

The geometric charged-lepton ladder gives a complementary constraint. $m_\mu^{\text{geom}}/m_e^{\text{geom}} = e^{2\alpha_{1p} + \sqrt{2}\pi}$ and $m_\tau^{\text{geom}}/m_\mu^{\text{geom}} = e^{2\alpha_{1p}}$ imply

$$\frac{(m_\mu^{\text{geom}})^2}{m_e^{\text{geom}} m_\tau^{\text{geom}}} = e^{\sqrt{2}\pi} = 12.2635110816. \quad (\text{C.18})$$

This relation eliminates the endpoint, torsion, dimensional anchor and bandwidth. It tests the Euclidean period insertion in the light family transfer. Equation (7.6) gives its pole

and running forms through G_ℓ and D_ℓ respectively. Its joint comparison with the sector relation uses the total factors $C_\ell = D_\ell G_\ell$ in (7.8).

The projection P_R of Section 4.4 restricts the source measurement before the trace is evaluated. It is held fixed in the reference spectral coordinates when taking the corresponding finite-source response. A transported boundary has the additional term in (C.38). The numerical tables use the acceptance threshold $R = 5$ and the stated source normalization. Equations (C.12) and (C.16) combine the generated shell support and generation count with a matched scale ratio into a quantitative bandwidth test.

Joint content and endpoint selection. Replicate the same one-generation content g times while keeping the bosonic content and τ with the projectors and mass shifts fixed. Linearity of the spectral trace gives

$$A_{g,R} = A_{B,R} + gA_{F,R}, \quad c_{g,R} = a_R + gb_R, \quad (a_R, b_R) = \frac{27}{4}(A_{B,R}, A_{F,R}). \quad (\text{C.19})$$

Here a_R, b_R denote endpoint coefficients. The mode coefficients a_j, b_j are defined in (C.7). Exactly g positive positions lie below the endpoint window if and only if

$$2g - \frac{1}{2} < (a_R + gb_R)^2 \leq 2g + \frac{3}{2}. \quad (\text{C.20})$$

The two bounds are the last retained position $2(g-1) + 3/2$ and the next position $2g + 3/2$ with the strict retention convention. This condition applies to the replication family (C.19).

The finite-domain coefficients follow from the integrals

$$I(a, b) = \int_0^{\sqrt{5\pi-a}} \left(\frac{u^2 + a}{u^2 + a + b} \right)^2 e^{-(u^2+a)/\pi} du, \quad 0 \leq a < 5\pi.$$

With $b_V = \tau^2/6$ and $b_F = 3\tau^2/8$ at $\tau = 1/50$ the projected towers give

$$\begin{aligned} a_5 &= \frac{27}{128\pi^3} [4I(0, 3/4) + 36I(8, 3/4) + 192I(4, 3/2 + b_V) \\ &\quad + 72I(4, 1 + b_V) + 24I(4, b_V)], \\ b_5 &= \frac{27}{128\pi^3} [56I(9/4, b_F) + 192I(25/4, b_F) + 336I(49/4, b_F)]. \end{aligned} \quad (\text{C.21})$$

The coefficients include the scalar multiplicities and twelve vector species and the odd-left/even-right assignment per generation. The interval calculation in Appendix D.1 gives $0.600 \leq a_5 \leq 0.602$ and $0.630 \leq b_5 \leq 0.632$.

Endpoint compatibility of the replication family. The three-copy content satisfies its endpoint and window conditions at $R = 5$. The same endpoint rules also admit one copy in the fixed-data family (C.19). The certified coefficient intervals give

$$c_1 \in [1.230, 1.234], \quad c_3 \in [2.490, 2.498].$$

Their squares lie in $(3/2, 7/2]$ and $(11/2, 15/2]$ respectively. These are the two plateaus in (C.20). Proposition C.2 selects the three-copy solution by imposing the joint screening condition. The numerical tables use its $R = 5$ endpoint. Appendix D.1 gives the complete-domain coefficients for comparison.

Uniform torsion response. Keep the species and projections fixed together with domain $R = 5$ or $R = \infty$ and vary only the specified mass shifts. Put $z = \tau^2$ and $b_j(z) = \beta_j + h_j z$ with

$$(\beta_j, h_j) = (3/4, 0), \quad (C_2/2, 1/6), \quad (0, 3/8)$$

for scalar and gauge entries together with fermion entries. Then $A_R(z)$ of (C.7) is completely monotone on $[0, \infty)$ and

$$-\frac{2}{6+z} \leq \frac{d \log A_R}{dz} \leq 0, \quad \left(\frac{6+z_1}{6+z_2} \right)^2 \leq \frac{A_R(z_2)}{A_R(z_1)} \leq 1 \quad (z_2 \geq z_1 \geq 0). \quad (\text{C.22})$$

The same bounds hold for $c_R(z) = 27A_R(z)/4$. With $\mathcal{B}_R = -A'_R(0)/2$ the linear remainder obeys

$$0 \leq A_R(z) - A_R(0) + 2z\mathcal{B}_R \leq \frac{z^2}{12} A_R(0). \quad (\text{C.23})$$

Set $X_j = u^2 + a_j$ and $r_j = h_j/(X_j + \beta_j)$. The lowest gauge and spinor denominators give $0 \leq r_j \leq 1/6$. Normalizing the positive $z = 0$ integrand to a probability measure ν_0 gives

$$A_R(z) = A_R(0) \int (1 + r_j z)^{-2} d\nu_0.$$

Differentiation proves complete monotonicity. The bounds follow from $r_j/(1 + r_j z) \leq 1/(6 + z)$ and $0 \leq (1 + x)^{-2} - 1 + 2x \leq 3x^2$. Bounded r_j and the Gaussian majorant justify differentiation and the complete-domain limit. At $\tau_0 = 1/50$ the relative Taylor remainder is at most 1.334×10^{-8} . For $0 \leq \tau \leq 0.1$ the finite ratio gives $0.9968 < c_{g,R}(\tau)/c_{g,R}(\tau_0) < 1.000134$. Applying this ratio to the certified intervals preserves both finite-domain plateaus in Eq. (C.20) and the three-copy full-domain plateau certified in Appendix D.1. Even independently varying the two torsions in this interval

$$c_{3,\infty}(\tau_\infty) - c_{3,5}(\tau_5) > 0.9968(2.6603141) - 1.000134(2.498) > 0.1534.$$

These bounds quantify the sensitivity to the torsion-dependent mass shifts while holding the response domain and source projections fixed.

C.2 The finite-content torsion response

Section 3.3 defines $\tau = (N_L - N_R)/(N_f \sum Y^2) = 1/50$ from the one-generation counts $N_L = 8$ and $N_R = 7$ with $N_f = 15$ and $\sum Y^2 = 10/3$. Replication and spatial weighting act differently on this ratio. The hypercharge scale is the normalised Y operator supplied by Ref. [4]. The homogeneous anomaly equations alone leave an overall nonzero rescaling of Y undetermined.

Replication and common-window normalisation. Let $\mathcal{H}_1 = \mathcal{M}_{\text{ch}}$ be the fifteen-dimensional charged response carrier for one generation. Set $\chi = +1$ on its eight left components and $\chi = -1$ on its seven right components. Use the given hypercharge operator Y . On $\mathcal{H}_g = \mathbb{C}^g \otimes \mathcal{H}_1$ let $\chi_g = I_g \otimes \chi$ and $Y_g = I_g \otimes Y$. Then

$$\bar{J}_g = \frac{\text{Tr } \chi_g}{15g} = \frac{1}{15}, \quad \bar{A}_g = \frac{\text{Tr } Y_g^2}{g} = \frac{10}{3}, \quad \frac{\bar{J}_g}{\bar{A}_g} = \frac{1}{50}. \quad (\text{C.24})$$

If a common positive spatial trace factor W multiplies the two extensive coefficients $J_g = W \text{Tr} \chi_g/15$ and $A_g = W \text{Tr} Y_g^2$ then $A_g \tau = J_g$ has the unique solution $\tau = 1/50$ independently of g and W . Tensor product trace factorisation gives $\text{Tr} \chi_g = g$ and $\text{Tr} Y_g^2 = 10g/3$. Substitution gives the trace ratios. Positivity of A_g makes the balance $A_g \tau = J_g$ unique.

The normalisations in (C.24) are per component for the chiral excess and per generation for the charge capacity. Both coefficients are derivatives of the same finite source.

One finite source for the chiral and charge responses. On each of the four left doublets let $t_a = \sigma_a/2$ and let $t_a = 0$ on the seven right singlets. For the equal-weight content trace set

$$\mathcal{W}_F(u, v, \mathbf{x}) = \log \frac{\text{Tr}_{\mathcal{H}_1} e^{u\chi + vY + \mathbf{x} \cdot \mathbf{t}}}{15}.$$

Here $u, v, \mathbf{x}$ are derivative coordinates. Apply the normalized trace Hessian formula in “Generated depolarization and the prepared response” in Ref. [4] to these specified charge blocks. $\text{Tr}_L Y = \text{Tr}_R Y = 0$ and $\text{Tr}_L Y^2 = 2/3$ and $\text{Tr}_R Y^2 = 8/3$. The cited formula therefore gives

$$\partial_u \mathcal{W}_F(0) = \frac{1}{15}, \quad \partial_v \mathcal{W}_F(0) = 0, \quad D^2 \mathcal{W}_F(0) = \text{diag} \left(\frac{224}{225}, \frac{2}{9}, \frac{2}{15} I_3 \right). \quad (\text{C.25})$$

The weak traces satisfy $\text{Tr}(t_a t_b) = 2\delta_{ab}$ and $\text{Tr} t_a = \text{Tr}(\chi t_a) = \text{Tr}(Y t_a) = 0$. Thus all mixed weak entries vanish. In particular the hypercharge second moment is already connected: its mean and mixed chiral covariance vanish. Use the normalisations $\bar{J} = \partial_u \mathcal{W}_F(0)$ and $\bar{A} = 15 \partial_v^2 \mathcal{W}_F(0)$. Thus $\tau = \bar{J}/\bar{A} = 1/50$. Both entries in this constitutive ratio are responses of one source. The factor 15 converts the normalised hypercharge Hessian to the per-generation charge capacity.

For positive weights w_L, w_R that are constant on the respective finite chiral subspaces let $E_L = (I + \chi)/2$ and $E_R = (I - \chi)/2$ of ranks 8, 7 on $\mathcal{H}_1$. These act on the finite content space whereas $P_\pm$ act on the spatial branches. Set $\rho_r = (w_L E_L + w_R E_R)/(8w_L + 7w_R)$ with $r = w_L/w_R$ and $\mathcal{W}_r(u, v) = \log \text{Tr}(\rho_r e^{u\chi + vY})$. The same readouts $J_r = \partial_u \mathcal{W}_r(0)$ and $A_r = 15 \partial_v^2 \mathcal{W}_r(0)$ give

$$\tau_r = \frac{8r - 7}{10r + 40}, \quad \tau_r - \frac{1}{50} = \frac{39(r - 1)}{50(r + 4)}, \quad \frac{d\tau_r}{dr} = \frac{39}{10(r + 4)^2} > 0. \quad (\text{C.26})$$

Thus the value $1/50$ is preserved exactly when $r = 1$. A common overall factor cancels. The response depends on relative branch weights. For $|r - 1| \leq \varepsilon < 1$ $|\tau_r - 1/50| \leq 39\varepsilon/[50(5 - \varepsilon)]$. Species-dependent weights within either block require their actual charge moments instead of this two-weight reduction.

The two spatial projections have different heat traces. Before additional mass insertions

$$T_\pm(t) = \text{Tr}(P_\pm e^{-tD^2}), \quad T_+(t) \sim 2e^{-9t/(4L^2)}, \quad T_-(t) \sim 6e^{-25t/(4L^2)} \quad (t \rightarrow \infty).$$

The finite content-response normalisation is therefore applied before the spatial projections. A torsion stationary equation formed from the projected trace includes the corresponding source pullback or compensation identity.

C.3 Geometric dynamics

In the dimensionless spectral convention of Section 3.4 the $J = 2$ Einstein–Cartan readout coefficient is

$$\lambda_{\text{EC}} = N_g \left(1 + \frac{\tau}{2}\right)^2 + 6 = 14 + 8\tau + 2\tau^2, \quad (\text{C.27})$$

the torsion-dressed $\mathfrak{su}(2)_L$ term $8(1 + \tau/2)^2$ plus the Lichnerowicz shift 6. The effective tensor readout coefficient is

$$\lambda_{\text{TT}} = 14 = 2N_R, \quad (\text{C.28})$$

the content double of the seven right-handed singlets. The squash amplitude is the broken-generator content

$$s_0 = n_{\text{broken}} \tau = 2\tau, \quad (\text{C.29})$$

with $n_{\text{broken}} = \dim \text{SU}(2)_R - \dim \text{U}(1)_R = 3 - 1 = 2$ the two broken generators absorbed as the Goldstone directions of the W_R bosons. Each contributes the torsion modulus. Its content ratio is

$$\frac{s_0}{N_R} = \frac{n_{\text{broken}}}{N_f \sum Y^2 N_R} = \frac{2}{15 \cdot (10/3) \cdot 7} = \frac{1}{175}, \quad (\text{C.30})$$

half the relative first-order shift $N_g \tau / 14 = 8\tau / 14$. The Euclidean-period factors enter as the amplitude $e^{1/(2\pi)}$ and Gaussian-window width $\sqrt{2\pi}$ and tensor ratio $r = (1/2\pi)^2$ in their respective readouts.

C.4 Weak-coupling normalisation and screening

The weak-coupling readout fixes the per-mode response norm to twice the unweighted norm with $I_{kv} = 2 \text{Vol}(RP^3) = 2\pi^2 L^3$ and uses numerator $16\pi^2$ and scale factor $(M_G/M_P)^2$. These are the absolute normalisation conventions of the readout. For comparison a pointwise orthonormal round Killing coframe has unweighted Gram matrix $\int \langle \theta^a, \theta^b \rangle dv = \pi^2 L^3 \delta_{ab}$. The response factor 2 is separate from the covering-space volume. For the dimensionless emergence input $c = kL$ use $\hat{I}_{kv}(c) = 2\pi^2 c^3$. These fixed conventions give

$$g_2^2 = \frac{16\pi^2 (M_G/M_P)^2}{2\pi^2 (kL)^3} = 8 \left(\frac{M_G}{M_P}\right)^2 (kL)^{-3}. \quad (\text{C.31})$$

The window-edge identity $kLM_G = M_P \sqrt{\pi}$ gives

$$g_2^2 = \frac{8\pi}{(kL)^5}, \quad \alpha_W = \frac{g_2^2}{4\pi} = \frac{2}{(kL)^5}. \quad (\text{C.32})$$

The proposition Matter coupling from the common transport in Section 8 of Ref. [4] supplies $\nabla_A^\rho = d + \rho_*(A)$ and its modal product coefficients. The absolute kinetic normalisation is specified above. If the mode frame is changed by an invertible constant matrix S its Gram matrix and coefficient column transform as $G' = S^\top G S$ and $a' = S^{-1}a$. The connection and $a^\top G a$ are unchanged. Consequently the mode frame and matter coefficients together with the quadratic norm must use one convention. The displayed coupling uses the response

normalisation specified above. In the common action, write the quadratic gauge and matter terms and its linear current vertex as

$$-\frac{Z_A}{4}F_{\mu\nu}F^{\mu\nu} + Z_\psi\bar{\psi}i\not{\partial}\psi + V_\psi A_\mu J^\mu.$$

Canonical fields give $g_{\text{can}} = V_\psi/(Z_\psi\sqrt{Z_A})$. Thus the gauge norm and the current vertex are evaluated together. For a constant internal mode c_0 , the common-action reduction gives $Z_A = \text{Vol}(RP^3)c_0^2/g_7^2$ and $V_\psi/Z_\psi = c_0$, hence $g_4^2 = g_7^2/\text{Vol}(RP^3)$, independently of c_0 . Here g_7^2 denotes the coefficient of this product-space action. Writing $g_7^2 = \gamma_7/(M_{\text{P}}^2 M_{\text{G}})$ and $L = c/M_{\text{G}}$ gives $g_4^2 = \gamma_7/(\pi c^5)$ after use of the window-edge identity. The weak-response normalization above corresponds to $\gamma_7 = 8\pi^2$. It fixes the absolute strength in the stated action convention; the c^{-5} dependence follows from the volume and scale relations. For repeated coefficient resolution on the fixed round realization the proposition Joint resolution of gauge and matter coefficients in Appendix A of Ref. [4] keeps the gauge and matter responses together under the specified heat evolution. The spinor heat traces defined above and the coupling running of Section D.6 retain their separate definitions. The bare Killing value is +0.345% above the reference coupling.

The weak response of the same finite source. Use the weak block of (C.25) already evaluated on the common finite trace. For g copies the normalized trace is still the same mean. The extensive response multiplies its Hessian by the dimension $15g$. $H_{F,ab}^{\text{ext}} = 2g\delta_{ab}$. For the three-dimensional adjoint source the normalized Hessian is $(2/3)\delta_{ab}$ and hence $H_{\text{ad},ab}^{\text{ext}} = 2\delta_{ab}$. The scalar ratio in the common generator convention is therefore

$$\delta_{\text{src}}(g) = \frac{C_2(\text{ad SU}(2))}{\sum_f T(R_f)} = \frac{2}{2g} = \frac{1}{g}. \quad (\text{C.33})$$

Common generator rescalings cancel. The ratio of the normalized Hessians is 5. Restoring their respective dimensions gives the extensive ratio $1/g$ used in the inverse weak-coupling relation.

Common screening coefficient and colour normalization. The finite trace and the screened coupling refer to the quadratic weak-source form in the same generator convention. After separating the torsion contribution its scalar correction is

$$\delta_{\text{col}} = \alpha_{\text{SM}}^{-1} - \alpha_W^{-1} + \frac{\pi\tau^2}{2}.$$

The adjoint and matter Hessians above evaluate this coefficient as $\delta_{\text{src}}(g) = 1/g$. The colour-normalized closure used in the screening rule assigns unit total correction to the N_c colour channels through $N_c\delta_{\text{col}} = 1$. Here $N_c = 3$ is inherited from the one-generation carrier of Ref. [4]. Both normalizations act on this single inverse weak-coupling coefficient. Requiring their agreement is the joint screening rule for the replication family. This determines the copy number before substitution of any three-copy Dynkin sum. The colour normalization has a simple operator form on the inherited irreducible colour carrier. A colour-invariant positive insertion of unit trace is $Q_{\text{col}} = I_{N_c}/N_c$ by Schur's lemma. Its

diagonal response is therefore $1/N_c$. The screening rule identifies this insertion with the scalar correction to the same weak-source quadratic form. Comparing that correction with the extensive matter response gives the compatibility condition below.

Proposition C.2 (Joint compatibility of the screening conditions). Fix the upstream three-colour one-generation carrier and the replication family of Section C.1. For every candidate positive integer g require the same correction

$$\delta_{\text{col}} := \alpha_{\text{SM}}^{-1} - \alpha_W^{-1} + \frac{\pi\tau^2}{2}$$

to satisfy both screening conditions $N_c\delta_{\text{col}} = 1$ and $\delta_{\text{col}} = \delta_{\text{src}}(g)$ with $N_c = 3$. Then the unique compatible copy number is $g = 3$. At $\tau = 1/50$ it also satisfies the $R = 5$ endpoint and three-position window conditions.

Proof. Each generation has four weak doublets. Thus $\sum_f T(R_f) = g(3+1)/2 = 2g$. Equation (C.33) and the colour-screening condition give $N_c/g = 1$. Thus $g = 3$. The coefficients of (C.21) give $c_3 = a_5 + 3b_5 \in [2.490, 2.498]$. Its square lies in the three-position plateau $(11/2, 15/2]$. $\square$

On the selected branch $C_2(\text{SU}(2)) = 2$ and the total fermion Dynkin index is $\sum_f T(R_f) = 6$. Thus the response check is

$$\delta_{\text{src}}(3) = \frac{C_2(\text{SU}(2))}{\sum_f T(R_f)} = \frac{2}{6} = \frac{1}{3} = \frac{1}{N_c}. \quad (\text{C.34})$$

The screened coupling then reads

$$\frac{1}{\alpha_{\text{SM}}} = \frac{1}{\alpha_W} + \frac{1}{N_c} - \frac{\pi\tau^2}{2}, \quad g_2(M_G) = \sqrt{4\pi\alpha_{\text{SM}}} = 0.508848. \quad (\text{C.35})$$

The term $-\pi\tau^2/2$ is the quadratic torsion response with $\tau^2 = 1/2500$. The squash and hypercharge matching also contain terms linear in τ . The source ratio and colour closure select the copy number and the endpoint plateau verifies its retained positions. Every quantity entering this compatibility calculation is evaluated before comparison with measured couplings.

Proposition C.3 (Replicated family module and the quadratic stress response). Let E_1 be the one-generation Hermitian matter bundle with the connection and source normalization inherited from Ref. [4]. Realize the replication rule of Section C.1 by $E_g = \mathbb{C}^g \otimes E_1$ and $\nabla_g = I_g \otimes \nabla_1$. The family factor is the least-dimensional nondegenerate faithful representation of $M_g(\mathbb{C})$ commuting with the fixed one-generation spatial action.

For $g = 3$ its coefficient space can be identified unitarily with

$$K = \text{Hom}_{H_{\text{iso}}}(S, F_0^+ \oplus F_1^+ \oplus F_2^+), \quad \dim K = 3, \quad H_{\text{iso}} \simeq \text{SU}(2), \quad S \simeq \mathbb{C}^2.$$

Here F_j^+ are the three selected positive Dirac shells and K carries the normalized intertwiner norm $\langle a, b \rangle = \frac{1}{2} \text{Tr}(a^\dagger b)$. At zero Higgs background the common quadratic operator and every family-independent response insertion have the form $Q_g = I_g \otimes Q_1$ and

$B_g = I_g \otimes B_1$. Therefore whenever the indicated trace exists

$$\text{Tr}_{E_g}[B_g f(Q_g)] = g \text{Tr}_{E_1}[B_1 f(Q_1)]. \quad (\text{C.36})$$

The identity holds for the Gaussian quadratic TT response including its complete first and second source variations on the common response domain.

Proof. The classification of finite-dimensional representations of $M_g(\mathbb{C})$ [48] shows that a faithful nondegenerate representation is a sum of defining modules. The least one has dimension g and is unique up to unitary equivalence. Tensoring it with the inherited one-generation bundle implements the stipulated replication. In each F_j^+ the isotropy spin $1/2$ occurs once. The normalized radial intertwiners R_j thus form an orthonormal basis of K . The projected character products and their Clebsch–Gordan normalization in Ref. [4] give the matrix units $R_i R_j^\dagger$ acting as $E_{ij} \otimes I_S$ on the radial isotypic source. A unitary identification of this coefficient module with $\mathbb{C}^3$ conjugates the kinetic form and all family vertices together. This preserves masses and mixing invariants together with traces. The spatial spinor modes remain those of E_1 in each summand.

Functional calculus and trace additivity prove (C.36). In a fixed orthonormal trace space write $G = Q_1^{-1}$. The determinant contribution has mixed variation proportional to

$$\partial_a \partial_b \text{Tr} \log Q_1 = \text{Tr}(G Q_{1,ab} - G Q_{1,b} G Q_{1,a}). \quad (\text{C.37})$$

The first term is the direct source term. Both terms and the derivatives of inserted projectors and measures together with Gaussian factors acquire the same factor g . Equivalently for $W = \log Z$ the connected response is $W_{ab} = \langle S_a S_b \rangle_c - \langle S_{ab} \rangle$. The signed species projection is applied to the full one-copy space in every summand. $\square$

The numerical endpoint fixes the $R = 5$ domain in its reference geometry. For a source-dependent domain the same proposition uses the full domain derivative. For example for a retained mode put $b(s) = \sqrt{R\Lambda(s)^2 - \lambda(s)} > 0$ and $I(s) = \int_{-b(s)}^{b(s)} f(\omega, s) d\omega$. Its mixed second derivative is

$$\begin{aligned} I_{ab} = & \int_{-b}^b f_{ab} d\omega + b_a(f_b(b) + f_b(-b)) + b_b(f_a(b) + f_a(-b)) \\ & + b_{ab}(f(b) + f(-b)) + b_a b_b(f_\omega(b) - f_\omega(-b)). \end{aligned} \quad (\text{C.38})$$

All source dependence of the measure and insertions is included in f . Fixed-rank spectral clusters admit differentiable Riesz projectors as long as their separating gaps remain open. At the reference endpoint the spatial cutoff is 5π in dimensionless units and the next vector eigenvalue is 16. Thus this condition holds in a neighbourhood. Equations (C.37) and (C.38) then preserve the copy factor. Bosonic channels keep their one-copy multiplicity. At nonzero Higgs background the family-dependent mass vertices enter Q_g and are retained explicitly in the corresponding response.

The torsion term combines the quadratic torsion contribution and Euclidean period with the conformal normalisation

$$\frac{\tau^2 \pi}{2} = (2\tau^2) (2\pi) \xi = \frac{4\pi \tau^2}{N_g}, \quad (\text{C.39})$$

The factor $2\tau^2$ is the quadratic coefficient in the Einstein–Cartan spectral readout $\lambda_{\text{EC}} = N_g(1 + \tau/2)^2 + 6 = 14 + 8\tau + 2\tau^2$. For the full torsion norm $T^a_{bc} = (\tau/L)\varepsilon^a_{bc}$ gives $L^2 T_{abc} T^{abc} = 6\tau^2$ in an orthonormal frame. The factor 2π is the Euclidean period and $\xi = 1/8 = 1/N_g$ is the conformal normalisation. The identity $N_g \xi = 1$ gives the second expression. The inverse-coupling relation above specifies the additive insertion and its sign.

Axial Killing-dual curvature normalisation. The hypercharge matching uses $\kappa(2\tau)$ at the breaking scale before running and the further content correction. Its geometric factor follows from the squashed metric

$$ds^2 = (1+s)\sigma_1^2 + (1+s)\sigma_2^2 + (1-2s)\sigma_3^2, \quad (\text{C.40})$$

at fixed round radius. Put $A = 1 + s$ and $B = 1 - 2s$ with $-1 < s < 1/2$. Fix the axial Killing generator X in the same scale for all s by choosing $\alpha_0 = X^{\flat_0} = \sigma_3$. Its flow rotates σ_1, σ_2 and preserves σ_3 . Thus it remains Killing for this axisymmetric family. Its metric dual is $\alpha_s = B\sigma_3$. Since $d\sigma_3$ is a fixed multiple of $\sigma_1 \wedge \sigma_2$ the pointwise two-form norm ratio is B^2/A^2 and the volume ratio is $A\sqrt{B}$. Therefore

$$\frac{\|d\alpha_s\|_{L^2(h_s)}^2}{\|d\alpha_0\|_{L^2(h_0)}^2} = \frac{B^{5/2}}{A} = \kappa(s)^{-2}, \quad \kappa^2(s) = \frac{1+s}{(1-2s)^{5/2}}. \quad (\text{C.41})$$

The covering factor and a fixed rescaling of X together with a fixed matrix-trace convention cancel in this ratio. For the same Abelian connection $\mathcal{A}_s = a_s \alpha_s T_Y$ with $T_Y = iY$ and a_s constant on the internal space one has $F_s = a_s d\alpha_s T_Y$. Preserving its internal curvature quadratic norm and choosing nonzero amplitudes of the same sign gives $a_s/a_0 = \kappa(s)$. The same connection acting on matter through the cited proposition of Ref. [4] obeys

$$\mathcal{A}_s(X) = B\kappa(s)\mathcal{A}_0(X), \quad \mathcal{A}_s(X/\sqrt{B}) = \sqrt{B}\kappa(s)\mathcal{A}_0(X). \quad (\text{C.42})$$

These evaluate the connection along a fixed and a unit axial direction respectively. If an amplitude varies the full curvature also includes $da_s \wedge \alpha_s$. The one-form norm ratio is $\|\alpha_s\|_{L^2(h_s)}^2/\|\alpha_0\|_{L^2(h_0)}^2 = AB^{3/2}$. In particular an amplitude $a(x)$ on the product space has curvature

$$F_s = d_x a \wedge \alpha_s T_Y + a d_y \alpha_s T_Y.$$

It is a four-dimensional scalar amplitude whose kinetic coefficient K_s and internal restoring coefficient C_s obey

$$\frac{K_s}{K_0} = AB^{3/2}, \quad \frac{C_s}{C_0} = \frac{B^{5/2}}{A}.$$

Under the curvature-preserving change $a = \kappa(s)a_0$ the kinetic ratio becomes $\kappa^2 K_s/K_0 = A^2/B$ and every linear matter vertex acquires the same amplitude factor. Canonical normalization therefore retains both Grams. For an external vector mode the coefficient of $F_{\mu\nu} F^{\mu\nu}$ and its matter current are projected from that mode in the common action, giving $g_{\text{can}} = V_\psi/(Z_\psi \sqrt{Z_A})$ as above. The hypercharge boundary rule combines the internal-curvature factor with the stated external-mode normalization and finite content correction. The constant-amplitude identity supplies its geometric factor with $\kappa(0) = 1$ and

$\kappa(2\tau) = 1.13183$. The mixing coefficient is corrected by the content ratio

$$\delta_{g_1} = -\tau r_{23} \sum Y^2 \Delta_f \xi, \quad (\text{C.43})$$

with $r_{23} = 3/(10\sqrt{3})$ and $\sum Y^2 \Delta_f \xi = (10/3)(3/2)(1/8) = 5/8$. This product combines the hypercharge moment and fermion conformal weight with the scalar conformal coupling used above. The colour matching rule is $g_3 = g_2(1 + \alpha_{\text{GUT}}^2/K)$ with $K = J(J+2)/d = 8/3$. It assigns the dimensionless squash eigenvalue $J(J+2) = 8$ over the internal dimension $d = 3$ to the long-root correction.

C.5 Two-loop gauge coefficients

We evaluate the $\overline{\text{MS}}$ gauge beta functions on the five inherited fermion representations. The minimal joint module and charged response quotient of Ref. [4] supply Q_L, L_L, u_R, d_R, e_R . Its result on the elementary doublet representation fixes the scalar type $(1, 2)_{1/2}$. The running uses the content of Section 3 with three copies of the fermion module and one complex Higgs doublet.

In the fixed convention $Y_H = 1/2$ and $Q = T_3 + Y$ Yukawa covariance and anomaly cancellation give $Y_Q = 1/6$ and $Y = (1/6, 2/3, -1/3, -1/2, -1)$ for the five inherited representations. The per-generation hypercharge moment is therefore

$$\sum Y^2 = 6 \cdot \frac{1}{6^2} + 3 \cdot \frac{4}{9} + 3 \cdot \frac{1}{9} + 2 \cdot \frac{1}{4} + 1 \cdot 1 = \frac{10}{3}. \quad (\text{C.44})$$

We use Eqs. (30)–(31) of Ref. [19] with the product-group substitutions of Eqs. (106)–(110) in its Section VI. For a running scale μ the loop convention is

$$\frac{dg_i}{d \log \mu} = \frac{b_i g_i^3}{16\pi^2} + \frac{g_i^3}{(16\pi^2)^2} \left(\sum_j B_{ij} g_j^2 - A_i y_t^2 \right).$$

We use $g_1 = \sqrt{5/3} g_Y$ with Weyl fermions and one complex Higgs doublet. In the two-loop scalar sums the latter is counted as its realification. The one-loop coefficients are

$$(b_1, b_2, b_3) = \left(\frac{41}{10}, -\frac{19}{6}, -7 \right), \quad (\text{C.45})$$

with $C_2(\text{SU}(2)) = 2$ and $C_2(\text{SU}(3)) = 3$ together with $T_f(\text{SU}(2)) = T_f(\text{SU}(3)) = 6$ and $T_H = 1/2$. For the two-loop substitution S_2 is fixed by $\text{Tr}(T_R^a T_R^b) = S_2(R) \delta^{ab}$. The fundamental indices are $1/2$ and the GUT-normalized abelian entries use $C_2 = (3/5)Y^2$. These conventions fix the Casimirs and Dynkin indices used in the product-group formula.

For each Weyl representation, the Casimirs and Dynkin indices are listed in the order

$$(C_2^{\text{U}(1)}, C_2^{\text{SU}(2)}, C_2^{\text{SU}(3)}, S_2^{\text{U}(1)}, S_2^{\text{SU}(2)}, S_2^{\text{SU}(3)}).$$

Their values are

$$\begin{aligned}
Q_L(3, 2)_{1/6} &: \left(\frac{3}{5} \frac{1}{36}, \frac{3}{4}, \frac{4}{3}, \frac{3}{5} \frac{1}{6}, \frac{3}{2}, 1 \right), \\
u_R(3, 1)_{2/3} &: \left(\frac{3}{5} \frac{4}{9}, 0, \frac{4}{3}, \frac{3}{5} \frac{4}{3}, 0, \frac{1}{2} \right), \\
d_R(3, 1)_{-1/3} &: \left(\frac{3}{5} \frac{1}{9}, 0, \frac{4}{3}, \frac{3}{5} \frac{1}{3}, 0, \frac{1}{2} \right), \\
L_L(1, 2)_{-1/2} &: \left(\frac{3}{5} \frac{1}{4}, \frac{3}{4}, 0, \frac{3}{5} \frac{1}{2}, \frac{1}{2}, 0 \right), \\
e_R(1, 1)_{-1} &: \left(\frac{3}{5}, 0, 0, \frac{3}{5}, 0, 0 \right),
\end{aligned} \tag{C.46}$$

and the Higgs scalar $(1, 2)_{1/2}$ contributes the real-scalar content $(\frac{3}{5} \frac{1}{4}, \frac{3}{4}, 0, \frac{3}{5}, 1, 0)$. Substitution of this table in the cited product-group formula gives

$$B = \begin{pmatrix} 199/50 & 27/10 & 44/5 \\ 9/10 & 35/6 & 12 \\ 11/10 & 9/2 & -26 \end{pmatrix}. \tag{C.47}$$

These standard coefficients for three fermion generations and one complex scalar doublet use the inherited charge normalisation. The abelian fourth moment checks $B_{11} = 199/50$. The same Weyl and scalar component counting in the top-Yukawa trace gives with $d(G_i) = \dim \text{adj}(G_i) = (1, 3, 8)$ for $G_i = \text{U}(1), \text{SU}(2), \text{SU}(3)$

$$A_i = \frac{6}{d(G_i)} [C_2^{(i)}(Q_L) + C_2^{(i)}(u_R)], \quad A = \left(\frac{17}{10}, \frac{3}{2}, 2 \right). \tag{C.48}$$

This agrees with the Yukawa term of the cited gauge beta function.

