

The Spectrum of a Compact Internal Space III: Finite-Window Dynamics of Cosmic Structure Formation

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ABSTRACT: Cosmic matter forms concentrations whose external force can often be described by their mass and spatial moments. Encounters and material exchange can make finer structure relevant again. We determine when field and object descriptions preserve that force and the subsequent evolution of specified observables while retaining unresolved influence in residual variables and memory. The compact internal-space construction supplies the matter carriers and collective source. Their coupled variations determine the macroscopic evolution. The applications examine homogeneous response reuse and a constrained nonuniform Einstein–matter perturbation together with autonomous expansion from a local constitutive action. In the expanding root-mean-square (RMS) application, the endpoint amplitude response fixes the constitutive flux. Its potential determines both the cold-matter force and the binding term in the expansion constraint. A finite-interval enclosure establishes noncrossing, positive reduced kinetic curvature and regular weak-field material-clock readings. Reconstruction in a common state space combines object replacement with moving spectral boundaries and truncated memory for a finite reference evolution. Spatial moments bound the force error. Cloud-in-cell assignment admits an exact eight-moment cell representation. A positive lower bound on the source derivative of the stationary constraint bounds the error in reusing its response. Integrable stability and defect bounds on a certified state tube then control the accumulated reconstruction error. A common modulus of continuity transfers this bound to observables over the prescribed interval. The first accepted window is the least certified member of a finite ordered family. Finite Galerkin examples retain positive density and the prescribed accuracy with reduced arithmetic work including verification. A cosmological particle calculation encloses the continuous-time trajectory at fixed mesh resolution. Its weakly perturbed cold-sheet reconstruction also gives full-frequency force bounds and continuous position and momentum equation-defect bounds. These estimates select sufficient retained structure for the prescribed interaction and observables. They control the re-entry of finer coordinates.

KEYWORDS: Cosmic structure formation; Particle-mesh methods; Adaptive coarse-graining; Memory effects; A posteriori error estimates; Validated numerics

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1 Introduction

Cosmic structure formation changes which features of matter determine its observable action. Small fluctuations admit a spectral description. Nonlinear transport couples those modes. Localized concentrations may act through their masses and positions with internal moments accounting for spatial structure. The matter evolution forms and deforms these concentrations. We determine when their influence on the surrounding matter can be represented by fewer variables over a finite interval and when finer structure must become explicit again.

A localized matter concentration illustrates the issue. When the force kernel varies little across its support the total mass and centre determine the leading external response. Higher moments measure the remaining influence of the internal distribution. Close encounters and tidal deformation can make those details important again. Material exchange can have the same effect. Field modes and object moments describe the same matter at different resolutions. Residual variables retain its unresolved influence. Passing between these descriptions requires a consistent transfer of the material source and control of its subsequent action. These two requirements lead to the reconstruction and force estimates.

The compact internal-space construction from minimal unbiased change supplies the physical setting. The source-preserving continuation constructs the internal response geometry and connection and determines the matter carriers [1]. Their complete relative spectral potential determines the restoring response of the collective source. The effective-coupling study evaluates the resulting scales and gives a common matter–source action [2]. The generated variables and their relations determine the allowed variations of that action. Its macroscopic fields express the resulting matter motion and geometric response with source exchange fixed by the same variation. Internal eigenvalues enter the matter coefficients and collective response. Spatial Fourier modes describe inhomogeneity of those matter fields in the cosmological realization. Their coupling through the same action connects the internal construction to structure evolution. We use its phase-clock readout and occupied expanding background with positive constrained kinetic and characteristic forms [2, Propositions “Congruence of a generated matter phase” and “Occupied common background and physical propagation”]. On this occupied background we solve the nonzero scalar metric and source constraints in canonical perturbations. The phase-congruence equation gives the material acceleration, including the drift from the evolving phase frequency. Spatial initial

data are specified for each realization. The cosmological particle and cold-sheet calculations use a declared background and cold-transport model. Source exchange and stresses quantify their relation to the generated matter equations. Bounds on these corrections enter the same equation defect as changes of spatial representation. Spectral gravitational response and preservation of complete source responses under coarse-graining are studied in Refs. [3, 4]. Here the question concerns the motion and changing representation of the matter carrying that response.

Cosmological perturbation theory calculates density–velocity mode coupling [5]. Halo models describe clustering through object populations and profiles [6]. The Press–Schechter construction relates collapse thresholds to a mass distribution [7]. These approaches provide comparisons at the field and object levels. Here a change of retained variables is governed by their influence on the specified matter evolution. Variables that cease to be explicit can still affect that evolution. Conserving the mass assigned to an object fixes one source moment. Predicting its force also requires control of the source’s spatial distribution. The present construction treats this change of variables within one specified matter evolution.

Physical smoothing and spectral retention play distinct roles. A weakly continuous centered isotropic convolution probability semigroup with unit covariance at the reference scale and square-root composition scaling determines the Gaussian smoothing law. The activity threshold selects the coefficients retained after smoothing. The cosmological nonlinear scale supplies a reference threshold. The error estimate determines sufficient retention. Motion of this threshold towards longer wavelengths describes the involvement of larger structures. Smaller structures can continue to contribute through their object variables and residual interactions. The information required for this feedback includes relative phase. Two fields with equal shell energies can produce different low-mode evolution through nonlinear mode coupling. Thus the residual description depends on the interaction and observables as well as on the amount of power outside the active window. The internal mode products and the spatial transport interactions each transfer response beyond a retained spectral set. The residual state keeps the part of this response that influences the retained matter. Re-entry restores explicit coordinates when that influence grows.

Projection methods express unresolved dynamics through memory [8]. A posteriori error analysis measures the defect of an approximate trajectory in its governing equation and propagates it by stability estimates [9, 10]. The retained variables themselves may change. A spectral region may be compressed into residual variables or localized matter into object moments. Reconstructing each representation in one fixed reference state space makes these changes comparable. Memory truncation and approximate forces contribute to the equation defect. Changes of representation may also produce reconstruction jumps. The resulting estimate propagates their combined effect on a certified tube about the reconstructed trajectory. Its stability factor measures how the specified dynamics amplifies a perturbation. Its defect measures the forcing introduced by the approximate description. The integrating-factor argument is classical. The common reconstruction allows changes of spectral retention and memory to be compared with source-response and object-representation errors in the same equation and norm.

A common modulus of continuity transfers the state bound to a specified family of observables. It quantifies which changes in the reconstructed matter can be distinguished by those readings. Within a finite ordered family the first window satisfying the estimate is the least certified choice. Finer resolution is restored when the retained description exhausts its allotted error. For discrete object catalogues the certified perturbation must preserve the selection and membership decisions. The same margin condition applies to ancestry choices. This gives a dynamical meaning to persistence of an object identity.

The force estimates make object replacement subject to this same test. A Taylor bound controls replacement of an extended source by its mass and spatial moments. For cloud-in-cell (CIC) assignment eight moments reproduce each cell source exactly. An estimate for grid-face crossings controls approximate replacement. This cellwise result retains the assignment phase on which the discrete force depends. Conservation of additive quantities follows separately from the particle partition. The force estimate additionally controls the spatial source distribution.

The common-source applications examine complementary aspects of this description. In homogeneous evolution the matter state and geometry respond to changes of the collective radius through their common action. The slope of the stationary source constraint controls the sensitivity of its solution and allows a reused response to be accepted at a prescribed accuracy. The nonuniform planar realization tests spatially varying matter–source coupling and propagation of the Einstein constraints. Each application fixes its action and initial data on an existence domain so that the estimates concern the interaction actually being evolved.

For the root-mean-square (RMS) application we use the nonrelativistic potential-action form of Ref. [11]. The compact-source covariance in Ref. [2, Appendix G] fixes the acceleration normalization on the decomposable occupation orbit. The squared-acceleration entry and positive Gaussian amplitude readout then determine the flux. We vary its potential together with cold matter, the homogeneous lapse and the scale factor. The resulting binding energy governs material acceleration and contributes to the mean expansion; its volume derivative supplies the binding part of the averaged pressure. The finite example retains one displacement mode and the complete periodic potential. It controls the constrained kinetic curvature and material-clock domain on a finite interval and records the orthogonal force driving additional displacement modes.

The finite Galerkin examples compare arithmetic work at a common accuracy with positive density and uniform error bounds. For a 4096-particle cosmological system the CIC force estimate and the time-reconstruction defect enclose the continuous-time trajectory at fixed mesh resolution. The initial cold sheet also defines a continuous matter distribution. In the weakly perturbed regime a displacement homotopy bounds its reconstructed Newtonian force at all spatial frequencies throughout the interval. Position and momentum residuals then bound the weak kinetic-equation defect. These estimates concern the specified reconstruction. The bounded-density stability results for Vlasov–Poisson [12, 13] instead propagate distances between continuous solutions. That comparison requires hypotheses on the exact solution as well.

The cosmological examples use the inherited background parameters and the numeri-

cal realization in the Coarse-Graining Genesis Framework [14]. The spectral envelope and endpoint convention fix the reference initial spectrum. CAMB and CLASS compute linear spectra from the Boltzmann hierarchy [15, 16]. HALOFIT and HMcode model nonlinear spectra using simulation calibration and halo structure [17, 18]. The estimates here control subsequent retention and object replacement for the chosen spectrum and evolution. Generated function dynamics [19] relates admitted operations and controlled limits to propagated regularity bounds and quadratic observations. Finite spectral computation [20] separates trajectory construction from independent residual checking and estimates the encoded work for its stated evolution problems. Here the stored trajectories and source enclosures supply the corresponding residual tests. The present estimates are proved with the reference equations and norms under the source-domain assumptions needed for the cosmological applications.

The state and observation map define the retained physical information. The common reconstruction estimate then connects changes of representation to force accuracy and observable evolution. The appendices contain the catalogue stability and finite-evolution estimates together with the cost analysis and continuous force calculations.

2 The coarse-grained cosmic state

The cosmic state must describe both an extended matter distribution and the concentrations that develop within it. In a field description the variables include the density contrast $\delta(x, a)$ and peculiar velocity $v(x, a)$. The Newtonian potential $\Phi(x, a)$ is determined by the source constraint. A phase-space description uses the one-particle distribution $f(x, u, a)$. Here x denotes comoving position and u denotes peculiar velocity. The scale factor is a . A localized concentration can also be described by its mass and motion with internal moments accounting for its spatial structure. The retained variables must reproduce its contribution to the observed evolution with the required accuracy.

Consider a matter concentration whose internal wavelengths are much shorter than the scale on which its external force is measured. Resolving every internal mode may then give little additional information about that force. The same internal modes can become important during a close encounter or under a rapidly varying tidal field. The appropriate state description therefore depends on the interaction and observation scales as well as on the instantaneous matter distribution. Resolved fields and residual variables retain its spatially extended response. Object coordinates describe localized concentrations within the same evolving system.

Nonlinear interactions make the choice dynamical. Equal readings at one time can lead to different readings later. Exact evolution on observable coordinates requires the update to preserve observational equivalence. Reconstruction in a common state norm measures departures from this exact closure.

2.1 Observable algebra and finite-window quotient states

At activity threshold Λ the field observables are built from retained coefficients of the Gaussian-smoothed field. Object and transfer observables describe identified matter ag-

gregates and their exchange channels. The smoothing length σ and the activity threshold are specified separately. The object-selection and persistence rules form part of the observation map. The Gaussian composition law has a generating construction in Ref. [21]. Allowed operations and controlled limits of generated functions are developed in Ref. [19]. Here the observables and reconstruction norm are specified for the stated matter evolution.

The underlying physical states belong to the generated branch. If $q \in \mathcal{Q}_{\text{gen}}$ denotes its admitted source and matter histories with the specified preparation data write their macroscopic representation as

$$\mathcal{P}_{\text{phys}} = \Gamma_{\text{gen}}(\mathcal{Q}_{\text{gen}}), \quad S_{\text{gen}}[q] = S[\Gamma_{\text{gen}}(q)]. \quad (2.1)$$

The source relations determine the variations of q and hence the physical field variations $D\Gamma_{\text{gen}}(q)\delta q$ [1, 2]. The observation map acts on these states. Its quotient describes which of their distinctions the selected readings resolve. A common normed space accommodates the finite reconstructions and their constraint residuals so that their distance from the chosen evolution can be estimated. Each field or cold-transport realization specifies its evolution equations and domain. The source model fixes their collective response.

Definition 2.1 (Observable algebra and quotient coordinates). At resolution Λ the generated observable algebra $\mathfrak{O}_{\Lambda}$ is the real algebra generated by a specified finite list of scalar readings assembled from

- (i) functionals of the Fourier coefficients of X_{σ} retained by the selector. These include power and shell content. Correlation products retain information about their joint configuration.
- (ii) object functionals of the generated catalogue. Mass and radius describe the selected material extent. Velocity scale and spin describe its motion. Growth coordinates record its evolution.
- (iii) transfer functionals between scale channels and object pairs.

Each generator is a function of the configuration with fixed selection and normalization conventions.

Two configurations carry the same finite-window reading when every observable in $\mathfrak{O}_{\Lambda}$ takes the same value on both. This defines an indistinguishability relation

$$X \sim_{\Lambda} Y \iff O(X) = O(Y) \text{ for all } O \in \mathfrak{O}_{\Lambda}. \quad (2.2)$$

The quotient class $[X]_{\Lambda}$ contains the configurations with the same readings at resolution Λ . Let $C_t X$ be the finite vector generating the observable algebra. The exact quotient coordinate is $z = C_t X$. Equality of these vectors is equivalent to equality of the whole algebra. A numerical state stores an enclosure or approximation to z together with a specified reconstruction $\mathcal{R}_t z$ and an error bound. The reconstruction selects a state against which the evolution error is measured.

This identification expresses the resolving power of the specified observations. Sources with the same mass and centre give the same monopole reading while an internal quadrupole distinguishes configurations seen by a more detailed force measurement. The material concentration evolves under the same matter equations. The observation map specifies which of its features the representation retains. Whether the omitted distinctions can subsequently affect the retained readings is a separate property of the evolution. Finite precision gives an enclosure or approximation of the coordinate vector. The chosen reconstruction specifies the representative used in the error estimate.

For approximate statements the reading family is a normalized subset $\mathfrak{O}$ with a common modulus of continuity on the certified state region. Normalization fixes the size of the observables in the uniform estimate. The full algebra is used for exact indistinguishability. Object labels are compared on regions where selection margins are certified for both states.

An exact evolution on these coordinates requires that the material representatives of a given class remain observationally indistinguishable. This is the condition that makes the reduced update determined by the stored state itself.

Proposition 2.2 (Descent of the generated update). Let $\Phi_{t,s}$ be the actual state evolution on its common domain. An exact update of quotient coordinates exists if and only if

$$C_s x = C_s y \implies C_t \Phi_{t,s} x = C_t \Phi_{t,s} y.$$

It is then unique on the image and composes wherever the underlying flow composes. In differentiable coordinates its generator satisfies

$$b_t(C_t x) = \partial_t C_t(x) + DC_t(x)F_t(x).$$

Proof. Define the update by $C_s x \mapsto C_t \Phi_{t,s} x$. The stated implication is exactly independence of the representative. Uniqueness and composition follow from the definition. Differentiating gives the generator formula. $\square$

Finite shell energies alone need not satisfy this criterion. The functions $u_{\pm} = \cos(Mx) \pm \cos((M+1)x)$ with $M > 2$ have identical low modes and all shell energies but the low component of $-\partial_x(u_{\pm}^2)/2$ is $\pm \sin x/2$. The two short wavelengths produce a long-wavelength beat through their quadratic interaction. Reversing their relative phase reverses that response while leaving both shell energies unchanged. Shell content therefore measures the amount of unresolved excitation whereas the interaction also probes how those excitations combine. Phase-sensitive coordinates or a retained tail state or a controlled memory approximation preserve the relevant part of this information. In the memory description the unresolved state carries both its initial condition and the effect of the forcing accumulated along the resolved history. When the retained coordinates do not admit an exact autonomous update their reconstruction is inserted into the same matter equation. The resulting defect measures the accuracy of the approximate evolution.

Two scale operations enter this description. The Gaussian map fixes the physical smoothed field while the activity selector chooses its retained coordinates. The computational state depends on the spectral retention and numerical precision together with the

chosen object replacement. The physical smoothing is

$$X_\sigma(a) = G_{\sigma(a)}X(a), \quad \widehat{G_\sigma X}(k, a) = \exp\left[-\frac{1}{2}\sigma^2(a)k^2\right] \widehat{X}(k, a). \quad (2.3)$$

The composition requirement has a direct scale interpretation. Successive unbiased smoothing steps are equivalent to one step with the summed variance. Centering preserves the mass centre of a localized source and isotropy assigns no preferred spatial direction. With finite covariance and square-root scaling these requirements select the kernel in (2.3).

Proposition 2.3 (Composition selects Gaussian coarse graining). Let μ_q with $q \geq 0$ be a weakly continuous convolution probability semigroup on $\mathbb{R}^d$. Assume centering and isotropy together with unit covariance at $q = 1$ and the composition scaling $\mu_q = (\sqrt{q} I)_* \mu_1$. Then its Fourier multiplier is $e^{-q|k|^2/2}$ uniquely.

Proof. The covariance expansion of the characteristic function is $\varphi_1(\xi) = 1 - |\xi|^2/2 + o(|\xi|^2)$. Composition and scaling imply $\varphi_1(\xi) = \varphi_1(\xi/\sqrt{n})^n$. Taking n to infinity gives $e^{-|\xi|^2/2}$. Scaling gives every q and uniqueness of characteristic functions identifies the measures. $\square$

Putting $q = \sigma^2$ gives (2.3) independently of the computational selector. The uniqueness statement uses square-root scaling together with the semigroup and covariance assumptions. It fixes the averaging operation within this composition class. The spatial distribution being averaged retains the correlations and initial spectrum of the matter state. For a periodic realization the same kernel is periodized. Its Fourier multiplier is restricted to the lattice $k = 2\pi n/L_{\text{box}}$. The mean mode is preserved by smoothing. Density contrast has zero mean. Any nonzero bulk velocity is retained separately from the positive-frequency window.

At scale factor a the exact reference decomposition is the tuple

$$\mathfrak{C}(a) = (X_\sigma(a), X_A(a), X_T(a), \sigma(a), \Lambda(a), k_{\text{box}}, \mathcal{M}_T(a), \mathcal{O}(a), \mathcal{G}(a)). \quad (2.4)$$

The symbol X denotes the selected field variables. Its Gaussian coarse-grained field is X_σ with active component X_A and residual tail X_T . The physical coarse-graining scale is $\sigma(a)$ and the active endpoint is $\Lambda(a)$. The quantity k_{box} is the lowest nonzero wavenumber of the specified periodic volume. The variables $\mathcal{M}_T$ describe residual moments and feedback. The catalogue $\mathcal{O}$ contains the object variables. The record $\mathcal{G}$ describes transfer across scales and between objects.

The full fields in (2.4) specify the reference decomposition. The stored coordinates $z = C_t X$ together with their finite reconstruction and error bound specify a reduced realization. The active field is advanced explicitly. Compressed tail variables approximate unresolved feedback. Object variables describe persistent concentrations of matter. A halo may be represented by its mass and position together with its velocity and spin and the required internal moments. Its replacement error is measured through the reconstruction defect and the resulting force error. Transfer records track changes between these descriptions.

Field and object coordinates refer to the same material. When an object replaces a resolved concentration its source contribution takes the place of the contribution assigned

to those field or particle variables. The membership and transfer records specify this reassignment so that mass and force are assembled once from the combined reconstruction. The tail X_T denotes unresolved spatial modes of that reconstruction. It has a different role from the physical residual component defined by the matter–source action. That component contributes to the material stress and may itself have resolved and unresolved spatial structure. A history term arises when the evolution of unresolved variables is eliminated. Its time dependence records their continuing influence on the retained state.

The information retained by an exact replacement is expressed by the Shannon chain rule [22]. For a discrete ensemble of finite-resolution states X with finite Shannon entropy suppose the retained field and object description is Y and the residual record is M both deterministic functions of X . If (Y, M) recovers X then

$$H(X) = H(Y, M) = H(Y) + H(M | Y). \quad (2.5)$$

The first equality follows from the bijection onto the encoded image. The second is the chain rule. For an approximate description $Z = (Y, M)$ the exact identity is instead $H(X) = H(Z) + H(X | Z)$. If a reconstruction of an ensemble on $d > 1$ possible states has error probability δ conditioning on the error event gives $H(X | Z) \leq h_2(\delta) + \delta \log(d - 1)$ where h_2 is the binary entropy. Thus recoverability controls lost information when a probability law and resolution are specified. The state-norm and force bounds certify their stated observables directly. Converting them to an entropy bound additionally requires this finite-resolution law or a spectral information-tail bound. Object and residual marginal entropies alone would omit their correlations.

The background parameters and smoothing scale are fixed when comparing successive states. Mass and velocity use the same conventions and object selection uses the same rules. The evolution acts on the retained coordinates together with the residual information required by their reconstruction.

2.2 Field residual and object carriers

At the field level two ordered operations split the state according to

$$X_\sigma(a) = G_{\sigma(a)}X(a), \quad X_A(a) = A_{\Lambda(a)}X_\sigma(a), \quad X_T(a) = (I - A_{\Lambda(a)})X_\sigma(a). \quad (2.6)$$

The active window is the interval

$$k_{\text{box}} \leq k \leq \Lambda(a), \quad k_{\text{box}} = \frac{2\pi}{L_{\text{box}}}, \quad (2.7)$$

in the angular-wavenumber convention. For the sharp selector the tail sector is $k > \Lambda(a)$. For a smooth selector it is the weighted complement $I - A_\Lambda$ including the transition overlap. The residual variables $\mathcal{M}_T(a)$ contain selected power fractions shell moments and transfers of this complement. Its reconstruction specifies how these quantities feed back into the retained equation.

Object carriers enter when a region of the field has acquired persistent identity. Such regions include haloes and subhaloes as well as filament segments and voids. Their coordinates record mass and size. Motion and environmental history supply the dynamical

information. These variables are often more economical than the full set of small-scale field modes from which the object formed. The object layer $\mathcal{O}(a)$ is therefore part of the state. Its entries carry reconstruction rules and growth indicators. Transfer channels specify their contribution to the active update.

The usefulness of such coordinates depends on the force they have to represent. If the interaction kernel varies slowly across a compact source total mass gives its leading contribution and the mass centre cancels the first spatial moment. Internal moments resolve the subsequent variation. This ordering explains why a localized concentration admits an effective source description for some observations. The internal degrees of freedom become explicit again when their effect on the retained force exceeds the assigned reconstruction accuracy.

The transfer record $\mathcal{G}$ describes changes between these representations. Halo growth changes the local mass distribution and velocity scale. Merger entries record mass and orbital motion. Spin and tidal alignment describe the angular-momentum geometry. Stripping returns bound material to a diffuse or substructure component. The force and reconstruction estimates determine the accuracy required of these records when they enter an active update.

2.3 Scale and mass bands

The same state can be read through prescribed scale bands. Let B_α denote a stored band in wavenumber space. With the mass convention c_M used for the endpoint the band is translated into a comoving size interval and a matter mass interval by

$$\begin{aligned} B_\alpha &= [k_{\alpha,\min}, k_{\alpha,\max}], \\ R_{\alpha,\min} &= \frac{c_M}{k_{\alpha,\max}}, \quad R_{\alpha,\max} = \frac{c_M}{k_{\alpha,\min}}, \\ M_\alpha(R) &= \frac{4\pi}{3} \bar{\rho}_{m,0} R^3. \end{aligned} \tag{2.8}$$

The reported band masses are $M_{\alpha,\min} = M_\alpha(R_{\alpha,\min})$ and $M_{\alpha,\max} = M_\alpha(R_{\alpha,\max})$. The assigned masses depend on the same fixed convention c_M as the endpoint mass.

A stored band may represent a spectral interval or a halo mass layer or a near-endpoint residual region or a selected object population. Its observables give band activity and residual moments together with object statistics. The band labels used in the figures are attached to generated quantities with explicit mass intervals. A band can be displayed as a dwarf or galaxy or group or cluster scale after the mass interval in (2.8) has been computed under the declared convention.

Comparing the endpoint with these bands locates the nonlinear frontier relative to the reported object populations. The inverse relation between wavenumber and size fixes the direction of this progression. An endpoint moving to smaller k reaches larger comoving volumes and larger band masses. Smaller structures can persist within those volumes and continue to contribute through the object and residual descriptions. The band mass in (2.8) is the mass assigned to a volume at the mean matter density. The mass of an individual identified object is obtained from its own membership or field reconstruction.

2.4 Normalization and observable stability

State errors are measured in a fixed norm with angular wavenumbers $2\pi n/L_{\text{box}}$. An amplitude change transforms the interaction coefficients as well as the norm. Object velocities and masses together with radii use the same units and selection convention in every comparison. These choices fix the meaning of the ratios R_h and R_v as well as B_h .

A density threshold or ranking rule or parent choice is locally stable when its separation margin exceeds the perturbation bound. The margin measures how far the configuration lies from a change of identity under that observation rule. Within such a region a small perturbation changes the measured mass or position continuously. Near a selection boundary the same perturbation can change the catalogue entry itself. The comparison then retains the event information. Lemma A.1 gives the corresponding Lipschitz regions. Uniform catalogue accuracy therefore applies on intervals throughout which the comparison tube satisfies these margin conditions. At a selection event reconstruction jumps control the state change. Differences in event times and discrete labels require separate enclosures. Appendix A collects these stability conditions and the notation.

3 Finite-window dynamics

The internal spectral construction supplies matter modes and their effective coefficients. Their occupation couples to a common source response. Structure formation concerns the spatially nonuniform occupation of these modes. A spatial Fourier wavenumber therefore labels a variation of the matter distribution whereas an internal eigenvalue labels a mode entering the matter and source response. Their coupling is carried by the matter–source action [2].

The generating operations fix the admissible physical variables and variations. Occupation and initial and boundary data specify the state. The potential, lapse and scale factor express its macroscopic response, and material clocks read durations along its evolution. At finite resolution only selected coordinates are explicit. Interactions can drive additional coordinates whose influence is retained in the equation defect and residual history or resolved by enlarging the window.

3.1 Generated matter and cold transport with clock readings

The cold transport description starts from the phase current of the matter–source action. We use the phase congruence and proper-time construction of Ref. [2, Proposition “Congruence of a generated matter phase”] and denote its internal conformal source by ζ reserving η for conformal time. For a retained mode $\varphi = re^{i\vartheta}$ set

$$M = e^{-3\zeta}, \quad m = m_0 e^\zeta, \quad X_\vartheta = -(\nabla\vartheta)^2 = m^2 - P > 0, \quad P = \frac{\nabla_\mu(M\nabla^\mu r)}{Mr}.$$

The future-directed velocity is $n_\mu = -\partial_\mu\vartheta/\sqrt{X_\vartheta}$. The conserved phase current defines a number density $\nu = Mr^2\sqrt{X_\vartheta}$. Its associated cold mass density $\rho_c = m\nu$ therefore obeys

$$\nabla_\mu(\nu n^\mu) = 0, \quad \nabla_\mu(\rho_c n^\mu) = \rho_c n^\mu \partial_\mu \zeta. \quad (3.1)$$

The source equation in the common action accounts for this energy exchange. The distinction between number and mass is physical. The phase current transports the occupation while the common source changes the mass carried by that occupation. A fixed-mass transport measure applies when the corresponding source variation vanishes. Otherwise the exchange contributes to the equation defect of the cold approximation.

Proposition 3.1 (Controlled cold-source interface). On a regular phase patch put $\epsilon = P/m^2$ and $\rho_0 = Mr^2m^2$ together with $h_\mu{}^\nu = \delta_\mu{}^\nu + n_\mu n^\nu$. Suppose $|\epsilon| \leq \epsilon_0 < 1$ and $|\partial_{\hat{a}}r|/(mr) \leq \delta$ in an orthonormal frame. The exact matter stress has the form $T_{\mu\nu} = \rho n_\mu n_\nu + \Pi_{\mu\nu}$ with $\rho = Mr^2X_\vartheta$ and satisfies

$$\begin{aligned} \frac{|\rho - \rho_0|}{\rho_0} &\leq \epsilon_0, & \frac{|\rho - \rho_c|}{\rho_c} &\leq \frac{\epsilon_0}{1 + \sqrt{1 - \epsilon_0}}, \\ \frac{|\Pi_{\hat{a}\hat{b}}|}{\rho_0} &\leq 3\delta^2 + \frac{\epsilon_0}{2}, & |n^\nu \nabla_\nu n_\mu + h_\mu{}^\nu \partial_\nu \zeta| &\leq \frac{|h_\mu{}^\nu \partial_\nu \epsilon|}{2(1 - \epsilon_0)}. \end{aligned} \quad (3.2)$$

Thus a constant-mass geodesic cold approximation has number-to-mass exchange rate bounded by $|n \cdot \nabla \zeta|$ and acceleration defect bounded by $|h \nabla \zeta| + |h \nabla \epsilon|/[2(1 - \epsilon_0)]$. For an evolution estimate obtained from stress conservation a bound on $\nabla^\mu \Pi_{\mu\nu}$ is additionally required in the norm used by the evolution operator.

Proof. Current conservation and $\partial_\mu m = m \partial_\mu \zeta$ give (3.1). The ratio $\rho/\rho_c = \sqrt{1 - \epsilon}$ gives the second density bound by rationalizing the difference from unity. The phase congruence identity gives

$$n^\nu \nabla_\nu n_\mu = -h_\mu{}^\nu \partial_\nu \zeta + \frac{h_\mu{}^\nu \partial_\nu \epsilon}{2(1 - \epsilon)}.$$

Finally $\Pi_{\mu\nu} = M \partial_\mu r \partial_\nu r - g_{\mu\nu} M[(\nabla r)^2 + r^2 P]/2$. Bounding the four orthonormal derivatives individually gives (3.2). Differentiating this stress decomposition identifies the stated stress-divergence term. $\square$

The cold approximation follows a timelike phase congruence with amplitude and source gradients supplying the corrections to mass transport. The bound on P/m^2 keeps the phase clock regular. The amplitude bound controls the stress carried by variations of the envelope. A small stress can still vary rapidly in space which is why its divergence enters an acceleration estimate separately.

Material force and the source rate. On a regular phase patch, the phase frequency determines the clock rate and the acceleration of the matter flow. Put

$$q_\vartheta = \sqrt{X_\vartheta} = m\sqrt{1 - \epsilon}, \quad \chi = \log(q_\vartheta/m_0) = \zeta + \frac{1}{2} \log(1 - \epsilon).$$

The exact congruence equation is $n^\nu \nabla_\nu n_\mu = -h_\mu{}^\nu \partial_\nu \chi$. For this force calculation let t denote background cosmic time and write a weak scalar metric as $ds^2 = -N^2 dt^2 + A^2 dx^2$, with $N = \sqrt{1 + 2\Phi}$ and $A = a\sqrt{1 - 2\Psi}$. The velocity measured by the orthonormal observers at fixed x is $v_o = (A/N) dx/dt$, and its derivative along the flow is $\mathfrak{D}_t = \partial_t + (N/A)v_o \cdot \nabla$. Its slow-motion equation takes the form

$$\mathfrak{D}_t v_o + H v_o = -\frac{\nabla \Phi_{\text{dyn}}}{a} - v_o \partial_t \chi + R_o, \quad \Phi_{\text{dyn}} = \Phi + \zeta + \frac{1}{2} \log(1 - \epsilon), \quad H = \dot{a}/a. \quad (3.3)$$

Appendix B.5 derives the exact velocity equation and an explicit weak-field bound for R_ϕ . The gradient of $\Phi + \chi$ determines the leading acceleration, while the time dependence of χ contributes to the peculiar-velocity drift. Both the conformal source and the amplitude correction enter this phase frequency. A quasistatic potential description applies when the drift term lies within the chosen force tolerance. For carriers with the same source exponents the leading source acceleration is independent of m_0 ; the amplitude term retains their finite-wavelength response.

For a finite spectral or particle realization the source and stress corrections are projected into its state norm. The mass-density correction enters the same equation defect. The chosen cold approximation also includes bounds on the weak-field and peculiar-velocity expansions. Velocity-dispersion estimates complete these bounds. The single-stream fluid description uses an interval on which its flow map remains invertible. A phase-space description retains velocity dispersion when needed. A principal characteristic speed of the constrained scalar action describes its short-wavelength propagation. The cold approximation instead requires the stress and source bounds in Proposition 3.1 at the scales being evolved.

For the cosmological realization of Section 2 the fixed-mass cold reference is represented by the phase-space distribution $f(x, u, \eta)$. The bounds above quantify the source and stress corrections to its collisionless transport and single-stream limit [5]. In conformal time this reference obeys

$$\partial_\eta f + \nabla_x \cdot (uf) - \nabla_u \cdot [(\mathcal{H}u + \nabla_x \Phi)f] = 0, \quad (3.4)$$

$$\nabla_x^2 \Phi = 4\pi G a^2 \bar{\rho}_m \delta, \quad (3.5)$$

$$\delta(x, \eta) = \frac{1}{\bar{\rho}_{m,0}} \int f(x, u, \eta) d^3u - 1.$$

Here $f d^3x d^3u$ is comoving mass so $\bar{\rho}_m = a^{-3} \bar{\rho}_{m,0}$. The velocity-divergence form preserves it under the peculiar-velocity drift. On scales and epochs where the single-stream limit applies the first two velocity moments give the pressureless fluid system

$$\partial_\eta \delta + \nabla \cdot [(1 + \delta)v] = 0, \quad (3.6)$$

$$\partial_\eta v + \mathcal{H}v + (v \cdot \nabla)v = -\nabla \Phi, \quad (3.7)$$

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho}_m \delta. \quad (3.8)$$

The phase-space and single-stream descriptions use different state variables with the potential fixed by the Poisson constraint. We write either evolution as

$$\partial_\eta X = \mathcal{N}[X], \quad (3.9)$$

where X denotes the resolved variables and $\mathcal{N}$ collects expansion and advection together with gravity in the field representation chosen for the computation.

The projection identities use conformal time η with dots denoting its derivatives. The error estimates use $t = \ln a$. $dt/d\eta = \mathcal{H} = aH(a)$. Thus $\mathcal{N}_t = \mathcal{N}_\eta/\mathcal{H}$ and $d\eta/da = 1/(a\mathcal{H})$.

Every forcing and residual is rescaled under this time change. Along each phase trajectory the measured proper time is

$$ds = \frac{d\vartheta}{\sqrt{X_\vartheta}}. \quad (3.10)$$

For the homogeneous comoving phase in the Friedmann–Lemaître–Robertson–Walker (FLRW) background $d\eta = ds/a$. Hence the clock chart $t = \ln a$ satisfies $dt/d\eta = \mathcal{H}$. Along moving trajectories the metric line element including the peculiar velocity relates their individual proper times to this background chart. Each vector field and its defect carries the same positive clock Jacobian. The phase reading belongs to the retained matter state and the common source and metric constraints determine its evolution together with that state. Changing from this proper-time reading to η or $\ln a$ changes the parametrization of the trajectory. Changing σ or Λ changes its spatial representation. Keeping these operations distinct allows one to compare physical evolution at different resolutions in a common time chart.

The covariant phase current carries a source-dependent mass; its fixed-mass cold approximation uses the exchange and stress bounds above. The root-mean-square (RMS) constitutive action retains a conserved comoving mass distribution as its macroscopic matter state. Comparison with a phase-current solution uses the source-exchange, amplitude-stress and acceleration bounds of Proposition 3.1. In each action the source coupling determines the material force.

3.2 A nonuniform common Einstein–matter evolution

Spatial gradients test whether the same source rule can support a nonuniform material configuration and its metric response. The auxiliary-radius action of Ref. [2, Proposition “Occupied common background and physical propagation”] also admits nonuniform initial data. In the common response units its matter Lagrangian is

$$-\frac{Z_E}{2}(\nabla E)^2 - \frac{M}{2}(\nabla\varphi)^2 - \frac{B}{2}|\varphi|^2 - U(E, \zeta), \quad M = e^{-3\zeta}, \quad B = m_0^2 e^{-\zeta},$$

with the inherited Einstein coefficient $P^2/2$ where $P = M_P/\mu$. The common response units measure lengths in μ^{-1} and fields in μ . The dimensionful condensate is $e = \mu E$. The retained matter field is $\varphi_{\text{phys}} = \mu\varphi$ and $m_{0,\text{phys}} = \mu m_0$. The dimensionless potential is $U(E, \zeta) = \mu^{-4}U_{\text{phys}}(\mu E, \zeta)$. The source convention agrees with that of the inherited action $L = L_{\text{ref}}e^{-\zeta}$. Its source variable is denoted by η in Ref. [2]. The potential U retains the complete unweighted relative potential with its constant conformal mode and original drive. Its feedback and vacuum subtraction are fixed by the inherited action. The common spectral-cutoff estimates of Ref. [1, Section 8 and Appendix E] bound its finite evaluation on compact condensed-branch intervals. The source and vector-field errors use the derivatives required on the actual state region with finite-sum and root errors included. For a spatial metric γ with volume density $v_\gamma = \sqrt{\det \gamma}$ and canonical matter momentum density p define

$$\mathcal{T} = \frac{|p|^2}{2v_\gamma^2 M}, \quad \mathcal{G}_s = \frac{M}{2}\gamma^{ij}\partial_i\varphi^\top\partial_j\varphi, \quad \mathcal{V} = \frac{B}{2}|\varphi|^2.$$

The local source equation and its two curvatures are

$$\begin{aligned} C_H &= U_{,\zeta} + 3(\mathcal{T} - \mathcal{G}_s) - \mathcal{V} = 0, \\ h_H &= U_{,\zeta\zeta} + 9\mathcal{T} + 9\mathcal{G}_s + \mathcal{V}, \quad h = U_{,\zeta\zeta} - 9\mathcal{T} + 9\mathcal{G}_s + \mathcal{V}. \end{aligned} \quad (3.11)$$

At fixed canonical matter data $h_H = \partial_\zeta C_H$ controls the sensitivity of the stationary source root. The fixed-velocity source Hessian h enters the reduced physical principal forms. The internal spectral gap controls the normalized response spectrum [1]. Source-root sensitivity and spacetime propagation are evaluated on the occupied background through h_H and the reduced principal forms respectively. Their bounds therefore refer to the same source law at different stages of the matter–geometry calculation. The spatial term follows from the same covariant kinetic contraction as the temporal term. Source elimination therefore retains the gradient contribution throughout the nonuniform evolution. The perturbation amplitude specifies the initial inhomogeneity. The source and geometric constraints then determine the compatible source value and spatial metric. The same constraints fix the mean expansion on the selected branch.

Proposition 3.2 (Constraint-compatible nonuniform initial data). Let r_0, p_0, E_0, H_0 be the occupied homogeneous initial data of Ref. [2, Proposition “Occupied common background and physical propagation”] such that $H_0 > 0$ and $\mathcal{T}_0 = \mathcal{V}_0 > 0$ together with $h_0 > 0$. On a three-torus choose an integer $k \neq 0$ and set

$$\begin{aligned} \varphi &= (r_0[1 + \varepsilon \cos(kx)], 0), \quad p = (0, p_0), \quad E = E_0, \quad p_E = 0, \\ \gamma_{ij} &= \psi^4 \delta_{ij}, \quad K^i_j = -H \delta^i_j. \end{aligned} \quad (3.12)$$

For sufficiently small $|\varepsilon|$ the common source and Einstein constraints uniquely determine a nearby positive periodic ψ with $\langle \psi \rangle = 1$ together with a positive constant H and the algebraic source ζ . These data have a unique local Einstein–matter development up to diffeomorphism. On a common nonzero interval the phase gradient is future timelike and the physical matter principal forms remain positive.

Proof. The matter momentum density is $-v_\gamma^{-1}(p_E \partial_i E + p^\top \partial_i \varphi) = 0$. Constant pure-trace K then solves the momentum constraint exactly. Since $h_H = h + 18\mathcal{T} > 0$ at the homogeneous datum (3.11) gives a smooth local source function of the remaining initial data. Put

$$\rho = \frac{p_E^2}{2v_\gamma^2 Z_E} + \frac{Z_E}{2} |\nabla E|_\gamma^2 + \mathcal{T} + \mathcal{G}_s + \mathcal{V} + U.$$

The Hamiltonian constraint and normalization become

$$-8\psi'' + \left(6H^2 - \frac{2\rho}{P^2}\right) \psi^5 = 0, \quad \langle \psi \rangle = 1. \quad (3.13)$$

At $\varepsilon = 0$ the solution is $\psi = 1$ with $H = H_0$. The source stationarity gives $\rho_{,\zeta} = C_H = 0$ hence at fixed canonical data $\rho_{,\psi} = -12\mathcal{T}_0$. The linearization of the augmented constraint is

$$(u, b) \mapsto \left(-8u'' + \frac{24\mathcal{T}_0}{P^2}u + 12H_0b, \langle u \rangle\right).$$

Its nonzero Fourier eigenvalues are $8j^2 + 24\mathcal{T}_0/P^2 > 0$. The mean equation and normalization determine the constant mode and b . The augmented operator is therefore invertible on periodic Hölder spaces. The implicit-function theorem constructs the asserted finite nonzero initial perturbations. In particular

$$\psi = 1 + \varepsilon \frac{\mathcal{V}_0}{2P^2k^2 + 6\mathcal{T}_0} \cos(kx) + O(\varepsilon^2), \quad H = H_0 + O(\varepsilon^2). \quad (3.14)$$

For the local principal calculation use a normal orthonormal frame. Set $q = (E, \varphi_1, \varphi_2)$ and $G = \text{diag}(Z_E, M, M)$. For the matter derivatives define $v = (0, \partial_\perp \varphi)$ and $w_n = (0, n^i \partial_i \varphi)$ for a unit spatial direction n . Eliminating the algebraic source in the quadratic action gives

$$K = G + \frac{9M^2}{h} vv^\top, \quad J_n = G - \frac{9M^2}{h} w_n w_n^\top, \quad B_n = -\frac{9M^2}{2h} (v w_n^\top + w_n v^\top). \quad (3.15)$$

These are the temporal and spatial principal coefficients together with the mixed coefficient. The sufficient inequality $h > 9M|\nabla \varphi|_\gamma^2 = 18\mathcal{G}_s$ ensures $K > 0$ and $J_n > 0$ for every n . It holds strictly at the homogeneous datum and throughout a neighborhood of the constructed data. Together with the positive Einstein wave block in generalized harmonic coordinates this gives a regular hyperbolic Cauchy system. Local existence and uniqueness follow from its energy estimate [23]. The common variational identity gives $\nabla_\mu T^{\mu\nu} = 0$ on the matter and source equations. The contracted Bianchi identity propagates the Einstein and harmonic constraints. The initial phase has positive temporal derivative and zero spatial derivative. Timelikeness and the source bracket persist on a sufficiently short common interval by continuity. The strict principal inequalities hold on that same interval. The phase then supplies the proper-time readout of (3.10). $\square$

Joint scalar response on the occupied background. We linearize the common action about its homogeneous occupied expanding solution. Dots denote derivatives with respect to background cosmic time. For a nonzero scalar Fourier mode use

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Psi)dx^2, \quad \kappa = |k|/a > 0.$$

The scalar Einstein constraints take their standard Newtonian-gauge form [24, Eqs. (23a), (23b) and (23d)]. Scalar perturbations of this homogeneous background have vanishing anisotropic stress at first order, giving $\Phi = \Psi$. Write $e_1 = \delta E$, $f = \delta \varphi$, $\pi_E = \delta p_E$ and $\pi_\varphi = \delta p$. These momenta are perturbations of the full canonical densities $p_E = v_\gamma Z_E \dot{E}/N$ and $p = v_\gamma M \dot{\varphi}/N$, where $v_\gamma = a^3(1 - 3\Psi)$ to first order. Define the background quantities and linear combinations

$$\begin{aligned} \mathcal{A} &= Z_E \dot{E}^2 + M|\dot{\varphi}|^2, & \mathcal{A}_\varphi &= M|\dot{\varphi}|^2, \\ \mathcal{L} &= Z_E \dot{E} e_1 + M \dot{\varphi}^\top f, \\ \mathcal{R}_0 &= \frac{\dot{E} \pi_E + \dot{\varphi}^\top \pi_\varphi}{a^3} + U_{,E} e_1 + B \varphi^\top f. \end{aligned}$$

Source stationarity gives $\delta\rho = \mathcal{R}_0 + 3\mathcal{A}\Psi$. The Einstein scalar constraints and the source equation have the explicit solution

$$\begin{aligned}\Psi &= -\frac{\mathcal{R}_0 + 3H\mathcal{L}}{2P^2\kappa^2 + 3\mathcal{A}}, \\ \delta\zeta &= \frac{-U_{,E\zeta}e_1 - 3\dot{\varphi}^\top\pi_\varphi/a^3 + B\varphi^\top f - 9\mathcal{A}_\varphi\Psi}{h_H}.\end{aligned}\tag{3.16}$$

For $P^2 > 0$ and $h_H > 0$ both denominators are positive. Thus the retained matter perturbation determines the metric and the internal source together. Substitution into the six real canonical perturbation equations gives a closed time-dependent linear system on each regular background interval. Appendix B.5 gives these equations and verifies propagation of the scalar constraints. The physical principal matrices are those in (3.15).