C.6 The top-Yukawa and Higgs-quartic beta functions

The top Yukawa and scalar running use the $\overline{\text{MS}}$ formulas of Sections IV.2–IV.3 of Ref. [19] specialised to the Standard Model in Eqs. (3) and (6) together with (9) and (10) of Ref. [20]. In this subsection $\lambda = \lambda_H$ is the Higgs quartic in $V = -m^2|H|^2 + \lambda|H|^4$ with $g_1 = \sqrt{5/3} g_Y$. The reference potential uses $(\lambda_{LX}/2)|H|^4$. Thus $\lambda = \lambda_{LX}/2$ and $\beta_\lambda = \beta_{\lambda, LX}/2$. For $X = y_t, \lambda$ define

$$\beta_X := \frac{dX}{d \log \mu} = \frac{\beta_X^{(1)}}{16\pi^2} + \frac{\beta_X^{(2)}}{(16\pi^2)^2}.$$

In the top-only approximation the Yukawa matrix is $Y_t = \text{diag}(0, 0, y_t)$. Its trace invariants are

$$\begin{aligned}
Y_2(S) &= \text{Tr}[3Y_t^\dagger Y_t] = N_c y_t^2, & H(S) &= \text{Tr}[3(Y_t^\dagger Y_t)^2] = N_c y_t^4, \\
\chi_4(S) &= \frac{9}{4} \text{Tr}[3(Y_t^\dagger Y_t)^2] = \frac{9}{4} N_c y_t^4,
\end{aligned} \tag{C.49}$$

and

$$Y_4(S) = \left(\frac{17}{20} g_1^2 + \frac{9}{4} g_2^2 + 8g_3^2 \right) y_t^2, \tag{C.50}$$

where the coefficients follow from $-3g^2\{C_2(F), Y\}$ with the Casimirs in Section C.5. The family count $n_g = 3$ is fixed by the joint content and endpoint rules of Proposition C.2.

The one-loop top Yukawa is

$$\frac{\beta_{y_t}^{(1)}}{y_t} = \left(\frac{3}{2} + N_c\right)y_t^2 - \frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 = \frac{9}{2}y_t^2 - \frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2, \quad (\text{C.51})$$

the Yukawa self-coupling $3/2$ plus the colour trace $Y_2(S) = N_c y_t^2$. The $U(1)$ coefficient $17/20$ is in the GUT normalisation and equals $-(17/12)g_Y^2$ in the hypercharge coupling. The two-loop top Yukawa is

$$\begin{aligned} \frac{\beta_{y_t}^{(2)}}{y_t} = & -12y_t^4 + \left(36g_3^2 + \frac{225}{16}g_2^2 + \frac{393}{80}g_1^2\right)y_t^2 - 108g_3^4 - \frac{23}{4}g_2^4 + \frac{1187}{600}g_1^4 \\ & + 9g_2^2g_3^2 + \frac{19}{15}g_1^2g_3^2 - \frac{9}{20}g_1^2g_2^2 + 6\lambda^2 - 12\lambda y_t^2, \end{aligned} \quad (\text{C.52})$$

Here the y_t^4 and $g_i^2 y_t^2$ coefficients include the $\chi_4(S)$ and $Y_4(S)$ traces above. The pure-gauge coefficients are those of Ref. [20, Eq. (6)] at $n_g = 3$. The one-loop Higgs quartic is

$$\beta_\lambda^{(1)} = 24\lambda^2 - 3\lambda\left(3g_2^2 + \frac{3}{5}g_1^2\right) + \frac{3}{8}\left(2g_2^4 + (g_2^2 + \frac{3}{5}g_1^2)^2\right) - 6y_t^4 + 12\lambda y_t^2, \quad (\text{C.53})$$

whose Yukawa part in this potential convention is $4Y_2(S)\lambda - 2H(S) = 12\lambda y_t^2 - 6y_t^4$. The two-loop Higgs quartic is

$$\begin{aligned} \beta_\lambda^{(2)} = & -312\lambda^3 + 36\lambda^2\left(3g_2^2 + \frac{3}{5}g_1^2\right) - 144\lambda^2 y_t^2 - \frac{73}{8}\lambda g_2^4 + \frac{117}{20}\lambda g_2^2 g_1^2 + \frac{1887}{200}\lambda g_1^4 \\ & + \left(\frac{17}{2}g_1^2 + \frac{45}{2}g_2^2 + 80g_3^2\right)\lambda y_t^2 - 3\lambda y_t^4 + 30y_t^6 \\ & - 32g_3^2 y_t^4 - \frac{8}{5}g_1^2 y_t^4 - \frac{9}{4}g_2^4 y_t^2 + \frac{63}{10}g_2^2 g_1^2 y_t^2 - \frac{171}{100}g_1^4 y_t^2 \\ & + \frac{305}{16}g_2^6 - \frac{289}{80}g_2^4 g_1^2 - \frac{1677}{400}g_2^2 g_1^4 - \frac{3411}{2000}g_1^6, \end{aligned} \quad (\text{C.54})$$

The conversion $\lambda_{LX} = 2\lambda$ multiplies the coefficients of λ^3 and λ^2 by four and two respectively and halves terms independent of λ . The displayed coefficients use $n_g = 3$ in Ref. [20, Eq. (10)]. The $\lambda y_t^2 g^2$ terms equal $10\lambda Y_4(S)$. All fermionic traces are evaluated in the top-only three-generation approximation.

C.7 Sector exponents and the Landau–Zener comparison

Scale transport and mode mixing. The proposition Homothetic scale transport of the spectral window in Ref. [4] preserves dimensionless Gaussian weights. The family transfer acts on the retained spectral coefficient module of Proposition C.3. Its diagonal weights and angular operators specify the mass construction.

The mass ladder attaches exponential suppression to the extrusion orders $n = \{0, 2, 4\}$. Its proposed non-adiabatic interpretation uses a Landau–Zener comparison [49] in addition to the pure homothetic flow $L(k) = C/k$ with the avoided-crossing normal form

$$H(t) = \begin{pmatrix} vt & \Delta_{LZ} \\ \Delta_{LZ} & -vt \end{pmatrix}, \quad P_{LZ} = e^{-\pi\Delta_{LZ}^2/|v|}, \quad (\text{C.55})$$

in the convention $i\partial_t\psi = H(t)\psi$ with real $v \neq 0$ and constant Δ_{LZ} . Here t is the parameter of this two-level comparison. Its physical realization parametrizes the source path by a material clock. The single-passage exponent is $\gamma = \pi\Delta_{\text{LZ}}^2/|v|$ and the diabatic survival amplitude has magnitude $e^{-\gamma/2}$ [50]. To derive the mass-amplitude factor $e^{-n\alpha_f}$ from a single passage the source trajectory must produce the operator above and the equality

$$\frac{\pi\Delta_{\text{LZ}}^2}{|v|} = 2n\alpha_f, \quad (\text{C.56})$$

with the transmitted mass vertex identified with the diabatic survival channel. The gap and sweep rate must be evaluated on that same trajectory. Multiple passages require their scattering matrices and relative phases. The mass construction uses the positive amplitude transfer $m_i \propto e^{-\alpha_f n_i}$ with the sector rates set by the content rules. The up-sector transfer relation subtracts the torsion deficit from the window width

$$\alpha_{\text{up}} = kL - 2\tau = 2.4535, \quad (\text{C.57})$$

where the window contribution is kL and the assigned deficit is $n_{\text{broken}}\tau = 2\tau$. This gives $m_t/m_c = e^{2\alpha_{\text{up}}}$. The down and lepton sectors descend by the step

$$\Delta = 6(1 - n_s)kL_{\text{CMB}}, \quad (\text{C.58})$$

the product of the six generators of the $\mathfrak{so}(4)$ isometry with the spectral tilt $1 - n_s$ and the microwave-pivot window kL_{CMB} . The down and lepton steps are split by the 9/8 hypercharge identity

$$\frac{9}{8} = \frac{1}{1 - (Y_d/Y_l)^2}, \quad (\text{C.59})$$

with $Y_d = |Y_{d_R}| = 1/3$ and $Y_l = |Y_{e_R}| = 1$. Thus the two contributions have ratio 9/8. Writing them as $(9/8)h$ and h the total lepton decrement 2Δ gives $(9/8)h + h = 2\Delta$ and hence $h = 16\Delta/17$. The indices are

$$\alpha_{\text{dn}} = \alpha_{\text{up}} - \frac{18}{17}\Delta, \quad \alpha_{\text{lp}} = \alpha_{\text{up}} - 2\Delta. \quad (\text{C.60})$$

The first-generation step of the down sector carries the colour dilution

$$\alpha_{\text{sd}} = \alpha_{\text{dn}} - \frac{kL_{\text{CMB}}}{6}, \quad (\text{C.61})$$

where the assigned colour dilution divides the microwave window by the six-generator count of $\mathfrak{so}(4)$. Substituting the stated tilt and pivot window gives the exact expression

$$\alpha_{\text{sd}} = kL \left[1 - \left(1 - \frac{\tau}{4}\right) \left(\frac{189\tau}{17} + \frac{1}{6}\right) \right] - 2\tau.$$

These rules determine the exponents used in the mass table. The generator counts and charge traces set the coefficients. Equal attenuation per source unit and its placement in a mass amplitude are the transfer choices summarized in Table 2. The consecutive sector steps also give the anchor-independent relation (7.3) by direct elimination of α_{up} and Δ .

The specified transfer generators give positive mass operators with a common left handed kinetic realization.

Proposition C.4 (Positive spectral transfer and the common mass vertex). Let P_0, P_1, P_2 be the orthogonal heavy and middle and light family projectors and put $c_C = kL_{\text{CMB}}$. Define

$$\begin{aligned} B_u &= 2\alpha_{\text{up}}P_1 + (2\alpha_{\text{up}} + 2c_C + \log 4)P_2, \\ B_d &= 2\alpha_{\text{dn}}P_1 + (2\alpha_{\text{dn}} + 2\alpha_{\text{sd}})P_2, \\ B_\ell &= 2\alpha_{\text{lp}}P_1 + (4\alpha_{\text{lp}} + \sqrt{2\pi})P_2. \end{aligned} \tag{C.62}$$

Then $T_f = e^{-B_f}$ is positive definite and its eigenvalues give the stated mass ratios relative to the heaviest member of each sector. The vertices $Y_u = y_u T_u$ and $Y_d = y_d U T_d$ use the same canonical left kinetic form with U the ordered angular rotation of Appendix C.13. Their singular values are the masses defined above times $\sqrt{2}/v$ and their left mixing matrix is U .

Proof. The three projectors are orthogonal and sum to the identity of the family module. Thus exponentiation acts separately on their real coefficients. In the up sector the eigenvalues are $1, e^{-2\alpha_{\text{up}}}, \frac{1}{4}e^{-2\alpha_{\text{up}}-2c_C}$. The other two sectors follow identically. Multiplication on the left by U leaves the down singular values unchanged and sets its left frame relative to the up frame.

The matrices implementing this construction are generated before mass diagonalization. In the normalized radial source basis write $\chi_k(x) = \sum_{m=-k}^k D_{mm}^k(x)$. In the inherited Clebsch–Gordan phase convention the intertwiner has entries

$$(R_j)_{(m,J,M),s} = \frac{C_{jm,1/2s}^{JM}}{\sqrt{j+1}}, \quad J = j + \frac{1}{2}, \quad M = m + s.$$

Inserting these coefficients into the normalized three-matrix-element integral on $SU(2)$ gives

$$\Pi_0 M_{\chi_1} \Pi_1 = \sqrt{2/3} R_0 R_1^\dagger, \quad \Pi_1 M_{\chi_1} \Pi_2 = \frac{3}{\sqrt{10}} R_1 R_2^\dagger, \quad \Pi_0 M_{\chi_2} \Pi_2 = \sqrt{3/5} R_0 R_2^\dagger.$$

Here $\Pi_j = R_j R_j^\dagger$. Dividing by these nonzero Clebsch–Gordan coefficients gives the family matrix units. Their anti-Hermitian combinations generate the ordered plane rotations including the complex phase. They act on the six-dimensional radial source within the complete $2 + 12 + 30 = 44$ dimensional three-shell carrier. The orthogonal complement is retained with zero family vertex. Moving the basepoint translates both characters and intertwiners unitarily. Thus the common kinetic form and two mass spectra together with the CP invariant in (C.66) transform covariantly. $\square$

In increasing mass order let E_{ab} denote these matrix units. The unitary matrix with the three CKM moduli of Appendix C.13 is

$$U = e^{\theta_{23}(E_{23}-E_{32})} e^{\theta_{13}(e^{-i\delta}E_{13}-e^{i\delta}E_{31})} e^{\theta_{12}(E_{12}-E_{21})},$$

with $s_{13} = |V_{ub}|$ and $s_{12} = |V_{us}|/\sqrt{1-s_{13}^2}$ together with $s_{23} = |V_{cb}|/\sqrt{1-s_{13}^2}$ and $\theta_{ij} = \arcsin s_{ij}$ and $\delta = 8\pi/21$. It gives $J = s_{12}s_{23}s_{13}c_{12}c_{23}c_{13}^2 \sin \delta$. The expressions in Appendix C.13 use its stated leading order in the small element $|V_{ub}|$.

This positive transfer is an operator realization of the hierarchy relation in the fixed generated algebra [2–4]. Its Landau–Zener interpretation concerns a further dynamical choice of a source trajectory with its gap and sweep rate.

C.8 The top-quark anchor

The Elementary doublet representation proposition of Ref. [4] gives the connector $T_H = I_C \otimes h_H$ with

$$h_H = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad H = (b, \bar{a})^\top, \quad h_H^\dagger h_H = \|H\|^2 I_2.$$

We use this representation in its inherited source Hilbert norm. On the radial representative $H = (0, r)^\top$ with $r \geq 0$ the finite matrix is $h_H = rI_2$. It gives

$$T_H = rI_C \otimes I_2, \quad \mathcal{D}_H^2 = r^2 I, \quad \frac{1}{2} \text{Tr}(dh_H^\dagger dh_H) = dr^2, \quad (\text{C.63})$$

where $\mathcal{D}_H = \begin{pmatrix} 0 & T_H \\ T_H^\dagger & 0 \end{pmatrix}$. For unit internal vectors paired by the inherited identity $C_0 = I_C \otimes I_2$ one obtains $\langle C_0 \eta, C_0 \eta \rangle = 1$. This is the geometric seed $y_0 = 1$. It applies in particular to the colour-triplet block of the top connector.

The seed is preserved under a common unitary comparison. If $C(s) = U_L(s)C_0U_R(s)^\dagger$ and $\eta_{L,R}(s) = U_{L,R}(s)\eta_{L,R}(0)$ then

$$\langle \eta_L(s), C(s)\eta_R(s) \rangle = \langle \eta_L(0), C_0\eta_R(0) \rangle = 1.$$

This identity preserves the elementary seed in its inherited norm. The readout $m_t = y_0 v / \sqrt{2}$ uses this seed and the window VEV. Equation (C.64) implements the corresponding signed spatial assignment. For responses involving the odd connector the complete carrier and neutral-feedback result of Ref. [4] applies before the charged compression is taken. The full response includes the effective source and readout together with the direct neutral term [15]. At $r > 0$ the full connector is invertible. Eliminating the neutral coordinate retains its induced source and direct response term together with the static charged-source trace.

The coupling of a spatial mass vertex to canonically normalized fields requires its own kinetic factors. With positive Gram matrices and the specified scalar normalization the matched matrix is

$$Y_{\text{can}} = Z_L^{-1/2} \mathcal{Y} Z_R^{-1/2} / \sqrt{Z_H}, \quad \mathcal{Y}_{ij} = \int \psi_{L,i}^\dagger \mathcal{O}_H \psi_{R,j} d\mu.$$

Numerical evaluation for fixed modes and sources checks the singular values and physical pairings under changes of basis [15]. For the spinor modes on the inherited compact spin bundle the spatial part of a constant radial connector is fixed by orthonormality

$$\int \psi_i^\dagger \psi_j d\mu = \delta_{ij}, \quad \int \psi_i^\dagger (rI) \psi_j d\mu = r\delta_{ij}.$$

The overlaps and nonconstant-source products are evaluated in the Clebsch–Gordan mode basis with the full product support retained before the chosen projection is applied [15]. The signs of the internal Dirac eigenvalues label the spatial spectral branches. Four-dimensional chirality enters through the assigned left and right fields. The kinetic coefficients and action vertex enter the canonical matching formula with the same inherited normalization. The inherited unit seed fixes the finite pairing above. The matched running Yukawa then follows its renormalization-group equation.

The signed spatial connector. Let $R_+ : \mathbb{C}^2 \rightarrow F_0^+$ and $R_- : \mathbb{C}^2 \rightarrow F_1^-$ be the normalized radial intertwiners for the signed eigenvalues $+3/(2L)$ and $-5/(2L)$. The complete positive and negative shells have dimensions two and six. The Gaunt integral of the inherited character is $R_-^\dagger M_{\chi_1} R_+ = I_2/\sqrt{3}$. Consequently

$$B_0 = \sqrt{3} \Pi_-^{\text{rad}} M_{\chi_1} \Pi_+^{\text{low}} = R_- R_+^\dagger, \quad B_0^\dagger B_0 = \Pi_+^{\text{low}}, \quad B_0 B_0^\dagger = \Pi_-^{\text{rad}}. \quad (\text{C.64})$$

This follows by inserting the normalized intertwiners in the projected product. Its four-dimensional negative-shell complement is annihilated. The family vertex $Y_f \otimes B_0$ therefore has the normalized infrared pairing Y_f and gives $vY_f/\sqrt{2}$ after symmetry breaking. Spatial Dirac signs and external chirality retain their distinct definitions.

The same construction has a smooth source continuation on an isolated pair of spectral clusters. If E is evaluation at the transported basepoint set $N_\pm = EP_\pm E^\dagger$ and $R_\pm = P_\pm E^\dagger N_\pm^{-1/2}$. Evaluation here is restricted to the finite smooth eigenspaces. On a source neighbourhood where these maps are twice differentiable and $N_\pm > 0$ the matrix $C = R_-^\dagger M_{\chi_1} R_+$ remains invertible near the round point. Multiplying the projected character kernel on its range by $R_- C^{-1} R_-^\dagger$ gives $B = R_- R_+^\dagger$. This fixes its derivatives and transforms covariantly under a common bundle-frame change. Source variations of the projectors and evaluation maps together with their Gram matrices are included.

Common action, flavour basis and renormalized boundary. For the action measure $V d\mu$ let $I_2 = \int |h|^2 d\mu$ and $I_4 = \int |h|^4 d\mu$. The separated action with scalar kinetic coefficient Z_H and quartic coefficient λ_{act} gives

$$y_{\text{can}} = \frac{y_{\text{act}}}{\sqrt{V Z_L Z_R Z_H}} \quad (h=1), \quad \lambda_{\text{can}} = \frac{\lambda_{\text{act}} I_4}{V Z_H^2 I_2^2}. \quad (\text{C.65})$$

The same left kinetic form occurs in the up and down vertices. Write their canonical matrices as Y_u, Y_d and diagonalize $H_f = Y_f Y_f^\dagger = L_f \text{diag}(y_{f,i}^2) L_f^\dagger$. Then $V_{\text{CKM}} = L_u^\dagger L_d$ for simple ordered spectra. In particular the basis-independent quantities are

$$|V_{ij}|^2 = \text{Tr}(P_i^u P_j^d), \quad \text{Im Tr}[H_u, H_d]^3, \quad (\text{C.66})$$

where P_i^f are the rank-one left spectral projectors. An arbitrary invertible change of the unnormalized common left basis induces the same unitary change in both canonical sectors. Thus these quantities and the singular values are unchanged. Degenerate eigenspaces admit only their total spectral projectors. The family dependence of the present construction

resides in its transfer factors. In increasing mass order the up and down diagonal factors are

$$D_u = \text{diag} \left(\frac{1}{4} e^{-2\alpha_{\text{up}} - 2c_{\text{CMB}}}, e^{-2\alpha_{\text{up}}}, 1 \right), \quad D_d = \text{diag} \left(e^{-2\alpha_{\text{dn}} - 2\alpha_{sd}}, e^{-2\alpha_{\text{dn}}}, 1 \right),$$

where $c_{\text{CMB}} = kL_{\text{CMB}}$. Their overall coefficients are the top and bottom readouts. The angular assignments specify the relative left frames. Proposition C.4 and (C.64) realize these data in one action with the same left kinetic form. Equations (C.65) and (C.66) give the forward test of a proposed common action including nonunitary changes of mode coordinates [15].

At a renormalization scale μ_0 finite quantum matching takes the form $\bar{Y}_f(\mu_0) = Y_{f,\text{can}} + \Delta Y_f(\mu_0)$ and $\bar{\lambda}(\mu_0) = \lambda_{\text{can}} + \Delta\lambda(\mu_0)$. The five-coupling evolution then acts on this renormalized boundary with explicit threshold maps. At a common scale and VEV convention the tree coefficients of the canonically normalized broken-phase action are

$$m_t^{\text{tree}} = \bar{y}_t v / \sqrt{2}, \quad (m_H^{\text{tree}})^2 = 2\bar{\lambda}_H v^2.$$

In this effective representation the conversion to physical mass observables additionally requires finite VEV matching and the corresponding self-energies at the specified order. The numerical tables use the geometric mass relations defined in Sections 7 and 6. The spatial integrals and canonical normalization together with joint evolution are evaluated from the supplied action coefficients [15].

Physical readouts and finite electroweak matching. Use the canonically normalized four-dimensional action with $V(H) = m^2 H^\dagger H + \lambda(H^\dagger H)^2$ and $g'^2 = 3g_1^2/5$. On $H = (0, (v+h)/\sqrt{2})^\top$ its stationary condition is $m^2 = -\lambda v^2$ and its quadratic operators give $m_t^2 = y_t^2 v^2/2$ and $m_H^2 = 2\lambda v^2$ and $m_W^2 = g_2^2 v^2/4$. The same covariant derivative couples $g_2 W_\mu^+ J^\mu / \sqrt{2} + \text{h.c.}$ where $J_\mu = \bar{\nu} \gamma_\mu P_L e$. Eliminating the massive W at small external momentum gives $-2J_\mu^\dagger J^\mu / v^2 = -4G_F J_\mu^\dagger J^\mu / \sqrt{2}$. Thus the generated v and top readouts together with the Higgs readout provide the tree-level physical conditions of this common action.

For a finite-order realization we impose these generated values as (G_F, M_t, M_H) with $v_F = (\sqrt{2}G_F)^{-1/2}$. No measured mass enters these conditions. Use the tadpole-free Landau-gauge $\overline{\text{MS}}$ convention of Refs. [51, 52] at a common scale Q . Let $\ell = 1/(16\pi^2)$ and $\delta r = \ell \Delta \tilde{r}^{(1)}$ and $\delta_t = \text{Re}(\ell \Delta s_t^{(1)})$ with $\delta_H = \text{Re}(\ell \Delta s_H^{(1)})$ and $T = \ell \Delta_1$ denote the corrections in the order displayed. These are the finite Fermi and top-pole corrections followed by the Higgs-pole and stationary-potential corrections. The pole kernels use the leading action coefficients with $y_b = 0$ and $|V_{tb}| = 1$ while neglecting the light Yukawas. They are given by corrected Eq. (4.3) of Ref. [51] and Eq. (2.46) of Ref. [53] together with Eqs. (2.2) and (2.20) of Ref. [26]. The loop integrals have the convention

$$A(x) = x(\log(x/Q^2) - 1), \quad B(s; x, y) = - \int_0^1 \log \frac{ux + (1-u)y - u(1-u)s - i0}{Q^2} du.$$

This branch gives $\Gamma = -\text{Im} \Delta s/M$ for a decaying pole.

Proposition C.5 (Finite matching of the generated physical conditions). Put $r_0 = v_F^2 > 0$ and $y_0 = \sqrt{2}M_t/v_F > 0$ and $\lambda_0 = M_H^2/(2r_0)$. Through one loop the solution for $r = v_L^2$ with the two running couplings and the potential coefficient is

$$\begin{aligned} r &= r_0(1 + \delta r), & v_L &= v_F(1 + \delta r/2), \\ \bar{y}_t &= y_0 - \frac{y_0}{2}\delta r - \frac{\delta_t}{y_0 r_0}, & \bar{\lambda} &= \lambda_0 - \lambda_0\delta r - \frac{\delta_H}{2r_0}, \\ \bar{m}^2 &= -\frac{M_H^2}{2} + \frac{\delta_H}{2} - T. \end{aligned} \tag{C.67}$$

Every correction on the right is evaluated at the leading coefficients. The equalities have remainders $O(\ell^2)$.

Proof. The four renormalization conditions are

$$r - r_0 - r_0\delta r = 0, \quad \frac{y_t^2 r}{2} + \delta_t - M_t^2 = 0, \quad 2\lambda r + \delta_H - M_H^2 = 0, \quad m^2 + \lambda r + T = 0.$$

Their leading Jacobian with respect to (r, y_t, λ, m^2) has determinant $2y_0 r_0^2 > 0$. Expanding these conditions to first order gives (C.67) and establishes local uniqueness order by order. In particular v_L and v_L^2 are expanded separately at this order. The value of G_F remains the imposed physical condition. $\square$

At the generated scale $Q = M_Z = 91.12484$ GeV the inputs are $v_F = 246.18970$ GeV and $M_t = 174.08240$ GeV together with $M_H = 124.98344$ GeV and $G_F = 1.16666255 \times 10^{-5}$ GeV⁻². The one-loop realization gives

$$\begin{aligned} \delta r &= 0.02008364, & v_L &= 248.66189 \text{ GeV}, \\ \bar{y}_t &= 0.98609417, & \bar{\lambda} &= 0.14526746, & \bar{m}^2 &= -8585.71890 \text{ GeV}^2. \end{aligned}$$

These are running coefficients matched to the stated generated physical conditions. The gauge couplings in the kernels are the leading common-scale coefficients. A finite conversion of the gauge inputs first changes these one-loop kernels at the next order. The strong input is the five-flavour coupling. Its threshold conversion is required at higher orders. The finite matching is distinct from the geometric mass comparison in Table 14 where the mass definitions are specified.

Top width from the matched action. Use the leading action coefficients of Proposition C.5 with $m_b = 0$ and $|V_{tb}| = 1$. Set $M_{W,0} = g_2 v_F/2$ and $x = M_{W,0}^2/M_t^2 < 1$. The absorptive part of the one-loop top pole gives the leading decay width

$$\Gamma_t^{(0)} = -\frac{\text{Im}[\ell \Delta s_t^{(1)}]}{M_t} = \frac{G_F M_t^3}{8\pi\sqrt{2}}(1-x)^2(1+2x). \tag{C.68}$$

The width is the rest-frame decay rate per unit proper time in natural units. It supplies an additional observable once the real matching conditions have fixed the action coefficients. With $t = M_t^2$ and $W = M_{W,0}^2$ the standard absorptive cut of the one-loop pole kernel [26,

Eq. (2.20)] gives $\text{Im } B(t; 0, W) = \pi(1 - x)$. The remaining terms are real at this order. Hence

$$\text{Im}[\ell \Delta s_t^{(1)}] = -\frac{g_2^2 t}{64\pi x} (1 - x)^2 (1 + 2x).$$

Using $W = g_2^2 v_F^2 / 4$ and $G_F = 1/(\sqrt{2} v_F^2)$ gives (C.68).

At the generated matching point $g_2 = 0.64821177$ and $M_{W,0} = 79.79153$ GeV. Equation (C.68) gives $\Gamma_t^{(0)} = 1.53445$ GeV and $\text{Im}[\ell \Delta s_t^{(1)}] = -267.12152$ GeV² in agreement with the complex pole calculation [15]. The one-loop absorptive self-energy corresponds to the Born $t \rightarrow bW$ decay rate. Relative QCD corrections of order α_s and electroweak radiative corrections enter through the two-loop absorptive pole and its consistent parameter conversion [1, 26]. The leading action mass $M_{W,0}$ is used throughout this comparison.

The construction also applies in a source neighbourhood. Let $\Phi(z, \theta) = 0$ denote the four conditions where θ varies the same generated modes and Gram matrices together with the vertices. On a fixed smooth threshold branch suppose the inverse Jacobian is uniformly bounded and the first-order solution has residual $\|\Phi(z^{(1)}(\cdot), \cdot)\|_{C^1(U)} = O(\ell^2)$. Implicit differentiation then gives $D_\theta z = -(D_z \Phi)^{-1} D_\theta \Phi + O(\ell^2)$. The C^1 residual controls both the value and its source derivative. The source variation includes the direct action term in (C.37). Neutral-mode elimination also retains its induced source and direct Schur contribution. These trace formulas use a common fixed trace space and even sources. Fermionic two-point projections use ordered graded left and right derivatives. For BRST-preserving source deformations and simple isolated complex poles the Nielsen identities give the corresponding gauge-independent pole conditions [54].

For an explicitly defined heavy-quark $\overline{\text{MS}}$ boundary the one-loop QCD part is

$$M_Q = \overline{m}_Q(\mu) \left[1 + \frac{\alpha_s(\mu) C_F}{4\pi} \left(4 + 3 \log \frac{\mu^2}{\overline{m}_Q(\mu)^2} \right) \right] + O(\alpha_s^2), \quad C_F = \frac{4}{3}.$$

The formula converts that running boundary to its perturbative pole definition [1]. The ladder mass itself retains the geometric definition in Section 7.

With $n_s^{\text{tilt}} \equiv 1 - n_s$ the bottom absolute mass is the geometric mean of the strange and top masses dressed by the window evolution

$$\frac{y_b}{y_t} = e^{-(2\alpha_{\text{dn}} - n_s^{\text{tilt}}(kL_{\text{CMB}} + 2\tau))}, \quad (\text{C.69})$$

so $m_b^2 = m_s m_t e^{n_s^{\text{tilt}}(kL_{\text{CMB}} + 2\tau)}$.

C.9 Charged-fermion masses

The electron exponent is the cascade-content number

$$N_e = (d + 1) \left(\sum Y^2 \Delta_f \right) = 4 \left(\frac{10}{3} \frac{3}{2} \right) = 20. \quad (\text{C.70})$$

The cascade uses $d + 1 = 4$ levels, each with the one-generation weight $\sum Y^2 \Delta_f = 5$. Its amplitude rule assigns the scalar generator $5kLI$ to each level. Thus their composition is $(e^{-5kL})^4 = e^{-N_e kL}$. The charge trace fixes the number of content units. The equal rate kL

per unit fixes the transfer used by the electron mass vertex. This source-to-amplitude rule is the electron entry of Table 2.

The electron mass is the Planck-anchored exponential chain with the low-scale squash correction

$$m_e = M_P e^{-20kL}(1 - s_0\kappa), \quad (\text{C.71})$$

where $s_0\kappa$ is the squash correction of Section C.4. This relation fixes the electron anchor directly. The muon-to-electron ratio carries the Euclidean-period factor

$$\frac{m_\mu}{m_e} = e^{2\alpha_{1p} + \sqrt{2\pi}}, \quad (\text{C.72})$$

the lepton ladder index plus the dimensionless entropy-minimum period $\sqrt{2\pi}$. The next lepton step is $m_\tau/m_\mu = e^{2\alpha_{1p}}$.