On the regular phase patch $r > 0$, $q_\vartheta > 0$, the same canonical perturbation determines the phase-frequency response. For $r = |\varphi|$, $\delta r = \varphi^\top f/r$ and $\delta\vartheta = (\varphi_1 f_2 - \varphi_2 f_1)/r^2$, one has

$$\delta\chi = \frac{\delta\nu}{\nu} + 3\delta\zeta - 2\frac{\delta r}{r}.$$

The phase charge density $\mathcal{Q} = \varphi_1 p_2 - \varphi_2 p_1 > 0$ gives $\delta\nu/\nu = \delta\mathcal{Q}/\mathcal{Q} + 3\Psi$. The first-order peculiar velocity and acceleration are

$$v = -\frac{\nabla\delta\vartheta}{aq_\vartheta}, \quad \dot{v} + (H + \dot{\chi}_0)v = -\frac{\nabla(\Psi + \delta\chi)}{a}.\tag{3.17}$$

Here $\chi_0 = \log(q_\vartheta/m_0)$ is evaluated on the homogeneous background. The resulting force contains its metric perturbation and the frequency change induced by the matter and source.

The common Einstein–matter equations also determine the local metric response. With signature $(-+++)$ a stationary weak metric $g_{00} = -(1 + 2\Phi)$ obeys to leading order on scales short compared with the background curvature radius

$$\Delta\Phi = \frac{\delta\rho + \delta S}{2P^2}, \quad S = \gamma^{ij}T_{ij}.\tag{3.18}$$

This is the 00 trace-reversed Einstein equation with the homogeneous contribution subtracted. In the cold limit $|\delta S| \ll |\delta\rho|$ it gives the Poisson normalization used in the pressureless model. The source and radial stresses remain in $\delta\rho$ and δS when that approximation is relaxed. The nonuniform common evolution constructed here uses (3.11) and (3.18) with one shared source stress. Its cold material acceleration also contains the source and amplitude gradients in Proposition 3.1. In the auxiliary-radius realization the algebraic source adjusts the matter coefficients, and its stress enters the Einstein metric response. The RMS action uses the endpoint susceptibility to define a macroscopic constitutive energy coupled to conserved cold matter. Its potential variation and material variation determine the field and force together.

3.3 An expanding finite window of the RMS source action

The endpoint amplitude response defines the local constitutive energy. We couple this energy to conserved cold matter and a homogeneous lapse and scale factor. The potential

responsible for material acceleration then contributes to the expansion constraint through its binding energy.

Endpoint covariance and amplitude response. The inherited compact-source read-out uses decomposable pure occupations $u \otimes v$, with independent invariant probabilities on $u, v \in S^2$. Its traceless source is

$$j(u, v) = \left(uu^\top - \frac{I_3}{3}, vv^\top - \frac{I_3}{3} \right).$$

Isotropy gives $\mathbb{E}j = 0$, while $\|uu^\top - I_3/3\|^2 = 2/3$ gives $\mathbb{E}\|j\|^2 = 4/3$. The directional mean vanishes while the mean-square source remains nonzero on this occupation orbit. With the inherited global correlation width $\sigma_C = c/H_0$, cyclic wavenumber $k_{\text{IR}} = (2\pi\sigma_C)^{-1}$ and acceleration conversion $c^2 k_{\text{IR}}$, the source variance is

$$\Sigma_{\text{IR}} = \frac{4}{3} \left(\frac{cH_0}{2\pi} \right)^2 = a_0^2, \quad \alpha_0 = \frac{a_0}{cH_0} = \frac{\sqrt{4/3}}{2\pi}. \quad (3.19)$$

These are the occupation orbit, endpoint and acceleration readouts of Ref. [2, Appendix G]. The reference endpoint fixes H_0 , the length unit and a_0 together. The instantaneous expansion rate evolves with the state while a_0 retains this reference-endpoint value.

Let ξ be the real Gaussian response coordinate with positive quadratic coefficient λ . Adding the source variance changes its normalized probability from a density proportional to $e^{-\lambda\xi^2/2}$ to one proportional to $e^{-(\lambda+\Sigma_{\text{IR}})\xi^2/2}$. Direct integration gives

$$Z(\lambda) = \frac{\mathbb{E}_\Sigma \xi^2}{\mathbb{E}_0 \xi^2} = \frac{\lambda}{\lambda + \Sigma_{\text{IR}}}, \quad A(\lambda) = \frac{\|\xi\|_{L^2(\mathbb{P}_\Sigma)}}{\|\xi\|_{L^2(\mathbb{P}_0)}} = \sqrt{Z(\lambda)}. \quad (3.20)$$

The Gaussian integral compares the squared response amplitudes through Z ; its positive square root compares their RMS amplitudes through A . For the spatial channel the squared-acceleration entry is $\lambda = (cH_0g)^2$. The positive RMS readout in (3.20) therefore gives $\mathcal{J}(g) = |g|g/\sqrt{g^2 + \alpha_0^2}$. The occupation orbit, quadratic spatial entry and RMS readout define the response class and fix its constitutive function, including the continuous value at $g = 0$.

Coordinates and durations are measured in c/H_0 and H_0^{-1} and densities in $M_P^2 H_0^2$. In this realization t is the dimensionless time parameter of the lapse action and dots denote derivatives with respect to it. At $N = 1$ the time coordinate is $t = H_0 T$ where T is cosmic proper time. Write $\tau = H_0 T$, so $d\tau = N dt$. The density coefficients are $m = 3\Omega_m$ and $\rho_v = 3\Omega_\Lambda$; m here is the comoving matter density. The matter budget combines the baryonic and residual endpoint fractions into one pressureless component with common velocity and density contrast. Cell averages have unit normalization. The primitive with $\Psi(0) = 0$ is uniquely determined by the flux:

$$\mathcal{J}(g) = \frac{|g|g}{\sqrt{g^2 + \alpha_0^2}}, \quad \Psi(g) = \frac{1}{2} \left(|g|\sqrt{g^2 + \alpha_0^2} - \alpha_0^2 \operatorname{arsinh} \frac{|g|}{\alpha_0} \right), \quad \Psi' = \mathcal{J},$$

Indeed $\Psi(g) = \int_0^{|g|} s^2/\sqrt{s^2 + \alpha_0^2} ds$. Its derivative is strictly increasing, with $\mathcal{J}'(g) = |g|(g^2 + 2\alpha_0^2)/(g^2 + \alpha_0^2)^{3/2} > 0$ for $g \neq 0$. The energy is strictly convex and has the

small-field form $\Psi(g) = |g|^3/(3\alpha_0) + O(|g|^5/\alpha_0^3)$. The resulting local field equation has the variational form studied in Ref. [11]. Here its function and acceleration normalization are supplied by (3.19) and (3.20).

At low acceleration the amplitude readout gives $\mathcal{J}(g) \sim |g|g/\alpha_0$. The flux is thus quadratic in field amplitude and its primitive is cubic. A planar source with amplitude D produces a flux proportional to D ; inversion gives $|g| \propto \sqrt{D}$ at a fixed point of nonzero flux shape and fixed background. Strict convexity gives uniqueness of the static field with fixed mean. Spatial regularity at flux zeros follows from the inverse constitutive law. Dynamical positivity is evaluated on the constrained physical momentum directions.

Common variation of source, matter and expansion. Retain the homogeneous lapse $N > 0$, scale factor $a > 0$, a periodic material map $X(q)$ with $J_X = \partial_q X > 0$, and a mean-zero periodic potential. The common nonrelativistic action per unit comoving volume is

$$S_R = \int dt \left[-\frac{3a\dot{a}^2}{N} - N\rho_v a^3 - Nm + \frac{ma^2}{2N} \langle \dot{X}^2 \rangle_q - Nm \langle \Phi(X) \rangle_q - 2Na^3 \left\langle \Psi \left(\frac{\partial_x \Phi}{a} \right) \right\rangle_x \right]. \quad (3.21)$$

The material density is $\rho(X(q)) = m/(a^3 J_X(q))$. Varying the full periodic potential and displacement gives

$$\begin{aligned} \partial_x \mathcal{J}(g) &= \frac{m}{2a^2} (J_X^{-1} - 1), & g &= \frac{\partial_x \Phi}{a}, & u &= \frac{a\dot{X}}{N}, & \mathcal{H}_R &= \frac{\dot{a}}{Na}. \\ \frac{1}{N} \frac{d}{dt} \left(\frac{a^2 \dot{X}}{N} \right) &= -\partial_x \Phi(X), & \frac{du}{d\tau} + \mathcal{H}_R u &= -g(X), \end{aligned} \quad (3.22)$$

The last equation is evaluated along a material trajectory. The coefficient m cancels from its inertial and potential terms. All retained cold matter therefore responds to this same acceleration. The density obtained from X obeys $\partial_\tau \rho + 3\mathcal{H}_R \rho + a^{-1} \partial_x(\rho u) = 0$. Mean-zero potential variations subtract the homogeneous source in the field equation. The potential equation and the material force thus use the same source coupling.

Eliminate this potential at fixed a, X . Periodic integration by parts gives its binding energy

$$\begin{aligned} W[a, X] &= m \langle \Phi(X) \rangle_q + 2a^3 \langle \Psi(g) \rangle_x = 2a^3 \langle \Psi(g) - g\mathcal{J}(g) \rangle_x, \\ \frac{\delta W}{\delta X} &= m \partial_x \Phi(X), & W_a &= 2a^2 \langle 3\Psi(g) - g\mathcal{J}(g) \rangle_x. \end{aligned} \quad (3.23)$$

The functional derivative is at fixed a , and W_a is at fixed material map. The stationary potential equation removes the implicit potential variation from both derivatives. Thus the field energy that determines the material force also determines the response to a change of physical volume. For a nonuniform source $W < 0$: gravitational binding lowers the energy relative to the homogeneous state. With $\Pi = ma^2 \dot{X}/N$ and $K = \langle \Pi^2 \rangle_q / (2ma^2)$, lapse

variation imposes $C = -p_a^2/(12a) + K + \rho_v a^3 + m + W = 0$, where $p_a = -6a\dot{a}/N$. Its homogeneous equations are

$$\begin{aligned} 3\mathcal{H}_R^2 &= \rho_v + \rho_c, & \frac{d\mathcal{H}_R}{d\tau} &= -\frac{\rho_c + p_c}{2}, \\ \rho_c &= \frac{m + K + W}{a^3}, & p_c &= \frac{2K - aW_a}{3a^3}, & \frac{d\rho_c}{d\tau} + 3\mathcal{H}_R(\rho_c + p_c) &= 0. \end{aligned} \quad (3.24)$$

To verify the exchange, the material Hamilton equations give $d(K + W)/d\tau = -2\mathcal{H}_R K + a\mathcal{H}_R W_a$. Differentiating the constraint gives the acceleration equation in (3.24). It also follows directly from scale-factor variation. The Hamiltonian NC preserves $C = 0$. The exchange law relates the material kinetic and binding energies to the mean expansion. The scale derivative W_a supplies the binding contribution to the pressure.

Retained material window. Choose

$$X(q) = q + \frac{D}{k} \sin(kq), \quad J_X = 1 + D \cos(kq), \quad 0 < D < 1.$$

The allowed displacement variations are now $\delta X = \sin(kq) \delta D/k$, while the potential continues to vary over all mean-zero periodic functions. The displacement restriction selects one material degree of freedom while variation of the full potential preserves the field response to that source. The condition $D < 1$ keeps the material map one-to-one and its density positive. The zero-mean condition fixes the additive potential freedom. Strict convexity fixes the remaining field for this source and constitutive law.

Proposition 3.3 (Exact periodic field and constrained finite dynamics). For $a > 0$ and $0 < D < 1$ the potential variation in (3.21) has a unique mean-zero weak solution. Its physical gradient $g = \partial_x \Phi/a$ is determined by

$$\mathfrak{f}(X(q)) = -\frac{mD}{2a^2k} \sin(kq), \quad g(X(q)) = \text{sgn}(\mathfrak{f}) \sqrt{\frac{\mathfrak{f}^2 + |\mathfrak{f}| \sqrt{\mathfrak{f}^2 + 4\alpha_0^2}}{2}}. \quad (3.25)$$

Define the eliminated binding energy and its derivatives by

$$\begin{aligned} W(a, D) &= 2a^3 \langle \Psi(g) - g\mathfrak{f} \rangle_q, \\ W_D &= \frac{ma}{k} \langle g \sin(kq) \rangle_q, & W_a &= 2a^2 \langle 3\Psi(g) - g\mathfrak{f} \rangle_q. \end{aligned} \quad (3.26)$$

Put $K = k^2 p_D^2 / (ma^2)$ and $R = \rho_v a^3 + m + W$. The lapse imposes

$$C = -\frac{p_a^2}{12a} + K + R = 0. \quad (3.27)$$

On an expanding branch with $R > 0$ elimination of this constraint gives the physical Hamiltonian in the scale-factor chart

$$H_a = \sqrt{12a} \sqrt{K + R}, \quad \frac{\partial^2 H_a}{\partial p_D^2} = \frac{\sqrt{12a} k^2 R}{ma^2 (K + R)^{3/2}} > 0. \quad (3.28)$$

The nonzero branch has smooth finite-dimensional evolution up to the first exit from these conditions. Its constraint and total energy balance propagate under the same evolution.

Proof. The potential equation is

$$\partial_x \mathcal{J}(g) = \frac{m}{2a^2} (J_X^{-1} - 1).$$

Multiplication by J_X and integration in q give the first equation in (3.25). Odd parity fixes the constant flux and ensures periodicity of Φ . Inversion of $\mathcal{J}$ gives its second equation. Strict convexity of Ψ with the constant potential removed proves uniqueness. Integration by parts in the source term gives the binding energy $2a^3 \langle \Psi - g\mathfrak{f} \rangle_x$. Its Jacobian factor $1 + D \cos(kq)$ has vanishing nonconstant average because the remaining integrand is a function of $|\sin(kq)|$. This gives (3.26). Variation of the occupied source gives W_D while differentiation of $\mathfrak{f} \propto a^{-2}$ gives W_a .

Since $\langle \dot{X}^2 \rangle_q = \dot{D}^2 / (2k^2)$ the momenta are $p_a = -6a\dot{a}/N$ and $p_D = ma^2\dot{D}/(2Nk^2)$. The Legendre transform is NC . In the unit-lapse chart its equations are

$$\begin{aligned} \dot{a} &= -\frac{p_a}{6a}, & \dot{D} &= \frac{2k^2 p_D}{ma^2}, \\ \dot{p}_D &= -W_D, & \dot{p}_a &= -\frac{p_a^2}{12a^2} + \frac{2K}{a} - 3\rho_v a^2 - W_a. \end{aligned} \quad (3.29)$$

Consequently $\dot{C} = \{C, C\} = 0$. Solving for the negative p_a branch yields (3.28). The endpoint substitution $kq = (\pi/2)u^2$ on a quarter cell removes the square-root endpoint in g . Its averaged energy is smooth in $a, D > 0$. Ordinary differential-equation existence and uniqueness therefore apply on compact subsets of this branch. Finally with

$$\rho_c = \frac{m + K + W}{a^3}, \quad p_c = \frac{2K - aW_a}{3a^3}, \quad H = \frac{\dot{a}}{a},$$

the canonical equations give $\partial_t(K + W) = -2HK + aHW_a$ and hence $\dot{\rho}_c + 3H(\rho_c + p_c) = 0$. $\square$

At $D = p_D = 0$ the same action gives $3H^2 = \rho_v + m/a^3$. At fixed a (3.25) is the complete periodic local RMS response to the retained cold source. The nonzero finite window therefore couples this response to an evolving homogeneous geometry. Its material equation is the allowed-mode projection of (3.22):

$$\ddot{D} + 2H\dot{D} = -\frac{2k}{a} \langle g(X(q)) \sin(kq) \rangle_q \quad (N = 1).$$

Inserting this finite evolution into the equations obtained by allowing arbitrary periodic displacements gives the acceleration defect orthogonal to the retained displacement mode

$$r_X(q) = \frac{g(X(q)) - 2\langle g \sin(kq) \rangle_q \sin(kq)}{a}, \quad \langle r_X \sin(kq) \rangle_q = 0. \quad (3.30)$$

The retained displacement obeys its projected equation exactly. The acceleration $-r_X$ drives the orthogonal displacement components in the enlarged variation space. Its L^∞ and L^2 norms measure the force available to excite these additional material modes. At a zero of the flux g has square-root behaviour. The periodic potential is $C^{1,1/2}$ and its field equation holds weakly with a smooth flux. These spatial defect norms are separate from the Sobolev trajectory estimates for the smooth reference systems. The construction is a nonrelativistic finite-window realization of (3.21). The covariant scalar realization of Proposition 3.2 is governed by its own common source stress.

Material clocks and observational readings. A clock carried by the material state reads duration along its trajectory, parametrized by t in the lapse action. Its macroscopic weak-field line element is

$$H_0^2 ds_{\mathcal{C}}^2 = (1 + 2\Phi) d\tau^2 - a^2 dx^2, \quad \frac{ds_{\mathcal{C}}}{dT} = \sqrt{1 + 2\Phi(X) - u^2}. \quad (3.31)$$

Here $ds_{\mathcal{C}}$ is the proper duration along a material trajectory in this metric. Expanding the clock-carrier action $-m_{\mathcal{C}} \int ds_{\mathcal{C}}$ and summing over conserved comoving mass gives the rest, potential and peculiar kinetic terms of (3.21) to first order in Φ and second order in u . This identifies the leading clock-rate correction associated with the material force. A carried clock with positive proper phase rate $\omega_{\mathcal{C}}$ accumulates

$$\vartheta_{\mathcal{C}}(T) - \vartheta_{\mathcal{C}}(T_0) = \int_{T_0}^T \omega_{\mathcal{C}}(T') \sqrt{1 + 2\Phi(X(T')) - u(T')^2} dT'.$$

The positive frequency belongs to the retained clock state. On a phase congruence $\omega_{\mathcal{C}} = \sqrt{X_{\vartheta}}$, this agrees with $ds_{\mathcal{C}} = d\vartheta/\sqrt{X_{\vartheta}}$. For that phase-current realization, the timelike-domain and cold-source conditions are those of Proposition 3.1. In the RMS action the carried clock records the conserved cold trajectory through the stated metric coupling. For $|2\Phi - u^2| \leq \eta_* < 1$, Taylor's theorem gives the explicit reading error

$$\frac{ds_{\mathcal{C}}}{dT} = 1 + \Phi - \frac{u^2}{2} + R_{\mathcal{C}}, \quad |R_{\mathcal{C}}| \leq \frac{(2\Phi - u^2)^2}{8(1 - \eta_*)^{3/2}}. \quad (3.32)$$

The remainder bounds the square-root expansion in the retained metric. Trajectory errors propagate through the reading map. Metric and cold-source approximations carry their own error terms.

The finite canonical state determines the density, velocity and expansion readings:

$$\begin{aligned} \frac{\rho(X(q))}{\bar{\rho}} &= \frac{1}{1 + D \cos(kq)}, & \delta_{\max} &= \frac{D}{1 - D}, & \delta_{\min} &= -\frac{D}{1 + D}, \\ \frac{v_{\max}}{c} &= \frac{2k|p_D|}{ma}, & \frac{H_{\text{phys}}}{H_0} &= -\frac{p_a}{6a^2}, & \bar{\rho} &= \frac{m}{a^3}. \end{aligned} \quad (3.33)$$

The density here is in the stated response units, and $H_{\text{phys}} = H_0 \mathcal{H}_{\text{R}}$. For light transported on the homogeneous flat metric the background redshift and distances are

$$1 + z = \frac{a_{\text{obs}}}{a_{\text{em}}}, \quad D_H(z) = \frac{c}{H_{\text{phys}}(z)}, \quad D_M(z) = c \int_0^z \frac{dz'}{H_{\text{phys}}(z')}. \quad (3.34)$$

The expansion in these background-distance readings includes the kinetic and binding contributions in (3.24) on the interval covered by the trajectory.

The observable estimates require explicit margins. For example, two positive expansion histories bounded below by $H_{\min} > 0$ on a common interval $0 \leq z \leq z_*$ satisfy

$$|D_M(z) - \tilde{D}_M(z)| \leq \frac{c}{H_{\min}^2} \int_0^z |H_{\text{phys}} - \tilde{H}_{\text{phys}}| dz'.$$

For two material states with $|2\Phi - u^2|, |2\tilde{\Phi} - \tilde{u}^2| \leq \eta_*$, their clock-rate difference is at most

$$\frac{2|\Phi - \tilde{\Phi}| + (|u| + |\tilde{u}|)|u - \tilde{u}|}{2\sqrt{1 - \eta_*}}.$$

Likewise $\delta_{\max}$ has derivative $(1 - D)^{-2}$. These estimates give observable moduli on a state tube with $D < 1$, $H_{\text{phys}} \geq H_{\min} > 0$ and $|2\Phi - u^2| \leq \eta_* < 1$. They transfer source and trajectory errors to the corresponding matter, clock and background readings.

3.4 Moving active window

For a specified matter evolution the window selects its retained spatial representation. Its motion changes that representation along the same material trajectory.

Gaussian smoothing prepares a regular spatial field on which the nonlinear interactions act. These interactions generate correlations and localized profiles through mode products. The composition law fixes the averaging kernel while the multiplication estimates determine the regularity available after interaction. For $\mathbf{A} = -\nabla^2$ on the periodic box and $\sigma > 0$

$$\|e^{\xi\sqrt{\mathbf{A}}}G_\sigma u\|_{H^s} \leq e^{\xi^2/(2\sigma^2)}\|u\|_{H^s}, \quad \xi \geq 0. \quad (3.35)$$

Indeed the maximum of $\xi|k| - \sigma^2|k|^2/2$ is $\xi^2/(2\sigma^2)$. For polynomially tame interactions satisfying the frequency triangle law binary multiplication preserves an analytic weight $e^{\xi\sqrt{\mathbf{A}}}$ with the stated derivative loss. A Gaussian weight $e^{\alpha\mathbf{A}}$ generally transfers to $e^{\alpha A/2}$ [19]. A common analytic bound on an evolution interval then controls its spectral tail on that interval. The finite-particle and piecewise CIC realizations use norms and force estimates appropriate to their equations and spatial regularity. The RMS model uses its weak constitutive field and displacement-force bounds.

Let $A_{\Lambda(\eta)}$ be the activity selector with endpoint $\Lambda(\eta)$

$$A_\Lambda X_\sigma(x) = \int \frac{d^3k}{(2\pi)^3} W_A\left(\frac{k}{\Lambda}\right) \hat{X}_\sigma(k) e^{ik \cdot x}, \quad (3.36)$$

where $W_A(y)$ is close to unity inside the active range and decreases across the frontier. A sharp selector gives an orthogonal spectral partition. A smooth selector as used in the field realization requires the overlap term in its norm budget.