C.10 Neutrino masses and mixing angles

The Weinberg dimension-five operator gives a mass proportional to v^2/Λ [55]. The present seesaw relation fixes the family scale and its coefficient so that

$$m_{\nu_3} = \frac{v^2(2\pi)^2}{k_{\text{GUT}}}(1 + s_0\kappa) = 0.0502 \text{ eV}, \quad (\text{C.73})$$

with the squash correction of Section C.4. In the common action this relation fixes the Majorana coefficient of mass dimension -1 to $(2\pi)^2(1 + s_0\kappa)/k_{\text{GUT}}$. The Euclidean-period factor therefore enters the normalization of the mass-generating vertex. In the convention $m_\nu = C_W v^2/\Lambda_W$ the same boundary relation is expressed by $C_W = 1$ and $\Lambda_W = k_{\text{GUT}}/(2\pi)^2$ before the explicit squash insertion. This fixes the absolute neutrino scale. The following hypercharge-trace ratios fix the relative eigenvalues

$$\frac{m_{\nu_1}}{m_{\nu_2}} = \frac{1}{\text{Tr}(Y^2)} = \frac{3}{10}, \quad \frac{m_{\nu_2}}{m_{\nu_3}} = \frac{1}{\sqrt{3} \text{Tr}(Y^2)} = \frac{3}{10\sqrt{3}}, \quad (\text{C.74})$$

With $r_{12} = 3/10$ and $r_{23} = 3/(10\sqrt{3})$ the ordered absolute-mass spectrum is

$$(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) = m_{\nu_3}(r_{12}r_{23}, r_{23}, 1). \quad (\text{C.75})$$

In the per-generation content normalization $r_{12} = (N_L - N_R)/\sum Y^2$ with $N_L - N_R = 1$. The lighter masses are $m_{\nu_2} = 0.0087 \text{ eV}$ and $m_{\nu_1} = 0.0026 \text{ eV}$ at the displayed precision. The same quadratic hypercharge moment that normalises the gauge sector sets the two hierarchy ratios. The angular readouts specify the mixing texture in this ordered mass basis. At fixed hierarchy ratios the common Majorana coefficient moves all three masses together and their squared differences scale quadratically. The summed mass and the atmospheric splitting thus test the same boundary normalization through different observables.

The absolute-mass texture gives the raw rest-mass split

$$\Delta m_{21, \text{raw}}^2 = m_{\nu_2}^2 - m_{\nu_1}^2 = 6.874 \times 10^{-5} \text{ eV}^2. \quad (\text{C.76})$$

The NuFIT 5.2 normal-ordering reference is $7.41 \times 10^{-5} \text{ eV}^2$ [45, 46]. The unrounded rest-mass result is $6.874155305644093 \times 10^{-5} \text{ eV}^2$ with signed residual -7.231% . In standard vacuum propagation the oscillation phases use these same mass-squared differences. See Eqs. (14.39)–(14.40) in Section 14.4 of the review “Neutrino Masses and Mixing and Oscillations” in Ref. [1].

The solar-channel model couples an environmental scalar source to the second mass eigenspace. Its spectral projector selects this direction in the generated mass operator and the window-tilt factor sets the source strength. In solar matter the scalar insertion and electron forward-scattering potential jointly determine the mass-state populations leaving the source region. After loss of coherence over the long propagation baseline detection reads these populations through their electron-flavour projections [56–59]. At unit source amplitude the second level is

$$m_{\nu_2}^{\text{osc}} = m_{\nu_2}(1 + n_s^{\text{tilt}}), \quad n_s^{\text{tilt}} \equiv 1 - n_s = \tau \frac{7}{4}, \quad (\text{C.77})$$

The vacuum spectrum corresponds to zero environmental source. The unit-source propagation readout gives

$$\begin{aligned} \Delta m_{21, \text{tilt}}^2 &= (m_{\nu_2}^{\text{osc}})^2 - m_{\nu_1}^2 = 7.412190 \times 10^{-5} \text{ eV}^2, \\ \Delta m_{31}^2 &= m_{\nu_3}^2 - m_{\nu_1}^2 = 2.511 \times 10^{-3} \text{ eV}^2. \end{aligned} \quad (\text{C.78})$$

The tilt factor uses the RP^3 Weyl-window ratio of Section C.14. We realize it as a specified scalar source in the propagation Hamiltonian. Scalar mass insertions and vector matter potentials have different energy and antineutrino transformations [1, 59]. Keeping the vacuum eigenvectors fixed the Hermitian scalar-source term is

$$\Delta H(E, x) = \frac{\zeta(x)}{2E} m_{\nu_2}^2 [(1 + n_s^{\text{tilt}})^2 - 1] P_2, \quad P_2 = U_{\cdot 2} U_{\cdot 2}^\dagger. \quad (\text{C.79})$$

At $\zeta = 1$ it implements the specified squared-mass shift exactly. At $\zeta = 0$ the rest-mass Hamiltonian is recovered. As a scalar self-energy it transforms as $\Delta \bar{H} = \Delta H^*$ for antineutrinos. Its E^{-1} dependence differs from an energy-independent vector forward-scattering potential. An environmental interpretation thus specifies the profile $\zeta(x)$ and its coupling together with its vanishing vacuum limit. Equation (C.79) identifies the operator tested by the scalar-source model. The unit-amplitude source comparison has residual $+0.030\%$ and is classified as Tier D. Its comparison with a fitted solar oscillation parameter uses the forward propagation and production average in Appendix E. Propagation with specified profiles retains the labelled rest spectrum [15]. The absolute masses and their sum together with the lightest mass used in the cosmological floor use the rest-mass spectrum above.

Correlated response of the three neutrino splittings. Let $K_\nu = U D^2 U^\dagger$ with the distinct rest masses above and $t = n_s^{\text{tilt}}$ and $d_\nu = m_{\nu_2}^2(2t + t^2)$. The generated second-state projector is the spectral polynomial

$$P_2 = \frac{(K_\nu - m_{\nu_1}^2 I)(K_\nu - m_{\nu_3}^2 I)}{(m_{\nu_2}^2 - m_{\nu_1}^2)(m_{\nu_2}^2 - m_{\nu_3}^2)}. \quad (\text{C.80})$$

For the scalar-source numerator $K_\nu + \zeta d_\nu P_2$

$$\Delta_{21}(\zeta) = \Delta_{21}(0) + \zeta d_\nu, \quad \Delta_{31}(\zeta) = \Delta_{31}(0), \quad \Delta_{32}(\zeta) = \Delta_{32}(0) - \zeta d_\nu. \quad (\text{C.81})$$

At $t = 7/200$ the common absolute mass scale cancels

$$\frac{d_\nu}{\Delta_{21}(0)} = \frac{407}{5200}, \quad \frac{\Delta_{21}(1)}{\Delta_{21}(0)} = \frac{5607}{5200}. \quad (\text{C.82})$$

Spectral interpolation gives (C.80). It acts as one on the second eigenvector and zero on the other two. Thus the source shifts only its squared eigenvalue. The exact mass ratios give $\Delta_{21}(0) = (1 - 9/100)m_{\nu_2}^2$ and $2t + t^2 = 2849/40000$. These yield the stated rational ratios.

In the absence of a noncommuting matter potential a varying profile has the exact propagator

$$S(L) = U \text{diag} \left(e^{-im_{\nu_1}^2 L/(2E)}, e^{-i(m_{\nu_2}^2 L + d_\nu \Xi)/(2E)}, e^{-im_{\nu_3}^2 L/(2E)} \right) U^\dagger, \quad \Xi = \int_0^L \zeta(x) dx.$$

For two profiles Duhamel's identity gives the transition-probability bound $|P - \tilde{P}| \leq d_\nu \|\zeta - \tilde{\zeta}\|_{L^1}/(2E)$. Subtracting the unobservable identity part of P_2 leaves an operator of norm $1/2$. Apply the unitary propagator bound to this operator. The algebraic splitting identities concern the scalar numerator. Appendix E.1 includes the electron matter potential in the propagation Hamiltonian.

The PMNS texture specifies the orientation of the neutrino mass eigenspaces relative to the charged-lepton frame in the common left-handed kinetic space. The mixing probabilities are the squared overlaps of these eigenspaces as in (C.66). The solar-angle relation ties the light-pair rotation to its mass ratio. The reactor angle weights the electron-flavour projection onto the third state by the Euclidean-period and internal-geometry factors. The atmospheric angle starts at equal muon and tau weights and receives a positive weak-isospin contribution

$$\begin{aligned} \sin^2 \theta_{12} &= \frac{m_{\nu_1}}{m_{\nu_2}} = \frac{3}{10} = 0.30, \\ \sin^2 \theta_{13} &= \left(\frac{1}{2\pi} \right)^2 \frac{\sqrt{3}}{2} = 0.02194, \\ \sin^2 \theta_{23} &= \frac{1}{2} + \frac{\text{Tr}(T_3^2)}{(2\pi)^2} = 0.5507. \end{aligned} \quad (\text{C.83})$$

Here $\text{Tr}(T_3^2) = 2$. These relations define the near-tribimaximal angular texture used in the comparison. The positive isospin term places θ_{23} in the upper octant.

C.11 The electroweak order parameter

The inherited complete condensate. The ‘‘Renormalized conformal spectral response’’ theorem of Ref. [4] supplies the complete conformal tower with its feedback and internal subtraction

$$\begin{aligned} a_j &= \frac{(2j+1)^2 - 1/4}{L^2}, \quad d_j = (2j+1)^2, \quad a_j \mapsto a_j + gE^2, \quad g > 0, \\ \mu_E &= \frac{8}{3}(R - R_c), \quad R = \frac{6}{L^2}. \end{aligned} \quad (\text{C.84})$$

The constant scalar shell is included. This unweighted relative response and the Gaussian TT trace have different spectral weights and source insertions. We apply the existence and stability theorems together with the gap theorem for the complete conformal response in the convention $Z_E = 1$.

In the dimensionless radius convention of Section 4.4 $L(k) = cM_G/k$ and $L_c = \sqrt{\pi}$. The certified $c > 2.491 > \sqrt{\pi}$ gives $L(k) > L_c$ for every $0 < k \leq M_G$ and $\mu_E(M_G) = 16(c^{-2} - \pi^{-1}) < -2.50$. Hence the complete condensate and positive scalar gap exist on this stationary branch for every $g > 0$. The result concerns stationary internal spectral states. Their relaxation requires a time evolution. The fixed length unit $\ell_* = 1/M_G$ recovers $L_{\text{phys}} = c/M_G$ at emergence. Dimensionful masses require the matching readout.

For explicit evaluation put $\zeta = gL^2$ and $\gamma_j = (2j+1)^2 - 1/4$. The proposition “Stationary equation for the conformal tower” of Ref. [4] gives the function

$$\mathcal{H}(z) = \sum_{j \geq 0} \frac{d_j z}{\gamma_j(\gamma_j + z)}, \quad z \geq 0. \quad (\text{C.85})$$

The cited proposition evaluates this sum and proves strict monotonicity. The stationary amplitude, radial mass and local quartic coefficient are

$$\begin{aligned} \mathcal{H}(z_*) &= -\frac{\mu_E}{2\zeta}, & E_*^2 &= \frac{z_*}{\zeta}, & m_\delta^2 &= \frac{4\zeta z_* \mathcal{H}'(z_*)}{Z_E}, \\ \lambda_{E,\text{spec}} &= \frac{\pi(\pi+2)}{4}\zeta^2, & \mathcal{M}_{\text{sc}}^2 &= \Delta_0 + m_\delta^2 I. \end{aligned} \quad (\text{C.86})$$

In particular the full scalar tower is $m_{l,\text{sc}}^2 = l(l+2)/L^2 + m_\delta^2$ for $l = 0, 2, \dots$. The constant mode gives its lowest mass followed by the $l = 2$ nonconstant shell. The inherited strict mass bound is

$$0 < m_\delta^2 < -\frac{2\mu_E}{Z_E}. \quad (\text{C.87})$$

The feedback coefficient g is defined in the E convention of the complete conformal response. The electroweak VEV is determined by the separate window-squared relation below.

Proposition C.6 (Joint amplitude and radius equilibrium). Retain the complete relative potential in fixed response units with its subtraction and fixed $g, L_c > 0$ and vary both $y = E^2 \geq 0$ and $L > 0$

$$F_L(y) = 8(L^{-2} - L_c^{-2})y + I(gL^2 y), \quad I(u) = \sum_j \beta_j \left[\frac{u}{\gamma_j} - \log \left(1 + \frac{u}{\gamma_j} \right) \right].$$

Assume $\beta_j \geq 0$ and $\gamma_j > 0$ together with $\sum_j \beta_j/\gamma_j^2 < \infty$ and $\sum_j \beta_j/\gamma_j > 2/(gL_c^4)$. Then the unique joint global minimum is

$$L_* = \sqrt{2}L_c, \quad I'(u_*) = \frac{2}{gL_c^4}, \quad y_* = \frac{u_*}{2gL_c^2} > 0. \quad (\text{C.88})$$

Its Hessian in (y, η) with $L \mapsto Le^{-\eta}$ is positive definite. After relaxing the amplitude its compression curvature is $64y_*/L_*^2 > 0$. The unweighted conformal tower satisfies the hypotheses.

Proof. Set $z = L^{-2}$ and $u = gL^2y$. Direct completion of the square gives

$$F_L(y) = I(u) - \frac{2u}{gL_c^4} + \frac{8u}{g} \left(z - \frac{1}{2L_c^2} \right)^2. \quad (\text{C.89})$$

The summability assumption gives $I'(0) = 0$ and $I''(u) = \sum_j \beta_j / (\gamma_j + u)^2 > 0$ and $I'(\infty) = \sum_j \beta_j / \gamma_j$. Thus the first two terms have a unique positive minimizer with negative value. The $y = 0$ boundary has value zero. At positive u the square vanishes only at the stated radius. At the joint minimum the Hessian in (u, z) is $\text{diag}(I''(u_*), 16u_*/g) > 0$. The change from (y, η) is nonsingular. Eliminating the amplitude variation gives the stated compression curvature. $\square$

The joint minimum determines the static collective response of the inherited potential. The amplitude deformation $A + gyI$ and the radius deformation of A are varied separately. On the amplitude-stationary branch at general $L > L_c$ the same derivatives give

$$\partial_\eta F_{Le^{-\eta}}(y_L)|_{\eta=0} = 16y_L(2L^{-2} - L_c^{-2}). \quad (\text{C.90})$$

These are the vacuum equilibrium and source force of the joint relative potential in its stated units. The finite TT endpoint c_R continues to be determined by its own sourced trace. In a common matter-geometry action the radius force enters the total source balance of (G.25) with the chosen physical density normalization.

The squash coordinate. The isometry breaking is driven by the $J = 2$ squash mode with order parameter φ of the internal geometry. Its dynamics is the Landau potential on the curvature axis

$$V(\varphi) = \frac{1}{2}\xi(R - R_c)\varphi^2 + \frac{\lambda}{4}\varphi^4, \quad (\text{C.91})$$

with $\xi = 1/8$ the conformal coupling and $R_c = 6/\pi$ the critical curvature. The effective mass squared is $m^2 = \xi(R - R_c)$. It is tachyonic for $R < R_c$ and hence the curvature crossing selects the condensed phase of this potential. The quartic coupling is fixed by the stationarity of the potential at the squash vacuum $\varphi_0 = 2\tau$

$$\frac{\partial V}{\partial \varphi} = 0 \quad \Rightarrow \quad \lambda = \frac{\xi(R_c - R_{\text{GUT}})}{(2\tau)^2} = 149, \quad (\text{C.92})$$

with $R_{\text{GUT}} = 6/L_{\text{GUT}}^2$ and $L_{\text{GUT}} = \sqrt{3}/\tau$ the isometry-breaking scale. The adopted parity-preserving torsion response is written as

$$L_{\text{torsion}} = aT^2 + bT^{bac}T_{abc} + c(T^a_{ab})^2. \quad (\text{C.93})$$

The torsion convention used here imposes $b = 4a$ and $c = -(a + b/3) = -(7/3)a$ with the emergence-scale normalisation $a = M_G^3/4$. Stationarity at $\varphi_0 = 2\tau$ then fixes the displayed squash quartic. These relations specify the torsion response used in this calculation. The electroweak VEV uses the separate window-squared relation

$$v = M_G \frac{3\alpha}{\pi} e^{-4\pi kL} (1 - s_0\kappa), \quad (\text{C.94})$$

with the squash correction $s_0\kappa$ of Section C.4. The prefactor is the product of the $J = 2$ squash factor $3\alpha/2$ with $\alpha = 1/(16\pi^2)$ and the Fourier factor $1/\pi$ together with the double-crossing factor 2. The electroweak amplitude is transported through creation and stabilization with the same generator $2\pi kLI$ at each crossing. Their composition gives $e^{-4\pi kL}$ and the hierarchy $v/M_G \sim 10^{-16}$. Equal crossing weights and their action on the amplitude specify this readout. Equation (C.10) relates it to the electron cascade. A Landau–Zener realization is specified by a source trajectory satisfying the amplitude-matching condition (C.56).

Coordinate consistency of the two Landau readouts. Write $\lambda_\varphi = \lambda$ and $\mu_\varphi = (R - R_c)/8$. The amplitude-coordinate rule in Ref. [4] transforms the entire response. If $E = a_{E\varphi}\varphi$ describes the same response and feedback without an overall rescaling matching the two drives and quartic Taylor coefficients requires

$$a_{E\varphi}^2 = \frac{3}{64}, \quad \lambda_E = \frac{4096}{9}\lambda_\varphi, \quad Z_E = \frac{64}{3}Z_\varphi, \quad g_E = \frac{64}{3}g_\varphi. \quad (\text{C.95})$$

The normalized radial mass is invariant and $Z_E = 1$ corresponds to $Z_\varphi = 3/64$. The amplitude-normalisation calculation in Ref. [4] gives the stationary-coordinate and kinetic matching. The squash potential uses its coefficient $\lambda = 149.145$ while the full spectral potential uses $\lambda_{E,\text{spec}}$ in the normalization above. The electroweak readout is the window-defined VEV v .

C.12 The Higgs quartic

The Higgs matching relation assigns a quartic restoring response to the canonically normalized electroweak amplitude. Its pseudo-dilaton parameterization combines a strong-sector coefficient with an additive contribution from the retained generations

$$\lambda_H = \frac{\lambda_{\text{dil}} + \sigma_{\text{SM}}}{32\pi^2} = \frac{12\pi + 3}{32\pi^2} = 0.129, \quad (\text{C.96})$$

The strong-sector matching assigns 4π per retained generation. Thus $\lambda_{\text{dil}} = N_{\text{gen}} 4\pi = 12\pi$. The additive content term is $\sigma_{\text{SM}} = N_{\text{gen}} = 3$. The integer 3 is fixed by the common copy count while the per-copy strength 4π and the factor $1/(32\pi^2)$ specify the effective matching and quartic normalization. These data define λ_H before the finite electroweak matching calculation. The strong-sector coefficient expresses the contribution assigned to the collective scalar channel. A related use of a collective scalar in mass generation occurs in the Nambu–Jona-Lasinio gap equation [60, Sec. III]. The effective matching here is fixed by the coefficients displayed above. Combining this quartic with the generated electroweak scale gives the tree-level Higgs mass

$$m_H = \sqrt{2\lambda_H} v = 124.98 \text{ GeV}, \quad (\text{C.97})$$

with a central-value deviation from the reference 125.2 GeV of -0.17% . This mass supplies M_H in Proposition C.5. The one-loop calculation determines the corresponding running quartic and potential coefficient together with the vacuum parameter in the stated scheme. The mass comparison therefore tests the generated tree-level relation and the finite matching establishes its quantum implementation through the calculated order. The quartic λ_H is distinct from the squash coefficient $\lambda = 149$ of Section C.11.

C.13 CKM mixing and CP

The Gatto–Sartori–Tonin relation motivates connecting the Cabibbo angle to a light-quark mass ratio [61]. Here the corresponding input is assigned through the right-singlet hypercharge ratio $|Y_{d_R}|/|Y_{u_R}| = 1/2$

$$\mathbf{r}_G = e^{-2\alpha_{\text{dn}}} \left(1 + \frac{|Y_{d_R}|}{|Y_{u_R}|}\right)^2 = e^{-2\alpha_{\text{dn}}} \left(\frac{3}{2}\right)^2 = 0.05015, \quad (\text{C.98})$$

and the Cabibbo element uses its square root

$$|V_{us}| = \sqrt{\mathbf{r}_G} = 0.2239. \quad (\text{C.99})$$

The Cabibbo overlap uses the down-sector transfer amplitude with the right-singlet charge factor $3/2$. The absolute-mass ratio $m_d/m_s = e^{-2\alpha_{\text{sd}}}$ of Section C.7 reads the first-generation diagonal transfer with step $\alpha_{\text{sd}} = \alpha_{\text{dn}} - kL_{\text{CMB}}/6$. These angular and diagonal source operations give values differing by 1.6% at the endpoint. Their respective roles are fixed by the relative frames and singular values of the common mass vertices. The remaining two quark-mixing assignments are

$$|V_{cb}| = \frac{1}{2}e^{-\alpha_{\text{up}}}(1 - s_0\kappa), \quad |V_{ub}| = \frac{1}{2}e^{-2\alpha_{\text{up}}}(1 + \tau\kappa), \quad (\text{C.100})$$

the adjacent-generation element carrying the squash correction $s_0\kappa$ and the cross-generation element the chiral correction $\tau\kappa$. The three-element product is numerically close to the weak cube

$$|V_{us}||V_{cb}||V_{ub}| \approx \alpha_W(v)^3, \quad (\text{C.101})$$

with $\alpha_W(v) = g_2(v)^2/(4\pi)$ the electroweak-scale weak coupling. The window-capacity coupling in Section 5.1 has value $2/(kL)^5$. The product is 2.45% below $\alpha_W(v)^3$ at the reported endpoint. This numerical comparison is not used to fix the ladder.

The angular source also fixes a complex relative phase. Together with the intergenerational overlaps this phase determines the basis-invariant CP-odd response (C.66) [62]. The phase assignment uses the left-right content ratio

$$\frac{N_L}{N_R} = \frac{8}{7}, \quad (\text{C.102})$$

eight left-handed Weyl components (four weak doublets) against seven right-handed singlet components per generation. The lepton phase assignment carries the undiluted ratio $\delta_{\text{PMNS}} \sim (8/7)\pi$ and the quark phase is diluted by the colour number

$$\delta_{\text{CKM}} = \frac{8}{7} \frac{\pi}{N_c} = \frac{8\pi}{21} = 68.57^\circ. \quad (\text{C.103})$$

The corresponding leading-order Jarlskog invariant [62] is

$$J = |V_{us}||V_{cb}||V_{ub}| c_{12}c_{23} \sin \delta = 3.15 \times 10^{-5}, \quad (\text{C.104})$$

where $c_{12} = \sqrt{1 - |V_{us}|^2}$ and $c_{23} = \sqrt{1 - |V_{cb}|^2}$. Corrections from c_{13} enter at relative order $|V_{ub}|^2$. The baryon-asymmetry ansatz combines this CP measure with the weak coupling and a content normalisation

$$\eta_B^{\text{raw}} = \frac{J\alpha_W(v)^2}{56} = 6.09 \times 10^{-10}, \quad (\text{C.105})$$

where $1/56 = 1/(N_L N_R) = \xi/N_R$ combines $\xi = 1/8$ with $N_R = 7$. The factor J carries the sign of the CP asymmetry and the suppression from flavour mixing. The weak and content factors convert this response into the raw baryon-number readout. Pairing it with the photon-temperature normalization gives the baryon density used in the cosmological calculation. Sakharov's conditions require baryon-number violation together with C and CP violation and departure from equilibrium [63]. The factor $1/56$ and the two powers of $\alpha_W(v)$ specify the weak and nonequilibrium weights of the present ansatz. A thermal-rate realization also specifies the baryon-violating processes and their kinetic evolution. Section 8.4 uses this raw asymmetry with the raw photon-temperature readout.

C.14 Cosmology

The cosmological readouts share one lightest-neutrino rest-mass floor. On the branch $0 \leq x = s_0 \kappa(s_0) < 1/4$ the density readout is

$$\rho_\Lambda = Y_u m_{\nu_1}^4 (1 - 4s_0 \kappa) = \frac{2}{3} m_{\nu_1}^4 (1 - 4s_0 \kappa), \quad (\text{C.106})$$

where $Y_u = 2/3$ is the up-quark hypercharge. The neutrino masses use the dimension-five operator with the coefficient specified in Section C.10

$$m_{\nu_3} = \frac{v^2 (2\pi)^2}{k_{\text{GUT}}} (1 + s_0 \kappa), \quad m_{\nu_1} = m_{\nu_3} \cdot \frac{m_1}{m_2} \cdot \frac{m_2}{m_3}, \quad (\text{C.107})$$

with the hierarchy ratios $r_{12} = m_1/m_2 = 3/10$ and $r_{23} = m_2/m_3 = 1/(\sqrt{3} \text{Tr}(Y^2))$. The same rest mass m_{ν_1} enters the vacuum floor and entropy span together with the photon temperature. All are expressed in common energy units.

Composition of the squash corrections. Put $x = s_0 \kappa(s_0)$. At fixed values of the remaining inputs define the quantities before the explicit squash corrections by

$$v_0 = M_G(3\alpha/\pi)e^{-4\pi kL}, \quad m_{3,0} = (2\pi)^2 v_0^2 / k_{\text{GUT}}, \quad m_{1,0} = r_{12} r_{23} m_{3,0}, \quad \rho_0 = Y_u m_{1,0}^4.$$

Substituting the corrected electroweak scale into the seesaw and vacuum-floor rules gives

$$\begin{aligned} v/v_0 &= 1 - x, & m_{\nu_3}/m_{3,0} &= (1 - x)^2(1 + x), \\ \frac{vm_{\nu_3}}{v_0 m_{3,0}} &= (1 - x)^3(1 + x), & \rho_\Lambda/\rho_0 &= (1 - x)^8(1 + x)^4(1 - 4x). \end{aligned} \quad (\text{C.108})$$

The last two factors are $1 - 2x + O(x^2)$ and $1 - 8x + O(x^2)$ respectively. At fixed seesaw inputs the local pair instead satisfies

$$(1 + x)^4(1 - 4x) = 1 - 10x^2 - 20x^3 - 15x^4 - 4x^5.$$

These identities follow by substituting $v = v_0(1 - x)$ into the stated seesaw formula and then using the same m_{ν_1} in the fourth-power vacuum floor. The correction first changes the electroweak amplitude, then the neutrino mass, and finally the vacuum density. The subscript 0 removes only the explicit x factors. The other inputs retain their dependence on τ . This dependence is included when taking a total derivative.

The endpoint entropy span combines a Gaussian variance ratio and a content offset. The Planck variance is M_P^2 and the vacuum-floor variance is $\sqrt{\rho_\Lambda}$. Since $\rho_\Lambda = M_P^4 \hat{\rho}_\Lambda$ the dimensional anchor cancels. The readout is

$$S_{\text{end}} = \ln \frac{M_P^2 \sqrt{2\pi + r_{23}}}{\sqrt{\rho_\Lambda}} = \ln \frac{\sqrt{2\pi + r_{23}}}{\sqrt{\hat{\rho}_\Lambda}} = 139.2537, \quad (\text{C.109})$$

with $r_{23} = 3/(10\sqrt{3})$. For a Gaussian of standard deviation σ the differential entropy is $H(\sigma) = \frac{1}{2} \ln(2\pi e \sigma^2)$. The endpoint span is defined as twice the entropy difference plus the single content offset $\ln \sqrt{2\pi + r_{23}}$

$$S_{\text{end}} = 2[H(M_P) - H(\rho_\Lambda^{1/4})] + \ln \sqrt{2\pi + r_{23}} = \ln \frac{M_P^2 \sqrt{2\pi + r_{23}}}{\sqrt{\rho_\Lambda}} = \ln \frac{\sqrt{2\pi + r_{23}}}{\sqrt{\hat{\rho}_\Lambda}}. \quad (\text{C.110})$$

The entropy difference uses the same energy reference unit at both ends, so the arbitrary logarithmic unit cancels. Its composition is exact. For positive Gaussian standard deviations $\sigma_a, \sigma_b, \sigma_c$

$$S(a, b) := 2[H(\sigma_a) - H(\sigma_b)] = 2 \ln(\sigma_a/\sigma_b), \quad S(a, c) = S(a, b) + S(b, c).$$

Intermediate Gaussian scales therefore telescope. For a specified positive C^1 variance path $V(t)$ twice the entropy difference from a to b is $-\int_a^b \partial_t \log V(t) dt$. This representation uses the variance-path derivative. The content offset is attached once to the full endpoint pair. Thus $S_{\text{end}} = \ln W$ with $W = \sqrt{2\pi + r_{23}}/\sqrt{\hat{\rho}_\Lambda}$ and the Hubble readout is

$$H_0 = M_P \sqrt{\pi} e^{-S_{\text{end}}}. \quad (\text{C.111})$$

The dark-energy fraction is

$$\Omega_\Lambda = \frac{\rho_\Lambda}{3H_0^2 M_P^2} = \frac{2}{3} + \frac{r_{23}}{3\pi} = 0.68504. \quad (\text{C.112})$$

Indeed $e^{-2S_{\text{end}}} = \rho_\Lambda/[M_P^4(2\pi + r_{23})]$ gives $H_0^2 = \pi\rho_\Lambda/[M_P^2(2\pi + r_{23})]$. Dividing by the critical density cancels the common mass floor and yields $(2\pi + r_{23})/(3\pi)$. For comparison the nonnegative scalar spectrum of the round internal space defines a normalized spectral state. In dimensionless variables let $\hat{\lambda}_j = 4j(j+1)$ and $d_j = (2j+1)^2$ and $u > 0$. Then

$$Z_{\text{sp}}(u) = \sum_{j \geq 0} d_j e^{-u \hat{\lambda}_j}, \quad S_{\text{sp}}(u) = \log Z_{\text{sp}}(u) - u \partial_u \log Z_{\text{sp}}(u), \quad \frac{dS_{\text{sp}}}{du} = -u \text{Var}_u(\hat{\lambda}). \quad (\text{C.113})$$

To verify the formula normalize each individual mode weight by Z_{sp} and substitute its logarithm in the Shannon sum. Quadratic growth of $\hat{\lambda}_j$ justifies differentiation. In particular $S_{\text{sp}} = H(p) + \sum_j p_j \log d_j$ with $p_j = d_j e^{-u \hat{\lambda}_j}/Z_{\text{sp}}$. A conformal constant shift gives

the same normalized state. On the complete spectrum the entropy diverges as $u \downarrow 0$ and tends to zero as $u \rightarrow \infty$. On a finite carrier its small- u limit is the logarithm of the carrier dimension. These limits follow respectively from divergence of the unweighted count and dominance of the simple ground mode.

Equation (C.113) describes the internal spectral state. The endpoint entropy S_{end} uses a Gaussian variance ratio and a single content factor containing r_{23} . Thus its definition is independent of the internal heat parameter u . For $A = p^2$ an amplitude window $e^{-\sigma^2 A/2}$ has heat parameter $\sigma^2/2$. Squaring that amplitude gives parameter σ^2 . These parameters have units of length squared and require a separate energy identification before they can be called a physical inverse temperature.

The endpoint residual is the remaining Hamiltonian fraction

$$\Omega_{\Sigma}^{\text{budget}} = 1 - \Omega_{\Lambda} - \Omega_b = 0.26554751. \quad (\text{C.114})$$

Photons and the three thermal rest-mass neutrino states occupy part of this budget. Equations (8.1) and (8.2) determine the remaining conserved cold fraction $\Omega_{\text{cold}} = 0.26403880$ for the cold-dark-matter comparison. The microwave temperature is the photon floor set by the lightest neutrino

$$T_{\text{CMB}}^{\text{raw}} = \frac{m_{\nu_1} r_{12}}{\pi} (1 - s_0 \kappa) (1 - \tau \Delta_s) = 2.7310 \text{ K}, \quad (\text{C.115})$$

with $r_{12} = 3/10$ and $\Delta_s = 1/2$ the scalar conformal weight. The insertion $(1 - s_0 \kappa)$ multiplies the corrected rest mass. Thus both enter the temperature response. The other factors are the content ratio $r_{12} = (N_L - N_R)/\sum Y^2$ and the period factor $1/\pi$ and the chiral correction $1 - \tau \Delta_s$. The formula is evaluated in energy units before conversion to Kelvin. The finite endpoint photon zero mode multiplies the raw floor by

$$C_{\gamma} = 1 - \frac{\tau}{\pi^2}, \quad T_{\text{CMB}} = C_{\gamma} T_{\text{CMB}}^{\text{raw}} = 2.72547 \text{ K}. \quad (\text{C.116})$$

The corrected temperature enters the CMB-monopole and linear CMB comparisons. The baryon-density readout uses the raw photon floor and raw Sakharov asymmetry

$$\Omega_b = \frac{\eta_B^{\text{raw}} n_{\gamma}^{\text{raw}} m_p}{\rho_{\text{crit}}}, \quad n_{\gamma}^{\text{raw}} = \frac{2\zeta(3)}{\pi^2} (T_{\text{CMB}}^{\text{raw}})^3,$$

The common raw convention fixes both photon number and baryon-to-photon ratio in this density.

Common-mass elimination in the cosmological readouts. Let $D = 2\pi + r_{23}$ and $A_{\gamma} = C_{\gamma}(r_{12}/\pi)(1 - \tau \Delta_s)$ and $T_E = k_B T_{\text{CMB}}$. For $0 \leq x < 1/4$ and positive $m_{\nu_1}, Y_u, A_{\gamma}$ the cosmological relations satisfy

$$\begin{aligned} M_P H_0 &= \sqrt{\frac{\pi Y_u}{D}} m_{\nu_1}^2 \sqrt{1 - 4x}, & T_E &= A_{\gamma} m_{\nu_1} (1 - x), \\ \Omega_{\Lambda} &= \frac{D}{3\pi}, & \frac{T_E^2}{M_P H_0} &= A_{\gamma}^2 \sqrt{\frac{D}{\pi Y_u}} \frac{(1 - x)^2}{\sqrt{1 - 4x}}. \end{aligned} \quad (\text{C.117})$$

Exponentiating the endpoint entropy gives $H_0^2 = \pi\rho_\Lambda/(M_{\text{P}}^2 D)$. Substitute the same fourth-power floor and divide the squared photon readout by $M_{\text{P}}H_0$. The mass and every upstream factor common to that mass cancel exactly. The cancellation holds for mass variations induced by v or the seesaw scale or their shared uncertainties. Appendix D.4 bounds the remaining dependence on x and distinguishes the normalized ratio from the full temperature–Hubble ratio.

The primordial readout assigns the variance of the generated window to the dimensionless power of the comoving curvature perturbation $\mathcal{R}$ at the comparison pivot. We use the Fourier convention

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} \Delta_{\mathcal{R}}^2(k),$$

with $\Delta_R^2 = \Delta_{\mathcal{R}}^2(k_*)$ and $n_s - 1 = d \log \Delta_{\mathcal{R}}^2 / d \log k|_{k_*}$. Thus the amplitude measures the variance per logarithmic wavenumber interval and the tilt specifies its local scale dependence [28]. The internal source rule fixes this tilt from the torsion modulus and projected RP^3 channel weights

$$1 - n_s = \tau \cdot \frac{7}{4} = 0.035, \quad (\text{C.118})$$

where the numerator sums the weights of the specified response channels and the denominator is the four-level window normalisation

$$\frac{7}{4} = \frac{1 + 2 + 1 + 3}{d + 1}, \quad d = 3.$$

The same rational value follows from the conformal-curvature identity

$$\frac{7}{4} = 1 + \xi R_{\text{LC}} L^2 = 1 + \frac{1}{8} 6.$$

Here 1, 2, 1, 3 are the weights in the order of the scalar and transverse-vector channels followed by the spinor-projected and effective three-branch tensor channels. The last channel includes a trace component and has weight 3. The rough TT bundle has weight 2. The 7/4 ratio and $\lambda_{\text{TT}} = 14$ specify the effective internal tensor catalogue used in this readout. For the scalar power the normalization assigns the spin- $\frac{1}{2}$ Gaussian zero-point variance to the curvature channel and multiplies it by the window-period attenuation and chiral response

$$\Delta_R^2 = \frac{1}{2} \left(\frac{1}{2\pi} \right)^2 e^{-2\pi k L_{\text{CMB}}} (1 - \tau\kappa) = 2.10 \times 10^{-9}, \quad (\text{C.119})$$

with $\Delta_0^2 = (1/2)(1/2\pi)^2$ the spin- $\frac{1}{2}$ zero-point factor and $e^{-2\pi k L_{\text{CMB}}}$ the attenuation of this quadratic response and $(1 - \tau\kappa)$ its chiral-squash correction. The microwave-pivot window is $k L_{\text{CMB}} = k L (1 - \tau/4)$. The tensor readout assigns the relative Euclidean-period weight to the same scalar variance. With Δ_t^2 denoting the power summed over the two external tensor helicities its ratio to the scalar power is

$$r = \left(\frac{1}{2\pi} \right)^2 = 0.02533, \quad (\text{C.120})$$

so the tensor amplitude is $\Delta_t^2 = r \Delta_R^2 = 5.32 \times 10^{-11}$. The observational reference is the BK18 bound $r_{0.05} < 0.036$ at 95% confidence [43]. Its likelihood evaluates the tensor-to-scalar ratio at $k_* = 0.05 \text{ Mpc}^{-1}$ with $n_t = 0$. Section 10.4 uses the same reference convention.

C.15 QCD scale extraction and composite-channel masses

The QCD running starts from $g_3(M_G)$ in Section 5.3. Section C.5 supplies the two-loop coefficients on the same representation content with $y_0 = 1$ held fixed.

The QCD extraction convention. Write $g_i(\mu)$ for this running solution and define the internal extraction point and coupling together with the correction by

$$\begin{aligned}\mu_{\text{ext}} &= \frac{v}{2} \sqrt{g_2(v)^2 + \frac{3}{5}g_1(v)^2}, \quad a_{\text{raw}} = \frac{g_3(\mu_{\text{ext}})^2}{4\pi}, \\ a_{\text{ext}} &= a_{\text{raw}}(1 - \epsilon), \quad \epsilon = \frac{2\tau\kappa}{8}.\end{aligned}\tag{C.121}$$

The extraction point is defined by the couplings at v . The electroweak Z -mass comparison uses the separate fixed point of Section 6. The five-flavour readout is

$$\begin{aligned}\Lambda_{\text{QCD}} &= \mu_{\text{ext}} \left(\frac{b_0 a_{\text{ext}}}{4\pi} \right)^{-p} e^{-q/a_{\text{ext}}}, \\ b_0 &= \frac{23}{3}, \quad b_1 = \frac{116}{3}, \quad p = \frac{b_1}{2b_0^2}, \quad q = \frac{2\pi}{b_0}.\end{aligned}\tag{C.122}$$

Equations (C.121) and (C.122) define the extraction convention. The five-flavour coefficient is assigned at μ_{ext} . Explicit heavy-quark decoupling converts the six-flavour boundary by threshold matching at m_t . The factor $(1 - \epsilon)$ acts on the coupling before the nonlinear extraction of Λ_{QCD} .

Composite labels and the mass readout. The scalar spectrum contains the constant mode and a first nonconstant shell of eigenvalue $8/L^2$. The composite-channel construction uses a separate mass operator with dimensionless labels

$$\hat{\lambda} = 2\hat{\lambda}_{\text{gluon}} + C_2(J) + nN_g\xi, \quad \hat{\lambda}_{\text{gluon}} = 4, \quad N_g\xi = 1.\tag{C.123}$$

Thus all terms have the same units. Geometric eigenvalues carry the common factor L^{-2} . For the normal two-leg reference the coclosed one-form spectrum of Ref. [4] gives the six-dimensional Killing carrier $\mathcal{K} = (1, 0) \oplus (0, 1)$ with $L^2\Delta_1^\perp|_{\mathcal{K}} = 4I$. The two vector responses form a common symmetric tensor carrier. The sum of their internal eigenvalues gives the shared two-leg contribution. The total-rotation Casimir resolves its composite channels. The additive two-leg/Casimir rule gives

$$\begin{aligned}L^2H_2 &= 8I + C_2^{\text{tot}}, \\ \text{Sym}^2\mathcal{K} &= 2(0, 0) \oplus (1, 1) \oplus (2, 0) \oplus (0, 2).\end{aligned}\tag{C.124}$$

With $C_2(j_L, j_R) = 2[j_L(j_L + 1) + j_R(j_R + 1)]$ the decomposition gives 8, 16, 20 with multiplicities 2, 9, 10. It verifies the reference levels 8, 16 on the Killing carrier. The tensor readout selects diagonal spin $J = 2$ in $(1, 1)$. The two singlets admit even and odd exchange combinations at the same reference value 8. The masses labelled 17, 18 therefore use the additional conformal-channel assignments in the four-channel operator below.

On the reported channel list $\mathcal{C} = (0^{++}, 2^{++}, 0^{-+}, 0^{++*})$ the absolute mass normalization uses the scalar reference $\hat{\lambda}_* = 8$ and $m_*/\Lambda_{\text{QCD}} = \hat{\lambda}_*$. Extending the same normalization to the resolved squared-mass operator gives $m_a^2 = \hat{\lambda}_* \hat{\lambda}_a \Lambda_{\text{QCD}}^2$. Thus the two-leg operator fixes the reference spectral label while this reference-to-mass map fixes the absolute hadronic unit. Together they give

$$D_{\mathcal{C}} = \text{diag}(8, 16, 17, 18), \quad M_{\mathcal{C}}^2 = 8\Lambda_{\text{QCD}}^2 D_{\mathcal{C}}. \quad (\text{C.125})$$

The channel ratios test the relative labels with the common scale cancelled. The scalar mass tests this normalization together with the coupling-to- Λ_{QCD} conversion. The labels 17, 18 retain the conformal-channel assignments stated above. For $\Lambda_{\text{QCD}} > 0$ the lowest mass in this four-channel assignment is

$$m_G = 8\Lambda_{\text{QCD}}, \quad (\text{C.126})$$

and the tower ratios are $2^{++}/0^{++} = \sqrt{2}$ and $0^{++*}/0^{++} = 3/2$ and $0^{-+}/0^{++} = \sqrt{17/8}$. The proton readout combines the constituent content and conformal-content correction

$$m_p = N_c \Delta_f \left(1 - \frac{1}{N_g^2 \Delta_s}\right) \Lambda_{\text{QCD}} \left(1 + \tau \kappa \sum Y^2 \Delta_s\right) = \frac{279}{64} \Lambda_{\text{QCD}} \left(1 + \frac{5\tau \kappa}{3}\right), \quad (\text{C.127})$$

where $1/(N_g^2 \Delta_s) = 1/32 = \xi/4$ follows from $N_g \xi = 1$ and $N_g \Delta_s = 2(d-1)$. Thus $N_c \Delta_f (31/32) = 279/64$. The effective hadronic response factor uses $\sum Y^2 \Delta_s = 5/3$ evaluated in Section C.17. The string tension and deconfinement temperature are

$$\sigma = \frac{\lambda_{\text{TT}}}{\pi} \Lambda_{\text{QCD}}^2 = \frac{14}{\pi} \Lambda_{\text{QCD}}^2, \quad T_d = \frac{\lambda_{\text{vector}}}{N_c} \Lambda_{\text{QCD}} (1 - \tau \kappa) = \frac{4}{3} \Lambda_{\text{QCD}} (1 - \tau \kappa), \quad (\text{C.128})$$

Here $\lambda_{\text{TT}} = 14$ is the effective coefficient of the three-branch tensor catalogue used in the confinement-scale relation. The rough-TT operator in Ref. [4] has minimum $6/L^2$ on its two-branch carrier. The chiral-squash factor $(1 - \tau \kappa)$ enters the deconfinement readout. The common scale cancels in the ratio

$$\frac{\sigma}{T_d^2} = \frac{14}{\pi} \frac{9}{16} (1 - \tau \kappa)^{-2} = \frac{126}{16\pi(1 - \tau \kappa)^2} = 2.6242, \quad (\text{C.129})$$

the content ratio $126/(16\pi) = 2.5068$ modified by the factor $(1 - \tau \kappa)^{-2}$. The common scale and the correction on its input coupling are specified by (C.121)–(C.122). Equation (C.125) summarises the four reported channel masses. The complete condensate spectrum of Ref. [4] is instead governed by $\Delta_0 + m_\delta^2 I$ including its constant mode. Their physical comparison uses a common source and scale normalization.

For a dynamical colour interpretation see the paragraph “A mass atom from a common cross response” in Appendix G of Ref. [4]. With Q a Hermitian scalar colour observable and ϕ the Hermitian collective scalar a common positive state and translation representation give the connected matrix spectral measure in a common normalization and the bound

$$\rho_{QQ}(\{m^2\}) \geq \frac{|\rho_{Q\phi}(\{m^2\})|^2}{\rho_{\phi\phi}(\{m^2\})}, \quad \rho_{\phi\phi}(\{m^2\}) > 0. \quad (\text{C.130})$$

Applying this criterion to the colour response requires a common representation of the full non-Abelian curvature and collective source with a nonzero cross atom for Q . A finite spectral approximation must retain an isolated eigenvalue and positive residue. Passage to the full theory requires convergence of the common response and uniform control of the surrounding spectral separation and residue. Identifying the lowest visible colour mass also requires exclusion of lower spectral support. The additional identification with $8\Lambda_{\text{QCD}}$ fixes the physical scale and source normalization. These conditions give a concrete dynamical test of the catalogue in (C.125).

C.16 Primordial nucleosynthesis

The abundance calculation combines the generated mass and coupling readouts with a radiation-era thermal history in the leading weak-rate and freeze-out approximation. Ref. [64, Secs. III.G–H and IV.C] reviews the radiative corrections and reaction-network treatment used in precision BBN. The Fermi constant is $G_F = 1/(\sqrt{2}v^2)$ and the axial coupling is assigned as

$$g_A = \frac{N_g \Delta_s}{\pi} = \frac{2(d-1)}{\pi} = \frac{4}{\pi}, \quad (\text{C.131})$$

using $N_g = 8$ and $\Delta_s = 1/2$. Here $N_g \Delta_s = 4$ is the content weight assigned to the axial response and division by π sets its period normalization. The weak rate uses this axial coefficient together with the canonically normalized vector current. The electromagnetic self-energy is

$$\Delta_{\text{EM}} = \left(1 - \frac{1}{2\pi}\right) \alpha_{\text{em}}(0) \Lambda_{\text{QCD}}. \quad (\text{C.132})$$

The low-energy coupling starts from the two-loop value $\alpha^{-1}(M_Z) = 128.207$ at the preliminary-scale readout of Section 5.2 and applies the one-loop QED threshold sum

$$\frac{1}{\alpha_{\text{em}}(0)} = \frac{1}{\alpha(M_Z)} + \frac{2}{3\pi} \sum_f Q_f^2 N_c^f \ln \frac{\mu_{\text{QED}}}{m_f}, \quad (\text{C.133})$$

with the charged leptons and c and b on the ladder masses and the u, d, s thresholds set to Λ_{QCD} . The logarithmic upper scale in this QED threshold sum is the auxiliary one-loop electroweak value

$$\begin{aligned} \mu_{\text{QED}} &= \frac{v}{2} \sqrt{g_2^{(1)}(v)^2 + \frac{3}{5} g_1^{(1)}(v)^2}, \\ \frac{1}{g_i^{(1)}(v)^2} &= \frac{1}{g_i(M_G)^2} + \frac{b_i}{8\pi^2} \ln \frac{M_G}{v}, \quad (b_1, b_2) = \left(\frac{41}{10}, -\frac{19}{6}\right). \end{aligned} \quad (\text{C.134})$$

This auxiliary scale defines the QED threshold integral whereas the electroweak mass comparison uses its self-consistent Z -mass fixed point. The threshold convention gives $\alpha_{\text{em}}^{-1}(0) = 137.0463$ with a +0.0075% deviation from the reference 137.036. The radiative correction is

$$\delta_R = 1 + \frac{1 - \tau}{8\pi}. \quad (\text{C.135})$$

The neutron-proton mass difference is

$$\Delta m_{np} = g_S^{\text{eff}}(m_d - m_u) - \Delta_{\text{EM}}, \quad g_S^{\text{eff}} = 1, \quad (\text{C.136})$$

with the quark masses from the ladder of Section C.7. The coefficient $g_S^{\text{eff}} = 1$ specifies the isovector response of this native mass readout. Matching to a renormalized QCD nucleon matrix element uses the isovector relation

$$\Delta m_{np}^{\text{QCD}} = g_S^{\mathcal{S}}(\mu)[m_d^{\mathcal{S}}(\mu) - m_u^{\mathcal{S}}(\mu)], \quad \Delta m_{np} = \Delta m_{np}^{\text{QCD}} - \Delta_{\text{EM}}, \quad (\text{C.137})$$

to first order in isospin breaking. The scalar charge and quark masses are evaluated in the same renormalization scheme $\mathcal{S}$ and at the same scale μ . Their reciprocal renormalizations preserve the product [65]. The electromagnetic term is defined as the proton-minus-neutron contribution in a fixed QCD–QED separation convention [66]. The generated mass ladder supplies the effective source masses used here with $g_S^{\text{eff}} = 1$. Its two contributions are 2.56131931 and 1.27635176 MeV. They give $\Delta m_{np} = 1.28496755$ MeV.

For a nucleon matching with isoscalar QCD mass M_0 and mean electromagnetic shift $\overline{\delta M}_{\text{EM}}$ the paired masses are $m_{p,n} = M_0 + \overline{\delta M}_{\text{EM}} \mp \Delta m_{np}/2$. Thus an independent scalar matrix element and isoscalar mass together with electromagnetic separation specify this QCD correspondence for the generated sources. The numerical calculation retains their common source dependence explicitly [15]. In terms of the generated ladder masses the coefficient to be determined by this matching is the combination

$$g_S^{\text{eff}} = g_S^{\mathcal{S}}(\mu) \frac{m_d^{\mathcal{S}}(\mu) - m_u^{\mathcal{S}}(\mu)}{m_d - m_u}. \quad (\text{C.138})$$

The scale dependence cancels between the renormalized scalar density and quark-mass difference. An independently calculated strong mass splitting therefore fixes this combination directly provided its electromagnetic separation is carried into the same matching. Appendix F propagates the published total nucleon splitting of Ref. [66] as a separate external comparison retaining the uncertainties from that calculation.

Proposition C.7 (Generated three-quark source channels). Use the inherited fundamental colour representation with Hermitian generators $\text{tr}(T^A T^B) = \delta^{AB}/2$. The normalized three-quark colour singlet is $|\epsilon\rangle = \epsilon_{abc}|abc\rangle/\sqrt{6}$. For T_r^A acting on leg r the three adjoint-source vectors $|\Omega_r\rangle = \sum_A T_r^A |\epsilon\rangle \otimes |A\rangle$ satisfy the global colour constraint and have Gram matrix

$$\langle \Omega_r | \Omega_s \rangle = 2\delta_{rs} - \frac{2}{3}, \quad \sum_r |\Omega_r\rangle = 0. \quad (\text{C.139})$$

Their span has dimension two. For the normalized scalar coefficient e_{00}^1 of the common matter connection multiplication of either lowest-shell spinor has squared norm $2/3$ in the $j = 1$ branch with $J = 3/2$ and $1/3$ in the $j = 1$ branch with $J = 1/2$ with zero projection onto the incoming $j = 0$ shell.

Proof. The decomposition $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$ gives the unique colour singlet. On this state $\sum_r T_r^A = 0$ and $\sum_A (T_r^A)^2 = 4I/3$. Permutation symmetry then gives $\langle \epsilon | \sum_A T_r^A T_s^A | \epsilon \rangle = -2/3$ for $r \neq s$. The adjoint transformation of $T_r^A |\epsilon\rangle$ cancels that of $|A\rangle$. The Gram eigenvalues are 0, 2, 2. The spinor product follows by coupling the right scalar index of e_{00}^1 to spin $1/2$. The squared Clebsch–Gordan coefficients are $2/3$ and $1/3$.

Orthogonality of the scalar shells eliminates the incoming projection. This is the common transport vertex given by the proposition “Matter coupling from the common transport” in Ref. [4]. $\square$

Thus the generated colour and internal product tensors supply explicit adjoint-source channels. The inherited electromagnetic charges give $\sum_i q_i^2 = 1$, $\sum_{i < j} q_i q_j = 0$ for uud and $2/3, -1/3$ for udd . These multiply the corresponding current kernels. The scalar charge measures how the nucleon energy responds to an isovector mass source at fixed mean light-quark mass. Proton and neutron responses enter with opposite isospin orientation. Thus their difference determines the strong contribution to the splitting [65]. This energy derivative is the Feynman–Hellmann response of the constrained baryon state [67]. It can be organized by a fixed projection P in the baryon sector after imposing the physical Gauss constraints. Write $Q = 1 - P$ and $V = H_{QP}$ and $\mathcal{R}_Q(E, s) = (E - H_{QQ}(s))^{-1}$ for the same sourced Hamiltonian differentiable in the source s . At a simple isolated eigenvalue outside the spectrum of H_{QQ} the reduced operator and its source derivative are

$$H_{\text{eff}}(E, s) = H_{PP}(s) + V^\dagger \mathcal{R}_Q(E, s) V, \quad H_{\text{eff}} u = E u, \quad \|u\| = 1, \\ \frac{dE}{ds} = \frac{\langle u, \partial_s H_{\text{eff}}(E, s) u \rangle}{1 + \|\mathcal{R}_Q(E, s) V u\|^2}. \quad (\text{C.140})$$

Indeed the eliminated component is $\mathcal{R}_Q V u$ and $\partial_E H_{\text{eff}} = -V^\dagger \mathcal{R}_Q^2 V$. Differentiation of the eigenvalue equation proves the second line. The denominator is the squared norm of the full state reconstructed from the retained component u and the eliminated component $\mathcal{R}_Q V u$. At fixed E the direct source insertion is $H_{PP,s}$ and the eliminated sector contributes $V_{,s}^\dagger \mathcal{R}_Q V + V^\dagger \mathcal{R}_Q V_{,s} + V^\dagger \mathcal{R}_Q H_{QQ,s} \mathcal{R}_Q V$. For $s = m_d^S(\mu) - m_u^S(\mu)$ at fixed mean quark mass the difference of the neutron and proton derivatives at the isospin-symmetric point determines the scalar coefficient in (C.137). Its evaluation uses the constrained baryon resolvent and its source insertions together with the same renormalized current kernels for the electromagnetic separation. The finite colour and spinor calculation fixes the factors in Proposition C.7 directly [15].

The neutron lifetime is

$$\tau_n = \frac{2\pi^3}{G_F^2 |V_{ud}|^2 (1 + 3g_A^2) m_e^5 f \delta_R}, \quad (\text{C.141})$$

using the Cabibbo approximation $|V_{ud}| = \sqrt{1 - |V_{us}|^2}$. The Coulomb-weighted phase-space integral is

$$f = \int_1^{W_0} F(1, W) W \sqrt{W^2 - 1} (W_0 - W)^2 dW, \quad W_0 = \frac{\Delta m_{np}}{m_e} > 1, \\ F(1, W) = \frac{2\pi\eta}{1 - e^{-2\pi\eta}}, \quad \eta = \frac{\alpha_{\text{em}}(0)W}{\sqrt{W^2 - 1}}. \quad (\text{C.142})$$

Put $d = W_0 - 1$ and $W = 1 + dx$ with $p = \sqrt{W^2 - 1}$ and $a = 2\pi\alpha_{\text{em}}(0)$. Cancelling the Coulomb endpoint factor gives

$$f = d^3 \int_0^1 \frac{aW^2}{1 - e^{-aW/p}} (1 - x)^2 dx.$$

The factor multiplying $(1-x)^2$ has the finite limit a at $x=0$. Adaptive quadrature of this regularized integral gives $f = 1.63492371128643$ with estimated absolute error 1.5×10^{-12} in 567 integrand evaluations and $\tau_n = 915.54$ s [15]. The common QCD scale enters Δm_{np} through the electromagnetic term. The decay phase space propagates this dependence into the lifetime and helium yield. The approximation is the static-nucleon weak rate with Coulomb and specified radiative factors. Precision recoil and weak-magnetism contributions are discussed in Ref. [64, Secs. III.G–H].

In natural units the freeze-out temperature is the positive solution of

$$\begin{aligned}\Gamma_{\text{weak}}(T_f) &= H(T_f), \\ \Gamma_{\text{weak}}(T) &= \frac{7\pi}{60}(1 + 3g_A^2)G_F^2 T^5 \left(1 + \frac{3\Delta m_{np}}{2T}\right), \\ H(T) &= \sqrt{\frac{4\pi^3 g_*}{45}} \frac{T^2}{M_{\text{Pl}}}, \quad M_{\text{Pl}} = \sqrt{8\pi} M_{\text{P}}.\end{aligned}\tag{C.143}$$

The thermal approximation takes $g_* = 10.75$ at freeze-out and $g_* = 3.36$ after electron-positron annihilation and $T_{\text{BBN}} = 0.08$ MeV. With

$$t(T, g_*) = \frac{0.301 M_{\text{Pl}}}{\sqrt{g_*} T^2}, \quad \Delta t = t(T_{\text{BBN}}, 3.36) - t(T_f, 10.75),\tag{C.144}$$

free neutron decay conserves the sum of neutron and proton numbers. Writing $r_f = \exp(-\Delta m_{np}/T_f)$ and $q = \exp(-\Delta t/\tau_n)$ the neutron fraction therefore evolves as $X_n = q r_f / (1 + r_f)$. Complete capture of the surviving neutrons gives

$$Y_p = \frac{2r_f q}{1 + r_f} = 0.24608, \quad (n/p)_{\text{BBN}} = \frac{r_f q}{1 + r_f(1 - q)}.\tag{C.145}$$

Both Δt and τ_n are in GeV^{-1} . Multiplying each by $\hbar = 6.582119569 \times 10^{-25}$ GeV s gives seconds without changing their ratio. The separate content assignment

$$N_{\text{eff}} = 3 + \frac{\sqrt{3}}{(2\pi)^2} = 3.0439\tag{C.146}$$

supplies the neutrino-number comparison and does not change the two specified thermal values of g_* .

C.17 Finite correction factors

With $s_0 = 2\tau$ and the curvature-norm factor κ of Section C.4 the finite corrections enter the masses and couplings at the following stages.

- $(1 - 4s_0\kappa)$ multiplies the fourth-power dark-energy floor of Section C.14. At fixed seesaw inputs it cancels the first-order explicit seesaw insertion. Including the dependence through v^2 gives Eq. (C.108).
- $(1 - s_0\kappa)$ multiplies the electroweak scale v of Section C.11 and the raw microwave floor $T_{\text{CMB}}^{\text{raw}}$ of Section C.14. The temperature also depends on the corrected neutrino mass.

- $(1 + s_0\kappa)$ is the seesaw level-transfer factor on the heaviest neutrino scale m_{ν_3} of Section C.10.
- $(1 - \tau\kappa)$ is the chiral-squash factor assigned to the deconfinement temperature T_d of Section C.15 and to the curvature amplitude Δ_R^2 of Section C.14.
- $(1 + \tau\kappa \sum Y^2 \Delta_s) = (1 + 5\tau\kappa/3)$ is the effective hadronic response factor of the proton mass m_p of Section C.15 with $\sum Y^2 = 10/3$ the hypercharge capacity and $\Delta_s = 1/2$ the scalar conformal weight.
- $(1 - s_0\kappa/N_g)$ is the Yukawa-difference factor multiplying α_s before the nonlinear extraction of Λ_{QCD} in Section C.15.
- $C_\gamma = 1 - \tau/\pi^2$ multiplies $T_{\text{CMB}}^{\text{raw}}$ in the microwave-monopole and linear CMB comparisons. The baryon density Ω_b uses the paired raw quantities $T_{\text{CMB}}^{\text{raw}}$ and η_B^{raw} of Sections C.14 and C.13.
- f is the Coulomb-weighted phase-space integral in the neutron lifetime of Section C.16.

The first six factors are expressed in terms of the isometry-breaking normalisation and squash amplitude together with the local response coefficients.

1. the content invariant $\tau = (N_L - N_R)/(N_f \sum Y^2) = 1/50$ of Section C.2.
2. the squash amplitude $s_0 = n_{\text{broken}}\tau = 2\tau$ with two broken $SU(2)_R/U(1)_R$ directions.
3. the inverse curvature-norm normalization $\kappa^2(s) = (1 + s)/(1 - 2s)^{5/2}$ of Eq. (C.41).
4. the local coefficient $c_Q = a_Q r_Q$ with amplitude fraction a_Q (geometric 1 or chiral 1/2) and content ratio r_Q . For an inverse-metric monomial with n_3 axial and n_{eq} equatorial factors the logarithmic derivative at the round point is $2n_3 - n_{\text{eq}}$. The shear has zero volume derivative at $s = 0$. Its finite- s volume follows from the squashed metric.

For each readout the local insertion $Q \rightarrow Q(1 + c_Q s_0\kappa)$ specifies the response at fixed upstream inputs with $s_0\kappa = N_g\tau\kappa/(d + 1)$. Its amplitude fraction and charge weight determine how that channel samples the common deformation. The total response follows by composing it with the change in its inputs. For example the explicit neutrino insertion acts on the dimension-five coefficient while the two powers of the generated VEV carry the electroweak response into the same neutrino mass. The channel factors have the following origins.

- v and V_{cb} carry the local factor $1 - s_0\kappa$ ($a_Q = 1$ and $r_Q = -1$). For $v = M_G e^{-\varphi}$ the exact stop increment is $-\ln(1 - s_0\kappa) = s_0\kappa + O((s_0\kappa)^2)$. Its finite error is bounded in Appendix D.4.

- m_{ν_3} has the explicit seesaw insertion $1 + s_0\kappa$ ($a_Q = 1$ and $r_Q = +1$) at fixed v, k_{GUT} . Substitution of the corrected v into $v^2(2\pi)^2(1 + s_0\kappa)/k_{\text{GUT}}$ gives the full response and the product vm_{ν_3} in Eq. (C.108).
- T_d and Δ_R^2 carry $-\tau\kappa$ and V_{ub} carries $+\tau\kappa$ ($a_Q = 1/2$ and $r_Q = \mp 1$). The chiral level couples to the chiral asymmetry $\tau = s_0/2$ (the amplitude fraction $a_Q = (N_L - N_R)/n_{\text{broken}} = 1/2$). The sign is $-$ for chiral restoration (T_d and Δ_R^2) and $+$ for the cross-generation element (V_{ub}).
- m_p carries $+\tau\kappa \sum Y^2 \Delta_s = +5\tau\kappa/3$ ($a_Q = 1/2$ and $r_Q = 5/3$). The effective hadronic response combines the chiral asymmetry and quadratic hypercharge moment with the scalar conformal weight.
- α_s carries $-s_0\kappa/N_g = -s_0\kappa \xi$ ($a_Q = 1$ and $r_Q = -1/N_g$). The effective correction between the fixed-Yukawa calculation at $y_0 = 1$ and the running-Yukawa calculation uses the conformal normalisation $\xi = 1/N_g$.

The amplitude fraction a_Q and the local fourth-power coefficient follow the broken-generator and chiral-asymmetry counts together with the $m_{\nu_1}^4$ count. The explicit seesaw insertion has $r_Q = +1$. Total scale derivatives apply the chain rule to all these input dependencies. The constituent and Yukawa sectors use the effective response maps

1. Hadronic response (m_p with $r_Q = \sum Y^2 \Delta_s = 5/3$). The finite-content response used by the mass readout is

$$\frac{\delta m_p}{m_p} = \tau\kappa\Delta_s \sum_f N_{\text{int}}(f)Y_f^2 = \frac{5\tau\kappa}{3}, \quad (C.147)$$

$$\sum_f N_{\text{int}}(f)Y_f^2 = 6\left(\frac{1}{6}\right)^2 + 3\left(\frac{2}{3}\right)^2 + 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{2}\right)^2 + 1 = \frac{10}{3}.$$

This is the product of the chiral asymmetry and curvature-norm factor with the scalar conformal weight and quadratic hypercharge moment.

2. Yukawa difference (α_s with $r_Q = -1/N_g = -\xi$). The top-Yukawa mixing coefficients in the two-loop gauge flow are given in Section C.5. The finite correction used with the fixed seed $y_0 = 1$ is

$$\alpha'_s = \alpha_s(1 - \epsilon), \quad \epsilon = s_0\kappa\xi = \frac{s_0\kappa}{N_g}, \quad \xi = \frac{d-2}{4(d-1)} = \frac{1}{8}. \quad (C.148)$$

For $0 \leq \epsilon < 1$ the logarithmic increment is $\ln(\alpha'_s/\alpha_s) = \ln(1 - \epsilon) = -\epsilon + O(\epsilon^2)$. The extraction map uses the finite multiplicative factor.

The numerical table uses these finite factors at the stated insertion points with the subsequent nonlinear dependence retained.

Table 18. Error-control ledger for the headline results. Entries identify the relevant error category. They are not additive uncertainty estimates.

Object	Numerical	Input propaga- tion	Truncation	Model sensitiv- ity
kL	quadrature and root bracket	content and τ	spectral-tail bounds	continuous cut- off $\mathcal{R}$
v	c_R and endpoint	G_N and c_R	two-loop running	matching scheme
m_e	c_R and transfer	τ and content	one-loop lepton matching	finite matching
m_t	c_R and y_0	G_N and c_R	top threshold	scheme
Γ_t	leading phase space	G_F, M_t, g_2	$\mathcal{O}(\alpha_s/\pi)$ and $\mathcal{O}(\alpha/\pi)$ omitted	bottom/CKM
τ_n, Y_p	weak-rate quadrature	m_{np} and G_F	finite-temperature network	BBN inputs
DESI	covariance and r_d	thermal history	fixed prediction	common-scale direction

D Error bounds and parameter sensitivity

The error estimates distinguish numerical error at a fixed response domain, changes of that domain and propagation of shared input uncertainties. The positive spectral sums permit explicit bounds on the endpoint. Their effect on the mass hierarchy follows from the same closed relations used to calculate the central values.

Table 18 identifies the error controls for the representative quantities: numerical integration, input propagation, perturbative truncation and response sensitivity. A numerical covariance additionally requires an input probability model.

The endpoint is dimensionless and independent of the Planck anchor at fixed content and source projections with a fixed response level. Newton’s constant enters the subsequent reconstruction of dimensional quantities through $M_P = (8\pi G_N)^{-1/2}$.

D.1 Gaussian spectral tails and endpoint enclosures

The endpoint is linear in the positive spectral coefficient A_R of Eq. (C.7). Bounds on the Gaussian tails therefore propagate directly to the endpoint and then to the mass hierarchy. Throughout this subsection the matter content, species projections, channel weights and torsion modulus are fixed. For the TT trace the mode coefficients satisfy $a_j, b_j, D_j \geq 0$ and $\sum_j D_j e^{-a_j/\pi} < \infty$. Thus changes in R compare the same spectral sum on different response domains.