For any linear activity selector A_Λ the additive reconstruction (2.6) of Section 2 is exact. The squared-norm budget obeys

$$\|X_\sigma\|^2 = \|X_A\|^2 + \|X_T\|^2 + \mathcal{O}_A, \quad \mathcal{O}_A(a) = 2 \operatorname{Re} \langle A_\Lambda X_\sigma, (I - A_\Lambda) X_\sigma \rangle, \quad (3.37)$$

where $\mathcal{O}_A$ is the selector overlap correction. It vanishes for the sharp orthogonal partition. For a self-adjoint selector with weights in $[0, 1]$ it is nonnegative and is positive precisely when the field has nonzero power at a strictly intermediate weight. Indeed expanding the squared norm of $X_A + X_T = X_\sigma$ gives (3.37). For a self-adjoint selector the spectral theorem expresses the cross term as the integral of $2w(1 - w)$ against the spectral measure of X_σ . This gives its sign and vanishing criterion including the zero overlap of an orthogonal partition.

The split rearranges the description of one material state. A mode whose active weight decreases retains its contribution through the residual component and a smooth transition shares that mode between the two components. The overlap term accounts for this shared amplitude in quadratic observables.

For a bounded linear reading R let $Q = R^*R$ and $\mathcal{Q}(X) = \|RX\|^2$. The same additive reconstruction gives

$$\mathcal{Q}(X_\sigma) = \mathcal{Q}(X_A) + \mathcal{Q}(X_T) + 2 \operatorname{Re}\langle X_A, QX_T \rangle. \quad (3.38)$$

The cross contribution is bounded in absolute value by $2\sqrt{\mathcal{Q}(X_A)\mathcal{Q}(X_T)}$. For a sharp selector it vanishes when Q commutes with that selector. A spatially weighted reading can couple the two spectral blocks in which case its cross contribution retains their relative phase. Equation (3.38) follows by expanding $\|RX_A + RX_T\|^2$. It is the quadratic-observation account associated with the additive state reconstruction [19]. The norm budget and the energy balance derived from a particular matter action each retain their own specified quadratic or variational form.

At fixed σ the exact coarse-grained evolution term is

$$\mathcal{F}_\sigma[X] = G_\sigma \mathcal{N}[X]. \quad (3.39)$$

For nonlinear matter dynamics this term depends on the underlying trajectory X . The chain-rule identity separates the evolution from smoothing-scale motion and selector motion before any feedback approximation is made.

Proposition 3.4 (Moving-window identity). Let X solve Eq. (3.9) and let the state path $X(\eta)$ and the smoothing scale $G_{\sigma(\eta)}$ together with the selector $A_{\Lambda(\eta)}$ be differentiable along the path in the norm in which the identity is read. Then the active field in Eq. (2.6) satisfies the exact chain-rule identity

$$\partial_\eta X_A = A_\Lambda \mathcal{F}_\sigma[X] + \dot{\sigma} A_\Lambda (\partial_\sigma G_\sigma) X + \dot{\Lambda} (\partial_\Lambda A_\Lambda) G_\sigma X. \quad (3.40)$$

If the resolved active model is denoted by $\widehat{\mathcal{N}}_\sigma[X_A]$ the same identity can be written as

$$\partial_\eta X_A = A_\Lambda \widehat{\mathcal{N}}_\sigma[X_A] + \mathcal{R}_X[X; X_A] + \dot{\sigma} A_\Lambda (\partial_\sigma G_\sigma) X + \dot{\Lambda} (\partial_\Lambda A_\Lambda) G_\sigma X. \quad (3.41)$$

where

$$\mathcal{R}_X[X; X_A] = A_\Lambda \mathcal{F}_\sigma[X] - A_\Lambda \widehat{\mathcal{N}}_\sigma[X_A]. \quad (3.42)$$

Proof. Differentiate $A_{\Lambda(\eta)} G_{\sigma(\eta)} X(\eta)$ with respect to η and use Eq. (3.9). Equation (3.41) follows by adding and subtracting $A_\Lambda \widehat{\mathcal{N}}_\sigma[X_A]$. $\square$

The first term of (3.40) is the matter evolution viewed through the current window. The other terms describe the change of that view. One varies the smoothing length and the other moves the boundary of explicit retention. They are fixed by the chosen representation and the underlying state. Omitting them would ascribe a change of resolution to the dynamics of matter. The field realizations use a fixed observation scale

$$\sigma(\eta) = \sigma_G, \quad \dot{\sigma} = 0, \quad (3.43)$$

so that active-window motion is carried by $\Lambda(\eta)$ while the term proportional to $\dot{\sigma}$ remains part of the general state equation.

At fixed time and coefficients write $\mathcal{N}[X] = f_0 + \mathcal{N}_{\sigma,1}X + \mathcal{N}_{\sigma,2}(X, X)$ with linear and bilinear parts as in the fluid realization. The coarse-grained evolution term decomposes as

$$\mathcal{F}_\sigma[X] = \mathcal{N}[X_\sigma] + \tau_\sigma[X], \quad \tau_\sigma[X] = G_\sigma \mathcal{N}[X] - \mathcal{N}[G_\sigma X], \quad (3.44)$$

where τ_σ is the subfilter stress. The Gaussian is injective but its inverse on the unrestricted range is unbounded. Stable evaluation of this stress from finitely many coefficients requires a bound on the unresolved contribution. Inserting the exact additive reconstruction of Eq. (2.6) into the evolution term gives the residual feedback expansion

$$\begin{aligned} \mathcal{R}_X &= A_\Lambda [\mathcal{N}_{\sigma,1}X_T + \mathcal{N}_{\sigma,2}(X_A, X_T) + \mathcal{N}_{\sigma,2}(X_T, X_A) + \mathcal{N}_{\sigma,2}(X_T, X_T)] \\ &\quad + A_\Lambda \tau_\sigma[X] + \text{model defect}, \end{aligned} \quad (3.45)$$

where the model defect is $A_\Lambda \mathcal{N}[X_A] - A_\Lambda \widehat{\mathcal{N}}_\sigma[X_A]$. To obtain this expansion substitute $X_\sigma = X_A + X_T$ in (3.44) and subtract the resolved model. The constant cancels and bilinearity gives the four displayed tail terms. A sharp projector commuting with the linear operator also annihilates $A_\Lambda \mathcal{N}_{\sigma,1}X_T$.

For a finite computation $A_\Lambda \widehat{\mathcal{N}}_\sigma[X_A]$ advances the resolved model and the residual record $\mathcal{M}_T$ supplies the coordinates of a closure map

$$\mathcal{R}_X[X; X_A] = \mathfrak{C}_X[X_A, \mathcal{M}_T; \Theta_r] + \epsilon_X, \quad (3.46)$$

where Θ_r specifies the chosen residual representation and ϵ_X is its error relative to the exact feedback above. A dimensionless measure of the error is

$$\mathcal{E}_{\text{cl}}(a) = \frac{\|\epsilon_X(a)\|_A}{\|A_\Lambda \widehat{\mathcal{N}}_\sigma[X_A](a)\|_A + \|\mathfrak{C}_X[X_A, \mathcal{M}_T](a)\|_A + \epsilon_0}, \quad (3.47)$$

with a fixed norm and positive normalization floor ϵ_0 . The overlap correction $\mathcal{O}_A$ enters the quadratic power budget independently of this equation defect. Tail power and shell moments determine the chosen spectral residual coordinates. Object observables complete the retained description. Their feedback error enters the reconstructed equation defect.

Equation (3.45) identifies the information needed by the closure. Mixed active-tail interactions and projected tail self-interactions enter alongside the stress created by physical smoothing. Their unrepresented contribution is ϵ_X in (3.46). This decomposition guides the choice of residual coordinates and the estimate of their feedback error. In particular short spatial variations can combine into an active long-wavelength response. Their contribution depends on relative phases as well as amplitudes. Eliminating the corresponding coordinates also retains their initial state and their subsequent forcing through the history dependence of the feedback. A finite set of residual moments approximates this influence. Its accuracy is determined by the reconstructed equation defect.

3.5 Endpoint motion and bandwidth with tail power

The smoothing length σ fixes the physical spatial averaging. The nonlinear endpoint supplies a reference for the change from field modes to object and residual descriptions. The accepted active boundary Λ also depends on the chosen observables and their tolerance while the numerical cutoff specifies the available spectral space. The reference nonlinear scale is defined by

$$\Delta^2(k_{\text{NL}}, a) = C_{\text{NL}}, \quad \Delta^2(k, a) = \frac{k^3 P(k, a)}{2\pi^2}. \quad (3.48)$$

The dimensionless power measures the variance per logarithmic wavenumber interval. An order-one level identifies spatial scales on which the small-contrast description becomes inadequate and an object or residual description becomes relevant. The value $C_{\text{NL}} = 1$ is used as the default convention with sensitivity to $C_{\text{NL}} = 0.5$ and $C_{\text{NL}} = 2$ reported in the numerical tables. Object-level comparisons use collapse thresholds and virial overdensities. C_{NL} fixes this reference level-set convention. At epochs where the level set has a least positive crossing in the specified spectral interval define

$$k_{\text{NL}}(a) = \min\{k : \Delta^2(k, a) = C_{\text{NL}}\}. \quad (3.49)$$

A continuous branch is followed between changes of crossing. For a tabulated spectrum the crossing is enclosed by adjacent samples and evaluated with the specified interpolation.

The corresponding real-space and mass scales are

$$R_{\text{NL}}(a) = \frac{c_M}{k_{\text{NL}}(a)}, \quad M_{\text{NL}}(a) = \frac{4\pi}{3} \bar{\rho}_{m,0} R_{\text{NL}}^3(a), \quad (3.50)$$

where c_M specifies the Fourier-to-real-space mass convention used in a given realization.

The reference endpoint is a level-set curve of the specified spectrum. Let

$$F(k, a) = \ln \Delta^2(k, a) - \ln C_{\text{NL}}. \quad (3.51)$$

Along the endpoint curve $F(k_{\text{NL}}(a), a) = 0$. When the spectrum is differentiable near k_{NL} differentiating this level-set condition gives

$$0 = \partial_{\ln a} \ln \Delta^2(k_{\text{NL}}, a) + \frac{d \ln k_{\text{NL}}}{d \ln a} \partial_{\ln k} \ln \Delta^2(k, a) \Big|_{k=k_{\text{NL}}(a)}. \quad (3.52)$$

Solving for the endpoint velocity gives

$$\frac{d \ln k_{\text{NL}}}{d \ln a} = - \frac{\partial_{\ln a} \ln \Delta^2(k, a)}{\partial_{\ln k} \ln \Delta^2(k, a)} \Big|_{k=k_{\text{NL}}(a)}, \quad \partial_{\ln k} \ln \Delta^2(k_{\text{NL}}, a) \neq 0, \quad (3.53)$$

on a branch with nonzero spectral slope. A vanishing slope makes this quotient formula inapplicable. The branch may still have a derivative determined by higher-order terms. Branch changes are treated as events. For tabulated spectra we use the discrete version

$$v_{\Lambda}(a_i) = \frac{\ln k_{\text{NL}}(a_{i+1}) - \ln k_{\text{NL}}(a_{i-1})}{\ln a_{i+1} - \ln a_{i-1}}, \quad (3.54)$$

with one-sided differences at the first and last stored epochs. The quantity v_Λ measures the logarithmic rate of change of this reference endpoint.

In the chosen background and growing-mode spectrum the endpoint moves from small comoving scales at high redshift toward lower wavenumbers. By (3.50) this motion increases the assigned nonlinear mass. Thus M_{NL} and v_Λ describe the mass scale and velocity of the frontier. The same level-set equation gives the forecast branch at $a > 1$ under the specified background. Its slowing follows from the reduced growth rate in the dark-energy-dominated regime. The decrease of k_{NL} therefore accompanies the growth of the characteristic spatial and mass scales. Smaller structures continue to contribute through their fields and object moments together with residual feedback. The reference logarithmic span is

$$S_{\text{AW}}(a) = \log \frac{k_{\text{NL}}(a)}{k_{\text{box}}}. \quad (3.55)$$

It measures the logarithmic range between the lowest nonzero wavenumber of the specified periodic volume and the nonlinear reference endpoint. It is the active bandwidth when $\Lambda = k_{\text{NL}}$. We use S_{AW} as a bandwidth coordinate. The Shannon entropy of a power distribution additionally depends on its normalized weights. A uniform distribution on N_A modes has entropy $\log N_A$. For a run whose numerical representation reaches k_{cut} the truncation deficit relative to the nonlinear reference scale is

$$D_S(a, k_{\text{cut}}) = \max \left\{ 0, \log \frac{k_{\text{NL}}(a)}{k_{\text{cut}}} \right\}. \quad (3.56)$$

The power fraction beyond this reference endpoint is

$$\Omega_{\text{tail}, P}(a) = \frac{\int_{k_{\text{NL}}(a)}^{k_{\text{max}}} \Delta^2(k, a) \, d \ln k}{\int_{k_{\text{box}}}^{k_{\text{max}}} \Delta^2(k, a) \, d \ln k}. \quad (3.57)$$

The same split can be applied to velocity or potential or phase-space moments when those fields are available.

In these formulas the reference span is read on the branch $k_{\text{NL}} \geq k_{\text{box}}$. The tail integral is over $[k_{\text{box}}, k_{\text{max}}] \cap (k_{\text{NL}}, \infty)$ with an empty interval contributing zero and its denominator is positive. Thus the tail fraction remains between zero and one when the endpoint leaves the sampled band.

The three diagnostics measure the reference logarithmic bandwidth and its unresolved range. Exterior spectral power records the excitation beyond that boundary. Feedback reconstruction converts the influence of these unresolved modes into an equation-defect bound.

3.6 Reconstruction error and source-response control

The accuracy of a reduced description is measured against the same material trajectory throughout the comparison. A residual map supplies the feedback in the retained equation. The reconstruction places this reduced evolution in the reference state space.

Definition 3.5 (Observable sufficiency and certified closure). Fix a reference trajectory X of (3.9) or of an explicitly specified finite spectral realization. Fix an observable family $\mathfrak{D} \subset \mathfrak{D}_\Lambda$ and an interval $[a_0, a_1]$. A finite-window advance $\tilde{X}$ computed from (3.41) with a fixed residual map

$$\mathfrak{C}_X : (X_A, \mathcal{M}_T, \Theta_r) \mapsto \text{active-sector residual feedback} \quad (3.58)$$

is ε -sufficient when

$$\sup_{O \in \mathfrak{D}} \sup_{a \in [a_0, a_1]} \left| O(X(a)) - O(\tilde{X}(a)) \right| \leq \varepsilon. \quad (3.59)$$

It has certified finite-window closure when a reconstruction and its propagated error estimate establish this inequality over the whole interval. Compression of the initial data contributes an initial error. The reference space and observable family remain fixed at the prescribed tolerance throughout the comparison.

The residual variables are

$$\mathcal{M}_T(a) = \{\Omega_{\text{tail}, P}, R_h, R_v, \mathcal{E}_T(s_a), \mathcal{I}_T(s_a), \mathcal{D}_T(s_a)\}_{s_a \in \mathcal{S}} \quad (3.60)$$

with the normalization in (3.47).

Here R_h and R_v compare object and residual-field velocity activity with the endpoint velocity scale. The variables $\mathcal{E}_T$ and $\mathcal{I}_T$ together with $\mathcal{D}_T$ are moments of residual content and its injection and drain. Their definitions depend on the field and object realization. They enter the feedback approximation through $\mathfrak{C}_X$. Additional coordinates are restored when the propagated error reaches its tolerance.

Changes of window or object representation alter the reconstruction while the reference vector field remains fixed. The estimate applies Gronwall's inequality [9] between reconstruction events and adds each jump in the same state norm. The common reconstruction space permits residual and temporal errors to enter the same observable bound as object replacement errors.

Theorem 3.6 (Reconstructed-state error propagation). Fix a finite spectral realization H_N with a norm and a specified monotone evolution parameter t . Let $x' = F_t(x)$ be its actual evolution. A reduced state of active and residual variables together with object coordinates z has a specified reconstruction $\hat{x} = \mathcal{R}_t z$ in the same H_N . Assume that this reconstruction is piecewise absolutely continuous. Define

$$r(t) = \hat{x}'(t) - F_t(\hat{x}(t)), \quad j_i = \|\hat{x}(t_i^+) - \hat{x}(t_i^-)\|.$$

Assume the tube lies in a compact subset of the open state domain and F satisfies the Caratheodory conditions there with integrable bounds $\ell, d \in L^1$ and finitely many reconstruction events on the declared interval. Suppose F_t is Lipschitz on a certified tube around $\hat{x}$ with nonnegative bound $\ell(t)$ and $\|r(t)\| \leq d(t)$. If the initial error is at most E_0 then

$$\|x(t) - \hat{x}(t)\| \leq E(t) := e^{\int_{t_0}^t \ell} \left[E_0 + \int_{t_0}^t e^{-\int_{t_0}^s \ell} d(s) \, ds + \sum_{t_i \leq t} e^{-\int_{t_0}^{t_i} \ell} j_i \right], \quad (3.61)$$

provided this bound lies strictly inside the tube throughout the interval. If $|O(x) - O(y)| \leq \omega_{\mathfrak{D}}(\|x - y\|)$ is a common nondecreasing modulus on the tube with $\omega_{\mathfrak{D}}(0) = 0$ the observable error is at most $\omega_{\mathfrak{D}}(E(t))$. For uniformly L -Lipschitz readings the bound is $LE(t)$.

Proof. Between events the error norm is absolutely continuous and its derivative is almost everywhere at most $\ell\|x - \hat{x}\| + d$. Integrate with the positive integrating factor and add the jump inequalities to obtain the bound. Apply it up to a first tube exit. Strict interior containment excludes that exit. Finite-dimensional local existence and continuation therefore give the whole certified interval. Apply the observable modulus last. $\square$

For each $O \in \mathfrak{D}$ let $\hat{O}_t$ denote its numerical evaluation and suppose these approximations satisfy $\sup_{O \in \mathfrak{D}} |O(y) - \hat{O}_t(y)| \leq \varepsilon_O(t)$ on the reconstructed state region. Then

$$\sup_{O \in \mathfrak{D}} |O(x(t)) - \hat{O}_t(\hat{x}(t))| \leq \omega_{\mathfrak{D}}(E(t)) + \varepsilon_O(t). \quad (3.62)$$

Insert $O(\hat{x})$ between the two readings and apply the triangle inequality. For a bounded linear reading C with $\|C - \hat{C}\| \leq \epsilon_C$ on the reconstructed range this is $\|Cx - \hat{C}\hat{x}\| \leq \|C\|E + \epsilon_C\|\hat{x}\|$. Thus the trajectory estimate controls the change in the physical state and the last term controls its measured representation [19, 20].

In particular the tube condition and

$$\sup_{t_0 \leq t \leq t_1} \omega_{\mathfrak{D}}(E(t)) \leq \varepsilon/2 \quad (3.63)$$

give the observable accuracy in Definition 3.5 with margin $\varepsilon/2$. Taking the two suprema in the theorem proves the claim and restriction to any subinterval starting at t_0 preserves the same bound. For approximate readings allocating $\sup_t \varepsilon_O(t) \leq \varepsilon/2$ to their evaluation gives total tolerance ε by (3.62).

The cosmological calculations use $t = \ln a$. The same theorem applies in the dimensionless proper-time chart $t = \mu s$ of the common-source example. For a fixed increasing reparametrization $t = t(\tau)$ the transformed field and defect are $F^{(\tau)} = (dt/d\tau)F^{(t)}$ and $r^{(\tau)} = (dt/d\tau)r^{(t)}$. The derivative bounds and integrals transform with the same factor. In (3.61) the initial term measures the discrepancy introduced by the first representation. The integral measures the accumulated mismatch of its evolution. Each jump measures a subsequent change of representation. The stability factor transports all three through the same matter dynamics. The observable modulus then expresses how sensitively the specified measurement responds to the resulting state displacement. This separates the accuracy of a material description from the precision with which it is observed.

Proposition 3.7 (Source-constraint residual and trajectory defect). Let the reference evolution in the chosen clock chart be

$$x' = F(t, x, \Gamma_t(x)), \quad C_H(t, x, \Gamma_t(x)) = 0.$$

On the state tube of Theorem 3.6 suppose C_H is continuously differentiable in its state and source arguments. Assume that the scalar source lies in a common bracket $[\zeta_-, \zeta_+]$ with

endpoint values of C_H of opposite strict signs and that $\partial_\zeta C_H \geq \kappa(t) > 0$ throughout that bracket. Let $C_{H,N}$ and F_N be finite evaluations with uniform bounds

$$|C_H - C_{H,N}| \leq \varepsilon_C(t), \quad \|F - F_N\| \leq \varepsilon_F(t).$$

Assume F is Lipschitz in the source with bound $L_\zeta(t)$. For a reconstructed path $\hat{x}$ and an approximate source $\tilde{\zeta}$ in the bracket define

$$r_C = C_{H,N}(t, \hat{x}, \tilde{\zeta}), \quad r_N = \hat{x}' - F_N(t, \hat{x}, \tilde{\zeta}).$$

Then

$$|\tilde{\zeta} - \Gamma_t(\hat{x})| \leq \frac{|r_C| + \varepsilon_C}{\kappa}, \quad (3.64)$$

$$\|\hat{x}' - F(t, \hat{x}, \Gamma_t(\hat{x}))\| \leq \|r_N\| + \varepsilon_F + \frac{L_\zeta}{\kappa}(|r_C| + \varepsilon_C). \quad (3.65)$$

The right-hand side of (3.65) is an admissible defect d in Theorem 3.6 provided its terms and the reduced-field Lipschitz bound are integrable.

Proof. The endpoint signs and strict monotonicity give a unique root. The mean value theorem between that root and $\tilde{\zeta}$ bounds their distance by $|C_H(t, \hat{x}, \tilde{\zeta})|/\kappa$. The finite-evaluation bound proves (3.64). Add and subtract $F_N(t, \hat{x}, \tilde{\zeta})$ and $F(t, \hat{x}, \tilde{\zeta})$ then use the source Lipschitz bound. This gives (3.65) for the same reference field at every source update. $\square$

The endpoint signs can themselves be certified by $C_{H,N}(\zeta_-) + \varepsilon_C < 0$ and $C_{H,N}(\zeta_+) - \varepsilon_C > 0$. If C_H is L_C -Lipschitz in the state and the fixed-source state Lipschitz bound is L_x the implicit-function formula gives a reduced-field bound $\ell \leq L_x + L_\zeta L_C/\kappa$. The canonical source equation of Ref. [2, Proposition ‘‘Occupied common background and physical propagation’’] provides a concrete instance.

$$C_H = U_{,\zeta} + 3\mathcal{T} - \mathcal{V}, \quad \partial_\zeta C_H = \mathfrak{h}_H = U_{,\zeta\zeta} + 9\mathcal{T} + \mathcal{V} = \mathfrak{h} + 18\mathcal{T}, \quad (3.66)$$

Here $\mathcal{T} = |p|^2/(2a^6 M)$ and $\mathcal{V} = B|\varphi|^2/2$ with $M = e^{-3\zeta}$ and $B = m_0^2 e^{-\zeta}$. The positive canonical curvature $\mathfrak{h}_H$ controls the root solve at fixed canonical data. The fixed-velocity curvature $\mathfrak{h}$ controls the physical kinetic reduction and has its own positive bound on that common background. These curvatures have distinct roles. The canonical slope measures how strongly the constraint fixes the source at the current state while the reduced kinetic form controls material propagation. In (3.64) a smaller constraint slope amplifies a given residual into a larger source displacement. Through L_ζ that displacement changes the matter evolution. Source reuse is accepted by the residual bound (3.65). A source is recomputed when that bound exhausts its assigned error budget. All bounds must hold between stored states. The finite spectral errors in C_H and in F enter the two displayed evaluation terms. An evaluation error already included in r_N is counted there once.