Proposition D.1 (Gaussian tail and endpoint enclosure). The finite frequency domain

and the full Gaussian coefficient satisfy

$$\begin{aligned}
0 &\leq A_\infty - A_R \leq B_R, \\
B_R &= \frac{1}{64\pi^2} \sum_j D_j e^{-a_j/\pi} \operatorname{erfc}\left(\sqrt{(R - a_j/\pi)_+}\right), \\
0 &\leq c_\infty - c_R \leq \frac{27}{4} B_R.
\end{aligned} \tag{D.1}$$

Proof. The nonnegative shifts give $0 \leq K_{\text{TT}} \leq 1$. For $a_j < R\pi$ the omitted interval starts at $\sqrt{R\pi - a_j}$. For $a_j \geq R\pi$ the entire mode is omitted. In both cases use

$$\int_v^\infty e^{-u^2/\pi} du = \frac{\pi}{2} \operatorname{erfc}(v/\sqrt{\pi}).$$

Sum the positive bounds and apply $c_R = 27A_R/4$. $\square$

The infinite sum defining B_R admits an explicit remainder bound. For the three-generation content, group the scalar and vector modes and the assigned spinor modes by their degree n . Their total weighted multiplicity is at most $196(n+2)^2$ and $a_j \geq n^2$. After summing through $n = N \geq 1$ the remaining contribution to B_R is bounded by

$$\frac{196}{64\pi^2} \frac{(N+3)^2 e^{-(N+1)^2/\pi}}{1 - r_N}, \quad r_N = \left(\frac{N+4}{N+3}\right)^2 e^{-(2N+3)/\pi} < 1. \tag{D.2}$$

Indeed $\operatorname{erfc} \leq 1$ and consecutive terms of $(n+2)^2 e^{-n^2/\pi}$ have decreasing ratio bounded by r_N for $n \geq N+1$. The constant 196 bounds the scalar and vector contributions together with the larger assigned chiral contribution by $4 + 96 + 96$.

There is also a strictly positive omitted-shell lower bound at $R = 5$. The vector shell $l = 3$ has $a_j = 16 > 5\pi$ and multiplicity 30. For the twelve gauge fields its total TT-weighted multiplicity is $12 \cdot 4 \cdot 30 = 1440$ and $b_j \leq 3/2 + \tau^2/6$. Hence

$$c_\infty - c_5 \geq \frac{1215}{8\pi^2} \left(\frac{16}{35/2 + \tau^2/6}\right)^2 e^{-16/\pi} > 0. \tag{D.3}$$

Retain this shell in the positive omitted sum. Bound its kernel below by $[16/(35/2 + \tau^2/6)]^2$ for every $u \geq 0$ and integrate $e^{-u^2/\pi}$ over the full half-line. At $\tau = 1/50$ the coarse estimates $b_j < 2$ and $\pi > 8/3$ with $e < 3$ and $\pi^2 < 10$ give

$$c_\infty - c_5 > \frac{120e^{-6}}{\pi^2} > \frac{120}{7290} > 0.016.$$

The complete-domain root is therefore strictly greater than the $R = 5$ root. Equation (D.1) bounds their difference for the same Gaussian kernel. Numerical error in c_R is controlled separately by the quadrature, mode summation and rounding bounds.

If a computed coefficient satisfies $|\tilde{A}_R - A_R| \leq \varepsilon_A$ with frequency integration and mode summation together with rounding included in ε_A then

$$\begin{aligned}
c_R &\in \frac{27}{4} [\tilde{A}_R - \varepsilon_A, \tilde{A}_R + \varepsilon_A], \\
c_\infty &\in \frac{27}{4} [\tilde{A}_R - \varepsilon_A, \tilde{A}_R + \varepsilon_A + B_R].
\end{aligned} \tag{D.4}$$

The first interval concerns the fixed-domain endpoint. The second also includes the positive contribution from the omitted domain.

Enclosing the joint-branch coefficients. For each of the eight integrals in (C.21) write $f(u) = g(u^2 + a)e^{-(u^2+a)/\pi}$ with $g(q) = [q/(q+b)]^2$. On its integration interval $q+b \geq 3/4$ and $u^2 < 20$ and $\pi > 3$. Hence

$$0 \leq g \leq 1, \quad |g'| \leq 8/3, \quad |g''| \leq 64/9, \quad |f''| \leq 726.$$

For an interval of length $U = \sqrt{5\pi - a} < 9/2$ the composite midpoint sum M_N therefore satisfies the explicit estimate

$$|I - M_N| \leq \frac{726(9/2)^3}{24N^2}. \quad (\text{D.5})$$

Using $N = 4096$ outward rational interval evaluation gives

$$a_5 \in [0.6010245, 0.6017579], \quad b_5 \in [0.6300616, 0.6313672],$$

which imply the wider bounds used in Eq. (C.20). Adding $a_5 + 3b_5$ before decimal rounding gives the coarse selection-level endpoint enclosure

$$c_{3,5} \in [c_-, c_+] = [2.4912094, 2.4958593], \quad c_+ - c_- = 0.0046499. \quad (\text{D.6})$$

This interval certifies the discrete $R = 5$ shell support. A finer production enclosure uses the homogeneous coefficient at $c = 1$. On each of 2048 cells per spectral row the trapezoid error is bounded by $h^3 \sup |f''|/12$. The cellwise derivative bound follows from

$$\begin{aligned} g'(q) &= \frac{2bq}{(q+b)^3}, \quad g''(q) = \frac{2b(b-2q)}{(q+b)^4}, \\ f''(u) &= e^{-q/\pi} \left[2 \left(g' - \frac{g}{\pi} \right) + 4u^2 \left(g'' - \frac{2g'}{\pi} + \frac{g}{\pi^2} \right) \right], \quad q = u^2 + a. \end{aligned}$$

The square-root enclosure $U \in [U_-, U_+]$ is included by integrating to U_+ and subtracting at most $U_+ - U_-$ from the lower bound, using $0 \leq f \leq 1$. Including all eight weighted rows gives

$$A_5 \in [0.36941244246332044290, 0.36941253772250165936].$$

The identities $c_5 = 27A_5/4$ and $F'_5(c) = -A_5/c^2$ transfer this single enclosure to the root and its slope. With outward rounding and a margin for strict endpoint signs the resulting bracket is

$$kL \in [2.493533986626, 2.493534629628], \quad (\text{D.7})$$

and propagating this interval through the closed mass relations gives

$$\begin{aligned} M_G &\in [1.7310762937, 1.7310767402] \times 10^{18} \text{ GeV}, \\ m_e &\in [0.5103512187, 0.5103577820] \text{ MeV}, \\ v &\in [246.1887479, 246.1908008] \text{ GeV}. \end{aligned} \quad (\text{D.8})$$

The coarse interval proves that the retained shell count is five while the narrow interval controls the precision of the comparison values. The BBN intervals are reported separately

in Appendix F. For reproducibility forty terms of the alternating arctangent series enclose π through

$$\pi = 16 \arctan(1/5) - 4 \arctan(1/239).$$

For $0 \leq x \leq 5$ enclose e^x by its degree-sixty positive Taylor sum and the remainder $243 \cdot 5^{61}/61!$ and invert to enclose e^{-x} . Integer square-root bounds enclose U . All arithmetic is rounded outwards on a 10^{-40} rational grid. These operations enclose the finite integrals entering Eq. (C.20) including all rounding errors.

Complete Gaussian coefficient certificate. The full coefficients use the same prefactor $P = 27/(128\pi^3)$ and mode integrands as (C.21) with all prescribed modes and $u \in [0, \infty)$. The bosonic rows are

degree	a	b	D
n even	$n(n+2)$	$3/4$	$4(n+1)^2$
n odd	$(n+1)^2$	$3/2 + 1/15000$	$64n(n+2)$
n odd	$(n+1)^2$	$1 + 1/15000$	$24n(n+2)$
n odd	$(n+1)^2$	$1/15000$	$8n(n+2)$

and the one-generation fermionic rows have $a = (n + 3/2)^2$ and $b = 3/20000$ and $D = 28(n+1)(n+2)$ for even n or $32(n+1)(n+2)$ for odd n . These define a_∞ and b_∞ respectively.

For an explicit finite certificate retain $n \leq 8$ and integrate on $0 \leq u \leq 8$ and use $N = 8192$ midpoint subintervals. The earlier bounds for $g(q)$ now give

$$|f''(u)| \leq 2310e^{-a/\pi}, \quad |I_{[0,8]} - M_N| \leq \frac{2310 \cdot 8^3}{24N^2} e^{-a/\pi}. \quad (\text{D.9})$$

Indeed $f'' = e^{-(u^2+a)/\pi} [2(g' - g/\pi) + 4u^2(g'' - 2g'/\pi + g/\pi^2)]$. Its bracket is bounded by $6 + 256 \cdot 9 = 2310$. The missing frequency contribution in each retained row satisfies

$$0 \leq \int_8^\infty f(u) du \leq e^{-a/\pi} \frac{\pi}{16} e^{-64/\pi},$$

using $u/8 \geq 1$ in the Gaussian integral.

For the remaining degrees $n \geq 9$ the bosonic total is bounded by $100(n+2)^2$ and the one-generation fermionic total by $32(n+2)^2$ while every eigenvalue coefficient is at least n^2 . Their respective contributions to the root coefficients are at most

$$\frac{27B}{256\pi^2} \frac{121e^{-81/\pi}}{1 - (12/11)^2 e^{-19/\pi}}, \quad B = 100 \text{ or } 32. \quad (\text{D.10})$$

The denominator uses the largest successive ratio of $(n+2)^2 e^{-n^2/\pi}$ for $n \geq 9$. Together with the bound on $u > 8$ this estimate controls the frequency and spectral truncations of the complete-domain integral.

Outward rational evaluation of the finite sums with the above positive tails added to their upper endpoints gives

$$\begin{aligned} a_\infty &\in [0.6897056, 0.6906685], \\ b_\infty &\in [0.6568695, 0.6574841], \\ c_{3,\infty} &\in [2.6603141, 2.6631205]. \end{aligned} \tag{D.11}$$

The last line is computed before decimal rounding of the first two. The rational grid and elementary enclosures are those used above. For exponent arguments greater than five halve the argument until it is in $[0, 5]$ and square the resulting interval the same number of times. The Taylor remainder and outward rounding are propagated through each squaring to give enclosures of the complete trace.

For the three-position test a coarser enclosure suffices. On each frequency subinterval $g(q) = (q/(q+b))^2$ increases and $e^{-q/\pi}$ decreases. Products of their opposite endpoint bounds therefore enclose the integral. Using 128 subintervals for each of the eight finite-domain and 26 complete-domain rows together with the positive tails above gives [15]

$$2.4724098408 < c_R < 2.7182303261, \quad 6.1128104212 < c_R^2 < 7.3887761056, \quad R \geq 5.$$

The inequalities follow from $c_5 \leq c_R \leq c_\infty$ and outward rational evaluation of the two endpoint integrals. The calculation uses 4352 row subintervals and 2720 distinct exponential enclosures with common Gaussian values reused. The enclosure places c_R^2 strictly between $11/2$ and $15/2$. The narrower intervals in (D.11) control the mass-scale comparison.

For downstream quantities $X(c) = C_0 c^p e^{-ac}$ at fixed other inputs the exact propagation is

$$\frac{X(c+\delta)}{X(c)} = \left(1 + \frac{\delta}{c}\right)^p e^{-a\delta}, \quad c + \delta > 0. \tag{D.12}$$

Its logarithmic derivative is $p/c - a$. Bound this derivative on the same certified interval or use the displayed exact ratio. All outputs sharing c inherit correlated uncertainty. For a quantity of mass dimension d proportional to M_P^d the anchor elasticity is $\partial \log X / \partial \log G_N = -d/2$.

Certified propagation at the production level. Keep the three-copy content and $R = 5$ with fixed τ and M_P . All remaining readout rules are also held fixed. For a convenient exact reference argument set $c_o = 2.49353$ and $X_o = X(c_o)$. The closed mass and density relations give

$$\begin{aligned} V(c) &:= \frac{v(c)}{v_o} = \frac{c_o}{c} e^{-4\pi(c-c_o)}, \\ \frac{m_e(c)}{m_{e,o}} &= e^{-20(c-c_o)}, \quad \frac{\rho_\Lambda(c)}{\rho_{\Lambda,o}} = V(c)^8. \end{aligned} \tag{D.13}$$

All three functions decrease on $[c_-, c_+]$. Evaluation at the two endpoints of (D.6) therefore gives Table 19. The decimal bounds can be checked with rational arithmetic. Use the Machin enclosure for π above and enclose e^x for $0 \leq x < 1/2$ by its degree-twenty-four Taylor sum with remainder at most $2x^{25}/25!$. Negative arguments are enclosed by reciprocal bounds. The arguments used here have magnitude below 0.05.

Table 19. Propagation of the coarse $R = 5$ selection enclosure at fixed remaining inputs. The reference values are evaluations at the exact decimal argument $c_o = 2.49353$. Each row encloses the same one-parameter curve. The three intervals are correlated. These selection-level enclosures accompany the production precision intervals of Eq. (D.8).

Ratio	Certified enclosure
v/v_o	$[0.9702, 1.0306]$
$m_e/m_{e,o}$	$[0.9544, 1.0476]$
$\rho_\Lambda/\rho_{\Lambda,o}$	$[0.7853, 1.2722]$

These are guaranteed output ranges for the stated quadrature certificate. The floating-point calculation supplies the central values used in the comparison tables. Its small comparison residuals and the certified coverage in Table 19 describe different properties of the calculation. Input uncertainties and finite matching together with changes of the response domain have their own propagation.

Response-domain sensitivity of the hierarchy. For this comparison keep the three-generation content and its projections fixed. Hold τ and M_P fixed together with $\alpha = 1/(16\pi^2)$. The derived scale $M_G = M_P\sqrt{\pi}/c$ changes with c . Here α is the fixed loop factor. The mass relations in Sections 6.3 and C.9 give

$$\frac{v_\infty}{v_5} = \frac{c_{3,5}}{c_{3,\infty}} e^{-4\pi(c_{3,\infty}-c_{3,5})}, \quad \frac{m_{e,\infty}}{m_{e,5}} = e^{-20(c_{3,\infty}-c_{3,5})}. \quad (\text{D.14})$$

The subscripts distinguish the two endpoint arguments of the same displayed closed forms. The finite-domain endpoint is enclosed by (D.6). Combining the unrounded endpoint enclosures and evaluating the above ratios with the same outward arithmetic certifies

$$\frac{v_\infty}{v_5} \in [0.10, 0.13], \quad \frac{m_{e,\infty}}{m_{e,5}} \in [0.03, 0.04]. \quad (\text{D.15})$$

Both ratios increase with $c_{3,5}$ and decrease with $c_{3,\infty}$, so opposite interval endpoints bound them. They quantify a change of integration domain at fixed remaining inputs. The numerical tables throughout this paper use $R = 5$.

D.2 Sensitivity of the hierarchy

The endpoint kL and torsion modulus τ follow from the spectral endpoint equation and normalized finite-source response of Section 2.3. Varying them measures the sensitivity of the dependent quantities to those response relations. The observed anchor G_N carries an experimental relative uncertainty of order 2×10^{-5} . An output proportional to M_P^d has elasticity $-d/2$ with respect to G_N ; dimensionless outputs are invariant under a common M_P rescaling. The structural constants 2, π , $3/2$, $1/8$ and the geometric factors remain exact.

Table 20 gives $d \ln O / d \ln p$ for $p = kL, \tau, M_P$. The strongly amplified $c = kL$ derivatives are evaluated analytically from Appendix C. At fixed content with fixed τ and M_P

Table 20. The local elasticity matrix with respect to kL or τ or M_{P} . Its entries describe infinitesimal relative changes. The large negative kL entries arise from the hierarchy factors $c^{-1}e^{-4\pi c}$ and e^{-20c} and their powers.

Quantity O	Elasticity $d \ln O / d \ln p$		
	$p = kL$	$p = \tau$	$p = M_{\text{P}}$
M_{G}	-1.0	0.0	+1.0
$g_2(M_{\text{G}})$	-2.5	0.0	0
m_t/m_c	+5.0	-0.1	0
v	-32.3	-0.1	+1.0
m_e	-49.9	-0.1	+1.0
m_{ν_3}	-64.7	-1.1	+1.0
ρ_{Λ}	-258.7	-4.5	+4.0
H_0	-129.3	-2.2	+1.0
Ω_{Λ}	0.0	0.0	0

the breaking scale $k_{\text{GUT}} = M_{\text{P}}\sqrt{\pi\tau}/\sqrt{3}$ and $\alpha = 1/(16\pi^2)$ are independent of c . Hence $v \propto c^{-1}e^{-4\pi c}$ and $m_e \propto e^{-20c}$ and

$$\begin{aligned}\mathcal{E}_c O &:= c \partial_c \ln O, & \mathcal{E}_c v &= -1 - 4\pi c, \\ \mathcal{E}_c m_{\nu_3} &= 2\mathcal{E}_c v, & \mathcal{E}_c \rho_{\Lambda} &= 2\mathcal{E}_c H_0 = 8\mathcal{E}_c v.\end{aligned}$$

At $c = 2.49353$ these give $\mathcal{E}_c \rho_{\Lambda} \simeq -258.67697694$ and $\mathcal{E}_c H_0 \simeq -129.33848847$. The other entries are rounded to the precision shown in the table. Tangent estimates scaled to a one-percent input change are -32% and -50% for v and m_e respectively and -259% for ρ_{Λ} . Finite variations of positive outputs use (D.12) or the defining expression for the observable. The listed mass ratio has an order-unity elasticity while $\Omega_{\Lambda} = 2/3 + r_{23}/(3\pi)$ has exactly zero elasticity.

For a threshold $M > 0$ at fixed spectral coefficient A_R the endpoint is $c = A_R/M$ and its threshold elasticity is exactly -1 . The threshold-only diagnostic in Table 21 holds the original $p = 2$ spectral trace fixed and replaces $4/27$ by

$$M(p) = \max_{0 \leq y \leq 1} y(1-y)^p = \frac{p^p}{(p+1)^{p+1}}, \quad c_{\text{thr}}(p) = \frac{A_R(2)}{M(p)}, \quad p > 0.$$

Thus $d \log c_{\text{thr}} / d \log p = -p \log[p/(p+1)]$. At $p = 2$ this gives the recorded $+0.81\%$ tangent response per one-percent change in p at fixed trace. A change of the actual kernel also varies the trace. Define $A_R(p)$ by replacing the square in (C.7) by the power p . Then

$$\frac{d \log c}{d \log p} = \frac{p A'_R(p)}{A_R(p)} - p \log \frac{p}{p+1}, \quad c(p) = \frac{A_R(p)}{M(p)}. \quad (\text{D.16})$$

Differentiating under the fixed-domain integral inserts $\log[(u^2 + a_j)/(u^2 + a_j + b_j)]$. Its weighted mean is strictly negative for the present positive-mass content and supplies the additional trace contribution to the kernel sensitivity.

Table 21. Local tangent diagnostics scaled to a one-percent input change. The first row varies the threshold $M(p)$ at fixed trace $A_R(2)$. The full kernel derivative is given by Eq. (D.16). Finite changes use the exact propagation formulas.

Convention (1% shift)	$\rightarrow kL$	$\rightarrow v$	$\rightarrow m_e$	$\rightarrow \rho_\Lambda$
Threshold $M(p)$ at $p = 2$	+0.8%	-26%	-40%	-210%
Threshold 4/27	-1.0%	+32%	+50%	+259%
Window kL	—	-32%	-50%	-259%

The extremum of Appendix C.1 gives the exact local identity $y(1-y)^2 = 4/27 - (y - 1/3)^2 + (y - 1/3)^3$. The change of this normalisation starts quadratically at $y = 1/3$ whereas the threshold-to-endpoint and endpoint-to-hierarchy maps retain their nonzero local elasticities. Equation (D.12) gives the corresponding finite output changes.

A separate comparison replaces the Gaussian weight by the hard spectral window $\Theta(\Lambda^2 - \omega^2 - \lambda_j)$ while retaining the response kernel [15]. It gives $kL = 1.09$ compared with 2.49 for the Gaussian profile. Its capacity $1.09^2 = 1.19 < 1.5$ lies below the first positive spinor position. This change of response weight is distinct from the FRG regulator $R_k(q^2) = Z_k(k^2 - q^2)\Theta(k^2 - q^2)$ of Eq. (1.2) of Ref. [68]. The Gaussian tail bounds above concern numerical evaluation at the fixed Gaussian weight.

D.3 Common-scale propagation in the QCD readout

For the retained map (C.122) write $\Lambda(\mu, a) = \mu(b_0 a/4\pi)^{-p} e^{-q/a}$. At fixed five-flavour coefficients the exact finite transfer is

$$\frac{\Lambda(\mu', a')}{\Lambda(\mu, a)} = \frac{\mu'}{\mu} \left(\frac{a'}{a}\right)^{-p} \exp\left[-q\left(\frac{1}{a'} - \frac{1}{a}\right)\right], \quad \mu, \mu', a, a' > 0. \quad (\text{D.17})$$

Its local differential is $d \log \Lambda = d \log \mu + (q/a - p)d \log a$. At the effective coupling $a = 0.11798585$ the coupling elasticity is approximately 6.6172. Thus applying $f = 1 - 2\tau\kappa/8$ to a_{raw} gives the scale factor $f^{-p} \exp[-q(1/f - 1)/a_{\text{raw}}]$. At $\tau = 1/50$ the scale multiplier is about 0.96325 whereas the coupling multiplier f is about 0.99434. If a certified input box has $0 < a_- \leq a \leq a_+ < q/p$ and $0 < \mu_- \leq \mu \leq \mu_+$ monotonicity gives $\Lambda \in [\Lambda(\mu_-, a_-), \Lambda(\mu_+, a_+)]$. The enclosure remains valid when the inputs are correlated.

Keep the catalogue labels and τ fixed. All of its masses and T_d have scale elasticity one and σ has elasticity two. In particular

$$\begin{aligned} \|M_C(\Lambda') - M_C(\Lambda)\| &= 12|\Lambda' - \Lambda|, \\ \|M_C^2(\Lambda') - M_C^2(\Lambda)\| &= 144|\Lambda'^2 - \Lambda^2|. \end{aligned} \quad (\text{D.18})$$

Both equalities follow from the largest listed label 18. Consequently the mass ratios and σ/T_d^2 are exactly independent of the common extraction scale at fixed τ . If an uncertainty model is assigned to shared upstream inputs θ their first-order covariance is $\text{Cov}(\log O) = J \text{Cov}(\theta) J^T$ with $J_{ai} = \partial_{\theta_i} \log O_a$. The common Λ contribution alone is proportional to vv^T where $v = (1, 1, 1, 1, 1, 2)$ for the four masses and T_d together with σ . Thus a shared extraction-scale uncertainty contributes a rank-one covariance to these six readouts.

D.4 Finite errors of the shared cosmological readouts

Use x, D, A_γ, T_E from Eq. (C.117). The dimensionless normalized ratio and its exact residual are

$$\begin{aligned}\hat{R} &:= \frac{T_E^2/(M_P H_0)}{A_\gamma^2 \sqrt{D/(\pi Y_u)}} = f(x) := \frac{(1-x)^2}{\sqrt{1-4x}}, \\ f(x) - 1 &= \frac{x^2(6-4x+x^2)}{\sqrt{1-4x}\{(1-x)^2 + \sqrt{1-4x}\}}.\end{aligned}\tag{D.19}$$

For $0 \leq x \leq X < 1/4$ differentiation and rationalization give

$$\begin{aligned}f'(x) &= \frac{6x(1-x)}{(1-4x)^{3/2}}, \quad f''(x) = \frac{6(1+2x^2)}{(1-4x)^{5/2}} \geq 6, \\ 3x^2 \leq f(x) - 1 &\leq \frac{6x^2}{\sqrt{1-4X}\{(1-X)^2 + \sqrt{1-4X}\}}.\end{aligned}\tag{D.20}$$

Since $f(0) = 1$ and $f'(0) = 0$, the lower bound follows by integrating $f'' \geq 6$ twice. In the exact residual the numerator is at most $6x^2$ and the denominator decreases on the stated interval to prove the upper bound. At $X = 1/20$ its coefficient is less than $15/4$. The recorded $\tau = 1/50$ gives $x^4 = 676/(25 \cdot 23^5) < (1/20)^4$. Hence the certificate applies. Here $x \simeq 0.04527327$ and $\hat{R} - 1 \simeq 0.00725839$. The normalized ratio starts at second order in x . The full ratio also contains $A_\gamma(\tau)$ and D together with Y_u . For two admissible endpoint tuples put $\delta \log q = \log(q'/q)$. Then

$$\begin{aligned}\delta \log \frac{T_E^2}{M_P H_0} &= 2\delta \log A_\gamma + \frac{1}{2}\delta \log D - \frac{1}{2}\delta \log Y_u + \log \frac{f(x')}{f(x)}, \\ \left| \log \frac{f(x')}{f(x)} \right| &\leq \frac{6X}{(1-X)(1-4X)}|x' - x|, \quad x, x' \in [0, X].\end{aligned}\tag{D.21}$$

The second line uses $(\log f)' = 6x/[(1-x)(1-4x)]$. The common mass cancels before uncertainty propagation. Likewise $\Omega_\Lambda = D/(3\pi)$ is exactly independent of the common mass and dimensional anchor at fixed content.

The logarithmic amplitude shift corresponding to $v/v_0 = 1 - x$ is $-\log(1 - x)$ with

$$\frac{x^2}{2} \leq -\log(1 - x) - x \leq \frac{x^2}{2(1 - x)}, \quad 0 \leq x < 1.$$

This follows from the positive logarithmic series and distinguishes the exact logarithmic change from its first-order insertion.

D.5 Uniform scalar-gap bounds

The results “Scale transport and a uniform scalar gap” and “Uniform truncation bounds on a condensed scale interval” of Ref. [4] apply to the complete scalar readout on $I = [k_{\min}, M_G]$ for $k_{\min} > 0$ with $L_k = cM_G/k$ and $c > \sqrt{\pi}$ and specified positive C^1 paths g_k, Z_k . They provide a common truncation and uniform positive gap once two-sided bounds for $g_k L_k^2$ and $8(L_k^2 - \pi)/(\pi g_k L_k^4)$ together with Z_k have been supplied. For the same potential

and weights with the same subtraction and normalization write $\Gamma = \inf_{k \in I} m_\delta^2(k) > 0$. The uniform truncation estimate then permits a finite N for which

$$\sup_{k \in I} |m_{\delta,N}^2(k) - m_\delta^2(k)| \leq \varepsilon_N < \Gamma/2, \quad \inf_{k \in I} m_{\delta,N}^2(k) \geq \Gamma/2.$$

The choice of N is uniform on I once the stated path bounds have been supplied. The carrier includes the constant scalar shell. The QCD catalogue has a separate scale dependence quantified in Section D.3.

D.6 The spectral-scale flow and the renormalisation group

The internal response fixes the normalization of the connection and matter vertices. Written in canonical fields these coefficients supply the boundary action whose quantum fluctuations determine the scale dependence of the effective couplings. The representation traces in the two-loop equations use the same retained field content as the spectral construction [19, 20]. The spectral boundary data are expressed at M_G with channel matching at k_{GUT} as specified in Section 5.2. Running and threshold matching then express this action at the lower comparison scales. Finite mass and vertex matching connects the geometric readouts to renormalized parameters as in (C.65) and the one-loop construction of Section 10.1 [51]. The production calculation uses the stated fixed Yukawa value whereas the five-coupling evolution transports the matched Yukawa and quartic coefficients together with the gauge couplings. Their perturbative accuracy is determined by the loop order and threshold treatment.

The “Distribution-valued Gaussian spectral completion” theorem in Ref. [4] supplies the field with the specified spectral covariance. The scalar estimates above control the boundary spectral functional. The running and matching errors are classified by their respective perturbative orders.

E Neutrino and quark-mixing relations

The neutrino masses and mixing matrices complete the flavour relations of Section 7. The lightest rest mass and the baryon-asymmetry estimate enter the cosmological densities in Section 8. Table 14 distinguishes the rest-mass assignments from the additional propagation and mixing assumptions.

E.1 Neutrinos

Appendix C.10 defines the Weinberg-operator normalisation and family scale and squash insertion in (C.73) with the hierarchy ratios in (C.74) and the ordered rest-mass spectrum in (C.75). The flavour construction combines this spectrum with the near-tribimaximal mixing-angle assignments in (C.83).

In the charged-lepton mass basis the columns of U give the orientations of the neutrino mass states relative to the flavours selected by the charged current. The solar angle sets the relative electron-flavour weights of the light pair. The reactor angle sets the electron weight of the third state and the atmospheric angle sets its muon–tau partition. With these

angles and the Dirac-phase readout of Appendix C.13 the numerical construction [15] gives a unitary lepton mixing matrix. The two Majorana phases remain free. For $D = \text{diag}(m_{\nu_i})$ the flavour mass texture and vacuum propagation numerator are

$$M_\nu = U^* D U^\dagger, \quad H_2 = U D^2 U^\dagger, \quad U^\top M_\nu U = D.$$

The Takagi relation fixes the correspondence between each mass and its column of U . The Majorana phases cancel from H_2 . Setting all phases to zero gives the real symmetric mass texture. The solar scalar source acts through the spectral projector in (C.80).

Solar production prepares an electron-flavour state. The scalar source changes the second mass-state energy while the electron matter potential acts along the electron-flavour direction. Their relative orientation determines the instantaneous propagation states and the conversion between them as the density changes [56, 57]. The Hermitian Hamiltonian is

$$H(E, x) = \frac{U D^2 U^\dagger}{2E} + V_{\text{SM}}(x) + \Delta H_\Sigma(E, x).$$

For the source in (C.79)

$$V_{\text{SM}} = \sqrt{2} G_F n_e(x) P_e, \quad P_e = \text{diag}(1, 0, 0), \quad \Delta H_\Sigma = \frac{d_\nu \zeta(x)}{2E} P_2.$$

The scalar channel P_2 and its tilt-dependent strength fix the mass-state response. An environmental realization specifies the scalar interaction and the map from the state of the medium to $\zeta(x)$. These data then determine the source profile in each propagation experiment. The mass-squared insertion produces the displayed E^{-1} term whereas coherent electron forward scattering supplies the vector potential [59]. The fixed interaction and map from the medium to its source apply to solar, terrestrial and antineutrino propagation. The correlated changes of all three splittings in (C.81) provide an additional test of that common coupling. Vacuum propagation uses $\zeta = 0$. The antineutrino Hamiltonian conjugates the two mass terms and reverses the vector matter potential. Given the solar source and electron-density profiles let $S_\odot(E; x)$ propagate from production point x to the solar surface and let $f_E(x)$ be the normalized production density. Propagation through the Sun sets the mass populations at its surface. When the distinct vacuum mass states decohere on the journey to Earth the detector receives an incoherent mixture of these populations [58]. Their production average and detection probabilities are

$$\begin{aligned} p_i(E) &= \int f_E(x) \left| [U^\dagger S_\odot(E; x)]_{ie} \right|^2 dx, \\ P_{ee}^{\text{day}}(E) &= \sum_i p_i(E) |U_{ei}|^2, \\ P_{ee}^{\text{night}}(E) &= \sum_i p_i(E) |[S_\oplus(E) U]_{ei}|^2. \end{aligned} \tag{E.1}$$

Here $S_\oplus$ includes the Earth's matter and any specified scalar profile. Experimental energy and path averages act on these probabilities. They define the solar observables of the scalar-source model. A source commuting with the vacuum numerator throughout a

matter-free trajectory changes only phases. After mass decoherence its phases leave the p_i unchanged. The noncommuting electron potential and the production average therefore enter a solar realization explicitly. The propagation calculation preserves probability and has the vacuum and two-flavour limits [15].

The raw and unit-source solar splittings are given in (C.76) and (C.78). Their residuals against the NuFIT 5.2 reference are -7.231% and $+0.030\%$ respectively [46]. The unit source entry evaluates the propagation numerator at $\zeta = 1$. The solar comparison averages the probabilities in (E.1) over production and detection with the fixed interaction and medium profile. Equation (C.81) gives the scalar projector and all three correlated splitting shifts and the cancellation of the common mass scale. Table 14 lists the rest-mass and unit-source values separately with the latter in Tier D. The absolute masses and their sum retain the rest-mass definition. In particular m_{ν_1} supplies the dark-energy density in Section 8.1.

The rest spectrum also determines

$$m_\beta^2 = \sum_i |U_{ei}|^2 m_i^2, \quad \max\left(0, 2 \max_i a_i - \sum_i a_i\right) \leq m_{\beta\beta} \leq \sum_i a_i, \quad a_i = |U_{ei}|^2 m_i.$$

The interval is the attainable range over the two Majorana phases. The rest spectrum and mixing angles above give $m_\beta = 0.0090585$ eV and $0 \leq m_{\beta\beta} \leq 0.0054362$ eV under the light-Majorana exchange interpretation [15]. Both quantities refer to the vacuum rest-mass texture.

E.2 CKM and CP

Appendix C.13 specifies the Cabibbo input $\mathbf{r}_G$ in (C.98) and its distinction from the mass ratio m_d/m_s . Equations (C.99) and (C.100) give the three mixing elements including the adjacent- and next-generation suppressions. Their common attenuation cancels in the mass-mixing combinations (7.18). The phase with colour dilution in (C.103) sets the complex relative orientation of the up and down mass sources. The Jarlskog invariant measures the CP-odd part that is preserved under rephasing of their modes [62]. Its leading expression (C.104) gives $J = 3.15 \times 10^{-5}$. The omitted c_{13} dependence enters at relative order $|V_{ub}|^2$. The raw asymmetry in (C.105) transfers this signed CP measure to the baryon readout with the weak weight $\alpha_W(v)^2$ where $\alpha_W(v) = g_2(v)^2/(4\pi)$ and the content normalisation $N_L N_R = 56$, giving $\eta_B^{\text{raw}} = 6.09 \times 10^{-10}$. This value accompanies the raw photon floor. The conversion $T_{\text{CMB}} = C_\gamma T_{\text{raw}}$ with $C_\gamma = 1 - \tau/\pi^2$ changes the photon number density by C_γ^3 . Holding the baryon number density fixed therefore gives the physical baryon-to-photon ratio $\eta_B = \eta_B^{\text{raw}} (1 - \tau/\pi^2)^{-3} = 6.12823 \times 10^{-10}$ while the baryon density is unchanged. Appendix C.13 specifies the content factors and their relation to Sakharov's conditions. This effective asymmetry enters the homogeneous baryon density.