The defect includes residual modelling and memory truncation together with changes of reconstruction. Temporal interpolation and numerical evaluation supply additional terms.

A normalized residual must first be multiplied by its denominator in (3.47). The contribution of the tail enters through its forcing in the reconstructed equation. Initial compression and reconstruction jumps enter E_0 and j_i . All errors are measured in the same state norm. The memory-truncation and hard-window contributions are derived in Appendix B. For catalogue readings the common modulus is supplied by the certified selection margins of Lemma A.1. Discrete event ambiguities are treated separately as described in Appendix A.

For active coordinates put $D_t = A_{\Lambda(t)} G_{\sigma(t)}$. Primes and dots in the finite-coordinate estimates denote derivatives with respect to t . Regard D_t as a map onto the retained coefficient space. On a fixed finite retained set let R_t be a differentiable right inverse $D_t R_t = I$ and write $y = D_t x$ together with $x = R_t y + e_T$. Direct differentiation gives

$$y' = D_t F_t(R_t y) + \dot{D}_t R_t y + c_t(y, m) + d_T, \quad (3.67)$$

$$d_T = D_t[F_t(x) - F_t(R_t y)] + \dot{D}_t e_T - c_t(y, m). \quad (3.68)$$

The term $\dot{D}_t e_T$ retains the interaction between boundary motion and reconstruction error. The right inverse acts only on the finite set of retained nonzero multipliers.

Write the realized vector field in fixed weighted coordinates as $F_t(x) = f_t + L_t x + B_t(x, x)$ including expansion and gravity. Let $\ell_N(t) = \|L_t\|$ and $\|B_t(u, v)\| \leq b_N(t) \|u\| \|v\|$. Expansion of the quadratic difference gives the same-norm bound $\ell(t) = \ell_N(t) + 2b_N(t)R$ on a ball of radius R . Weighted Frobenius norms of the finite matrix and interaction tensor provide explicit upper bounds in Euclidean coordinates. These bounds are computed from the coefficients of the chosen matter evolution. For (3.67) include the derivative of c_t and $\|\dot{D}_t R_t\|$. With fixed σ a selector equal to one for $k \leq \Lambda$ and to $\exp[-\log^2(k/\Lambda)/(2\sigma_T^2)]$ above Λ with transition width $\sigma_T > 0$ gives

$$\|\dot{D}_t R_t\| \leq |\partial_t \log \Lambda| \frac{\max_{k \leq k_{\text{cut}}} [\log(k/\Lambda)]_+}{\sigma_T^2}.$$

The bound depends on the endpoint velocity and retained spectral range. Amplitude rescaling also transforms the interaction tensor.

Between consecutive updates the estimate has the form $E_{n+1} \leq \rho_n E_n + \iota_n$ where $\rho_n = \exp(\int_{t_n}^{t_{n+1}} \ell)$ and ι_n contains the propagated defect and reconstruction jumps. Its coefficients come from the actual finite vector field and the chosen observable modulus. This is the local counterpart of the certificate propagation in Ref. [19]. The factor $\rho_n \geq 1$ propagates previously accumulated error. The defect bound determines the admissible retention and step size.

4 Reduced evolution and window selection

A reduced evolution advances the retained variables and reconstructs their residual feedback. Acceptance depends on the propagated equation defect. For a localized concentration of matter this asks whether its effect on the surrounding state is still represented with the required accuracy as its internal structure changes. The retained description can therefore vary during evolution while the reference matter equations remain fixed. The field and particle realizations use a common step and window acceptance test.

4.1 Reduced state and cosmological background

At an epoch a_n the reduced coordinates z_n represent the active field and residual variables together with object moments of (2.4). Their reconstruction $\mathcal{R}_{a_n} z_n$ lies in the reference space used by Theorem 3.6. An approximate advance is accepted when the propagated bound remains inside the state tube and the observable tolerance. Otherwise the retained space is enlarged or the step is shortened.

The spectral-field and particle examples use a spatial matter spectrum with an envelope normalized at the endpoint and a specified background growth law. The matter and vacuum densities come from the companion calculation [2] and its numerical realization [14]. The same calculation supplies the endpoint entropy span and neutrino mass floor with the Hubble scale. The field and particle calculations use the supplied full-precision parameters recorded with each realization. Each record specifies a numerical value with its inherited readout and units. It also identifies the reference epoch. The neutrino mass floor refers to the rest-mass spectrum of Ref. [2]. In the species-resolved background its thermal occupation energies are evaluated from that same spectrum and the specified momentum distribution. The spectral envelope and finite-window endpoint convention are part of the reference cosmological model with the entropy parameter fixing the endpoint span. The internal spectrum supplies the matter and source parameters. The spatial spectrum specifies how the initial inhomogeneity is distributed among wavelengths. These are different inputs to the cosmological realization. The amplitudes and phases of the spatial perturbations belong to the specified initial data on this background.

The single-component reference assigns the entire non-vacuum endpoint budget to one pressureless component.

$$\Omega_m \equiv \Omega_m^{\text{eff}} = \Omega_b + \Omega_\Sigma \simeq 0.314956, \quad \bar{\rho}_m(a) = \rho_{\text{crit},0} \Omega_m^{\text{eff}} a^{-3}. \quad (4.1)$$

The merge identifies the component velocities and density contrasts and uses this same density in the background and Poisson equation. This defines the effective reference problem for the reported growth and particle trajectories together with their error bounds. The common velocity makes the merged component a single transported material measure. It allows expansion and clustering to be compared with window selection within one evolution with the same mass normalization in each calculation.

The resolved endpoint budget of Refs. [2, 14] is

$$\Omega_\Sigma = \Omega_{\Sigma,\text{cold}} + \Omega_\gamma + \Omega_\nu, \quad (4.2)$$

where Ω_ν denotes the total neutrino energy fraction at the present epoch. In a species-resolved evolution the expansion is determined by the total homogeneous density while the Poisson source contains the perturbations of the species included in the clustering model. In particular the resolved background is

$$E_{\text{res}}^2(a) = \Omega_\Lambda + (\Omega_b + \Omega_{\Sigma,\text{cold}})a^{-3} + \Omega_\gamma a^{-4} + \frac{\bar{\rho}_\nu(a)}{\rho_{\text{crit},0}}. \quad (4.3)$$

Writing the merged expansion as $E_{\text{eff}}^2 = \Omega_\Lambda + \Omega_m^{\text{eff}} a^{-3}$ subtraction gives

$$E_{\text{res}}^2(a) - E_{\text{eff}}^2(a) = \Omega_\gamma(a^{-4} - a^{-3}) + \frac{\bar{\rho}_\nu(a)}{\rho_{\text{crit},0}} - \Omega_{\nu,0}a^{-3}, \quad (4.4)$$

where $\Omega_{\nu,0} = \bar{\rho}_\nu(1)/\rho_{\text{crit},0}$. Both backgrounds therefore have the same normalization at $a = 1$. Photon redshifting and the neutrino transition between relativistic and nonrelativistic motion determine their subsequent difference. The merged pressureless background of (4.1) defines the computations reported here. A comparison with the resolved system reconstructs the individual species on a common state space. Their background difference changes the expansion coefficients of the transport equation while their density perturbations change its gravitational source. The two contributions are evaluated in the same state norm and propagated as equation defects by Theorem 3.6. The reported numerical enclosures concern the merged reference with its specified initial data and force. The spatial inhomogeneity has an envelope whose normalization fixes the present nonlinear scale.

$$P_{\text{sp}}(k, a) = A_* D^2(a) \left(\frac{k}{k_*} \right)^{n_s} W_{\text{UV}}(k), \quad (4.5)$$

$$W_{\text{UV}}(k) = \left[1 + \left(\frac{k}{k_{\text{UV}}} \right)^\alpha \right]^{-\beta}, \quad (4.6)$$

$$\Delta^2(k_{\text{NL},0}, 1) = C_{\text{NL}}. \quad (4.7)$$

In this spectrum the amplitude A_* is fixed by (4.7). The numerical examples use $k_{\text{NL},0} = 0.2 h \text{ Mpc}^{-1}$ and $k_{\text{UV}} = 20 h \text{ Mpc}^{-1}$ with $\alpha = 2$ and $\beta = 2$. The envelope controls ultraviolet decay. The finite box and declared cutoff specify the sampled spectral carrier. The growth factor is computed from

$$D(a) = \frac{\mathcal{D}(a)}{\mathcal{D}(1)}, \quad \mathcal{D}(a) = \frac{5\Omega_m}{2} E(a) \int_0^a \frac{da'}{a'^3 E^3(a')}, \quad E(a) = (\Omega_m a^{-3} + \Omega_\Lambda)^{1/2}. \quad (4.8)$$

This expression is used for the past branch and the present epoch as well as the forecast branch with $a > 1$. The spectral parameters and growth law are specified with the numerical data [14].

The spectrum used by the field generator is the Gaussian coarse-grained spectrum

$$P_\sigma(k, a) = |G_{\sigma(a)}(k)|^2 P_{\text{sp}}(k, a), \quad \Delta_\sigma^2(k, a) = \frac{k^3 P_\sigma(k, a)}{2\pi^2}. \quad (4.9)$$

The activity transition is applied after this Gaussian smoothing as prescribed by (2.6). The envelope sets the distribution of spatial power. Gaussian smoothing specifies its resolution. The activity weight determines which resolved coefficients are advanced explicitly. The Gaussian composition law fixes the smoothing family under its stated assumptions. It leaves the choice of the initial power spectrum as part of this realization.

At each epoch we evaluate the endpoint and band powers in the same units. Residual moments and object observables use these conventions. The band decomposition of Section 2.3 resolves these quantities by wavenumber and mass.

4.2 Three-dimensional spectral realization

The higher-resolution run realizes the finite-window state on a periodic three-dimensional grid. It isolates the relation between spectral weighting and the locations and sizes of visible peaks. Their amplitudes are evaluated in the same realization. Using the same seeded modes across epochs permits this relation to be followed as the endpoint moves. For grid size N and box length L_f with cell spacing $\Delta x = L_f/N$ the generated signed spectral field is

$$\delta_N(x, a) = \frac{c_N(a)}{N^3} \sum_{k \in \mathcal{K}_N} W_A(k, a) P_\sigma^{1/2}(k, a) \xi_k e^{ik \cdot x}, \quad (4.10)$$

where $\mathcal{K}_N$ is the discrete Fourier lattice and ξ_k is the seeded phase-amplitude field with the Hermitian symmetry required for a real field. The positive factor $c_N(a)$ imposes the target sample variance. The 512³ realization has $\delta_{\min}(z=0) \simeq -13.835$. It is therefore a signed spectral field used to study window–peak relations. The particle realization uses a positive matter measure. In this field experiment a peak records where the retained modes interfere constructively. Its variation measures how the selected spatial description organizes the field. The spectrum P_σ is defined in (4.9). The activity transition weight realizing A_Λ is

$$W_A(k, a) = \begin{cases} 1, & 0 < k \leq \min[k_{\text{NL}}(a), k_{\text{cut},f}], \\ \exp\left[-\frac{1}{2} \left(\frac{\ln[k/k_{\text{NL}}(a)]}{\sigma_T}\right)^2\right], & k_{\text{NL}}(a) < k \leq k_{\text{cut},f}, \\ 0, & k > k_{\text{cut},f}. \end{cases} \quad (4.11)$$

The field is normalized at each stored epoch to the variance implied by the windowed spectrum

$$\sigma_{\delta,f}^2(a) = \int_{k_{\min,\text{tab}}}^{k_{\text{cut},f}} \Delta_\sigma^2(k, a) W_A^2(k, a) d \ln k. \quad (4.12)$$

The factor $c_N(a)$ is $\sigma_{\delta,f}(a)$ divided by the unscaled sample standard deviation. Here $k_{\min,\text{tab}} = 10^{-4} h \text{ Mpc}^{-1}$ is the lowest positive wavenumber in the supplied spectral table. The normalization uses the positive tabulated spectrum with trapezoidal quadrature in $\ln k$. The periodic realization has its own discrete box modes. Both the physical smoothing and the activity transition contribute to the target variance. The realization uses $N = 512$ and $L_f = 512 h^{-1} \text{ Mpc}$ with cell spacing $\Delta x = L_f/N$ and Nyquist wavenumber $k_{\text{Ny}} = \pi/\Delta x$. Its reference nonlinear endpoint at the present epoch is $0.2 h \text{ Mpc}^{-1}$. Shell powers and active and residual fractions are evaluated over 24 epochs. Peak statistics use the same epoch grid.

Objects are extracted from high-amplitude peaks of the generated field. With threshold $\nu\sigma_{\delta,f}$ for $\nu > 0$ the selected peak set is

$$\mathcal{P}(a_n) = \{x_i \in \mathcal{G}_N \mid \delta_N(x_i, a_n) > \nu\sigma_{\delta,f}(a_n), \\ \delta_N(x_i, a_n) \geq \delta_N(x_j, a_n) \text{ for neighbouring } x_j\}. \quad (4.13)$$

A finite patch around each selected peak gives the object mass and size proxies. Local gradients determine the velocity and spin proxies. For the support S_h of cells in that patch

with $\delta_N > \nu\sigma_{\delta,f}$ the mass readout is

$$M_h^{\text{fld}} = \bar{\rho}_{m,0}(\Delta x)^3 \sum_{x_i \in S_h} [1 + \delta_N(x_i, a_n)]_+, \quad [q]_+ = \max(q, 0), \quad (4.14)$$

where $\bar{\rho}_{m,0}$ is the mean comoving matter density used for unit conversion. On the selected support $1 + \delta_N > 1$ so the positive-part operation leaves the summands unchanged. This mass is a local peak observable of the signed field.

For growth links the preceding catalogue is restricted to the current peak's mass band and ordered by peak rank. The current peak's snapshot rank selects an entry in that list capped at its last available entry. Thus the matching can be many-to-one. Mass ratios and the associated finite-difference indicators describe growth and formation. Balance and stripping are read from the same band-and-rank matching. Where no matched nonzero mass difference is available the initialized local-activity proxy is retained. Particle-member overlap ancestry is instead supplied by the PM calculation. Neighbour strengths and spin-alignment diagnostics supplement the shell powers and growth statistics. The full numerical parameters and these observables are supplied in the supplementary material [14].

4.3 Particle-mesh dynamics

Material labels permit a different reading of structure growth. The same mass can be followed as it enters or leaves or is reassigned between localized concentrations. This identifies object histories through transported members and connects catalogue changes to the underlying mass measure.

The particle realization advances equal-mass particles with fixed labels α [25]. Its variables are comoving position x_α and $u_\alpha = dx_\alpha/da$. Here u_α is a scale-factor derivative whereas u in (3.4) is peculiar velocity. At epoch a_n the state is

$$Y_{\text{PM}}(a_n) = \{x_\alpha(a_n), u_\alpha(a_n), m_\alpha\}_{\alpha=1}^{N_p}, \quad (4.15)$$

with fixed $m_\alpha = m_p$. The mesh density is obtained by cloud-in-cell assignment

$$1 + \delta_{\text{PM}}(x_g, a_n) = \frac{1}{\bar{n}_g} \sum_{\alpha=1}^{N_p} W_{\text{CIC}}(x_g - x_\alpha(a_n)), \quad (4.16)$$

with $\bar{n}_g = N_p/N_m^3$. The normalized mesh potential is solved in Fourier space by

$$\Phi_{\text{PM}}(k, a_n) = -\frac{\delta_{\text{PM}}(k, a_n)}{k^2}, \quad \Phi_{\text{PM}}(0, a_n) = 0. \quad (4.17)$$

The acceleration field follows from the same spectral representation

$$g_i(k, a_n) = -ik_i \Phi_{\text{PM}}(k, a_n), \quad g_\alpha(a_n) = \mathcal{I}_{\text{CIC}}[g](x_\alpha), \quad (4.18)$$

where $\mathcal{I}_{\text{CIC}}$ denotes interpolation back to particle positions. With comoving coordinates measured in $h^{-1}\text{Mpc}$ the normalized potential and its gradient carry squared-length and length units respectively. The dimensional gravitational coefficient is removed from

Eq. (4.17) and included in the evolution prefactor. CIC assignment and interpolation set the spatial smoothing of the mesh force. The examples use the finite Fourier modes of the Poisson solve without an additional softening length. Between two stored scale factors the finite comparison model uses a kick-drift-kick update

$$u_\alpha^{m+1/2} = u_\alpha^m + \frac{\Delta a}{2} Q(a_{m+1/2}) g_\alpha^m, \quad (4.19)$$

$$x_\alpha^{m+1} = x_\alpha^m + \Delta a u_\alpha^{m+1/2} \pmod{L_{\text{PM}}}, \quad (4.20)$$

$$u_\alpha^{m+1} = u_\alpha^{m+1/2} + \frac{\Delta a}{2} Q(a_{m+1/2}) g_\alpha^{m+1}. \quad (4.21)$$

The kick-drift-kick sequence is second order on subintervals with a sufficiently smooth force and no event. CIC cell crossings and coefficient corners require event treatment or explicit step-refinement verification. The state at the end of each step is the post-kick pair (x_α, u_α) . The finite comparison model uses

$$Q(a) = \frac{q_{\text{PM}}}{\max(a, a_{\text{floor}})}, \quad a_{\text{floor}} = 10^{-4}. \quad (4.22)$$

This prefactor defines the finite system $x_a = u$, $u_a = Q(a)g$. Its initial positions are potential displacements of a uniform particle lattice as in the Zel'dovich construction [26] with a specified amplitude and initial velocity multiplier.

The cosmological realization follows by transforming (3.7) with $u = x_a$.

$$u_a = -\left(\frac{3}{a} + \frac{E'(a)}{E(a)}\right)u + \frac{3\Omega_m(a)}{2a^2}g.$$

The expansion determines both the damping and the force coefficient.

The momentum variable

$$p = a^3 E(a)u, \quad \frac{dx}{da} = \frac{p}{a^3 E(a)}, \quad \frac{dp}{da} = \frac{3\Omega_{m,0}}{2a^2 E(a)}g$$

absorbs the damping exactly into the momentum definition. A symmetric kick-drift-kick step integrates the two displayed coefficients over its half-kick and full-drift intervals. The initial growing-mode velocity is $u_0 = f(a_0)\psi/a_0$ where ψ is the specified displacement and $f = d \log D / d \log a$. Thus the background fixes both the force coefficient and the growing-mode initial velocity. This calculation uses the pressureless collisionless reference component of (4.1). The internal-space calculation supplies its endpoint budget. The background and initial growing mode together with the force normalization therefore refer to the same effective component.

Objects in the PM layer are connected components of cells above the stored density threshold with member particles assigned from the same CIC cell map. For an object h let $I_h(a_n)$ be its set of particle labels. The ancestry weight from a candidate parent p to a descendant d is

$$\omega_{p \rightarrow d} = \frac{|I_p(a_i) \cap I_d(a_j)|}{|I_d(a_j)|}, \quad a_i < a_j. \quad (4.23)$$

The selected parent is the candidate with the largest $\omega_{p \rightarrow d}$. The PM object fate coordinate is then

$$B_{h,\text{PM}}^{p \rightarrow d} = t_{\text{proxy},d} \frac{\ln M_d - \ln M_p}{T_d - T_p}, \quad (4.24)$$

with $M_h = m_p |I_h|$. It is the logarithmic growth coordinate evaluated with T the proper cosmic time and t_{proxy} an extent-based time. If R and M are the numerical comoving radius and mass in kpc/h and $M_\odot/h$ respectively define $t_{\text{proxy}} = \sqrt{R^3/(GM)}$ using G expressed in kpc and $M_\odot$ together with the chosen time unit. The corresponding physical time is $a^{3/2} t_{\text{proxy}}/h$ and the same factor converts the growth statistic. These radii measure the spatial extent of the selected component under the connectivity rule.

Neighbouring PM objects are described by strength and alignment diagnostics. For a source p and target q define

$$A_{p \rightarrow q}^{\text{PM}} = \frac{M_p/M_q}{|r_{pq}|^3}, \quad (4.25)$$

$$\tau_{p \rightarrow q}^{\text{PM}} = A_{p \rightarrow q}^{\text{PM}} r_{pq} \times (u_q - u_p), \quad (4.26)$$

$$G_{p \rightarrow q}^{\text{PM}} = \frac{\tau_{p \rightarrow q}^{\text{PM}} \cdot J_q}{|\tau_{p \rightarrow q}^{\text{PM}}| |J_q|}. \quad (4.27)$$

Here r_{pq} is the periodic separation vector and J_q is the target spin proxy obtained from member particle offsets and relative velocities. The pair $(A_{p \rightarrow q}^{\text{PM}}, G_{p \rightarrow q}^{\text{PM}})$ characterizes the geometry of neighbouring objects at an epoch. The strength has inverse-volume units while the alignment is dimensionless.

4.4 Tail activity variables

When a peculiar velocity field is carried by the generated state the field-level activity ratio is

$$R_v(a) = \frac{\text{rms}(P_{>k_{\text{NL}}} v_{\text{pec}})}{v_{\text{vir}}(M_{\text{NL}}, a)}. \quad (4.28)$$

Here the Hubble flow is removed and the weighting convention is recorded. For the object layer the halo-level version is

$$R_h(a) = \frac{\left[\frac{\sum_h M_h \sigma_{v,h}^2 W_h}{\sum_h M_h W_h} \right]^{1/2}}{v_{\text{vir}}(M_{\text{NL}}, a)}. \quad (4.29)$$

Here $\sigma_{v,h}$ is the stored halo velocity scale and W_h is the object selection weight. The default selection is $M_h \leq M_{\text{NL}}$ with $W_h = 1$ for selected objects and $W_h = 0$ otherwise. Smooth mass windows are recorded when used. The selected total mass must be positive. An empty selection is recorded as an undefined reading. Mass-cut sensitivity uses the same selection and velocity conventions. The virial normalization is

$$R_{\text{vir}}(M, a) = \left[\frac{3M}{4\pi \Delta_h(a) \rho_{\text{ref}}(a)} \right]^{1/3}, \quad v_{\text{vir}}(M, a) = \left[\frac{GM}{R_{\text{vir}}(M, a)} \right]^{1/2}. \quad (4.30)$$

The mass-radius conversion requires a reference density and an overdensity. The virial scaling analysis of Bryan and Norman [27] provides a standard comparison for these conventions. The velocity estimator and mass weighting are fixed in each comparison. The PM objects are connected components of thresholded mesh cells as specified in Section 4.3. This is a grid connectivity rule. Particle friends-of-friends [28] provides a related comparison based on particle linking lengths.

The normalization by $v_{\text{vir}}(M_{\text{NL}}, a)$ compares field and object velocity activity with the same endpoint mass scale. Each ratio retains its specified weighting and selection convention.

4.5 Residual moments and re-entry

During structure growth part of the spatial field can be represented by aggregate quantities while its interactions remain relevant to the surrounding matter. Residual moments describe the distribution of this unresolved content across scales. Re-entry is required when the retained moments no longer give an adequate account of its force or transport feedback. The decision is consequently tied to the evolved observables and their accuracy as well as to the amount of power outside the window.

Let the scale shells be indexed by j with representative wavenumbers k_j and logarithmic width $\Delta \ln k_j$. Let $J(a)$ denote the shell containing the active endpoint. For a field diagnostic with shell contribution $E_j(a)$ define the Laplace shell moments

$$\mathcal{E}_J(s, a) = \sum_{j>J(a)} E_j(a) e^{-s(j-J(a))}, \quad s \geq 0. \quad (4.31)$$

For a generated transfer budget the analogous moment is

$$\mathcal{I}_J(s, a) = \sum_{j>J(a)} I_j(a) e^{-s(j-J(a))}, \quad (4.32)$$

where I_j records net injection into the shell. The corresponding drain moment is

$$\mathcal{D}_J(s, a) = \sum_{j>J(a)} D_j(a) e^{-s(j-J(a))}. \quad (4.33)$$

The value $s = 0$ gives the total tail content. Positive s weights the shells nearest the endpoint more strongly. A finite set of these moments provides residual diagnostics. Their adequacy for feedback is tested through the reconstruction error. Different s values resolve how the content is distributed with distance from the active boundary. The injection and drain moments also indicate whether that distribution is being replenished or depleted. Relative phases or other variables needed by the actual interaction remain part of the reconstruction when shell totals alone do not determine the feedback.

The residual content can also be split by a reference profile. For a scale ratio $q_j = k_j/k_{\text{NL}}(a)$ define

$$\Omega_j(a) = \frac{E_j(a)}{\sum_{\ell} E_{\ell}(a)} \quad (4.34)$$

and the floor contribution

$$\Omega_T^{\text{floor}}(Q, a) = \sum_{j>J(a)} \chi_G\left(\frac{q_j}{Q}\right) \Omega_j(a), \quad \chi_G(y) = \frac{1}{2} \operatorname{erfc}\left(\frac{\ln y}{\sqrt{2} \sigma_{\log}}\right). \quad (4.35)$$

This is a Gaussian profile in logarithmic scale used to resolve the tail budget. Its dimensionless width $\sigma_{\log} > 0$ and scale ratio $Q > 0$ are fixed profile parameters. The physical smoothing length σ enters the field averaging. Using the same positive shell weights to evaluate $\Omega_{\text{tail},P} = \sum_{j>J} \Omega_j$ the excess activity is

$$\Omega_T^{\text{exc}}(Q, a) = \Omega_{\text{tail},P}(a) - \Omega_T^{\text{floor}}(Q, a). \quad (4.36)$$

The reference and excess contributions are nonnegative since $0 \leq \chi_G \leq 1$ and add to the total tail fraction. The ratios are defined for positive total shell energy. The zero-energy state is treated by its vanishing unnormalized moments. This floor is a reference partition of the measured tail power. Its parameters specify the comparison profile. The weights entering both parts are those of the same material state. The physical residual fraction Ω_{Σ} in the source budget has a separate origin and is fixed by its own source readout.

Let $E_j \geq 0$ with $T = \sum_j E_j \in (0, \infty)$ and $\bar{\mathcal{E}}_J = \mathcal{E}_J/T$ for fixed shell boundaries. Then $0 \leq \bar{\mathcal{E}}_J(s) \leq \bar{\mathcal{E}}_J(0) \leq 1$. For $s > 0$ it is completely monotone and log-convex when nonzero. Its derivative moment matrices

$$\left[(-1)^{p+q} \partial_s^{p+q} \bar{\mathcal{E}}_J(s)\right]_{p,q=0}^m$$

are positive semidefinite. At fixed state a one-shell boundary move gives

$$\mathcal{E}_{J+1}(s) = e^s \mathcal{E}_J(s) - E_{J+1}.$$

The normalized bounds follow from positive weights at most one. Termwise differentiation is justified for $s > 0$ since every polynomial in $j - J$ times its exponential weight is bounded. A quadratic form in the moment matrix is $T^{-1} \sum_{j>J} E_j e^{-s(j-J)} |\sum_p c_p (j - J)^p|^2 \geq 0$. Its two-by-two minor gives log-convexity. Shifting the series index gives the boundary identity.

These relations constrain the distribution of residual power. Its effect on the evolution depends additionally on the interaction tensor and on the state norm used for reconstruction.

Proposition 4.1 (Residual-state feedback bound). For the actual finite vector field $F = Lx + B(x, x) + f$ let $\|B(u, v)\| \leq b_N \|u\| \|v\|$ in the chosen state norm and let D be the linear coordinate map. If v is a reconstruction with $\|x - v\| \leq T_e$ then

$$\|D[F(x) - F(v)]\| \leq \|D\| \{ \|L\| T_e + b_N (2\|v\| T_e + T_e^2) \}. \quad (4.37)$$

For a moving reconstruction add $\|\dot{D}\| T_e$ and separately bound any declared closure-map or memory error.

Proof. Set $e = x - v$ and expand $F(x) - F(v) = Le + B(v, e) + B(e, v) + B(e, e)$. The linear and bilinear operator bounds give (4.37). The moving-coordinate contribution is bounded by $\|\dot{D}e\| \leq \|\dot{D}\| T_e$. $\square$

For diagonal weighted Euclidean norms replacing omitted coefficients by zero gives the certified tail norm

$$T_e^2 = \sum_{\text{omitted}} w_k^2 |\hat{x}_k|^2.$$

Thus (4.37) supplies a derivative-error bound from omitted amplitudes measured in the state norm and the finite interaction tensor.

A continuous boundary probe is

$$\mathcal{L}_h(s, \Lambda) = \frac{\int_{\Lambda}^{k_{\max}} \Delta^2(k) e^{-s \log(k/\Lambda)/h} d \log k}{\int_{k_{\min}}^{k_{\max}} \Delta^2(k) d \log k},$$

with h the logarithmic spectral sampling interval. If the spectrum is continuous at Λ and bounded on the integration interval with $s > 0$ a change of variable gives $\mathcal{L}_h(s, \Lambda) \sim h \Delta^2(\Lambda)/(sT)$ as $h \downarrow 0$ when $\Delta^2(\Lambda) > 0$. Thus $\mathcal{L}_h$ measures localization at the endpoint on a sampling-dependent width. Fixed physical shell widths are used when comparing residual contents across resolutions.

The tail activity ratio R_T depends on the available state variables. For a velocity field it is R_v . For the object layer it is R_h . For spectrum tables it can be the excess spectral tail Ω_T^{exc} normalized by the residual floor. The activity channel is fixed throughout each comparison.

A residual band or object class is reconsidered for active retention when an activity threshold is crossed.

$$\frac{|\Lambda(a) - \Lambda(a_n)|}{\Lambda(a_n)} > \theta_{\Lambda}, \quad (4.38)$$

$$R_T(a) > R_{\max}, \quad (4.39)$$

$$\frac{|\mathcal{I}_T(a)|}{\mathcal{D}_T(a)} > \theta_T, \quad (4.40)$$

$$B_h(a) > B_{\max} \quad \text{for a tracked object class.} \quad (4.41)$$

Here a_n is the last endpoint-update epoch and R_T is the chosen tail activity ratio. The quantities $\mathcal{I}_T$ and $\mathcal{D}_T$ are the tail injection and drain budgets. The object indicator B_h is its logarithmic mass-growth rate multiplied by its internal dynamical time. These thresholds locate renewed activity. The interval error test determines whether the resulting representation is accepted. For example a slowly varying compact object may remain represented by its moments while contributing continuously to the force. Rapid mass exchange or a growing residual interaction prompts a new assessment of that representation. Restoring coordinates changes which details are explicitly evolved with the accumulated state error carried into the new representation.

Proposition 4.2 (Re-entry and the least certified window). On a finite reference spectral space order candidate retained windows by their discrete spectral thresholds. Suppose each candidate supplies on the proposed step tube bounds $\bar{\ell}, \bar{d} \geq 0$ together with a current error bound E_n and an end-of-step reconstruction-event bound j_n . Accept it only when

$$B_n(h) = e^{\bar{\ell}h} (E_n + h\bar{d}) + j_n$$

lies inside the tube and $\omega_{\mathcal{D}}(B_n(h)) \leq \varepsilon$. If the acceptance set is nonempty the first accepted candidate is the least certified window in this ordered list. Each accepted interval including its endpoint event is ε -sufficient.

Proof. The expression bounds (3.61) on the whole step since all budget terms are non-negative. Induction over accepted steps proves the error bound including event jumps. A nonempty finite ordered acceptance set has a least element. Monotonicity of the actual errors in window size is unnecessary. $\square$

The least certified window is determined by this candidate list and its error bounds. Its minimality expresses certified sufficiency for a fixed physical question. Actual error and total work remain separate quantities in the comparison. Among the tested descriptions it is the first whose retained variables and feedback control the prescribed readings. Changing those readings or their tolerance can change the selected window for the same state. The acceptance set is nonempty locally when the full finite state is available and its current error bound lies strictly inside both the tube and the observable tolerance while consistent time refinement makes its numerical defect tend to zero. At full retention there is no compression defect so sufficiently short steps pass by continuity. An enlargement is initialized from an enclosed state. If needed the interval is recomputed from the preceding enclosed state. The estimate applies to the specified reference H_N . A comparison with a larger reference includes the additional spectral tail in the defect.

The four activity thresholds determine when to evaluate this interval test. Zero drain is assigned infinite activity when injection is nonzero and zero when both vanish. Positive and negative injections are recorded separately or their absolute sum is used. For each monitored continuous activity use separated admission and release thresholds with gap $\gamma > 0$. If its rate is at most V a return transition takes at least γ/V by the rate bound. Jumps are processed once with their reconstruction budget. This lower bound on the time between transitions depends explicitly on V .

4.6 Mode counts and arithmetic work

Let k_{ref} be a conservative reference cutoff and let Λ_A be the largest active endpoint used by the active-window calculation over the interval. The inactive logarithmic span is

$$\Delta S = \log \frac{k_{\text{ref}}}{\Lambda_A}. \quad (4.42)$$

For isotropic Fourier retention in three spatial dimensions leading mode counting gives the state-size or memory fraction

$$\rho_N \simeq \exp(-3\Delta S) = \left(\frac{\Lambda_A}{k_{\text{ref}}} \right)^3. \quad (4.43)$$

For computations whose timestep or update cadence is controlled by the active endpoint the leading evolution-work fraction is

$$\rho_C \simeq \exp[-(3+q)\Delta S] = \left(\frac{\Lambda_A}{k_{\text{ref}}} \right)^{3+q}, \quad q \geq 0. \quad (4.44)$$

The exponent 3 counts spatial degrees of freedom. The exponent q records the time or refresh cost associated with the highest active scale. The value $q = 0$ corresponds to a scale-independent number of updates. Transport-controlled refresh has $q \simeq 1$. An explicit diffusive update restriction gives $q \simeq 2$. Fast Fourier transform (FFT) work includes its logarithmic factor. These scalings count evolved modes and updates at a common accuracy. The complete comparison also includes residual evaluation and the error estimate

$$\rho_{\text{total}} = \frac{\mathcal{C}_{\text{active}} + \mathcal{C}_{\text{residual}} + \mathcal{C}_{\text{bound}}}{\mathcal{C}_{\text{ref}}}. \quad (4.45)$$

The spectral-field realization measures the storage of selected observables. For the Galerkin examples Proposition B.1 counts the arithmetic work of both evolution and error estimation including every attempted step.

A reduction in total work requires the residual and error-evaluation costs to remain below the work removed from the active evolution. All terms must be compared at the same prescribed accuracy. Such a saving is possible when the influence of omitted coordinates on the retained observables is small or sufficiently slow to admit less frequent evaluation. The residual calculation retains that influence and the propagated bound determines how long the separation remains adequate for the chosen observables.

5 Object dynamics and numerical results

As matter becomes concentrated its effect on the surrounding field can often be described by fewer variables than its internal configuration. The relevant criterion is the change in the force produced by this replacement. A particle partition tracks the same material through object moments and cloud-in-cell (CIC) assignment. Force estimates quantify the resulting change of representation. The numerical examples examine collective-source feedback and evolution with reduced spatial variables. They also relate a computed trajectory to a continuous matter distribution.

5.1 Object representations and material transfer

An object representation retains a localized region through its mass and position with selected spatial moments. These variables are ordered by their contribution to the surrounding force. Total mass fixes the monopole. The mass centre removes the first spatial moment. Higher moments describe the effect of the internal distribution. A compact region can therefore admit an accurate object description for one set of force observations while requiring finer resolution for another. The partition and kernel estimates quantify this dependence on the region and interaction kernel at the required accuracy.

For an object h at epoch a_n the physical object description is

$$\mathcal{O}_h(a_n) = (h, x_h, v_h, M_h, R_{\text{vir},h}, \sigma_{v,h}, J_h, \mathcal{E}_h, \mathcal{L}_h)_{a_n}. \quad (5.1)$$

Here x_h and v_h give the object position and peculiar velocity. The mass and virial radius are M_h and $R_{\text{vir},h}$ under the declared mass convention. The internal velocity scale is $\sigma_{v,h}$ and the angular momentum is J_h . Environmental variables are recorded in $\mathcal{E}_h$. The links

$\mathcal{L}_h$ specify ancestry. This is the object component of $\mathfrak{C}(a_n)$. The numerical realizations specify which entries they estimate and which radius and velocity proxies they use.

Object growth and stripping transfer material between localized structures and their environment. Small net growth indicates slowly varying mass. Binding and internal stability also depend on the velocity and force fields. Resolving these transfers at object scale gives a physical interpretation of the active-tail exchange in (3.41).

Definition 5.1 (Object representation and material balance). An object representation specifies a spatial selection rule with particle membership and ancestry. The retained object variables determine its reconstruction. Physical splitting and merging change the material configuration and are recorded through membership and ancestry. Loss of catalogue identity also depends on the selection and matching rules. The contribution of an object representation to the evolution is evaluated through the reconstructed force with an error bound in the state norm.

For a change of representation at fixed physical state let ΔC_α denote the change of mass or momentum or angular momentum attributed to carrier $\alpha \in \{\text{field}, \text{object}, \text{tail}\}$ with a common origin for angular momentum. Material balance requires

$$\Delta C_{\text{field}} + \Delta C_{\text{object}} + \Delta C_{\text{tail}} = 0. \quad (5.2)$$

Physical forces and fluxes contribute separately during evolution.

In a particle representation the carrier weights partition the material source. Assigning a particle to an object transfers its contribution to that object's reconstructed source. The combined source uses each material contribution once. Ancestry records which material persists across changes in the detected components. This separates a change of description at fixed state from the physical transport that occurs between epochs.

The particle realization uses connected overdensity regions and maximum-overlap ancestry. The spectral-field realization uses the peak and band-and-rank observables of Section 4.2. For a nonnegative object measure with the specified moments the force estimate gives a sufficient accuracy condition for replacement.

The particle partition gives an exact realization of (5.2). The subsequent force estimate controls the effect of replacing each selected component by its retained moments.

Proposition 5.2 (Carrier partition and transfer bounds). At a fixed particle state let nonnegative carrier weights satisfy $\sum_\alpha w_{\alpha i} = 1$. For an additive particle quantity c_i put $C_\alpha = \sum_i w_{\alpha i} c_i$. Changing weights without changing the particles preserves $\sum_\alpha C_\alpha$ for mass and momentum as well as angular momentum about the same origin. This is relabelling conservation. Physical forcing and expansion enter separately during evolution.

For disjoint detected objects of equal-mass particles let $N_O = \sum_h n_h$ and $N_D = N - N_O$. Then $\Delta N_O + \Delta N_D = 0$. If only selected parent links are stored let S_{known} be their shared-particle count with no duplicate intersections. The full object-to-object shared count satisfies

$$S_- := \max(S_{\text{known}}, N_O^- + N_O^+ - N) \leq S \leq S_+ := \min(N_O^-, N_O^+).$$

Diffuse-to-object and object-to-diffuse counts are $N_O^+ - S$ and $N_O^- - S$ giving intervals and an exact net transfer.

Proof. Sum the partition of unity for each particle then subtract the two labellings. The intersection of the old and new object-particle sets contains all recorded disjoint intersections. Inclusion-exclusion and containment give its other bounds. Its two set differences are the stated transfers. $\square$

Proposition 5.3 (Object replacement in the generated force). Let the declared finite spectral force have kernel K_N . A nonnegative object measure ρ_h of mass M_h with arithmetic mass centroid q_h in a fixed unwrapped coordinate chart and support radius $r_{\text{sup},h}$ has monopole replacement error

$$\left\| \int K_N(x - y) d\rho_h(y) - M_h K_N(x - q_h) \right\| \leq \frac{1}{2} C_N M_h r_{\text{sup},h}^2,$$

where C_N bounds $\|D^2 K_N\|$ along the relevant segments. For the finite Gaussian-smoothed periodic kernel a global bound is $C_N \leq \sum_k |k|^2 |\hat{K}_N(k)|$.

Proof. Expand the kernel at $x - q_h$. The first-order term integrates to zero by the centroid definition. Its integral Taylor remainder gives the bound. Differentiate the finite Fourier sum twice for C_N . $\square$

For an arbitrary stored centre retain $m_{1,h} = \int (y - q_h) d\rho_h$ and $m_{2,h} = \int |y - q_h|^2 d\rho_h$. The same proof gives the general bound $C_{1,N} \|m_{1,h}\| + C_{2,N} m_{2,h}/2$. An angular-average periodic centre need not annihilate $m_{1,h}$. If only mass and radius are known use $C_{1,N} M_h r_{\text{sup},h} + C_{2,N} M_h r_{\text{sup},h}^2/2$. The support radius is distinct from the dimensionless activity ratio R_h and from a virial radius. Summing the object errors and multiplying by the actual force/time prefactor and the finite-state embedding constant gives the defect for Theorem 3.6. For m pointwise vector samples the Euclidean embedding constant is $\sqrt{m}$. For their normalized root-mean-square norm it is one. Weighted spectral norms use the corresponding finite transform norm. Applying the estimate along a trajectory requires stepwise enclosures of the masses and support radii. The extent entering this estimate is enclosed by the support radius. Exhaustion of the replacement-error budget triggers finer resolution under Proposition 4.2.

The estimate expresses the physical scale separation behind object replacement. A force kernel that varies slowly across the object is insensitive to most of its internal structure. The centroid cancels the linear variation leaving the kernel curvature multiplied by the second spatial moment. Expansion of the object or increased variation of the resolved force makes these internal degrees of freedom more important and tightens the admissibility of the monopole description.

For the particle-mesh calculation the assignment phase enters the kernel explicitly. Let w_g denote periodic CIC basis functions of spacing h . Let $K_{b,g-\ell}$ be the specified grid-force kernel and put $\mu_i = N_g m_i / M_{\text{tot}}$. The interpolated force is

$$G_b[X](x) = \sum_{g,\ell} w_g(x) K_{b,g-\ell} \left[\sum_i \mu_i w_\ell(X_i) - 1 \right]. \quad (5.3)$$

The constant Fourier multiplier vanishes. The resulting pair kernel is $\mathcal{K}_b(x, y) = \sum_{g, \ell} w_g(x) K_{b, g-\ell} w_\ell(y)$. Within each cell CIC weights are linear in each of the three coordinates. Their products span eight monomials so eight corresponding moments determine the assigned source exactly. This finite set follows from the spatial assignment rule. Retaining it preserves the particles' position dependence relative to the grid which a cell mass alone loses.

Proposition 5.4 (Cell moments and CIC source replacement). In a cell of lower vertex c put $t_d = (X_d - c_d)/h - 1/2$ and retain the eight moments $m_{c,S} = \sum_{i \in c} \mu_i \prod_{d \in S} t_{i,d}$ for $S \subset \{1, 2, 3\}$. The corner source is exactly

$$D_{c,\epsilon} = \sum_S 2^{|S|-3} \prod_{d \in S} (2\epsilon_d - 1) m_{c,S}, \quad \epsilon \in \{0, 1\}^3. \quad (5.4)$$

For an object with nonnegative weights and a reference point q in a fixed periodic lift write $r_i = X_i - q$ and $M = \sum_i \mu_i$. Define

$$A_d = \max_b \|\Delta_d K_b\|_\infty / h, \quad B_{de} = \max_b \|\Delta_d \Delta_e K_b\|_\infty / h^2, \quad J_d = \max_b \|\Delta_d^2 K_b\|_\infty / h.$$

If $C_d(q, r) = \sum_{t \in \mathcal{T}_d} (1 - t)$ sums the crossings of $q + tr$ for $0 \leq t \leq 1$ through d -normal grid faces then

$$\begin{aligned} \sup_x \|G_O(x) - G_{M\delta_q}(x)\|_\infty &\leq \sum_d A_d \left| \sum_i \mu_i r_{i,d} \right| + \sum_{d < e} B_{de} \sum_i \mu_i |r_{i,d} r_{i,e}| \\ &\quad + \sum_d J_d \sum_i \mu_i |r_{i,d}| C_d(q, r_i). \end{aligned} \quad (5.5)$$

At an initial face the derivative is taken from its assigned half-open cell including a jump at $t = 0$ if the segment leaves into the other cell.

Proof. Expanding $\prod_d [1/2 + (2\epsilon_d - 1)t_d]$ gives (5.4). Periodic assembly gives the entire source. Inside a cell the second derivative along a source segment contains only mixed derivatives. Taylor's integral remainder with weight $1 - t$ gives the second term of (5.5). The normal derivative jumps at a face by at most J_d whereas tangential derivatives agree. Integrating these jumps gives the third term. Simultaneous crossings may be resolved one coordinate at a time. Sum the linear terms before taking absolute values to retain the actual first moment. Nonnegative target interpolation has norm one. $\square$

The bound holds with a fixed observation point. Evolving the moments requires their position-momentum moments and face fluxes. Changing the target representation also contributes its reconstruction defect. For unit weights and C occupied cells direct assignment requires $24N$ elementary source operations whereas the moment assembly requires $12N + 40C$ excluding common coordinate work and grouping. The present $N = C = 4096$ trajectory uses direct assignment. At a fixed configuration the eight moments are sufficient for the grid source. During evolution particles carry momentum through the cell faces so

the moment description must also retain or bound the resulting transport. Source sufficiency at one epoch and sufficiency for subsequent dynamics are thus distinct uses of the same representation.

The partition identity preserves the additive quantities at a change of representation. The force bound controls the resulting dynamical approximation through Theorem 3.6. In the particle realizations considered here particles carry the material motion and enter the force update. Object moments provide an alternative reconstruction of that same source. The force estimates quantify the force error produced by this replacement while the particle histories determine the material transferred between objects.