F Primordial-nucleosynthesis relations

The masses and couplings of Sections 5, 8 and 9, with the Cabibbo input of Appendix C.13, give a reduced estimate of the primordial helium abundance from weak freeze-out. Appendix C.16 specifies the thermal history and weak rates used in this estimate. The

Table 22. Common weak-decay and helium calculation. The mass difference and lifetime references are from the PDG 2024 neutron listing [1]. Residuals compare central values. The reduced weak and capture calculation has no assigned statistical theory covariance.

Quantity	Calculated	Reference	Residual
Δm_{np} (MeV)	1.28497	1.29333236(46)	−0.647%
τ_n (s)	915.54	878.4 ± 0.5	+4.23%
Y_p	0.246082	0.245	+0.44%

precision calculation of Ref. [64, Secs. III.G–H and IV.C] includes radiative and finite-temperature weak-rate corrections and a nuclear reaction network. Here the abundance is estimated from neutron freeze-out and decay.

F.1 Primordial helium

The calculation uses $g_A = 4/\pi$ and $G_F = 1/(\sqrt{2}v^2)$ and the Cabibbo approximation $|V_{ud}| = \sqrt{1 - |V_{us}|^2}$ at leading order in $|V_{ub}|^2$. The native isovector response is $g_S^{\text{eff}} = 1$. The mass difference $d = \Delta m_{np}$ includes the electromagnetic subtraction in Eq. (C.136). The low-energy QED coupling and radiative factor are specified in Eqs. (C.133) and (C.135). The electron mass in the weak rates and decay phase space is $m_e := M_e = G_e m_e^{\text{geom}}$ with the displayed calculation using $G_e = 1$ in Eq. (7.1). With these inputs the Coulomb-weighted phase space (C.142) gives the static-nucleon lifetime (C.141) and the weak-rate balance (C.143) determines T_f . With $r_f = \exp(-d/T_f)$ and $q_{\text{dec}} = \exp(-\Delta t/\tau_n)$, baryon number conservation and complete capture of the surviving neutrons give

$$X_n = \frac{r_f q_{\text{dec}}}{1 + r_f}, \quad Y_p = 2X_n,$$

with the neutron-to-proton ratio given by Eq. (C.145). The separate neutrino-number assignment (C.146) can also be written as $N_{\text{eff}} = 3(1 + \delta_N)$ with $\delta_N = \sqrt{3}/[3(2\pi)^2]$. Here Δt is the elapsed time from weak freeze-out to the adopted nucleosynthesis temperature measured by the homogeneous comoving clock. A rest-frame width is the decay rate with respect to proper time s through $P(s) = \exp[-\int \Gamma(s') ds']$ in natural units. For constant Γ_n the conversion to seconds gives $\tau_n = \hbar/\Gamma_n$. The same clock convention is used for the decay interval and lifetime. The calculation gives $T_f = 0.75394$ MeV and the three linked readouts in Table 22 [15]. The two-species decay equations $\dot{n} = -n/\tau_n$ and $\dot{p} = n/\tau_n$ give the same abundance. The two thermal values of g_* are held fixed when applying the N_{eff} assignment in the time-temperature relation (C.144).

F.2 Common-input response

Write $d = \Delta m_{np}$ and $q = d/m_e$ and $f = f(q, \alpha_{\text{em}})$. The mass and lifetime differentials are

$$\begin{aligned} dd &= (m_d - m_u) dg_S^{\text{eff}} + g_S^{\text{eff}}(dm_d - dm_u) - d\Delta_{\text{EM}}, \\ d \log \tau_n &= -2d \log G_F - 2d \log |V_{ud}| - \frac{6g_A^2}{1 + 3g_A^2} d \log g_A - d \log \delta_R \\ &\quad - 5d \log m_e - \frac{f_q dq + f_\alpha d\alpha_{\text{em}}}{f}. \end{aligned} \quad (\text{F.1})$$

For the retained freeze-out kernel let C_f multiply the weak rate with $C_f = 1$ in the calculation. Its root obeys $C_f G_F^2 (1 + 3g_A^2) M_P T_f^2 (T_f + 3d/2) = \text{constant}$. At fixed relativistic degrees of freedom

$$d \log T_f = -\frac{T_f + 3d/2}{3(T_f + d)} d \log [C_f G_F^2 (1 + 3g_A^2) M_P] - \frac{dd}{2(T_f + d)}. \quad (\text{F.2})$$

If t_f and t_B denote the two radiation-era times then fixed capture temperature gives $d\Delta t = \Delta t d \log M_P + 2t_f d \log T_f$. Writing $x = d/T_f$ and $r = e^{-x}$ then yields

$$d \log Y_p = -\frac{dx}{1 + r} - \frac{d\Delta t}{\tau_n} + \frac{\Delta t}{\tau_n} d \log \tau_n. \quad (\text{F.3})$$

These derivatives retain the change of the freeze-out abundance as well as subsequent decay.

Order the columns as g_S^{eff} , Δ_{EM} , g_A , δ_R and G_F , and the rows as τ_n and Y_p . The central logarithmic Jacobian is

$$J_{\text{weak}} = \begin{pmatrix} -13.0288 & 6.49251 & -1.65890 & -1 & -2 \\ -6.69407 & 3.33578 & -1.41785 & -0.223789 & -1.70938 \end{pmatrix}.$$

The electromagnetic subtraction and the Coulomb factor also share α_{em} . Their common variation is composed before propagating an input covariance. With an independently specified Σ_θ the covariance of the logarithmic outputs is $J\Sigma_\theta J^\top$. Interval input errors instead give first-order bounds through $|J|$ with nonlinear remainders controlled over the selected interval. The weak rates, hadronic matching and nuclear capture approximations require their own error estimates [64]. The reported lifetime residual compares central values. Its statistical interpretation depends on these theory errors as well as the experimental uncertainty.

The response calculation [15] evaluates f once and then integrates f_q, f_α together on the accepted adaptive partition with local refinement for the derivatives. The recorded calculation uses 567 central-integrand evaluations and 576 derivative-integrand evaluations. The remaining chain is differentiated analytically. Independent displaced-input calculations check the displayed five-input Jacobian. The lifetime and helium responses have different weights. Their joint comparison tests more than either abundance or decay separately.

Table 23. Finite mass-difference diagnostic. The second calculation uses the experimental neutron–proton mass difference as an external condition. Both columns use the same weak and capture approximations.

Quantity	Generated mass difference	Experimental mass difference
f	1.63492371	1.70563405
T_f (MeV)	0.75394339	0.75240100
τ_n (s)	915.54351	877.58790
Y_p	0.24608238	0.24071583

F.3 Mass-difference diagnostic

At fixed electron mass and other weak inputs the phase-space response gives

$$\left. \frac{\partial \log \tau_n}{\partial \log d} \right|_{m_e, \alpha_{\text{em}}, G_F, V_{ud}, g_A, \delta_R} = -\frac{qf_q}{f} = -6.53633. \quad (\text{F.4})$$

The -0.647% mass-difference residual thus corresponds locally to a lifetime shift of about $+4.23\%$ with the same sign and magnitude as the reported lifetime residual. To test this sensitivity over the finite displacement we replace only d by the registered experimental value 1.29333236 MeV and propagate it through the phase space and freeze-out root together with the decay interval. Table 23 compares this externally conditioned calculation with the generated chain [15]. All other inputs and the capture temperature and weak-rate approximations are held fixed.

The beta-phase-space enclosure gives

$$\tau_n \in [915.5097706, 915.5853079] \text{ s}, \quad Y_p \in [0.2460803490, 0.2460848926].$$

These ranges enclose the phase-space evaluation at the displayed inputs. The weak-rate and nuclear-capture model spread is assessed separately.

The conditioned lifetime has a central-value residual of -0.0925% while the helium residual is -1.7486% . The mass difference therefore accounts for most of the lifetime residual within this diagnostic. Its simultaneous effect on freeze-out and neutron survival is essential for assessing the whole chain. The replacement probes the total mass difference. The strong isovector contribution and electromagnetic subtraction require independent information in a common separation convention. Propagating only the registered mass-difference uncertainty gives 0.00203 s for the lifetime and 2.94×10^{-7} for Y_p at the conditioned point. These are input-uncertainty contributions. The physical matching and rate errors remain those discussed above.

F.4 Independent QCD–QED input

The lattice calculation of Ref. [66, Table 1] gives $\Delta m_{np} = 1.51(16)(23)$ MeV with statistical and systematic errors respectively. Its separated contributions are $\Delta m_{np}^{\text{QCD}} = 2.52(17)(24)$ MeV and $\Delta m_{np}^{\text{QED}} = -1.00(07)(14)$ MeV. The separation uses $\Delta_{\text{QED}} M_\Sigma = 0$ as specified in Sec. 10.3 of its supplementary material. Propagating the published total

Table 24. Finite propagation of the external lattice total mass difference through the fixed weak and capture model. The statistical and systematic error sizes are varied separately. The last row uses their sum as an input half-width. It has no assigned coverage probability. The helium ranges follow the positive-factor bounds (F.5).

Input variation	d (MeV)	τ_n (s)	Y_p
Statistical	[1.35, 1.67]	[181.54, 665.67]	[0.05293, 0.20524]
Systematic	[1.28, 1.74]	[142.81, 939.05]	[0.03429, 0.24937]
Sum of half-widths	[1.12, 1.90]	[86.20, 2330.93]	[0.009830, 0.355290]

splitting and its errors retains their correlated determination. The separately rounded contributions serve to identify the source convention.

With all other weak and thermal inputs fixed the total central value gives $\tau_n = 331.476$ s and $Y_p = 0.116457$. Positivity of the phase-space integral and monotonicity of the freeze-out root give finite bounds for this propagation.

Finite mass-difference ranges. Fix the weak inputs and capture temperature and let $m_e < d_- \leq d \leq d_+$ with capture after freeze-out throughout. Write $T_\pm = T_f(d_\pm)$ and $\tau_\pm = \tau_n(d_\pm)$ with $b_\pm = t_B - t_f(d_\pm)$ and $z_\pm = (1 + e^{d_\pm/T_\pm})^{-1}$. Then

$$\tau_+ \leq \tau_n(d) \leq \tau_-, \quad 2z_+e^{-b_-/\tau_+} \leq Y_p(d) \leq 2z_-e^{-b_+/\tau_-}. \quad (\text{F.5})$$

The phase integral has a positive weight multiplying $(q - W)^2$ on $1 < W < q = d/m_e$. Increasing d increases the integral and decreases τ_n . Equation (F.2) gives $T'_f(d) = -T_f/[2(T_f + d)] < 0$. Hence d/T_f increases, the initial neutron fraction decreases and the positive decay interval $b = t_B - t_f$ also decreases. Combining $z_+ \leq z \leq z_-$ with $b_+/\tau_- \leq b/\tau_n \leq b_-/\tau_+$ gives the bounds.

The three ranges and their centre require seven distinct phase-space integrals. Shared endpoints are reused. The estimated quadrature contributions to the displayed endpoints are below 10^{-9} s for the lifetime and 6×10^{-14} for Y_p . These numerical estimates are separate from the input ranges and the weak-rate and capture approximations. The combined lattice-input range is wider than the correction indicated by the lifetime residual. Resolving that correction requires the combination (C.138) and the electromagnetic contribution in a common separation convention. The production weak and cosmological values remain those in Table 22.

F.5 Continuous weak evolution

With the same weak and hadronic inputs continuous freeze-out follows from the six forward and reverse neutron–proton processes. Fermi occupation and blocking factors are retained in the static-nucleon approximation [15]. The neutron fraction is evolved according to

$$\dot{X}_n = -\lambda_{n \rightarrow p} X_n + \lambda_{p \rightarrow n} (1 - X_n)$$

coupled to the photon–electron thermal bath with a specified neutrino decoupling history. The zero-temperature decay integral reproduces the lifetime above. Independently integrated rates obey detailed balance at a common temperature and zero chemical potentials.

With equilibrium initial data at 2 MeV and instantaneous neutrino decoupling at 1.2 MeV the example evolved to 0.5 MeV gives $X_n = 0.19335$. This value describes the neutron fraction at 0.5 MeV before nuclear assembly. The helium abundance quoted above uses the freeze-out and decay approximation with complete neutron capture. Evolution through nuclear assembly additionally depends on the nuclear reaction rates.

Weak-fraction bounds under continuous cooling. Let x increase along cooling with $dt/dx > 0$ and set $a_+ = (dt/dx)\lambda_{p \rightarrow n} \geq 0$ and $a_- = (dt/dx)\lambda_{n \rightarrow p} \geq 0$ and $A = a_+ + a_-$. For integrable rates the solution of $X'_n = a_+(1 - X_n) - a_-X_n$ stays in $[0, 1]$ if its initial value does. For an absolutely continuous approximation $\hat{X}$ with defect $R = \hat{X}' - a_+(1 - \hat{X}) + a_-\hat{X}$ its error obeys

$$|\hat{X}(x) - X_n(x)| \leq e^{-\int_{x_i}^x A} |\hat{X}(x_i) - X_n(x_i)| + \int_{x_i}^x e^{-\int_s^x A} |R(s)| ds.$$

The vector field points into $[0, 1]$ at both endpoints. Subtracting the exact and approximate equations gives $e' + Ae = R$. The integrating factor proves the error bound. The bound controls the integration error for fixed weak rates and thermal history. The quadrature and evolution tolerances are specified in Ref. [15].

G The acceleration scale and gravitational dynamics

The compact-source angular covariance sets the endpoint acceleration variance. The Gaussian susceptibility compares squared response amplitudes; its positive RMS readout determines the spatial flux. The resulting constitutive energy has the nonrelativistic variational form of Ref. [12]. On a bounded Lipschitz domain with H^1 Dirichlet data and an H^{-1} source, strict convexity gives a unique weak solution. Coupling the same potential to pressureless trajectories makes the source respond to the field. On a fixed smooth domain the joint action gives their energy balance throughout smooth evolution before caustic formation.

The covariant analysis retains matter stress, source exchange and the phase clock congruence in a common action. An auxiliary collective radius closes its source balance on an occupied expanding background. The independent-foliation action defines a separate effective realization of the local constitutive law with the congruence varied as an additional field. The kinetic form is reduced on the constraint and variation space of each action.

G.1 The endpoint acceleration variance

Set the terminal coarse-graining length to $\sigma_C = c/H_0$ where c denotes the speed of light throughout this appendix. The centered maximum-entropy displacement law with variance σ_C^2 has the Gaussian spectral multiplier

$$W_{\text{IR}}(\Delta) = \exp\left(-\frac{1}{2}\Delta^2\sigma_C^2\right).$$

The Euclidean-period normalization used here assigns the wave number

$$k_{\text{IR}} = \frac{1}{2\pi\sigma_C} = \frac{H_0}{2\pi c}.$$

The angular factor is evaluated on the compact source of Section 4.5 of Ref. [11] under “Visible tensor modes of the compact source”. Its vertex normalization and occupation source are given there in Eqs. (4.27) and (4.28) respectively. On the scalar shell $\mathcal{H}_2 \simeq \mathbb{R}^3 \otimes \mathbb{R}^3$ the normalized source in the two orthogonal five-dimensional compact TT spaces is

$$\begin{aligned}\widehat{j}(N) &:= j_{\text{hom}}(N)/\alpha_2 = ((\rho_L)_0, (\rho_R)_0), \\ \rho_L &= \text{Tr}_R N, \quad \rho_R = \text{Tr}_L N, \quad B_0 = B - \frac{1}{3} \text{tr}(B) I_3.\end{aligned}$$

Here $N \geq 0$ is a real occupation matrix with $\text{Tr} N = 1$ and division by the inherited normalization α_2 defines the angular source. Unit trace fixes the total occupation while resolving its distribution among the internal angular modes. In the Hilbert–Schmidt norm

$$\|\widehat{j}(N)\|^2 = \text{tr} \rho_L^2 + \text{tr} \rho_R^2 - \frac{2}{3} \in [0, 4/3]. \quad (\text{G.1})$$

The upper equality holds exactly for decomposable pure occupations $N = |u \otimes v\rangle\langle u \otimes v|$ with $|u| = |v| = 1$. Each marginal then has purity one. Conversely, two pure marginals force this product occupation. Equal occupation $N = I_9/9$ gives zero source.

On this decomposable orbit, independent uniform $u, v \in S^2$ induce its unique $SO(3) \times SO(3)$ -invariant reference probability by transitivity of the compact group action. With $Q_u = uu^\text{T} - I_3/3$ the angular fourth moment

$$\mathbb{E}(u_i u_j u_k u_l) = \frac{\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}}{15}$$

gives the following covariance in orthonormal bases of the two TT spaces

$$\begin{aligned}\mathbb{E} N &= I_9/9, \quad \mathbb{E} \widehat{j} = 0, \\ \text{Cov}(\widehat{j}) &= \frac{2}{15} I_{10}, \quad C_{\text{TT},3} := \mathbb{E} \|\widehat{j}\|^2 = \frac{4}{3}.\end{aligned} \quad (\text{G.2})$$

Uniform orientations cancel the directional source in the ensemble mean while their squared amplitudes add. The factor $4/3$ is the mean square angular fluctuation on the decomposable occupation orbit. The mean occupation and this covariance describe different moments of the same specified ensemble. Assigning acceleration amplitude $c^2 k_{\text{IR}}$ to the endpoint wave number gives

$$\Sigma_{\text{IR}} = C_{\text{TT},3} (c^2 k_{\text{IR}})^2 = \frac{4}{3} \left(\frac{c H_0}{2\pi} \right)^2 = a_0^2, \quad a_0 = \frac{c H_0}{2\pi} \sqrt{\frac{4}{3}}. \quad (\text{G.3})$$

The Hubble length that enters the homogeneous endpoint therefore also sets the local acceleration normalization. Here H_0 labels the common reference endpoint. The local law retains this value of a_0 while the instantaneous expansion rate evolves with the macroscopic state. With the content factors, decomposable occupation orbit and period normalization fixed, a change of the common neutrino mass normalization changes H_0 and a_0 together while the photon temperature follows its linear mass readout. These are linked endpoint normalizations at the same reference epoch. The resulting scale is $a_0 = 1.204 \times 10^{-10} \text{ m s}^{-2}$. Its central-value difference from the empirical $g_\dagger = 1.2 \times 10^{-10} \text{ m s}^{-2}$ is $+0.36\%$. The radial-acceleration fit of Eq. (4) of Ref. [69] gives random and systematic uncertainties of 0.02 and 0.24 respectively in the same units. We use Eq. (G.3) as the acceleration normalization in the local response.

G.2 The endpoint variance and the local response

For $\lambda_{\text{IR}} > 0$ the retained one-mode quadratic form is

$$S_2[q] = \frac{1}{2}(\lambda_{\text{IR}} + \Sigma_{\text{IR}})q^2.$$

The unscreened and endpoint variances are $\langle q^2 \rangle_0 = 1/\lambda_{\text{IR}}$ and $\langle q^2 \rangle_{\text{IR}} = 1/(\lambda_{\text{IR}} + \Sigma_{\text{IR}})$. Define the variance ratio by

$$Z_{\text{IR}} = \frac{\langle q^2 \rangle_{\text{IR}}}{\langle q^2 \rangle_0} = \frac{\lambda_{\text{IR}}}{\lambda_{\text{IR}} + \Sigma_{\text{IR}}}. \quad (\text{G.4})$$

The quasi-static spatial entry is the squared acceleration

$$\lambda_{\text{IR}} = |\nabla\Phi|^2, \quad y = \frac{|\nabla\Phi|}{a_0}.$$

The same-source response identity of Proposition 3.1 of Ref. [11] under “Response of the sourced equations” expresses the response in terms of its kernel and source together with the readout and direct term. For the quadratic form above and the same linear source hq the normalized Gaussian integral gives exactly

$$\mathcal{W}_{\Sigma}(h) - \mathcal{W}_{\Sigma}(0) = \frac{h^2}{2(\lambda_{\text{IR}} + \Sigma_{\text{IR}})}, \quad \frac{\partial_h^2 \mathcal{W}_{\Sigma}}{\partial_h^2 \mathcal{W}_0} = Z_{\text{IR}}.$$

Thus Z_{IR} is the normalized susceptibility and compares squared amplitudes under the same source. The spatial response reads their positive RMS amplitude ratio, giving

$$\mu(y) = \sqrt{Z_{\text{IR}}} = \sqrt{\frac{|\nabla\Phi|^2}{|\nabla\Phi|^2 + a_0^2}} = \frac{y}{\sqrt{1 + y^2}}. \quad (\text{G.5})$$

Equation (G.5) is the standard MOND interpolation function used in Section 3.1 of Ref. [70]. Here the RMS response fixes its form and Eq. (G.3) fixes a_0 . Within the response class defined by this occupation orbit, quadratic spatial entry and RMS readout, these expressions determine the local constitutive law. The Gaussian variance is defined for $\lambda_{\text{IR}} > 0$. The corresponding flux extends continuously to zero with $\mu(0) = 0$. For $y \ll 1$ $\mu(y) \sim y$. Thus the flux is asymptotic to $|\nabla\Phi|\nabla\Phi/a_0$. The spatial integral of its leading cubic primitive is invariant under $x \mapsto sx$ with $\Phi \mapsto \Phi$ in three spatial dimensions. This is the scale invariance of the leading deep-infrared field functional. For a plane-symmetric source at fixed geometry, Gauss integration makes the flux proportional to the source amplitude. Its inversion then gives a field proportional to the square root of that amplitude in the low-acceleration regime. The cubic energy and this nonlinear source response follow from the same amplitude law.

The endpoint flux is

$$\mathcal{J}_i = \mu\left(\frac{|\nabla\Phi|}{a_0}\right) \nabla_i \Phi.$$

Its Gauss law is

$$\nabla \cdot \left[\mu\left(\frac{|\nabla\Phi|}{a_0}\right) \nabla\Phi \right] = 4\pi G \left(\rho_b + \rho_{\Sigma}^{\text{free}} \right). \quad (\text{G.6})$$

This is the nonrelativistic field equation of Bekenstein and Milgrom in Eq. (3) of Ref. [12] with the stated μ and source density. For this choice of flux the Dirichlet problem follows from a strictly convex energy with quadratic growth at large gradients. This growth gives coercivity, while strict monotonicity gives uniqueness including fields whose gradient vanishes.

Proposition G.1 (Linearized flux and unique weak Dirichlet solution). Let $a_0 > 0$ and $\mathcal{J}(g) = \mu(|g|/a_0)g$ for $g \in \mathbb{R}^3$. For $g \neq 0$ put $y = |g|/a_0$ and $P_{\parallel} = gg^T/|g|^2$ and $P_{\perp} = I - P_{\parallel}$. The flux derivative is

$$\begin{aligned} \mathbf{M}(g) &:= D\mathcal{J}(g) = \frac{y}{\sqrt{1+y^2}}P_{\perp} + \frac{y(y^2+2)}{(1+y^2)^{3/2}}P_{\parallel}, \\ \mu(y)I &\leq \mathbf{M}(g) \leq 2\mu(y)I. \end{aligned} \tag{G.7}$$

It extends continuously with $\mathbf{M}(0) = 0$. Hence the positive linearized operator $-\nabla \cdot [\mathbf{M}(\nabla \bar{\Phi})\nabla]$ is uniformly elliptic on backgrounds with $|\nabla \bar{\Phi}| \geq g_{\min} > 0$ almost everywhere.

More generally let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Fix $\Phi_b \in H^1(\Omega)$ and $f = 4\pi G(\rho_b + \rho_{\Sigma}^{\text{free}}) \in H^{-1}(\Omega)$. There is a unique $\Phi \in \Phi_b + H_0^1(\Omega)$ satisfying

$$\int_{\Omega} \mathcal{J}(\nabla \Phi) \cdot \nabla \eta \, d^3x = -\langle f, \eta \rangle \quad \text{for every } \eta \in H_0^1(\Omega).$$

Proof. Differentiating the flux gives Eq. (G.7). The longitudinal-to-transverse eigenvalue ratio is $(y^2+2)/(y^2+1) \in [1, 2]$. Its radial primitive is

$$\mathcal{F}(g) = \frac{a_0^2}{2}[y\sqrt{1+y^2} - \text{arsinh } y], \quad \nabla_g \mathcal{F} = \mathcal{J}(g).$$

The matrix $\mathbf{M}$ is positive definite off zero. Integrating it along the segment joining distinct g, h gives $(\mathcal{J}(g) - \mathcal{J}(h)) \cdot (g - h) > 0$. The segment can meet zero at at most one parameter value. Thus $\mathcal{F}$ is strictly convex. Also $\mathcal{F}(g) = \int_0^{|g|} \mu(t/a_0)t \, dt$. Since $0 \leq \mu \leq 1$ and $\mu(t/a_0) \geq 1/\sqrt{2}$ for $t \geq a_0$

$$\frac{|g|^2 - a_0^2}{2\sqrt{2}} \leq \mathcal{F}(g) \leq \frac{|g|^2}{2}.$$

For $u \in H_0^1(\Omega)$ the primitive defines the minimization functional

$$\mathcal{E}(u) = \int_{\Omega} \mathcal{F}(\nabla(u + \Phi_b)) \, d^3x + \langle f, u \rangle.$$

The displayed growth bound and Poincaré's inequality make $\mathcal{E}$ coercive while convexity makes it weakly lower semicontinuous. The standard direct method and strict-monotonicity uniqueness theorem for quasilinear Dirichlet problems [71] therefore apply. $|\mathcal{J}(g)| \leq |g|$ gives the stated Euler equation. At vanishing field the constitutive law is degenerate: $D\mathcal{J}(0) = 0$ and the leading primitive is $|g|^3/(3a_0)$ while strict convexity still gives uniqueness. $\square$

G.3 The spherical response

For an isolated spherical baryonic mass with $\rho_\Sigma^{\text{free}} = 0$ and vanishing external field define

$$x = \frac{a_N}{a_0}, \quad y = \frac{a}{a_0}, \quad a_N = \frac{GM_b}{r^2}.$$

Integration of Eq. (G.6) outside the source gives

$$x = y\mu(y) = \frac{y^2}{\sqrt{1+y^2}}.$$

For $x > 0$ the positive solution is

$$y^2 = \frac{x^2 + x\sqrt{x^2 + 4}}{2}, \quad \frac{a}{a_N} = \left[\frac{1 + \sqrt{1 + 4/x^2}}{2} \right]^{1/2}.$$

For $x \gg 1$ it approaches $a/a_N = 1 + 1/(2x^2) + \mathcal{O}(x^{-4})$. In the isolated deep-infrared limit $x \ll 1$

$$a(r) = \frac{\sqrt{GM_b a_0}}{r}, \quad v^4 = GM_b a_0.$$

The mass-asymptotic-velocity relation is the usual isolated deep-MOND limit in Eq. (7) of Ref. [12] with the acceleration scale fixed here by Eq. (G.3).

Fixed-parameter rotation-curve comparison. The SPARC comparison reported in Ref. [15] applies the algebraic acceleration map of Section G.3 pointwise to the tabulated baryonic mass models of Ref. [72]. It fixes the stellar mass-to-light ratios at $\Upsilon_{\text{disk}} = 0.5$ and $\Upsilon_{\text{bulge}} = 0.7$ in solar units at $3.6 \mu\text{m}$ and retains points with $V_{\text{obs}} > 20 \text{ km s}^{-1}$ and $\sigma_V/V_{\text{obs}} < 0.1$ and uses the endpoint a_0 above. For its 2790 retained points the reported standard deviation about the mean of $\log_{10}(g_{\text{obs}}/g_{\text{pred}})$ is 0.144314 dex. The statistic tests the algebraic acceleration relation on the observed mass models at fixed stellar normalization. The nonspherical field equation (G.6) is evaluated separately with its spatial source and boundary data. The comparison data and the three dimensional field calculation are specified in Appendix B.

G.4 The endpoint-source decomposition

Equation (G.6) can be written in Newtonian-source form by defining

$$\rho_{\text{dark,eff}} = \rho_\Sigma^{\text{free}} + \rho_{\text{pol}}, \quad \rho_{\text{pol}} = \frac{1}{4\pi G} \nabla \cdot [(1 - \mu) \nabla \Phi].$$

It follows identically that

$$\nabla^2 \Phi = 4\pi G \left(\rho_b + \rho_\Sigma^{\text{free}} + \rho_{\text{pol}} \right).$$

At each time the transported free density $\rho_\Sigma^{\text{free}}$ and the baryonic density supply the local field source. Solving the nonlinear boundary problem gives Φ and its polarization flux then determines ρ_{pol} . Thus the free density carries the material transport history while the

polarization follows the changing source through its field response. An isolated baryonic source sets $\rho_\Sigma^{\text{free}} = 0$ whereas a region containing cold matter retains the transported free density. The homogeneous cold mean is $\bar{\rho}_\Sigma = \rho_{\text{crit}}\Omega_{\text{cold}}$ at the reference epoch with Ω_{cold} fixed by (8.2). In the dust realization of Section 8 initial data with that mean are transported by (8.6). This fixes $\rho_\Sigma^{\text{free}}$ on the local domain once the flow and the initial density are specified. For a periodic cosmological cell the free-source mean is $\bar{\rho}_\Sigma \propto a^{-3}$. On a bounded physical region Ω the complete source identity is

$$\langle \rho_{\text{dark,eff}} \rangle_\Omega = \langle \rho_\Sigma^{\text{free}} \rangle_\Omega + \frac{1}{4\pi G|\Omega|} \int_{\partial\Omega} (1 - \mu) \nabla \Phi \cdot n \, dS. \quad (\text{G.8})$$

The second term vanishes for a periodic polarization flux or an imposed zero net polarization flux. Otherwise the boundary data determine it. For weak solutions the same statement uses the normal trace whenever the flux admits such a trace. With zero net polarization flux the polarization redistributes the local effective source while preserving the mean free-source budget. On a general bounded region its net contribution is fixed by the boundary flux.

The quasistatic energy on a physical region is

$$E_{\text{qs}}[\Phi] = \frac{1}{4\pi G} \int_\Omega \mathcal{F}(\nabla \Phi) \, d^3x + \int_\Omega (\rho_b + \rho_\Sigma^{\text{free}}) \Phi \, d^3x, \quad (\text{G.9})$$

where $\mathcal{F}$ is the convex primitive in Proposition G.1. Variation at fixed boundary data gives Eq. (G.6). For a background $\bar{\Phi}$ and a fluctuation ϕ its second variation is

$$\delta^2 E_{\text{qs}}[\bar{\Phi}](\phi, \phi) = \frac{1}{4\pi G} \int_\Omega \nabla \phi \cdot \mathbf{M}(\nabla \bar{\Phi}) \nabla \phi \, d^3x.$$

Thus the ambient field fixes the local response tensor. Its transverse and longitudinal eigenvalues are those in Eq. (G.7). For a nonzero ambient field the larger longitudinal coefficient gives a larger flux variation for gradient perturbations along the field than across it. The environment thus supplies a preferred direction in a non-spherical source problem. At a vanishing static field the cubic energy controls the leading response. These coefficients and the source decomposition are evaluated in Ref. [15]. For cosmological applications Section 8.5 fixes the homogeneous expansion and mean cold density. The local problem additionally requires the source flow and ambient field together with boundary data. Its response to that ambient field is the second variation displayed above.

Plane-symmetric cosmological transport. The relation between the transported density and its mean can be evaluated exactly for plane-symmetric Euler–Poisson flow. On a periodic cell set

$$x_1 = q_1 - \frac{AD(t)}{k} \sin(kq_1), \quad x_2 = q_2, \quad x_3 = q_3, \quad |AD| < 1.$$

Then $J = 1 - AD \cos(kq_1)$ and $\rho = \bar{\rho}_\Sigma / J$. In comoving coordinates Poisson’s equation and the characteristic equation are satisfied by

$$\partial_{x_1} \Phi = 4\pi G a^2 \bar{\rho}_\Sigma \frac{AD}{k} \sin(kq_1), \quad \ddot{D} + 2H\dot{D} = 4\pi G \bar{\rho}_\Sigma D.$$

Indeed $\partial_{x_1} = J^{-1} \partial_{q_1}$ gives $\partial_{x_1}^2 \Phi = 4\pi G a^2 (\rho - \bar{\rho}_\Sigma)$. Substitution in $\ddot{x} + 2H\dot{x} = -a^{-2} \nabla_x \Phi$ gives the growth equation. The identity $\rho J = \bar{\rho}_\Sigma$ fixes the cell mean exactly. The numerical characteristic evolution is checked against the Einstein–de Sitter solution $D \propto a$ and stops at $|AD| = 1$ [15]. Here the potential obeys the cosmological Poisson equation for the density contrast. The corresponding local constitutive problem uses the flux in Eq. (G.6).

G.5 Joint source and potential evolution

The quasistatic functional also determines a dynamical coupling to nonrelativistic matter. We use the variational formulation of Ref. [12] with the primitive $\mathcal{F}$ and acceleration normalization obtained above. On the physical time interval under consideration G and a_0 are constant. Let $X_a(t, q)$ be the trajectory map of a pressureless source component from a fixed reference domain Q_a onto the common physical domain $\mathcal{D}$ with reference mass measure $\rho_{a,0}(q) d^3q$. Its density and velocity obey

$$\rho_a(t, X_a(t, q)) \det D_q X_a(t, q) = \rho_{a,0}(q), \quad v_a(t, X_a(t, q)) = \dot{X}_a(t, q).$$

The components may include the free cold source and a pressureless baryonic component. Set $\rho = \sum_a \rho_a$. The joint action is

$$S[X_a, \Phi] = \int dt \left\{ \sum_a \int_{Q_a} \rho_{a,0}(q) \left[\frac{|\dot{X}_a|^2}{2} - \Phi(t, X_a) \right] d^3q - \frac{1}{4\pi G} \int_{\mathcal{D}} \mathcal{F}(\nabla \Phi) d^3x \right\}. \quad (\text{G.10})$$

Proposition G.2 (Source feedback and boundary energy balance). Suppose the trajectory maps are smooth orientation-preserving diffeomorphisms on a time interval. Suppose the densities and potential are sufficiently smooth for the following integrations by parts and Φ has prescribed Dirichlet data on the fixed smooth domain $\mathcal{D}$. Let $\Omega \subseteq \mathcal{D}$ be a fixed smooth region on which the energy is evaluated. Stationarity of (G.10) with fixed trajectory endpoints and compactly supported spatial variations gives

$$\begin{aligned} \partial_t \rho_a + \nabla \cdot (\rho_a v_a) &= 0, \\ (\partial_t + v_a \cdot \nabla) v_a &= -\nabla \Phi, \\ \nabla \cdot \mathcal{J}(\nabla \Phi) &= 4\pi G \rho. \end{aligned} \quad (\text{G.11})$$

Define

$$E(t) = \sum_a \int_{\Omega} \frac{1}{2} \rho_a |v_a|^2 d^3x + \int_{\Omega} \rho \Phi d^3x + \frac{1}{4\pi G} \int_{\Omega} \mathcal{F}(\nabla \Phi) d^3x. \quad (\text{G.12})$$

Then its boundary balance is

$$\frac{dE}{dt} = - \int_{\partial\Omega} \sum_a \rho_a (v_a \cdot n) \left(\frac{|v_a|^2}{2} + \Phi \right) dS + \frac{1}{4\pi G} \int_{\partial\Omega} \partial_t \Phi \mathcal{J}(\nabla \Phi) \cdot n dS. \quad (\text{G.13})$$

In particular for $\Omega = \mathcal{D}$ impermeable matter boundaries and time-independent Dirichlet data conserve E .

Proof. Differentiating the pushforward identity gives continuity. Variation of X_a in the kinetic and source terms gives $\ddot{X}_a = -\nabla\Phi(t, X_a)$. The potential variation uses $\nabla_g\mathcal{F} = \mathcal{J}$ and gives the third equation. With $j = \sum_a \rho_a v_a$ the kinetic balance has volume term $-\int_\Omega j \cdot \nabla\Phi$. The derivative of the source energy has volume terms $\int_\Omega j \cdot \nabla\Phi + \int_\Omega \rho \partial_t\Phi$ and boundary term $-\int_{\partial\Omega} \Phi j \cdot n$. Finally

$$\frac{d}{dt} \frac{1}{4\pi G} \int_\Omega \mathcal{F}(\nabla\Phi) = \frac{1}{4\pi G} \int_{\partial\Omega} \partial_t\Phi \mathcal{J} \cdot n - \int_\Omega \rho \partial_t\Phi.$$

The volume terms cancel to leave (G.13). $\square$

The result applies to smooth evolution before trajectory crossing. At each time the transported density and boundary data determine the unique potential of Proposition G.1. Equation (G.13) combines the work exchanged between matter and the potential with transport across the boundary. The volume exchange cancels, leaving the matter flux and potential work at the boundary. For $\Omega = \mathcal{D}$ and zero Dirichlet data the field equation rewrites the potential part of E as $(4\pi G)^{-1} \int_\Omega [\mathcal{F}(g) - g \cdot \mathcal{J}(g)] d^3x$. Its Newtonian quadratic limit gives the usual negative gravitational potential energy.

Finite material-flow realization. A plane-symmetric variational approximation [15] assigns a fixed mass m_i per unit transverse area to each affine material cell of length ℓ_i . With $\kappa = 4\pi G$ the field flux is $j = C + \kappa M(x)$ where $M(x)$ is cumulative mass and the Dirichlet potential difference fixes C . Define $\mathcal{H}(j) = gj - \mathcal{F}(g)$ with $j = \mathcal{J}(g)$. The relaxed potential energy is

$$U = \frac{b_R j_R - b_L j_L - \sum_i \ell_i \bar{\mathcal{H}}_i}{\kappa},$$

where b_L, b_R are the boundary potentials and the bar denotes the cell average. The consistent kinetic matrix of cell i is $m_i \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} / 6$. Variation at an interior node gives the force $(\bar{\mathcal{H}}_{i-1} - \bar{\mathcal{H}}_i) / \kappa$. The boundary term in (G.13) becomes $(\dot{b}_R j_R - \dot{b}_L j_L) / \kappa$.

For a finite calculation choose units with $\kappa = 1$ on the interval $[0, 1]$ with $a_0 = 0.3$ and $b_L = b_R = 0$. Eight cells start at $x_i = i/8$ with

$$m_i = 0.1 \Delta x_i [1 + 0.2 \cos(2\pi \bar{x}_i) + 0.1 \sin(2\pi \bar{x}_i)], \quad v_i(0) = 0.02 \sin(\pi x_i),$$

where $\bar{x}_i$ is the initial cell midpoint and the end nodes are fixed. The nonlinear field is resolved at each velocity-Verlet step. 24-point Gaussian quadrature evaluates the cell energies after removal of the square-root behavior at zero flux. At $t = 0.4$ halving the time step from 0.02 to 0.01 reduces the maximum energy-balance residual from 3.86×10^{-11} to 9.65×10^{-12} . The final nodal positions differ by 2.66×10^{-7} and all cell lengths exceed 0.1168. The two evolutions require 21 and 41 field solutions respectively. These tests compare time integration at fixed material cell resolution. Independent variations of the relaxed energy check the force; time dependent boundary data test the boundary work.

G.6 Generated matter dynamics and covariant response

G.6.1 Canonical matter evolution under an internal source

The canonical matter action fixes the response to a time-dependent internal source. Let $A = D_B^\dagger D_B + \mathcal{R}_g/8$ be its positive internal operator with the spatial connection B held fixed. The conformal-source identity of the proposition “Local energy charge on the conformal scalar branch” in Ref. [11] gives

$$g_\eta = e^{-2\eta} g, \quad dv_{g_\eta} = e^{-3\eta} dv_g, \quad A_{g_\eta, B} = e^{5\eta/2} A e^{-\eta/2}.$$

Here the conformal transformation acts on the three-dimensional internal space. Pulling the canonical time kinetic term and the internal quadratic form back to $L^2(dv_g)$ gives

$$L_\eta = \frac{1}{2} [\dot{\varphi}^\dagger M_\eta \dot{\varphi} - \varphi^\dagger B_\eta \varphi], \quad M_\eta = e^{-3\eta}, \quad B_\eta = e^{-\eta/2} A e^{-\eta/2}. \quad (\text{G.14})$$

The real quadratic form is understood when complex mode coordinates are used. The temporal kinetic term is that of the canonically normalized matter fields of Ref. [4]. The internal volume and conformal covariance determine its source dependence. A finite calculation uses Hermitian source matrices on a retained carrier and the matrix functions in (G.14). These matrix functions define the finite source family. The retained support of the stated static derivatives is checked separately in Appendix B.

Proposition G.3 (Canonical matter evolution and source work). On a finite carrier let $A > 0$ and let $\eta(t)$ be a continuously differentiable Hermitian source history. With canonical momentum $p = M_\eta \dot{\varphi}$ define

$$z = \begin{pmatrix} \varphi \\ p \end{pmatrix}, \quad \mathbb{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \mathbb{K}_\eta = \begin{pmatrix} B_\eta & 0 \\ 0 & M_\eta^{-1} \end{pmatrix}. \quad (\text{G.15})$$

The evolution $\dot{z} = \mathbb{J} \mathbb{K}_\eta z$ preserves the canonical symplectic form. For a positive semidefinite ensemble second moment $W = \langle z z^\dagger \rangle$ it preserves $W \geq 0$ and satisfies

$$E = \frac{1}{2} \text{Tr}(\mathbb{K}_\eta W) \geq 0, \quad \dot{E} = \frac{1}{2} \text{Tr}(\dot{\mathbb{K}}_\eta W). \quad (\text{G.16})$$

At a fixed source the squared frequencies are the eigenvalues of $e^\eta A e^\eta$. At $\eta = 0$ a stationary occupation $N \geq 0$ with $[N, H] = 0$ and $H = \sqrt{A}$ is represented by

$$W_0 = \begin{pmatrix} H^{-1} N & 0 \\ 0 & H N \end{pmatrix}, \quad E_0 = \text{Tr}(N H). \quad (\text{G.17})$$

For a source direction $P_a = P_a^\dagger$ its initial generalized force is $J_a = \text{Tr}(N H P_a)$ equal to the first variation of the occupied static spectral energy.

Proof. Both M_η and B_η are positive definite. Thus the Hamiltonian $\mathcal{H} = z^\dagger \mathbb{K}_\eta z / 2$ has positive kinetic and potential quadratic forms. If S is the fundamental solution with

$\dot{S} = \mathbb{J}\mathbb{K}_\eta S$ and $S(t_i) = I$ then $S^\dagger \mathbb{J} S = \mathbb{J}$ and $W(t) = S(t)W(t_i)S(t)^\dagger$. This proves positivity and gives

$$\dot{W} = \mathbb{J}\mathbb{K}_\eta W + W(\mathbb{J}\mathbb{K}_\eta)^\dagger.$$

Cyclicity of the trace cancels the two evolution terms in $\dot{E}$ to give (G.16). At a fixed source canonical normalization gives

$$M_\eta^{-1/2} B_\eta M_\eta^{-1/2} = e^\eta A e^\eta,$$

which is the operator in the static energy response. Substitution of (G.17) proves $E_0 = \text{Tr}(NH)$. At zero source

$$\partial_a M_\eta^{-1} = 3P_a, \quad \partial_a B_\eta = -\frac{1}{2}(P_a A + A P_a).$$

Consequently $J_a = \text{Tr}[(\partial_a \mathbb{K})W_0]/2 = \text{Tr}(NHP_a)$. Differentiating $H_\eta^2 = e^\eta A e^\eta$ and using $[N, H] = 0$ gives the same derivative of $\text{Tr}(NH_\eta)$. $\square$

Here the stationary matrix N specifies the occupied mode weights on the retained carrier including their overall normalization in the energy and source force. The angular calculation uses unit trace to isolate the distribution of a fixed total occupation. The phase-space matrix W instead collects coordinate and momentum second moments and their correlations so that the occupied input can be followed as the source changes.

This action fixes the time dependence of canonical normalization. For $R = M_\eta^{1/2}$ with $q = R\varphi$ and $C = \dot{R}R^{-1}$

$$L_\eta = \frac{1}{2} \left[(\dot{q} - Cq)^\dagger (\dot{q} - Cq) - q^\dagger e^\eta A e^\eta q \right]. \quad (\text{G.18})$$

For a spatially uniform twice differentiable source this becomes

$$\ddot{q} + \left[e^{2\eta} A + \frac{3}{2}\ddot{\eta} - \frac{9}{4}\dot{\eta}^2 \right] q = 0.$$

The additional terms follow from the evolving internal volume. The canonical phase-space evolution retains the correlations between coordinates and momenta and between both frequency branches. For a piecewise constant source it is evaluated by the exact finite propagators $\exp(\mathbb{J}\mathbb{K}_j \Delta t_j)$. Energy is conserved within each segment. A source jump contributes $\text{Tr}[(\mathbb{K}_{j+1} - \mathbb{K}_j)W]/2$. This gives a source work check for finite histories of the kind used in Ref. [3].

A clock assigns an increasing readout to successive states of a generated trajectory. In a monotone chart τ_{clk} the clock Jacobian is $\mathcal{N}_{\text{clk}} = dt/d\tau_{\text{clk}} > 0$. Writing primes for derivatives in this chart gives exactly

$$S_\eta = \frac{1}{2} \int d\tau_{\text{clk}} \left[\mathcal{N}_{\text{clk}}^{-1} \varphi'^\dagger M_\eta \varphi' - \mathcal{N}_{\text{clk}} \varphi^\dagger B_\eta \varphi \right], \quad z' = \mathcal{N}_{\text{clk}} \mathbb{J}\mathbb{K}_\eta z. \quad (\text{G.19})$$

The canonical momentum is $p = M_\eta \varphi' / \mathcal{N}_{\text{clk}}$ and the generator in the new chart is $\mathcal{N}_{\text{clk}} \mathcal{H}_\eta$. For a fixed clock chart $\mathcal{N}_{\text{clk}}$ is its prescribed Jacobian. The matter variation holds that Jacobian fixed. A phase readout supplies such a chart on each interval where it is monotone

along the trajectory. The symplectic identity and source-work law are preserved with $E' = \text{Tr}(\mathbb{K}'_\eta W)/2$.

The causal linear response follows from the same phase evolution. Write $K_a = \partial_a \mathbb{K}|_0$ and $K_{ab} = \partial_a \partial_b \mathbb{K}|_0$ and $X_0 = \mathbb{J} \mathbb{K}_0$. For a perturbation proportional to $e^{-i\zeta t}$ with $\text{Im} \zeta > 0$ set

$$\mathcal{L}_0 V = X_0 V + V X_0^\dagger, \quad F_b = \mathbb{J} K_b W_0 + W_0 (\mathbb{J} K_b)^\dagger.$$

Linearization gives

$$\chi_{ab}(\zeta) = \frac{1}{2} \text{Tr}(K_{ab} W_0) + \frac{1}{2} \text{Tr}[K_a (-i\zeta - \mathcal{L}_0)^{-1} F_b]. \quad (\text{G.20})$$

The first term is the direct source variation. Both terms are retained when evaluating the response. For one oscillator of frequency ω with occupation n and a uniform source they give the independent check

$$\chi(\zeta) = 5n\omega + \frac{16n\omega^3}{\zeta^2 - 4\omega^2}, \quad \chi(0) = n\omega.$$

For general source directions the causal response resolves the finite ensemble's evolution as well as the instantaneous spectral variation.

G.6.2 Covariant source exchange and metric constraints

Let $s^a(x)$ be the real coefficients of the internal source $\eta(x) = \sum_a s^a(x) P_a$ on the finite carrier. Use real field coordinates including real and imaginary parts when the original carrier is complex and signature $(-+++)$. The covariant matter action is

$$S_m[g, \varphi, s] = -\frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \varphi^\top M(s) \partial_\nu \varphi + \varphi^\top B(s) \varphi \right], \quad (\text{G.21})$$

with the same M and B as in (G.14). These matrices act on the retained internal mode coefficients. Internal deformation changes their kinetic norms and spectral frequencies. The external metric contracts the spacetime derivatives and variation with respect to it gives the matter stress tensor. Thus internal geometry enters macroscopic propagation through $M(s)$ and $B(s)$ and their common variation determines the exchange with the collective source. The homogeneous flat-space reduction is the canonical action above.

Source work and total stress conservation. Changes of the common source alter both the kinetic form and restoring coefficients of matter. Their energy cost is the work exchanged with that source. The same variation determines its contribution to the matter stress identity. For (G.21) define $\mathcal{E}_\varphi = (\sqrt{-g})^{-1} \delta S_m / \delta \varphi$ and $Q_a = (\sqrt{-g})^{-1} \delta S_m / \delta s^a$. Then

$$T_{\mu\nu}^m = \partial_\mu \varphi^\top M \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} \left[g^{\rho\sigma} \partial_\rho \varphi^\top M \partial_\sigma \varphi + \varphi^\top B \varphi \right],$$

$$Q_a = -\frac{1}{2} \left[g^{\mu\nu} \partial_\mu \varphi^\top M_{,a} \partial_\nu \varphi + \varphi^\top B_{,a} \varphi \right], \quad (\text{G.22})$$

$$\nabla_\mu (T_m)^\mu{}_\nu = \mathcal{E}_\varphi^\top \partial_\nu \varphi + Q_a \partial_\nu s^a. \quad (\text{G.23})$$

This identity holds before imposing the matter equation. On $\mathcal{E}_\varphi = 0$ its right-hand side reduces to $Q_a \partial_\nu s^a$. The cancellation of this exchange also uses the common source equation below. In homogeneous flat space the matter equation gives, per unit external spatial volume, $\dot{E}_m = -Q_a \dot{s}^a = J_a \dot{s}^a$ where $J_a = \partial \mathcal{H}_m / \partial s^a$ at fixed canonical data. Metric and source variation give (G.22) while $\mathcal{E}_\varphi = \nabla_\mu (M \partial^\mu \varphi) - B \varphi$. Differentiating the displayed stress tensor cancels the two derivatives of the kinetic contraction and leaves $\mathcal{E}_\varphi^\top \partial_\nu \varphi + Q_a \partial_\nu s^a$. This is the source-dependent specialization of the common-action Noether identity in Ref. [11]. In the homogeneous limit $T^0_0 = -E_m$ gives the last identity. It agrees with (G.16).

The metric constraints follow from the same action. For lapse $\mathcal{N} > 0$ with shift β^i and spatial metric h_{ij}

$$\begin{aligned} \pi &= \frac{\sqrt{h}}{\mathcal{N}} M (\dot{\varphi} - \beta^i \partial_i \varphi), \\ \mathcal{H}_\perp^m &= \frac{\pi^\top M^{-1} \pi}{2\sqrt{h}} + \frac{\sqrt{h}}{2} \left[h^{ij} \partial_i \varphi^\top M \partial_j \varphi + \varphi^\top B \varphi \right], \quad \mathcal{H}_i^m = \pi^\top \partial_i \varphi. \end{aligned} \quad (\text{G.24})$$

Thus $H_m = \int (\mathcal{N} \mathcal{H}_\perp^m + \beta^i \mathcal{H}_i^m) d^3x$. Positivity of M and B gives a positive matter energy for the normal observer and the principal matter symbol is $Mg^{\mu\nu} k_\mu k_\nu$.

For independent generated source fields with action S_s the common system includes the source equation and both metric constraints

$$\mathcal{E}_a^s + Q_a = 0, \quad \mathcal{H}_\perp^g + \mathcal{H}_\perp^s + \mathcal{H}_\perp^m = 0, \quad \mathcal{H}_i^g + \mathcal{H}_i^s + \mathcal{H}_i^m = 0. \quad (\text{G.25})$$

The first equation cancels the source exchange in (G.23) against that of S_s . The total stress is then conserved on the matter equations and enters the Einstein equation. In particular the homogeneous stationary occupied branch at zero source requires $\langle \mathcal{E}_{\text{uniform}}^s \rangle = \text{Tr}(NH)$. This is the collective source force required by the calculated matter energy. The inherited complete condensate supplies a static radius force through (C.90). Its joint amplitude–radius equilibrium and positive Hessian are given in Proposition C.6. A spacetime realization uses that potential with its physical density normalization and the temporal source terms in the same action.

A generated source can instead be a functional $s = \Gamma[g, q]$ of retained physical fields. For a diffeomorphism-equivariant Γ variation of $S_s[g, q] + S_m[g, \varphi, \Gamma[g, q]]$ gives

$$\mathcal{E}_q = \mathcal{E}_q^s + (D_q \Gamma)^* Q, \quad T_{\mu\nu}^{\text{tot}} = T_{\mu\nu}^s + T_{\mu\nu}^m - 2[(D_{g^{-1}} \Gamma)^* Q]_{\mu\nu}. \quad (\text{G.26})$$

The adjoints include integrations by parts in the covariant volume measure when Γ contains derivatives. Variations have compact support or the stated fixed boundary data. These expressions follow directly from the first variation and retain the metric dependence of the source. Diffeomorphism equivariance then gives the Noether identity for the complete substituted action. On the equations of the retained fields it yields

$$\nabla_\mu (T_{\text{tot}})^\mu{}_\nu = 0. \quad (\text{G.27})$$

The metric dependence of the generated source contributes to the total stress through (G.26), so source feedback and stress conservation follow from the same allowed variation.

G.6.3 Generated variation and finite-window control

Write $\mathcal{P}(g, q, \varphi) = (g, q, \varphi, \Gamma[g, q])$ for the full field map into a parent action S . Its quadratic variation retains both chain-rule terms

$$\delta^2(S \circ \mathcal{P})[u, v] = \delta^2 S[DPu, DPv] + \delta S[D^2\mathcal{P}(u, v)].$$

The second term vanishes when the background is stationary with respect to all parent variables. The spectral constructions and controlled limits of Refs. [2, 3] provide the operations and convergence estimates for representing generated source directions. For the field map $\mathcal{P}$ in (G.26) the native variational theory is

$$S_{\text{gen}} = S \circ \mathcal{P}, \quad \mathcal{E}_{\text{gen}} = (DP)^* \mathcal{E}_S = 0. \quad (\text{G.28})$$

Its constraints and gauge identities together with its evolution follow from this action. At a native stationary background its quadratic action uses both terms of the second-variation chain rule above. An equivariant field map gives the corresponding Noether identity and the associated metric-constraint propagation on the native equations.

When a generated system is realized as an invariant subsystem of a specified parent evolution its phase-space embedding Γ also satisfies

$$C_{\text{parent}}(\Gamma(q)) = 0, \quad F_{\text{parent}}(\Gamma(q)) = D\Gamma(q)F_{\text{gen}}(q). \quad (\text{G.29})$$

Here C_{parent} collects the parent constraints and the second identity establishes preservation of the image by that evolution.

For a Lagrangian depending on first time derivatives and a derivative-free field map the velocity Hessian of the native action before solving its auxiliary equations is

$$K_{\text{pull}} = (DP)^* K_{\text{full}} DP.$$

It includes those fields' kinetic terms and their cross terms. Algebraic auxiliary fields are eliminated from this common quadratic action. If their invertible block is C and their velocity mixing is B_{aux} the reduced velocity block is $A - B_{\text{aux}}C^{-1}B_{\text{aux}}^*$. Linear lapse and shift multipliers impose their constraints before the remaining physical directions are selected and gauge directions are factored out. Denote the resulting kinetic form by K_{phys} . A derivative-dependent map is treated with its full temporal derivative structure and degeneracy conditions. The projection and interaction conditions of Ref. [3] then apply to these actual operations.

Choose a positive definite reference Gram matrix G_{phys} on the retained physical directions. If an approximation $\hat{K}$ to the complete reduced form satisfies

$$\|G_{\text{phys}}^{-1/2}(\hat{K} - K_{\text{phys}})G_{\text{phys}}^{-1/2}\| \leq \varepsilon_K,$$

its least generalized eigenvalue greater than ε_K certifies positive kinetic energy on those physical directions. An eigenvalue smaller than $-\varepsilon_K$ certifies a negative physical direction. The error budget includes the generated map and complete quadratic-action assembly. It also includes mixed-field terms and the constraint solution together with omitted-mode errors when passing to a larger space. Gradient stability and hyperbolicity are evaluated from the full principal symbol of the same reduced action.

G.6.4 Uniform compression, lapse and phase clocks

The spatially uniform source permits an exact comparison with a metric lapse. For prescribed $\eta(t)$ set $q = e^{-\eta}\varphi$ and $P = e^\eta p$. The action and Hamiltonian become

$$\begin{aligned} L_\eta &= \frac{e^{-\eta}}{2} |\dot{q} + \dot{\eta}q|^2 - \frac{e^\eta}{2} q^\top A q, \\ H'_\eta &= \frac{e^\eta}{2} (P^\top P + q^\top A q) - \dot{\eta} P^\top q. \end{aligned} \quad (\text{G.30})$$

The substitution $\varphi = e^\eta q$ gives the first line. The canonical identity $p^\top d\varphi = P^\top dq + P^\top q d\eta$ gives the second. The first term of H'_η is the lapse coupling $\mathcal{N} = e^\eta$ of the reference oscillator. The second is the associated time-dependent dilation coupling. It is fixed by the same transformation.

When η is varied together with q the velocity Hessian of this matter action is

$$\frac{\partial^2 L_\eta}{\partial(\dot{q}, \dot{\eta})^2} = e^{-\eta} \begin{pmatrix} I & q \\ q^\top & q^\top q \end{pmatrix}, \quad K_{\text{vel}} \begin{pmatrix} -q \\ 1 \end{pmatrix} = 0. \quad (\text{G.31})$$

Its rank equals the number of real matter coordinates and its canonical constraint is $P_\eta - q^\top P = 0$. The collective source action and the metric constraints in (G.25) determine the remaining physical evolution. Equations (G.26) and (G.30) specify the source-force and dilation terms to retain when deriving a common metric realization.

Proposition G.4 (Congruence of a generated matter phase). Retain a degenerate pair of real scalar coordinates as one complex mode of reference squared mass $m_0^2 > 0$. For a source uniform on the internal carrier write $\varphi = r e^{i\vartheta}$ with $r > 0$ and $M = e^{-3\eta}$ and $m^2 = m_0^2 e^{2\eta}$. Its covariant action is

$$S_m = -\frac{1}{2} \int \sqrt{-g} M [(\nabla r)^2 + r^2 (\nabla \vartheta)^2 + m^2 r^2] d^4 x. \quad (\text{G.32})$$

Define the radial derivative term $P = \nabla_\mu (M \nabla^\mu r) / (Mr)$ and assume $X = -(\nabla \vartheta)^2 > 0$. On the radial and phase equations

$$X = m^2 - P, \quad n_\mu = -\frac{\partial_\mu \vartheta}{\sqrt{X}}, \quad \nabla_\mu (Mr^2 \sqrt{X} n^\mu) = 0. \quad (\text{G.33})$$

With $h_\mu{}^\nu = \delta_\mu{}^\nu + n_\mu n^\nu$ and $\epsilon = P/m^2$ the acceleration is

$$A_\mu = n^\nu \nabla_\nu n_\mu = -h_\mu{}^\nu \partial_\nu \eta + \frac{h_\mu{}^\nu \partial_\nu \epsilon}{2(1-\epsilon)}. \quad (\text{G.34})$$

The exact stress and compression source decompose as

$$\begin{aligned} T_{\mu\nu} &= \rho n_\mu n_\nu + \Pi_{\mu\nu}, & \rho &= Mr^2 X, \\ \Pi_{\mu\nu} &= M \partial_\mu r \partial_\nu r - \frac{1}{2} g_{\mu\nu} M [(\nabla r)^2 + r^2 P], & Q_\eta &= -\rho + \frac{M}{2} [3(\nabla r)^2 + r^2 P]. \end{aligned} \quad (\text{G.35})$$

Along every future-directed integral curve the phase reads proper time according to

$$\frac{d\vartheta}{ds} = \sqrt{X}, \quad ds = \frac{d\vartheta}{\sqrt{X}}. \quad (\text{G.36})$$

On a contractible spacetime patch a unit scalar clock $dT = X^{-1/2}d\vartheta$ exists precisely when

$$d(X^{-1/2}d\vartheta) = 0 \iff A_\mu = 0. \quad (\text{G.37})$$

Proof. Substitution into the original quadratic action gives (G.32). Its amplitude and phase variations give (G.33). The exact one-form $\sqrt{X}n_\mu = -\partial_\mu\vartheta$ implies $A_\mu = -h_\mu{}^\nu\partial_\nu\log\sqrt{X}$. Using $X = m^2(1-\epsilon)$ proves (G.34). Substituting the polar fields into (G.22) and applying the radial equation gives (G.35). The definition of n gives $n^\mu\partial_\mu\vartheta = \sqrt{X}$. This proves (G.36). The one-form $X^{-1/2}d\vartheta = -n_\mu dx^\mu$ is closed exactly when the component of dX orthogonal to $d\vartheta$ vanishes. By (G.34) this is $A_\mu = 0$. The Poincaré lemma supplies T on the stated patch and its unit norm follows from the definition of X . $\square$

The variable-mass cold-source approximation is controlled by the radial derivative terms. If $|\epsilon| \leq \epsilon_0 < 1$ and $|\partial_{\hat{a}}r|/(mr) \leq \delta$ in an orthonormal frame then with $\rho_0 = Mr^2m^2$

$$\frac{|\rho - \rho_0|}{\rho_0} \leq \epsilon_0, \quad \frac{|\Pi_{\hat{a}\hat{b}}|}{\rho_0} \leq 3\delta^2 + \frac{\epsilon_0}{2}, \quad \frac{|Q_\eta + \rho|}{\rho_0} \leq 6\delta^2 + \frac{\epsilon_0}{2}.$$

The acceleration remainder is bounded separately by $|h\nabla\epsilon|/[2(1-\epsilon_0)]$. For homogeneous amplitude and compression with a homogeneous phase in an FLRW geometry the congruence is comoving and geodesic. A constant compression and controlled radial derivative terms give the usual cold-source limit. For a static rest phase $\vartheta = \omega t$ $m^2 - P = \omega^2/\mathcal{N}^2$. The phase-dominated limit gives $\eta + \log\mathcal{N} = \log(\omega/m_0)$ and $A_i = \partial_i\log\mathcal{N} = -\partial_i\eta$. Thus the same matter carrier supplies homogeneous and accelerated congruences with their source exchange. These relations use the amplitude and phase equations of (G.32). A coupled gravitational realization obtains the corresponding matter and source equations together with the metric equations by varying its complete action. The metric provides the common macroscopic representation of this relation between phase and proper time together with the source response.

G.6.5 A closed realization with an auxiliary collective radius

In a lowest-derivative realization of the inherited relative response the amplitude and matter pair are dynamical while the collective radius obeys its local stationary equation. Let $\mu > 0$ be the fixed response mass unit with $e = \mu E$ the dimension-one amplitude and $L_{\text{ref}} = \sqrt{2}L_C$. The physical density and radius are

$$U(e, \eta) = \mu^4 [F_{L_{\text{ref}}e^{-\eta}}((e/\mu)^2) - F_*] + \rho_\Lambda, \quad L = L_{\text{ref}}e^{-\eta}, \quad (\text{G.38})$$

where F_* is the joint minimum in Proposition C.6 and $\rho_\Lambda \geq 0$. The subtraction is independent of the fields and uses the common vacuum normalization of Ref. [11]. The constants

$g > 0$ and $Z_E > 0$ retain the amplitude convention of the complete condensate. For a retained degenerate pair with reference squared mass $m_0^2 > 0$ put

$$M = e^{-3\eta}, \quad B = m_0^2 e^{-\eta}, \quad \mathbf{X} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^\top \partial_\nu \varphi, \quad \varphi = (\varphi_1, \varphi_2)^\top.$$

The pair can be taken from two real component directions of the existing scalar doublet carrier with the constant internal harmonic and vanishing fixed internal connection. This invariant internal eigenspace has $A = m_0^2 I$ and $m_0^2 = 3\mu^2/(4L_{\text{ref}}^2)$. The retained macroscopic fields are $g_{\mu\nu}$ and e together with φ_1 and φ_2 and the auxiliary radius. Their action is

$$S_{\text{ar}} = \int \sqrt{-g} d^4x \left[\frac{M_{\text{P}}^2}{2} R - \frac{Z_E}{2} (\nabla e)^2 + M\mathbf{X} - \frac{B}{2} |\varphi|^2 - U(e, \eta) \right]. \quad (\text{G.39})$$

The macroscopic Einstein term has the inherited positive normalization. The radius has an algebraic equation while e and φ carry the temporal response. All variations including metric variation are taken in this one action. In this initial-value construction H_{ini} denotes the expansion rate at $t = 0$. The Hamiltonian constraint fixes it from the chosen occupied state and its physical density normalization.

Proposition G.5 (Occupied common background and physical propagation). For the complete unweighted conformal potential the action (G.39) has a nonempty family of expanding spatially flat FLRW solutions with nonzero matter phase charge. There is a time interval on which the radius is a smooth generated function of the retained fields and their first derivatives. The Einstein constraints propagate and the three physical scalar kinetic coefficients and squared characteristic speeds are positive. The two tensor polarizations have the Einstein kinetic normalization and unit characteristic speed.

Proof. Eliminating the radius requires control of its stationary equation both at fixed field derivatives and at fixed canonical data. These derivatives also determine the radius contribution to the physical kinetic form. The radius equation and its derivative at fixed matter derivatives are

$$\begin{aligned} \mathcal{C} &= U_{,\eta} + 3M\mathbf{X} - \frac{B}{2} |\varphi|^2 = 0, \\ \mathfrak{h} &= \mathcal{C}_{,\eta} = U_{,\eta\eta} - 9M\mathbf{X} + \frac{B}{2} |\varphi|^2. \end{aligned} \quad (\text{G.40})$$

Thus the source is determined together with the matter and metric. Where $\mathfrak{h} > 0$ the implicit-function theorem gives $\eta = \Gamma[g, e, \varphi, \nabla\varphi]$. The substituted action retains first derivatives only. Its source exchange cancels against the e, η sector by the common-action identity (G.23).