5.1.1 Object growth rate

In the object formulas T is proper cosmic time distinct from the logarithmic evolution coordinate $t = \ln a$. For a halo of mass M_h with virial radius $R_{\text{vir},h}$ and logarithmic mass-growth rate $g_h = d \ln M_h / dT$ define

$$t_{\text{dyn},h} = \left(\frac{R_{\text{vir},h}^3}{GM_h} \right)^{1/2}, \quad g_h = \frac{d \ln M_h}{dT}, \quad B_h = t_{\text{dyn},h} g_h. \quad (5.6)$$

The dimensionless number B_h compares the rate of object growth with the internal dynamical time. It is positive for net growth and negative for net mass loss. Slowly changing mass gives a value close to zero. Binding and stability depend on the internal dynamics in addition to this instantaneous growth rate. $B_h = 1$ marks equal instantaneous mass-growth and dynamical time scales. It provides a reference for growth classification. The relaxation rate is a separate property of the object dynamics. From two linked epochs $p \rightarrow d$

$$g_h^{p \rightarrow d} = \frac{\ln M_d - \ln M_p}{T_d - T_p}, \quad B_h^{p \rightarrow d} = t_{\text{dyn},d} g_h^{p \rightarrow d}. \quad (5.7)$$

For a multi-progenitor event the main branch supplies the object value and the individual links are stored as transfer entries with weights

$$\mu_{p \rightarrow d} = \frac{M_p}{\sum_{p' \in \text{Prog}(d)} M_{p'}}. \quad (5.8)$$

We classify the continuous growth coordinate by

$$\text{forming } B_h > 1, \quad \text{balanced } 0 \leq B_h \leq 1, \quad \text{stripping } B_h < 0. \quad (5.9)$$

These classes compare the rate of exchange with the time on which the object responds to its own gravity. In the forming class the local mass-growth time is shorter than the dynamical time. The balanced class has slower nonnegative growth. Stripping denotes a net outward mass transfer. The designation “balanced” refers to this mass budget. Mechanical equilibrium additionally concerns the internal stresses and velocities under the applied forces. The masses and times along the selected branch specify (5.7). The numerical results use the same class thresholds and declared link quality and mass convention for the finite-difference growth statistic

$$B_{\text{frac}} = t_{\text{proxy},d} \frac{M_d - M_p}{M_d \Delta T},$$

with the extent-based time defined in Section 4.3. At fixed time convention the fractional and logarithmic mass differences agree only to first order in fractional growth. The logarithmic coordinate is also evaluated on every stored positive-mass parent link. New objects without a parent are excluded from that readout. The figures use B_{frac} .

5.1.2 Transfer diagnostic

For runs that include generated density and shape information with spin readings a transfer channel is assigned to a source p and target q . The source p may be a progenitor or an environmental shell or a Fourier band of the tidal field. The target q is the descendant object whose angular momentum is considered. Peebles [29] analysed angular-momentum acquisition by gravitational torques. White [30] clarified the role of nonspherical material regions in the leading-order growth and Catelan and Theuns [31] calculated nonlinear Lagrangian corrections. Here the local force expansion gives an instantaneous channel reading. Use physical offsets $r_\alpha = a(x_\alpha - x_q)$ in a fixed periodic chart and physical angular momentum about the mass centre. At a fixed epoch the physical tidal tensor of the source channel is

$$T_{ij}^{(p)}(x_q) = a^{-2} \left[\partial_i \partial_j \Phi^{(p)} - \frac{1}{3} \delta_{ij} \nabla^2 \Phi^{(p)} \right]_{x=x_q}, \quad (5.10)$$

and the target inertia tensor is

$$I_{ij}^{(q)} = \sum_{\alpha \in q} m_\alpha (r_\alpha^2 \delta_{ij} - r_{\alpha i} r_{\alpha j}), \quad (5.11)$$

where derivatives of Φ are with respect to comoving position. The associated tidal-torque vector is

$$\tau_i^{p \rightarrow q} = \epsilon_{ijk} I_{jl}^{(q)} T_{lk}^{(p)}. \quad (5.12)$$

Expand the physical source acceleration about the target mass centre. Its constant term gives no torque while the linear term gives the inertia contraction in (5.12). The isotropic trace cancels by antisymmetry. If the Hessian of that acceleration is bounded by C the neglected torque is at most $C \sum_\alpha m_\alpha |r_\alpha|^3/2$. The dimensionless transfer strength and alignment are

$$A_{p \rightarrow q} = t_{\text{dyn},q} \frac{|\tau^{p \rightarrow q}|}{|J_q|}, \quad G_{p \rightarrow q} = \frac{\tau^{p \rightarrow q} \cdot J_q}{|\tau^{p \rightarrow q}| |J_q|}. \quad (5.13)$$

These quantities require $|J_q| > 0$. Alignment also requires $|\tau^{p \rightarrow q}| > 0$. Zero vectors are recorded as undefined directions. $A_{p \rightarrow q}$ is the angular-momentum change predicted by the instantaneous torque over one target dynamical time divided by $|J_q|$. The cosine $G_{p \rightarrow q} \in [-1, 1]$ measures the torque's alignment with the spin. Both belong to the transfer component $\mathcal{G}$ of the retained state.

Ancestry links describe mass changes between epochs. Tidal channels describe angular-momentum forcing at a fixed epoch. Their statistics are evaluated separately.

The peak-neighbour diagnostics measure geometric strength and alignment. For the field run define

$$A_{p \rightarrow q}^{\text{fld}} = \frac{\min(M_p, M_q)}{\max(M_p, M_q) d_{pq}^3}, \quad \tau_{p \rightarrow q}^{\text{fld}} = A_{p \rightarrow q}^{\text{fld}} s_{pq} \times (v_q^{\text{proxy}} - v_p^{\text{proxy}}).$$

Here d_{pq} is the minimum-image periodic distance. The field diagnostic uses the direct coordinate difference $s_{pq} = x_q - x_p$ for the cross product whereas the PM diagnostic uses the minimum-image vector. The saved field alignment is the cosine between this vector and its target spin proxy. Thus these diagnostic conventions including the periodic chart are part of their definitions. A^{fld} and A^{PM} have inverse-volume units unlike the physical A in (5.13). The saved field and PM transfer rows all have nonzero torque and spin proxies so their alignment cosines are defined. For general data zero-vector rows must be excluded from directional statistics with the same selection rule at each epoch.

The tensor construction estimates angular-momentum forcing from the source potential and target moments. The neighbour diagnostics measure the geometry of the selected pairs at a fixed epoch. Their strengths and alignments quantify this geometry. A torque flux uses the source potential and target inertia through the tensor construction.

5.2 Coupled matter and source evolution

The homogeneous and planar realizations use the occupied common matter–source action. Homogeneous occupation permits a uniform estimate of how source–response reuse affects matter and geometry. Spatial dependence adds stress gradients and metric feedback within that same action. The local RMS realization uses the constitutive action of Section 3.3 and its cold-matter reduction. There the full periodic field drives both expansion and a finite displacement window whose omitted force components measure the effect of that material reduction.

5.2.1 Common-source evolution with certified response reuse

The collective source is determined by the evolving matter and geometry. Recomputing its full spectral response at every numerical node may be unnecessary when the stationary branch changes slowly. This example asks how long a previously evaluated response remains adequate for the same physical evolution. The source–constraint residual measures the departure from stationarity and the positive slope of the stationary source constraint converts it into a bound on the change in the collective response.

The occupied homogeneous action and initial data of Ref. [2, Proposition “Occupied common background and physical propagation”] give a finite example in which the collective source evolves with the matter and geometry. We use its dimensionless canonical coordinates $x = (a, E, p_E, \phi_1, \phi_2, p_1, p_2, H/\mu)$ the maximum norm and the clock $\tau = \mu s$. The parameters are $g = 1$ and $Z_E = 2$ with $L_{\text{ref}}^2 = 2\pi$. In the response units the matter coefficient is $m_0^2 = 3/(8\pi)$ and the initial radius satisfies $L_0^2 = 3\pi$. The physical normalization uses $\mu = M_G$ and $P = M_P/\mu$ together with $\rho_\Lambda = \Lambda M_P^2$ from the common parameter calculation [14]. The canonical source equation is (3.66). It is eliminated in the same action throughout the comparison. The numerical reference uses the finite canonical initial data of that calculation. Their floating-point Friedmann residual is -3.47×10^{-18} in the common energy units. The trajectory bound treats the encoded values as exact initial data. Uncertainty in their physical determination enters the initial-error term.

Both reconstructions use 64 equal segments on $0 \leq \tau \leq 10^{-2}$ with $h = 1/6400$. On segment j the reconstructed state is $\hat{x}(\tau) = x_j + (\tau - \tau_j)v_j$ and the source ζ_j is held fixed.

The first calculation solves for the source at every node. The second retains its value and spectral response while an outward bound $b_j \geq |C_H(x_j, \zeta_j)|$ satisfies the dimensionless budget 2.5×10^{-5} . A refresh supplies a new proposed root and an independent spectral enclosure of its residual. Both procedures approximate the same complete implicit vector field $\mathcal{F}(x) = F(x, \Gamma(x))$.

The spectral response can be bounded with a shared tail moment. Write $c_n = (2n + 1)^2 - 1/4$ and $d_n = (2n + 1)^2$ and define

$$J_N(u) = \sum_{n=0}^N \frac{d_n u}{c_n(c_n + u)}, \quad S_{2,N} = \frac{\pi(\pi + 2)}{8} - \sum_{n=0}^N \frac{d_n}{c_n^2}.$$

For $u \geq 0$ the complete response satisfies

$$J_N(u) + uS_{2,N} - \frac{u^2 S_{2,N}}{c_{N+1}} \leq J(u) \leq J_N(u) + uS_{2,N}, \quad J = I'. \quad (5.14)$$

Indeed each omitted term equals $d_n u / c_n^2 - d_n u^2 / [c_n^2(c_n + u)]$. Summation and $c_n \geq c_{N+1}$ give the bound. The derivative has the analogous tail interval $[S_{2,N} - 2uS_{2,N}/c_{N+1}, S_{2,N}]$. The calculation uses $N = 64$. At a refresh it stores the value and derivative enclosures. Between refreshes the Taylor remainder is bounded by $\frac{1}{2} \sup |J''| |u - u_j|^2$ with $\sup |J''| \leq 2J'(0)/(c_0 + u_-)$ on the certified spectral interval. Thus reuse requires no new mode sum at an accepted node.

Let q_j bound $\|v_j - F(x_j, \zeta_j)\|_\infty$ including the spectral enclosure and rounding and let $B_j = \|v_j\|_\infty$. The uniform bound κ controls the canonical source curvature. The fixed-state source derivative is bounded by L_ζ and the Lipschitz constant of $\mathcal{F}$ by ℓ . Proposition 3.7 gives on every segment

$$d_j = q_j + \frac{L_\zeta b_j}{\kappa} + \ell h B_j, \quad E_{j+1} = \frac{E_j + h d_j}{1 - h\ell}, \quad h\ell < 1. \quad (5.15)$$

The last term in d_j bounds the change of the complete vector field along the linear segment. The recurrence follows from Theorem 3.6 and $e^{h\ell} \leq (1 - h\ell)^{-1}$. It bounds the entire segment. The initial error is included in E_0 .

The inherited invariant box gives a positive source bracket and finite derivative bounds for both paths. If L_C bounds the state derivative of C_H the within-segment source error is at most $(b_j + L_C h B_j)/\kappa$. For the internal radius $L = L_{\text{ref}} e^{-\zeta}$ this is combined with $\|D\Gamma\| \leq L_C/\kappa$ and the state error. The phase $\vartheta = \arg(\phi_1 + i\phi_2)$ has a bounded derivative on the same box where $\phi_1 > 0$. These estimates give uniform radius and clock-readout errors as well as the scale-factor error. Source updates affect its recorded approximation. The reference source remains the unique stationary root along the common evolution. The radius and phase bounds apply the reading estimate (3.62) on this same box. Thus the effect of source reuse is followed from the constraint residual through the matter trajectory to the geometric and clock readings.

Outward evaluation on the common box gives $\kappa > 0.21440886$ and $L_C/\kappa < 9.359250$ with $L_\zeta < 1.423579$ and $\ell < 23.148721$. The Lipschitz constants for radius as a function of source and for phase as a function of state are less than 3.172948 and 2.973200 respectively.

	Source at every node	Residual-driven reuse
Uniform state and scale-factor error	4.029×10^{-6}	4.861×10^{-6}
Uniform radius error	5.778×10^{-4}	9.200×10^{-4}
Uniform phase error (radians)	1.198×10^{-5}	1.446×10^{-5}
Source root solves	64	4
Analytic response evaluations in root solves	829	52
Source-reuse budget tests	0	63
Spectral shell-pair evaluations	4160	260
Interval additions	19776	4149
Interval multiplications	24000	8565
Interval divisions	9984	2169
Interval negations	1600	1564
Rational point conversions	27717	4131
Absolute-bound evaluations	1219	1283

Table 1. Two reconstructions of the same common-source evolution on $0 \leq \mu s \leq 0.01$ each with 64 segments. The error bounds are rounded upward. The interval counts include advancement and residual evaluation together with source-reuse decisions and error propagation. Analytic root proposals and spectral enclosures are counted separately.

The initial values lie in the inherited initial enclosure and both reconstructions remain in the common box. The residual-driven calculation refreshes at nodes 0, 17, 34, 51. Table 1 gives the resulting full-interval bounds and arithmetic counts [14].

The scale factor and internal radius are read from the same evolving state as the phase. Their simultaneous bounds quantify the effect of infrequent response updates on geometry and matter. The reference branch continues to respond at every time. The refreshes regulate its numerical resolution.

The work comparison includes unsuccessful reuse decisions and subsequent refreshes as well as interval certification. Both evolutions share the same initialization and invariant-box construction. Their counts and integer precision are given in Appendix B.3. At the stated accuracy response reuse reduces both source elimination and the interval arithmetic used to certify the reconstructed trajectory.

5.2.2 A constrained planar matter–source perturbation

Spatial gradients test the common-source mechanism beyond homogeneous occupation. They enter the collective-source constraint and the stress that drives the geometry while the metric in turn changes matter transport. A planar perturbation retains this feedback with a single spatial coordinate and allows its constraint propagation to be compared at different resolutions. Its initial geometry is obtained with the matter source from the constraint equations.

The nonuniform construction of Proposition 3.2 is evaluated with $\varepsilon = 10^{-3}$ and $k = 1$ on a domain of period 2π . It uses the same generated coefficients as the homogeneous example. Coordinates and fields are expressed in the same response units. The conformal constraint determines the initial metric and expansion together with the source. Evolution

uses the planar metric

$$g_{\alpha\beta}dx^\alpha dx^\beta = -d\tau^2 + A(\tau, x)^2 dx^2 + B_\perp(\tau, x)^2(dy^2 + dz^2),$$

$$K^i_j = \text{diag}(K_x, K_\perp, K_\perp), \quad v_\gamma = AB_\perp^2.$$

This is a synchronous chart on the common solution. With $K = K_x + 2K_\perp$ and $W = U + \mathcal{V}$ together with $\mathcal{G}_{\text{tot}} = Z_E E_x^2 / (2A^2) + \mathcal{G}_s$ its Einstein equations are

$$\begin{aligned} \dot{A} &= -K_x A, & \dot{B}_\perp &= -K_\perp B_\perp, \\ \dot{K}_x &= {}^{(3)}R^x_x + K K_x - \frac{2\mathcal{G}_{\text{tot}} + W}{P^2}, & \dot{K}_\perp &= {}^{(3)}R^y_y + K K_\perp - \frac{W}{P^2}, \end{aligned} \quad (5.16)$$

where

$${}^{(3)}R^x_x = -\frac{2}{A^2} \left(\frac{B''_\perp}{B_\perp} - \frac{A'B'_\perp}{AB_\perp} \right), \quad {}^{(3)}R^y_y = -\frac{1}{A^2} \left(\frac{B''_\perp}{B_\perp} + \frac{B'^2_\perp}{B_\perp^2} - \frac{A'B'_\perp}{AB_\perp} \right).$$

The canonical field equations are

$$\begin{aligned} \dot{E} &= \frac{pE}{v_\gamma Z_E}, & \dot{p}_E &= \partial_x \left(\frac{B_\perp^2}{A} Z_E E_x \right) - v_\gamma U_{,E}, \\ \dot{\varphi} &= \frac{p}{v_\gamma M}, & \dot{p} &= \partial_x \left(\frac{B_\perp^2}{A} M \varphi_x \right) - v_\gamma B \varphi. \end{aligned} \quad (5.17)$$

Equation (3.11) is solved at every spatial evaluation. The source responds both to occupation and to spatial gradients. The resulting stress determines the metric update.

The calculation uses 17 Fourier collocation points and eight classical Runge–Kutta steps on $0 \leq \tau \leq 0.01$. Two independent resolution comparisons use 16 time steps and then 33 spatial points. The initial expansion is $H = 0.05851942090$ with a nonconstant conformal factor and source. At the final epoch $\langle v_\gamma \rangle^{1/3} = 1.0005851390$ and the source ranges from -0.2024419505 to -0.2023005166 . The maximum sampled Hamiltonian residual is below 10^{-12} . The momentum and source residuals are below 10^{-13} and 10^{-14} respectively in the common units. The minimum sampled eigenvalues of both physical principal forms exceed 1.83. The characteristic roots are real and the phase norm X_ϑ exceeds 0.0793. The mean phase charge $\langle \varphi_1 p_2 - \varphi_2 p_1 \rangle$ is conserved within floating-point accuracy. Along its moving phase trajectories the clock obeys

$$\frac{d(\mu s)}{d\tau} = \frac{\sqrt{X_\vartheta}}{\partial_\tau \vartheta}, \quad \frac{dx}{d\tau} = -\frac{\partial_x \vartheta}{A^2 \partial_\tau \vartheta}.$$

The two refinement differences are below 10^{-12} in the maximum state norm and the retained Fourier coefficients respectively. These are numerical constraint and resolution checks for the nonuniform realization. Its local continuum existence and constraint propagation follow from Proposition 3.2. The outward full-interval trajectory bounds in Section 5.2.1 refer to the homogeneous reference.

The stored calculation includes all three evolutions and their initial constraint solves [14]. Together they use 160 right-hand-side evaluations and 6883 algebraic source solves.

They also use 61792 analytic source-response evaluations. The 6883 potential quadratures require 144543 restoring-function evaluations. These are included in the computation count. The initial metric check also compares its first Fourier coefficient with (3.14). Setting $\varepsilon = 0$ reduces the same planar equations to the occupied homogeneous system and agrees with its independently integrated ordinary differential equations. This limit connects the homogeneous and nonuniform realizations through the same field equations. At nonzero perturbation the source variation and metric response along with the transported phase give complementary observations of one coupled solution. The small constraint and refinement residuals measure the consistency of this spatial realization.

5.2.3 Autonomous expansion with the exact periodic RMS field

This initial-value problem retains one cold-displacement mode and the homogeneous scale factor. The complete periodic potential supplies the binding energy and its derivatives for the canonical evolution, together with the force in the omitted displacement directions.

We evaluate Proposition 3.3 with $\Omega_m = 0.3149557035$ and $\Omega_\Lambda = 1 - \Omega_m$ together with $\alpha_0 = 0.1837763$ retaining the full precision of the generated inputs [14]. The acceleration ratio is formed from the same acceleration and Hubble readouts as the local response. The initial displacement and Fourier window are $D_0 = 10^{-3}$ and $k = 100$. The initial scale factor and momentum are $a(0) = 1$ and $p_D(0) = 0$. The initial expanding momentum is fixed by $p_a(0) = -\sqrt{12R(1, D_0)}$. These are initial data and resolution choices for a nonzero cold perturbation.

The flux formula determines the periodic potential at every state without a Fourier truncation. Its energy averages are evaluated by Gauss–Legendre integration after $kq = (\pi/2)u^2$ which regularizes the square-root endpoint. The canonical system is integrated on $0 \leq t \leq 0.05$ using 64 quadrature nodes and relative tolerance 2×10^{-10} . A second calculation uses 128 nodes and relative tolerance 2×10^{-12} . Table 2 gives the resulting coupled evolution.

Quantity	Initial value	Value at $t = 0.05$
Scale factor a	1	1.0506801910
Displacement D	0.001	0.00112259217
Expansion H/H_0	0.9999999995	0.9780525628
Momentum p_D	0	2.49917×10^{-7}
Maximum peculiar speed divided by c	0	5.03483×10^{-5}
Physical Hamiltonian Hessian	2.11670×10^4	1.86589×10^4
$\ r_X\ _{L^2(dq)}$	1.22974×10^{-4}	1.18028×10^{-4}

Table 2. Coupled expansion and displacement with local RMS response from (3.21). Time and acceleration use the units specified in Section 3.3. The last row measures the orthogonal acceleration defect relative to arbitrary periodic displacements. The periodic potential is solved at all spatial frequencies.

Existence and physical kinetic positivity on $0 \leq t \leq 0.05$ follow from an outward rational enclosure of the canonical evolution in

$$1 \leq a \leq 1.06, \quad D_0 \leq D \leq 0.002, \quad 0 \leq p_D \leq 8 \times 10^{-7}, \quad -6.6 \leq p_a \leq -5.9. \quad (5.18)$$

The inverse periodic flux bounds the force and the derivatives of its binding energy. Their integrated bounds keep the exact solution inside this box. The boundary estimates are derived in Appendix B.4. They yield

$$R > 2.99999997, \quad \frac{\partial^2 H_a}{\partial p_D^2} > 1.5665 \times 10^4, \quad |g| < 0.001318, \quad \|r_X\|_{L^\infty} < 0.003954. \quad (5.19)$$

The final bound follows from $\|r_X\|_{L^\infty} \leq 3\|g\|_{L^\infty}$. These bounds refer to the exact decimal encodings of the supplied inputs. Uncertainty in their upstream construction is a separate input enclosure. Applied to (3.33), the same exit box keeps the density contrast between $-1/501$ and $1/499$, the peculiar speed below $1.70 \times 10^{-4}c$, and the mean expansion above $0.875H_0$. It bounds $|\Phi| < 9 \times 10^{-5}$ and therefore keeps the weak-field material clock strictly positive with $|2\Phi - u^2| < 1/5000$. The proof of these uniform reading bounds is included in Appendix B.4. The field equation has zero spatial defect by (3.25). The nonzero r_X quantifies the acceleration defect in the enlarged model with arbitrary periodic displacements; its negative is the drive omitted by the material window. The positive Hamiltonian Hessian certifies the retained physical momentum direction throughout the interval. The density and clock bounds keep the same trajectory within the material and reading domains.