To construct an occupied background choose

$$\sqrt{2}L_c < L_0 \leq 2L_c, \quad z_0 = L_0^{-2}, \quad z_c = L_c^{-2}, \quad \eta_0 = -\log(L_0/L_{\text{ref}}). \quad (\text{G.41})$$

Let $y_0 > 0$ be the unique amplitude stationary point at L_0 and set $u_0 = gL_0^2 y_0$ and $e_0 = \mu\sqrt{y_0}$. Its equation is $gL_0^2 I'(u_0) = 8(z_c - z_0)$. The required occupied source density is

$$\mathcal{A}_0 = -U_{,\eta}(e_0, \eta_0) = 16\mu^4 y_0 (z_c - 2z_0) > 0. \quad (\text{G.42})$$

At proper time $t = 0$ take

$$\begin{aligned} a &= 1, \quad \dot{e} = 0, \quad \varphi = (r_0, 0), \quad \dot{\varphi} = (0, \omega_0 r_0), \\ r_0^2 &= \frac{\mathcal{A}_0}{B_0}, \quad \omega_0^2 = \frac{B_0}{M_0}, \quad H_{\text{ini}}^2 = \frac{U(e_0, \eta_0) + \mathcal{A}_0}{3M_{\text{P}}^2}. \end{aligned} \quad (\text{G.43})$$

with $H_{\text{ini}} > 0$. The initial matter kinetic and potential densities are both $\mathcal{A}_0/2$. Thus $\mathcal{C} = 0$ exactly. The amplitude force also vanishes initially. Since $U \geq \rho_\Lambda \geq 0$ these data satisfy the Hamiltonian constraint with positive H_{ini} and homogeneity satisfies the momentum constraint. Direct differentiation of the same potential gives the strict margin

$$\mathfrak{h}_0 = \mu^4 [32y_0(4z_0 - z_c) + 4u_0^2 I''(u_0)] > 0. \quad (\text{G.44})$$

Indeed $z_c/4 \leq z_0 < z_c/2$ and $I''(u_0) > 0$. The concrete choice $L_0^2 = 3L_c^2$ has $\mathfrak{h}_0 = 2\mathcal{A}_0 + 4\mu^4 u_0^2 I''(u_0) > 2\mathcal{A}_0$.

With comoving momenta $p_e = a^3 Z_E \dot{e}$ and $p = a^3 M \dot{\varphi}$ the common background satisfies the closed system

$$\begin{aligned} \dot{e} &= \frac{p_e}{a^3 Z_E}, & \dot{p}_e &= -a^3 U_{,e}, \\ \dot{\varphi} &= \frac{p}{a^3 M}, & \dot{p} &= -a^3 B \varphi, \\ \dot{a} &= aH, & \dot{H} &= -\frac{Z_E \dot{e}^2 + M |\dot{\varphi}|^2}{2M_{\text{P}}^2}, \\ 0 &= U_{,\eta} + \frac{3|p|^2}{2a^6 M} - \frac{B}{2} |\varphi|^2. \end{aligned} \quad (\text{G.45})$$

At fixed canonical data the last equation has derivative

$$\mathfrak{h}_H = U_{,\eta\eta} + 9\mathcal{T} + \mathcal{V} = \mathfrak{h} + 18\mathcal{T} > 0, \quad \mathcal{T} = \frac{|p|^2}{2a^6 M}, \quad \mathcal{V} = \frac{B}{2} |\varphi|^2. \quad (\text{G.46})$$

For fixed $g > 0$ and canonical data with $a > 0$ and $y = (e/\mu)^2 > 0$, the radius equation has exactly one root $\eta \in \mathbb{R}$. Put $z = L^{-2} = L_{\text{ref}}^{-2} e^{2\eta}$ and $u = gy/z$. The complete potential in Proposition C.6 gives

$$\mu^{-4} U_{,\eta} = 16zy - 2uI'(u), \quad \mu^{-4} U_{,\eta\eta} = 32zy + 4uI'(u) + 4u^2 I''(u) > 0.$$

Writing $Q_2 = \sum_j \beta_j / \gamma_j^2 < \infty$, its spectral sum obeys

$$0 \leq I'(u) \leq Q_2 u, \quad I'(u) \geq \frac{\beta_{j_0}}{2\gamma_{j_0}} \quad \text{for } u \geq \gamma_{j_0},$$

where any term with $\beta_{j_0} > 0$ may be used. At fixed canonical data $\mathcal{T}$ is proportional to $e^{3\eta}$ and $\mathcal{V}$ to $e^{-\eta}$, with nonnegative coefficients. Thus $\mathcal{C}_H := U_{,\eta} + 3\mathcal{T} - \mathcal{V}$ is strictly increasing, tends to $-\infty$ as $\eta \rightarrow -\infty$ and to $+\infty$ as $\eta \rightarrow +\infty$. The implicit-function theorem gives a smooth radius function on this domain of canonical data. Substituting it into (G.45) gives a smooth finite-dimensional system and hence a unique local solution. The positivity of $\mathfrak{h}_H$ makes the canonical elimination regular; $\mathfrak{h}$ enters the physical kinetic form. For the

occupied initial data, $H > 0$ and $\mathfrak{h} > 0$ persist on a nonzero interval by continuity. The energy density satisfies

$$\rho = \frac{Z_E}{2}\dot{e}^2 + \mathcal{T} + \mathcal{V} + U, \quad \dot{\rho} = -3H(Z_E\dot{e}^2 + 2\mathcal{T}).$$

Consequently $d(3M_{\text{p}}^2 H^2 - \rho)/dt = 0$. This proves propagation of the background Hamiltonian constraint and the common source balance. At the occupied initial state differentiation of the source equation also gives its strictly positive initial compression rate

$$\dot{\eta}_0 = \frac{9H_{\text{ini}}\mathcal{A}_0}{\mathfrak{h}_0 + 9\mathcal{A}_0} > 0, \quad \dot{L}_0 = -L_0\dot{\eta}_0. \quad (\text{G.47})$$

Here $\dot{e}_0 = 0$ and $\varphi_0^\top \dot{\varphi}_0 = p_0^\top \dot{p}_0 = 0$ and so the explicit scale-factor derivative of $3\mathcal{T}$ is $-18H_{\text{ini}}\mathcal{T}_0$. Division by $\mathfrak{h}_H$ proves the formula.

The physical quadratic action follows by eliminating the auxiliary radius and metric constraints. The radius momentum is zero. Its preservation gives the algebraic source constraint. Their Poisson bracket is nonzero by (G.46). Thus this pair removes the radius phase-space pair. Around the homogeneous solution put $q = (e, \varphi_1, \varphi_2)$ and $v = \dot{\varphi}$. At fixed lapse the terms involving the radius perturbation and the field velocities are

$$\frac{1}{2}\dot{Q}^\top \mathcal{G} \dot{Q} + \delta\eta b^\top \dot{Q} - \frac{\mathfrak{h}}{2}(\delta\eta)^2, \quad \mathcal{G} = \text{diag}(Z_E, M, M), \quad b = (0, -3Mv_1, -3Mv_2)^\top,$$

up to terms with at most one derivative on the physical perturbations. Eliminating $\delta\eta$ gives the positive rank-one contribution $bb^\top/\mathfrak{h}$. There is no analogous spatial-gradient term on a homogeneous background.

For nonzero spatial Fourier modes use spatially flat gauge. The linear momentum constraint determines the lapse perturbation as

$$\delta\mathcal{N} = \frac{Z_E\dot{e}Q_e + Mv^\top Q_\varphi}{2M_{\text{p}}^2 H}. \quad (\text{G.48})$$

It contains no $\dot{Q}$. The scalar shift multiplies this constraint in the quadratic action. Its terms cancel after substitution. The remaining lapse terms affect the zero- and one-derivative coefficients. Including both matter and Einstein variations gives the complete scalar principal action

$$S_{\text{sc}}^{(2)} = \frac{1}{2} \int a^3 dt d^3x \left[\dot{Q}^\top \mathcal{K} \dot{Q} - \frac{1}{a^2} \partial_i Q^\top \mathcal{G} \partial_i Q + \text{lower-derivative terms} \right], \quad (\text{G.49})$$

$$\mathcal{K} = \mathcal{G} + \frac{bb^\top}{\mathfrak{h}}.$$

This agrees with the general constrained scalar-field action in Ref. [73, Eqs. (41), (45)]. Here the lapse constraint and the auxiliary Schur complement have been evaluated for the actual response coefficients. Both $\mathcal{K}$ and $\mathcal{G}$ are positive definite. The generalized characteristic eigenvalues are

$$c_1^2 = c_2^2 = 1, \quad c_\parallel^2 = \frac{\mathfrak{h}}{\mathfrak{h} + 9M|v|^2} > 0. \quad (\text{G.50})$$

For $L_0^2 = 3L_c^2$ the initial nontrivial speed obeys $c_{\parallel}^2 > 2/11$. The tensor action is $M_P^2 \int a^3 [\dot{\gamma}_{ij}^2 - a^{-2}(\partial_k \gamma_{ij})^2] dt d^3x/8$ for transverse traceless γ_{ij} . These are all the propagating principal blocks after the common constraints have been removed. $\square$

A certified interval of common evolution. Set $L_c^2 = \pi$ and $g = 1$ with $Z_E = 2$ together with $m_0^2/\mu^2 = 3/(8\pi)$ and $L_0^2 = 3\pi$ in Proposition G.5. For $P = M_P/\mu \geq 1$ and $0 \leq \rho_\Lambda/\mu^4 \leq 10^{-3}$ its occupied solution exists throughout $0 \leq \mu t \leq 10^{-2}$. On this interval

$$\frac{\mathfrak{h}}{\mu^4} > 0.04769800, \quad c_{\parallel}^2 > 0.19180880, \quad 0.23578722 \leq \frac{d\vartheta}{d(\mu t)} \leq 0.34430776. \quad (\text{G.51})$$

The proof bounds the vector field and the algebraic source root on a common phase-space box including the infinite spectral tail. Use dimensionless fields $E = e/\mu$ and $\phi = \varphi/\mu$ with momenta $\hat{p}_e = p_e/\mu^2$ and $\hat{p} = p/\mu^2$ together with $w = PH/\mu$ and $\alpha = e^\eta$. Dots in the following table denote derivatives with respect to μt . Outward interval evaluation gives the box and initial enclosures together with absolute derivative bounds below. All bounds are uniform for $P \geq 1$.

Coordinate	Box	Initial enclosure	$ \dot{x} $ bound
a	[0.99, 1.01]	1	0.12120000
E	[0.101, 0.105]	[0.10249012, 0.10352021]	0.00206123
$\hat{p}_e$	[-0.004, 0.004]	0	0.11170501
ϕ_1	[0.347, 0.354]	[0.34925525, 0.35276545]	0.00189878
ϕ_2	[-0.004, 0.004]	0	0.11709085
$\hat{p}_1$	[-0.003, 0.003]	0	0.05510890
$\hat{p}_2$	[0.178, 0.185]	[0.18099847, 0.18281760]	0.00062270
w	[0.06, 0.12]	[0.07709848, 0.10895794]	0.01116963

On this box the canonical source equation $\mathcal{C}_H := U_\eta + 3\mathcal{T} - \mathcal{V} = 0$ has

$$\frac{\mathcal{C}_H(\alpha = 0.79)}{\mu^4} < -0.00264330, \quad \frac{\mathcal{C}_H(\alpha = 0.85)}{\mu^4} > 0.00401790.$$

Its positive derivative $\partial_\alpha \mathcal{C}_H = \mathfrak{h}_H/\alpha$ gives a unique source in $0.79 < \alpha < 0.85$. The spectral evaluations retain $j = 0, \dots, N$ with $N = 64$. In the notation of (C.85) they use

$$0 \leq \mathcal{H}(u) - \mathcal{H}_N(u) \leq r_N u, \quad 0 \leq \mathcal{H}'(u) - \mathcal{H}'_N(u) \leq r_N, \quad r_N = \frac{8}{9(2N+1)}.$$

These bounds follow from $\gamma_j \geq 3(2j+1)^2/4$ and the integral bound on $\sum_{j>N} (2j+1)^{-2}$. Monotonicity of $\mathcal{H}$ and decrease of $\mathcal{H}'$ enclose the full response on each argument interval. Eight distinct spectral endpoints require 520 shell evaluations with the response and its derivative evaluated together. They yield the displayed source brackets and (G.51) on the entire box [15]. The initial roots obey $u_0 \in [0.099, 0.101]$ and $u_* \in [0.10, 0.12]$. The initial energy enclosure uses the joint-minimum inequality $I(u_*) \geq 0$ and $I(u_0) \leq \mathcal{H}'(0)u_0^2/2$ where $\mathcal{H}'(0) = \pi(\pi+2)/8$.

For each coordinate the derivative bound multiplied by 10^{-2} is less than half the distance from its initial enclosure to the box boundary. Integration of these bounds excludes

a first exit before $\mu t = 10^{-2}$. The source root and its derivatives remain regular there. Thus the ordinary differential equations continue to that time. This proves the interval claim; (G.49) and (G.52) give its physical kinetic and propagation bounds and its clock bounds.

The radius map has a finite-record construction on every compact neighborhood with $0 < h_- \leq \mathfrak{h} \leq h_+$. For fixed retained data and a bracket containing its root the iteration

$$\eta_{n+1} = \eta_n - \frac{\mathcal{C}(\eta_n)}{h_+}$$

is a contraction with factor at most $1 - h_-/h_+$. It gives a geometric error bound for the same source map Γ . Its domain and limit are recorded together as in Refs. [2, 3].

Conserved phase charge and the matter clock. In this auxiliary radius realization the common action depends on rotational invariants of the retained real pair. Its global phase rotation gives the conserved charge

$$\mathcal{Q} = a^3 M(\varphi_1 \dot{\varphi}_2 - \varphi_2 \dot{\varphi}_1) = a^3 M r^2 \dot{\vartheta} > 0. \quad (\text{G.52})$$

The homogeneous matter equations are $\partial_t(a^3 M \dot{\varphi}_i) + a^3 B \varphi_i = 0$. Multiplying the second by φ_1 and the first by φ_2 and subtracting gives $\dot{\mathcal{Q}} = 0$. The initial data above have $\mathcal{Q} > 0$. Thus the phase remains a monotone clock on the regular solution interval. Its rate $\dot{\vartheta} = \mathcal{Q}/(a^3 M r^2)$ is determined by the conserved charge, physical volume, compression and matter amplitude. The evolution parameter orders the states; the matter phase supplies the clock reading. Its proper time readout is $ds = d\vartheta/\dot{\vartheta} = dt$ in the unit-lapse chart. Other positive clock charts give the same trajectory and rates by (G.19). This realization determines the matter history, occupied collective radius, expanding metric and phase clock readout. Its positive kinetic and characteristic forms hold on the interval where $H > 0$ and $\mathfrak{h} > 0$. The density normalization and retained fields and algebraic radius equation in (G.39) specify this dynamical branch. The production TT endpoint is determined by the sourced trace defined in Section 4.4.

G.6.6 An effective realization with an independent foliation

An independent foliation field χ gives a separate covariant realization of the local constitutive law in (G.10). It specifies a macroscopic congruence. The matter clock T describes the pressureless source trajectories. The auxiliary-radius dynamics retain the field content of (G.39) and the distance calculation uses the Einstein-matter thermal background of Section 8.5. For the foliation realization use signature $(-+++)$ with $c = 1$ and $M^2 = M_{\text{P}}^2 = (8\pi G)^{-1}$ and retain the constant endpoint normalization a_0 of (G.3). For a timelike gradient of χ define

$$n_\mu = -\frac{\partial_\mu \chi}{\sqrt{-g^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi}}, \quad A_\mu = n^\nu \nabla_\nu n_\mu, \quad Y = A_\mu A^\mu. \quad (\text{G.53})$$

The function in the action is fixed by the same primitive $\mathcal{F}$ as the local energy

$$\mathcal{A}(Y) = 2\mathcal{F}(\sqrt{Y}) - Y = a_0^2 \left[y \sqrt{1 + y^2} - \text{arsinh } y - y^2 \right], \quad y = \frac{\sqrt{Y}}{a_0}, \quad (\text{G.54})$$

$$S = \frac{M^2}{2} \int d^4 x \sqrt{-g} [R - 2\Lambda - 2\mathcal{A}(Y)] + S_{\text{m}}. \quad (\text{G.55})$$

Here $\mathcal{F}(\sqrt{Y})$ denotes its radial value and S_m contains the canonical radial field and pressureless source together with any visible matter under consideration. The preferred-foliation action and its covariant variation follow the construction of Ref. [13]. Cosmological clock dynamics in this class of theories are also discussed in Ref. [74]. Equation (G.54) specifies the full acceleration function from the present local response. The independent foliation and its initial and boundary data are the effective field content of this realization. The static construction determines $\mathcal{F}$ and its spatial response. The constrained dynamics of this field content provide a comparison for generated variations retaining the same foliation sector.

Proposition G.6 (Local and cosmological limits of the foliation action). The action (G.55) has the following properties.

(i) Every smooth Einstein–matter solution admitting a smooth geodesic foliation is a solution of the enlarged system with that foliation. In a static metric with vanishing shift the choice $\chi = t$ satisfies the clock equation and gives (G.6) in the leading weak-field nonrelativistic limit.

(ii) Consider a spatially flat expanding or contracting dust–vacuum background with

$$3M^2H^2 = \rho + \rho_\Lambda, \quad -2M^2\dot{H} = \rho, \quad \dot{\rho} = -3H\rho,$$

where $\rho > 0$ and $H \neq 0$, ρ_Λ is constant and the radial field is at a stationary condensate with $m_\phi^2 > 0$. For each nonzero Fourier mode $\kappa = |k|/a$ the constrained quadratic scalar action consists of the Einstein radial–cold-source action and one additional mode with positive kinetic coefficient and conserved conjugate momentum. The effective cold density perturbation obeys the Einstein cold-source equations for every value of this momentum. The tensor quadratic action is the Einstein tensor action.

Proof. The identities $\mathcal{A}(0) = 0$ and

$$\mathcal{A}_Y = \mu(\sqrt{Y}/a_0) - 1, \quad \frac{\partial \mathcal{A}}{\partial A_\mu} = 2\mathcal{A}_Y A^\mu$$

show that the added action and its first variation vanish when $A_\mu = 0$. The Einstein–matter equations and the clock equation therefore hold on every smooth geodesic-foliation branch. This is a local statement on a domain where the foliation remains smooth.

With $K_{\mu\nu} = (g_\mu^\alpha + n_\mu n^\alpha)\nabla_\alpha n_\nu$ the clock equation can be written as

$$\nabla_\mu [n^\mu \nabla_\nu (\mathcal{A}_Y A^\nu)] = \frac{1}{2} n^\mu \nabla_\mu \mathcal{A} + \mathcal{A}_Y A^\mu A^\nu K_{\mu\nu}. \quad (\text{G.56})$$

In the static foliation $A_i = D_i \log N$ and $K_{ij} = 0$. All time derivatives in (G.56) vanish. With $N = 1 + \Phi$ at leading order the gravitational and source Lagrangian is

$$-M^2 [|\nabla\Phi|^2 + \mathcal{A}(|\nabla\Phi|^2)] - \rho\Phi = -\frac{\mathcal{F}(\nabla\Phi)}{4\pi G} - \rho\Phi.$$

Its variation is (G.6). The independent matter variation gives the nonrelativistic geodesic equation appearing in (G.11).

For the cosmological background choose $\chi = t$ and $T = t$. Both congruences are geodesic. Thus the background equations above follow directly. In spatially flat gauge write $N = 1 + n$ and $T = t + \pi$ and $\chi = t + \sigma$. To first order $A_i = \partial_i(n - \dot{\sigma})$. Consequently the added quadratic Lagrangian divided by a^3 is $C(\dot{\sigma} - n)^2/2$ where $C = 2M^2\kappa^2$. It contains no linear shift perturbation. The shift constraint of the Einstein radial-cold-source action therefore remains

$$n = B\pi, \quad B = \frac{\rho}{2M^2H}.$$

Eliminating the dust auxiliary variable gives the first-order scalar action apart from the canonical radial fluctuation Q_r

$$S_c^{(2)} = \int dt a^3 \left[p(\dot{\pi} - B\pi) - \frac{D}{2}\pi^2 + \frac{C}{2}(\dot{\sigma} - B\pi)^2 \right], \quad D = \rho\kappa^2 + \frac{3\rho^2}{2M^2} > 0. \quad (\text{G.57})$$

The Einstein radial and dust constraints used here are those of the common normalized matter action [11]. Let $P = a^3p$ and $Q_\chi = a^3C(\dot{\sigma} - B\pi)$. Since σ is cyclic $\dot{Q}_\chi = 0$. The Hamiltonian and canonical one-form are

$$H_c^{(2)} = B(P + Q_\chi)\pi + \frac{a^3D}{2}\pi^2 + \frac{Q_\chi^2}{2a^3C}, \quad (\text{G.58})$$

$$P d\pi + Q_\chi d\sigma = R d\pi + Q_\chi d\zeta_\chi, \quad R = P + Q_\chi, \quad \zeta_\chi = \sigma - \pi. \quad (\text{G.59})$$

This transformation is independent of time. Set $r = R/a^3$. Integration by parts in $R\dot{\pi}$ and elimination of π give

$$S_{\text{scal}}^{(2)} = \int dt a^3 \left\{ \frac{1}{2}\dot{Q}_r^2 - \frac{1}{2}(\kappa^2 + m_\delta^2)Q_r^2 + \frac{[\dot{r} + (3H + B)r]^2}{2D} + \frac{C}{2}\dot{\zeta}_\chi^2 \right\}. \quad (\text{G.60})$$

The kinetic coefficients $1, D^{-1}, C$ are all positive on the stated domain. Lapse variation identifies the foliation density as $\delta\rho_\chi = Q_\chi/a^3$. Its linear momentum density and spatial stress vanish. Thus $r = p + Q_\chi/a^3$ is the effective cold density perturbation. Its equation and its metric constraints are the Einstein cold-source equations. Matching the initial r and velocity therefore matches the cosmological scalar metric response. The branch $Q_\chi = 0$ is invariant. The extra equation is

$$a^3C\dot{\zeta}_\chi = Q_\chi, \quad \zeta_\chi(t) = \zeta_\chi(t_i) + \frac{Q_\chi}{2M^2|k|^2} \int_{t_i}^t \frac{dt'}{a(t')}.$$

Finally A_i has only a scalar perturbation at first order. The added term contributes no tensor quadratic term. $\square$

The additional scalar has zero spatial restoring term at quadratic order and hence $c_s^2 = 0$. Its conserved momentum partitions the total cold density between the matter and foliation sectors. The usual gravitational growth of the cold density is retained. This result establishes the kinetic signs and linear evolution on the specified dust-vacuum background.

The static and cosmological reductions use different macroscopic foliations with their respective constraints eliminated. In the static sector with bounded $\nabla\phi$

$$\mathcal{J}(\varepsilon\nabla\phi) = \frac{\varepsilon|\varepsilon|}{a_0}|\nabla\phi|\nabla\phi + O(|\varepsilon|^4), \quad D\mathcal{J}(0) = 0.$$

This quadratic leading flux is the static reduction of (G.55). The cosmological reduction retains the independent clock fluctuation and gives (G.60). These are background-specific reductions of the nonlinear field and foliation equations with the initial and boundary data chosen in each branch.

G.6.7 The independent foliation on an accelerated background

The stationary branch of the independent-foliation action permits an explicit reduction of its scalar velocity form. We use the leading slow-motion action of Ref. [14] with the matter configuration held fixed when taking this velocity Hessian. Put $\chi = t + \sigma$ and $\Phi = \bar{\Phi} + \phi$. The acceleration invariant depends at this order on $\Xi = \Phi - \dot{\sigma} + \frac{1}{2}|\nabla\sigma|^2$. On a static background define

$$H(x) = \mu(y)I + y\mu'(y)e \otimes e, \quad y = \frac{|\nabla\bar{\Phi}|}{a_0}, \quad e = \frac{\nabla\bar{\Phi}}{|\nabla\bar{\Phi}|}. \quad (\text{G.61})$$

Proposition G.7 (Reduced velocity form of the independent foliation). Let Ω be a bounded Lipschitz domain with fixed Dirichlet perturbations and suppose H is bounded and uniformly positive on Ω . Write $D = -\Delta$ and $L = -\nabla \cdot H \nabla$ as maps from $H_0^1(\Omega)$ to its dual. Equip $H_0^1(\Omega)$ with the gradient norm. Eliminating the potential gives the scalar velocity operator

$$K_{\text{red}} = 2M^2(DL^{-1}D - D). \quad (\text{G.62})$$

For variations in this reduced foliation sector with the other physical variables fixed and $\sigma = Bq$ the induced velocity form is $K_{\text{gen}} = B^*K_{\text{red}}B$. A finite set of independent directions $b_i \in H_0^1(\Omega)$ therefore has matrix

$$(K_{\text{gen}})_{ij} = 2M^2 \{ \langle Db_i, L^{-1}Db_j \rangle - \langle b_i, Db_j \rangle \}. \quad (\text{G.63})$$

These pairings use the Sobolev duality associated with the Dirichlet forms. In particular $0 < H \leq I$ implies $K_{\text{red}} \geq 0$.

Proof. The part of the quadratic Lagrangian containing the potential and foliation velocity is

$$-M^2 [\langle \phi, D\phi \rangle + \langle \phi - \dot{\sigma}, (L - D)(\phi - \dot{\sigma}) \rangle].$$

The remaining terms contain no squared velocities of σ . Potential variation gives $L\phi = (L - D)\dot{\sigma}$. Uniform ellipticity supplies its unique Dirichlet solution. Substitution gives $M^2\langle \dot{\sigma}, (DL^{-1}D - D)\dot{\sigma} \rangle$. Restricting $\dot{\sigma}$ to $B\dot{q}$ proves the matrix formula. If $L \leq D$ in quadratic-form order then $L^{-1} \geq D^{-1}$ on the dual space. This proves the last assertion. $\square$

For a constant background and a nonzero wavevector put $h = \hat{k}^T H \hat{k}$. The reduced Lagrangian is $M^2|k|^2(1 - h)|\dot{\sigma}_k|^2/h$. For the RMS law

$$h_{\parallel} = \frac{y(y^2 + 2)}{(1 + y^2)^{3/2}}, \quad h_{\parallel} > 1 \iff y > y_* := \sqrt{\frac{\sqrt{5} - 1}{2}} = 0.786151377\dots \quad (\text{G.64})$$

Thus the independent foliation has a negative longitudinal velocity direction above this threshold in the static slow-motion branch. Equation (G.62) retains the spatially varying background coefficients in this same approximation.

Radial specialization. For a smooth spherical static solution with regular perturbations at the centre and fixed outer potential put $\lambda(r) = \mu(y(r)) + y(r)\mu'(y(r))$. For radial velocities with gradient compactly supported in a region where $\lambda > 0$

$$L_{\text{kin}}^{(2)} = 4\pi M^2 \int r^2 \frac{1 - \lambda(r)}{\lambda(r)} |\partial_r \dot{\sigma}|^2 dr. \quad (\text{G.65})$$

Every open shell on which $\lambda > 1$ supports a strictly negative radial velocity direction. In the vacuum exterior of a positive spherical source of mass M_s and support radius R_s such a shell exists whenever $R_s < r_*$ where

$$r_* = \left(\frac{y_*^2}{\sqrt{1 + y_*^2}} \right)^{-1/2} \sqrt{\frac{GM_s}{a_0}} = 1.434632715 \dots \sqrt{\frac{GM_s}{a_0}}.$$

Radial integration of the potential constraint using regularity at the centre gives $\phi' = (\lambda - 1)\dot{\sigma}'/\lambda$. Substitution into the quadratic action proves (G.65). The potential perturbation is constant on any region where $\lambda = 0$ and the velocity gradient vanishes. Its additive constant is fixed by the outer boundary. A nonconstant smooth bump supported in a shell with $\lambda > 1$ makes the integral strictly negative. The exterior Gauss law reads $GM_s/(a_0 r^2) = y^2/\sqrt{1 + y^2}$. The right-hand side is a strictly increasing function of $y > 0$. Evaluation at y_* gives the stated radius.

Constitutive restriction in the independent-foliation action. For this action the kinetic restriction can be expressed directly in terms of the constitutive law independently of its detailed interpolation. For the independent-foliation action (G.55) nonnegative scalar velocity energy in every constant accelerated background requires

$$0 < \mu(y) \leq 1, \quad 0 < (y\mu(y))' \leq 1.$$

A differentiable response with $0 < \mu(y) < 1$ and $1 - \mu(y) = o(1/y)$ at large y cannot satisfy these inequalities for every $y > 0$. For the RMS response the smallest coefficient normalized by $2M^2 D$ is

$$\inf_{y, \hat{k}} \frac{1 - h(y, \hat{k})}{h(y, \hat{k})} = \frac{3\sqrt{6}}{8} - 1 = -0.081441346 \dots, \quad y = \sqrt{2}, \quad \hat{k} = e. \quad (\text{G.66})$$

The transverse and longitudinal eigenvalues of H are μ and $(y\mu)'$. Positive elliptic response and nonnegative reduced foliation kinetic energy require both to lie in $(0, 1]$. Put $d(y) = y(1 - \mu(y))$. The longitudinal condition gives $d' \geq 0$ whereas the stated large- y condition gives $d(y) \rightarrow 0$ with $d(y) > 0$. This is a contradiction. For the RMS law $h'_{\parallel} = (2 - y^2)/(1 + y^2)^{5/2}$. Thus its maximum is $4\sqrt{2}/(3\sqrt{3})$ at $y = \sqrt{2}$. It exceeds the transverse eigenvalue. This proves (G.66).

Within an extension retaining this reduced foliation block the static primitive fixes the coefficient of the spatial acceleration invariant. The independent foliation turns that same coefficient into a velocity form after potential reduction. Additional generated fields contribute through their common action and constraints. On a direction attaining (G.66) its

contribution in the same normalization must be at least $1 - 3\sqrt{6}/8$ to obtain a nonnegative form provided that direction remains in the reduced physical space with the same normalization. The bound is necessary on that retained direction. The foliation kinetic function in Ref. [74] illustrates additional dynamical data. Its coefficients and its background and perturbation equations enter the common matching.

Finite-direction control of the independent foliation. For the independent-foliation diagnostic of (G.63) the retained vectors b_i are independent and $G_{ij} = \langle b_i, Db_j \rangle$ is its positive gradient metric. The corresponding matrix bound certifies this restricted foliation form. Its potential-solve error can be estimated directly. For $Lw = Db$ and a computed $\hat{w}$ with residual $r = Db - L\hat{w}$ coercivity $\langle u, Lu \rangle \geq \alpha \|\nabla u\|^2$ gives

$$\|w - \hat{w}\|_{H_0^1} \leq \alpha^{-1} \|r\|_{H^{-1}}.$$

This provides the potential-solve contribution for that response. All its source columns use the same factorization of L .

On the same fixed finite space Gaussian transport acts on a quadratic form by congruence. Its multipliers $e^{-t\lambda_j}$ are nonzero. Thus it preserves the inertia of that form. In the unrestricted Fourier algebra at $y = 1$ the directions $(1, \pm 1, 0)$ have $h = 5/(4\sqrt{2}) < 1$ whereas their sum direction has $h = 3/(2\sqrt{2}) > 1$. The product of their cosine modes contains that sum mode. Closure under these Fourier products therefore reaches directions of both signs in this independent-foliation form. Its cosmological reduction in Proposition G.6 concerns the geodesic dust–vacuum background.

Finite mode verification. The finite calculation [15] uses an Einstein–de Sitter background in units $M^2 = 1$. Here $a = t^{2/3}$ and $H = 2/(3t)$ and $\rho = 4/(3t^2)$ with $|k| = 4$ and $1 \leq t \leq 3$. For initial values $(p_0, \pi_0, \sigma_0) = (-0.01, 0.003, -0.002)$ at $t = 1$ the exact solutions are

$$\pi(t) = \pi_0 t, \tag{G.67}$$

$$p(t) = \frac{C_0}{t^3} - \frac{4\pi_0 |k|^2}{5t^{4/3}} - \frac{8\pi_0/3 + Q_\chi}{t^2}, \quad C_0 = p_0 + \frac{4\pi_0 |k|^2}{5} + \frac{8\pi_0}{3} + Q_\chi, \tag{G.68}$$

$$\sigma(t) = \sigma_0 + \pi_0(t - 1) + \frac{3Q_\chi}{2|k|^2}(t^{1/3} - 1). \tag{G.69}$$

Integrations for $Q_\chi = 0$ and 0.007 are checked against these expressions at 25 times. A third integration with $(p_0, Q_\chi) = (-0.017, 0.007)$ compares the same initial effective cold density as the zero-charge solution. The sampled maximum errors in p and σ across the three integrations are 1.73×10^{-12} and 1.44×10^{-17} respectively. For equal initial effective sources the sampled maximum differences in r and n are 1.21×10^{-12} and 1.74×10^{-18} . The three integrations use a total of 720 right-hand-side evaluations and 795 background evaluations including the 75 output samples. These are finite numerical checks of (G.57) complementary to the action and constraint calculation above.

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Declarations

Conflict of interest. The author declares no competing interests.

Data availability. The calculations use the Coarse Graining Genesis Framework [15]. Appendix B identifies the production calculation of 24 October 2026 and the archived fixed-background thermal evaluation [35] with their source and data checksums, numerical conventions and comparison inputs. The software repository is given in Ref. [15]. The article is deposited at <https://doi.org/10.5281/zenodo.22067469>.

The Lean proof library [18] provides kernel checked declarations for the indexed clauses of the inherited source and basic physical interfaces including source exchange and centre trace cancellation, the charge lattice and the global group kernel. It also checks covariance under common frame changes, compensation under finite resolution changes, phase charge algebra and tensor identities in the local five coefficient class. The theorem index records the statement mappings and comparison classes; the transitive axiom audit records their dependencies. The article derives the covariant source work and total stress identity. Physical calibration and the charged-lepton conversion G_ℓ have the analytic or numerical status specified at their defining relations.

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