The two numerical calculations agree within 1.2×10^{-14} in the maximum state norm. Sampled lapse-constraint residuals are below 10^{-13} and the independently evaluated periodic field equation has an L^2 residual below 10^{-15} . Its flux inversion is checked directly. The energy and derivative identities (3.26) are also compared with an independent quadrature of the binding energy. The two evolutions use 64 right-hand-side evaluations and 100 exact field reconstructions requiring 9600 quadrature-node evaluations. At the 34 saved states, the same potential supplies the canonical derivatives and material readings. The additional algebraic readings use 102 floating-point square roots and 34 hypotenuse evaluations. Spatial diagnostics use 139264 phase samples. The uniform certificate evaluates the rational exit inequalities once; its material-clock bounds add two outward interval square roots and require no further field reconstruction or evolution steps. The growth of D and the nonzero momentum in Table 2 describe the material response to this common evolution. The orthogonal acceleration defect identifies the additional displacement modes driven by the same field. It thus gives a dynamical meaning to enlargement of the material window.

5.3 Finite-dimensional evolution at controlled accuracy

The Galerkin examples examine whether the nonlinear feedback of omitted modes is small enough to permit fewer evolved variables at a fixed accuracy. Both calculations approximate the same reference evolution so their work comparison measures the effect of the active-window reduction at that accuracy.

We apply the a posteriori estimate of Appendix B.2 to the alias-free matter system with N positive Fourier modes. The reduced and full Galerkin evolutions approximate the same initial-value problem on the same time grid with uniform tolerance $\varepsilon = 10^{-4}$ in the joint Fourier ℓ^1 norm. For $N = 8$ with $T = 0.2$ and $h = 1/640$ the admissibility

	$N = 8$ 128 steps		$N = 32$ 50 steps	
	Reduced	Full	Reduced	Full
Evolution multiplications (floating point)	2693	47232	250	304050
Error-estimate multiplications (direct integer)	7445	41699	1300	203365
Error-recurrence operations (rational)	645	640	250	250
Polynomial-construction operations (rational)	44	344	10	5408
Upper bound on direct integer bit length	168	284	132	721
Upper bound on scalar rational bit length	1312	2048	1048	5480

Table 3. Arithmetic work at uniform tolerance $\varepsilon = 10^{-4}$. Each reduced evolution is compared with the full Galerkin system on the same time interval and grid using the same error estimator. Direct integer multiplications and scalar rational operations are counted separately. The last two rows bound the sizes of their intermediate integers. The counts measure arithmetic evaluation with the indicated integer bit lengths.

condition selects two positive modes at the initial step and retains them for the subsequent evolution. The reduced and full error bounds are 8.666×10^{-5} and 8.643×10^{-5} respectively over all 128 steps. For $N = 32$ with $T = 0.05$ and $h = 10^{-3}$ a single positive mode suffices throughout the 50 steps. The corresponding bounds are 7.004×10^{-5} and 4.885×10^{-5} . The continuation estimate also gives $1 + \delta > 0.97$ in both examples.

Arithmetic work. The quadratic coupling of M retained modes has support only up to $\min(2M, N)$. Consequently the defect of the embedded reduced state can be evaluated without advancing the omitted modes. Direct evaluation of either the retained convolution or this defect requires $O(M^2)$ coefficient operations per attempted step compared with $O(N^2)$ for the full Galerkin convolution. Evaluation of the error bound further requires exact integer arithmetic and a scalar rational recurrence. Its bit complexity depends on the sizes of the integer operands in addition to the number of arithmetic operations.

Table 3 compares these costs at the same prescribed accuracy. Floating-point operations in the evolution and exact arithmetic in the error estimate are listed separately. Polynomial coefficients and repeated monomials are combined before evaluation. The one-time construction of these expressions is reported separately. The $N = 8$ reduction rejects one trial step with a single mode accounting for its five additional rational operations in the error recurrence. All rejected trials are included in the counts. The additional arithmetic associated with additions and divisions as well as conversions and window selection is specified in the supplementary material [14].

The reduction in both convolution work and exact-arithmetic work is a consequence of the restricted spectral support on the specified intervals. Each error bound is evaluated from the reduced state and the reference vector field. An independent high-accuracy integration provides a separate numerical check.

At a common error tolerance the relevant comparison is between the complete work expressions for the reduced and full evolutions. The retained convolution controls the gain when the selected windows remain small and the combined cost of selection and reconstruction with certification remains subleading. Proposition B.1 in Appendix B gives the corresponding operation and bit-work bounds including every rejected attempt and error

update. Together with the counts in Table 3 they specify this comparison independently of elapsed time.

5.4 Spatial realizations and matter observables

5.4.1 Cosmological realizations

The spatial realizations examine how a changing window reorganizes spectral power and localized structure. The signed-field construction isolates the relation between a prescribed spectrum and its active window as measured by the selected peaks. The particle construction supplies a positive matter measure and permits material membership to be followed between epochs. Their catalogues consequently answer different questions. Peak statistics describe the geometry of the windowed field whereas particle intersections describe transfers of identified material.

Four related sets of observations describe the spectral realization. The active endpoint and its length and mass scales locate the retained spectral range. Shell fractions measure the distribution of power. Peak masses and radii r_{peak} together with the growth coordinate B_{frac} describe selected spatial concentrations. Neighbour strengths and spin alignments describe their relative geometry. These readings use the same field and epoch grid.

For the spectral-field power budget write $\mathcal{P} = \Delta^2 G^2$ where G is the Gaussian multiplier and use the positive tabulated wavenumbers with logarithmic trapezoidal quadrature. The three integrals are

$$\begin{aligned} I_A &= \int_{0 < k \leq \min(k_{\text{NL}}, k_{\text{res}})} \mathcal{P} \, d \ln k, & I_T &= \int_{k_{\text{NL}} < k \leq k_{\text{res}}} \mathcal{P} W_A^2 \, d \ln k, \\ I_{\text{all}} &= \int_{k > 0} \mathcal{P} \, d \ln k, & \epsilon_P &= \frac{\max(0, I_{\text{all}} - I_A - I_T)}{I_A + I_T + \varepsilon_f}. \end{aligned} \quad (5.20)$$

Each integral is restricted to the supplied spectral table. $\varepsilon_f = 10^{-12}$ is the positive denominator floor. The numerator measures power outside the retained active and transition parts. Equation (3.47) in contrast measures the norm of a defect in an evolution equation.

The spectral-field calculation uses 512^3 cells in a $512 h^{-1} \text{Mpc}$ box at 24 epochs. It gives 48 spectral shells and 384 selected peaks per epoch together with peak-neighbour relations and the normalized power budget (5.20). Table 4 compares the stored observables with the density snapshots over the complete epoch grid. This is a storage comparison for the selected observables. The evolution-work comparison at controlled accuracy is given in Section 5.3.

The large particle catalogue follows material membership using 64^3 particles on a 128^3 mesh over the same epochs with $Q(a) = 0.35 / \max(a, 10^{-4})$. Its object statistics and ancestry are summarized in Table 4. This prescribed force law differs from the cosmological force with Hubble damping used for the trajectory error bounds. Numerical parameters and data are supplied in Ref. [14].

5.4.2 Spectral-field history

The spectral realization follows the specified background through the present epoch and into $a > 1$. In Table 5 the nonlinear endpoint moves to lower wavenumbers hence to larger

Table 4. Sizes and stored observables of the two realizations each at 24 epochs. The field snapshot estimate uses one four-byte density value per cell. The saved size counts the CSV observation tables. Storage units are 1 MiB = 2^{20} and 1 GiB = 2^{30} with 1 MB = 10^6 bytes. The particle comparison uses $Q(a) = 0.35/\max(a, 10^{-4})$. Its object and link counts are summed over all epochs.

Quantity	Value
Spectral-field realization. 512^3 cells	
One raw density snapshot	536 870 912 bytes (512 MiB)
All raw density snapshots	12 884 901 888 bytes (12 GiB)
Stored observation tables	15 738 667 bytes (15.01 MiB)
Stored-to-raw ratio	0.0012214813
Particle comparison	
Particles	$64^3 = 262144$
Force mesh	128^3
Epochs	24
Object entries	837
Particle-overlap ancestry links	401
Neighbour relations	2508
Density-assignment mass discrepancy	8.81×10^{-13} – 1.07×10^{-10}
Stored observables	3.57 MB

comoving lengths and masses under (2.8). Figure 1 shows these linked histories over all 24 epochs. The table places them on the same time axis as the field standard deviation and residual-tail power. It also reports the object balance coordinate and transfer alignment with the power budget. The motion towards smaller k_{NL} corresponds to larger spatial regions entering the nonlinear scale convention. Small-scale power continues to contribute through the transition and tail. The history therefore records a redistribution among resolved scales and retained residuals together with the changing geometry of the peaks.

Active-window, scale and mass history

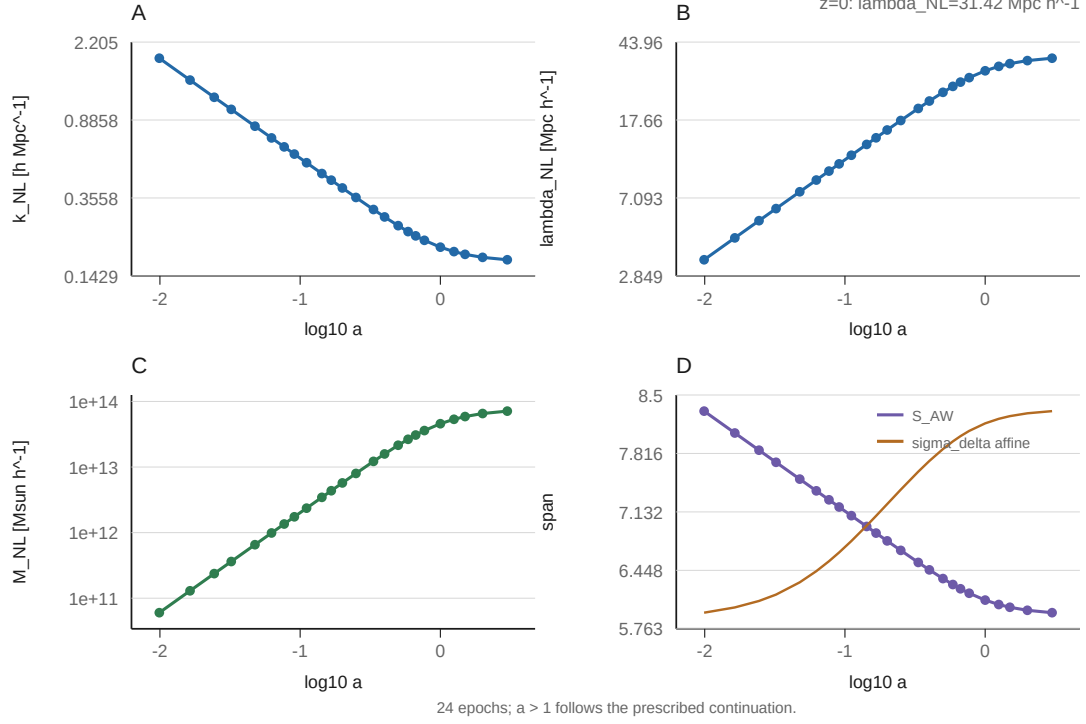


Figure 1. Active-window length and mass histories over all 24 epochs including the forecast branch. Panels A–C show k_{NL} and the comoving wavelength $\lambda_{\text{NL}} = 2\pi/k_{\text{NL}}$ together with M_{NL} under the stored c_M convention. The wavelength is distinct from the mass radius c_M/k_{NL} . At the present epoch it is $31.42 h^{-1} \text{Mpc}$. Panel D compares the logarithmic span S_{AW} with the density standard deviation affinely rescaled to the displayed span range solely to compare their shapes. All panels use $\log_{10} a$ as the horizontal coordinate.

The wavelength and mass curves are two readouts of the same moving endpoint. Their linked evolution expresses the change of reference scale under (2.8). The comparison with the field variance concerns the shapes of the histories because the variance in panel D has been affinely rescaled. The values at representative epochs are given in Table 5.

Table 5. Representative epochs from the spectral-field realization. The columns give the active endpoint and nonlinear mass with the field standard deviation and residual-tail power fraction. The remaining columns report the median object balance coordinate and median absolute transfer alignment together with the normalized power budget.

z	k_{NL} $h \text{ Mpc}^{-1}$	M_{NL} $M_{\odot} h^{-1}$	$\sigma_{\delta,f}$	F_T	$B_{\text{frac},50}$	$ G^{\text{fld}} _{50}$	ϵ_P
100	1.826	6.01×10^{10}	0.109	0.000	29.86	0.589	2.87
10	0.595	1.74×10^{12}	0.928	0.775	13.03	0.596	3.57
3	0.358	7.99×10^{12}	1.781	0.928	2.475	0.571	8.17
1	0.257	2.15×10^{13}	2.333	0.956	0.381	0.631	18.7
0	0.200	4.58×10^{13}	2.682	0.966	0.294	0.591	39.5
-0.667	0.173	7.12×10^{13}	2.849	0.970	0.038	0.606	63.3

As the endpoint moves towards smaller wavenumbers a growing fraction of the field power lies above it. Figure 2 follows this redistribution. A larger nonlinear length therefore coexists with substantial short-wavelength content. The power outside the active and transition contributions is measured by ϵ_P . Its growth identifies spectral content whose effect must be retained through the interaction and residual response.

Residual spectral power

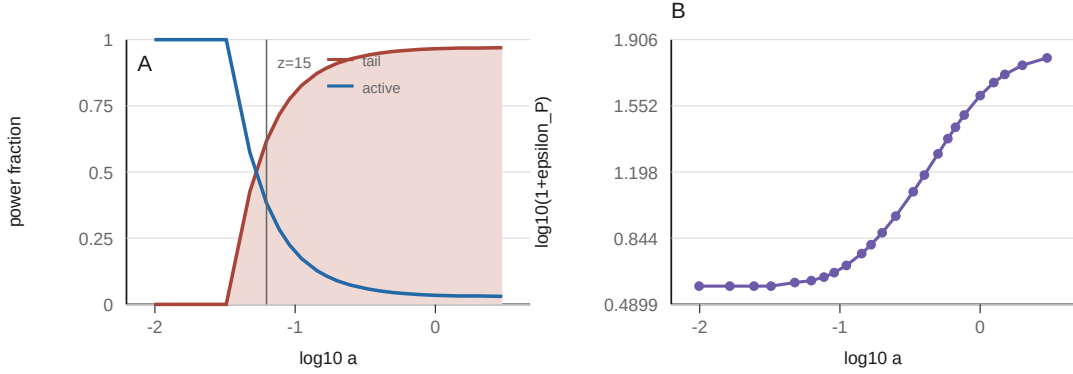


Figure 2. Residual activity. Panel A shows active and tail fractions of the generated field power with the first stored tail-majority epoch marked at $z = 15$. Panel B shows the transformed spectral-power budget $\log_{10}(1 + \epsilon_P)$ defined in (5.20).

Table 6 gives the first sampled epochs at which the stated conditions hold. Tail onset and tail dominance precede unit field variance followed by a majority of balanced peaks at the present epoch. These are first sampled crossings on the common epoch grid. Their temporal resolution is set by that sampling. The same table identifies the start of the stored forecast branch.

Table 6. First sampled epochs satisfying the listed conditions in the spectral-field realization. Object classes use the specified band-and-rank matching.

Marker	z	a	Selected quantity	Value
Tail onset	20	0.0476	F_T	0.427
Tail majority	15	0.0625	F_T	0.617
Unit field variance	8	0.111	$\sigma_{\delta,f}$	1.083
Balanced-object majority	0	1.000	f_{bal}	0.729
Forecast branch	-0.2	1.25	z	-0.2

At $z = 0$ the selected peaks have median mass $1.27 \times 10^{14} M_{\odot} h^{-1}$ and median radius $3.32 h^{-1} \text{Mpc}$ with median $B_{\text{frac}} = 0.294$. The balanced fraction at this epoch is 0.729 while the stripping fraction is 0.253. The median growth coordinate and the class fractions describe complementary aspects of the selected population. Figure 3 resolves the change in these fractions over all stored epochs. Since the peaks are matched by band and rank the decline of the median growth coordinate describes the slowing change of these catalogue statistics. Material accretion and stripping are read from the particle membership histories.

Peak growth classification

Scale-band and rank matching

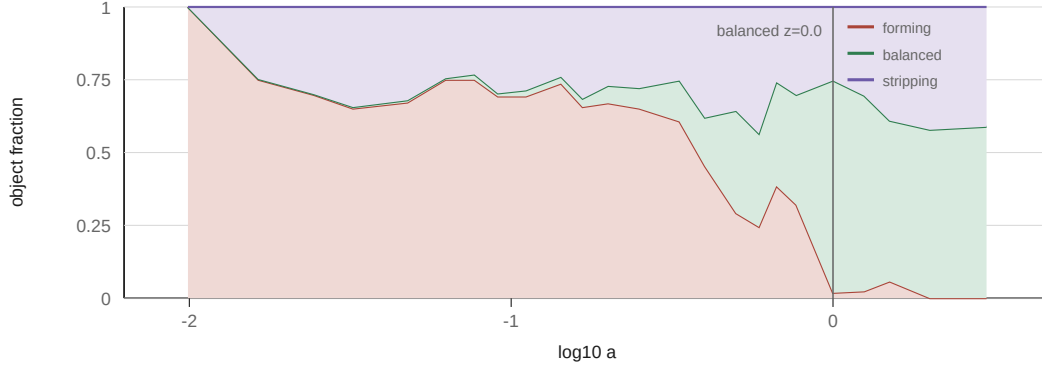


Figure 3. Fractions of selected peaks in the forming and balanced bins and the stripping bin of B_{frac} using the thresholds in (5.9). The marker locates the first stored epoch at which the balanced fraction exceeds one half. The bins use the fractional-growth statistic and the band-and-rank matching of Section 4.2. Logarithmic growth is a separate readout of linked positive-mass peaks. Particle ancestry uses the overlap in (4.23).

The balanced majority at the present epoch coexists with peaks in the other growth classes. The population fractions retain this variation which is compressed into a single value by a median. Their time history relates the changing peak catalogue to the same endpoint and spectral power history.

Figure 4 follows the geometry of neighbouring peaks. The median absolute alignment remains of order 0.6 over the representative epochs in Table 5. The strength combines a mass ratio with the inverse cube of the pair separation whereas the alignment measures a

direction cosine involving the velocity and spin proxies. Their separate histories distinguish the magnitude of this geometric diagnostic from its orientation.

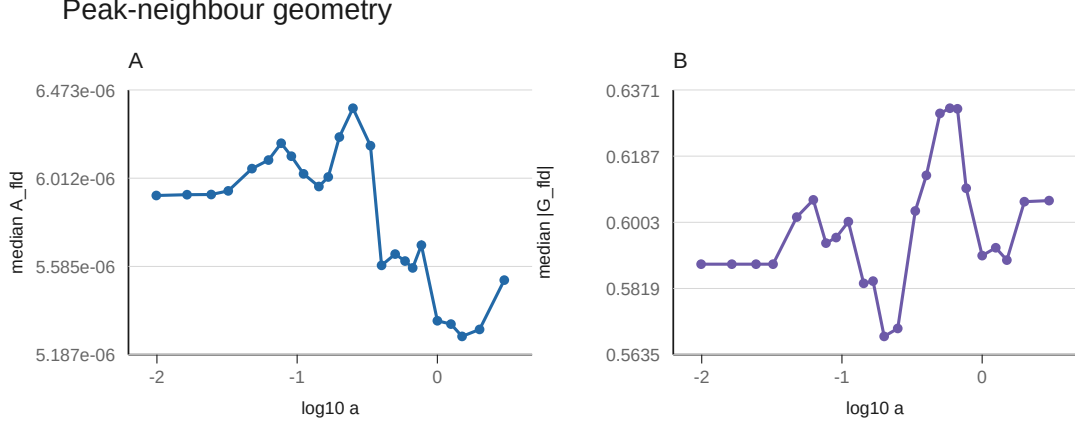


Figure 4. Peak-neighbour diagnostics in the field run. Median inverse-volume strength $A_{p \rightarrow q}^{\text{fid}}$ (A) and median absolute alignment $|G_{p \rightarrow q}^{\text{fid}}|$ (B). These quantities use the stored peak separations and velocity proxies together with spin directions as specified in the text.

The distributions at $z = 0$ in Figure 5 resolve information compressed by the time-dependent summary statistics. The shell distribution locates the power within the spectrum. The mass-growth scatter retains variation among individual peaks while the alignment histogram retains the spread among neighbouring pairs. These three views connect spectral content with peak and pair statistics in one realization.

Three-dimensional field relations

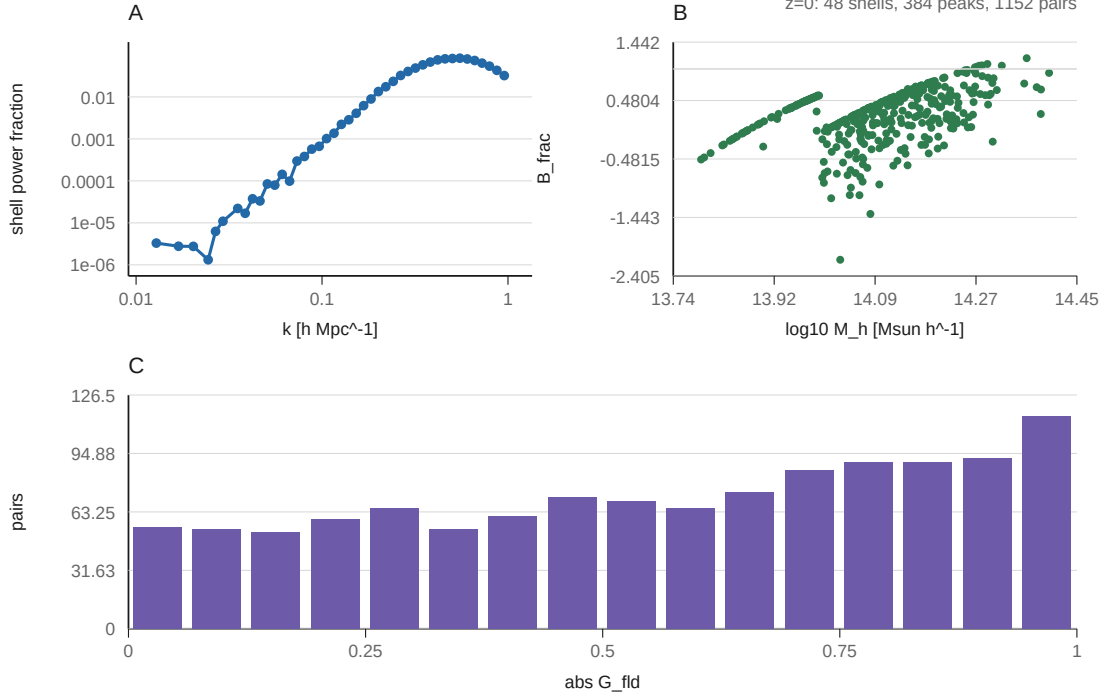


Figure 5. Field relations at $z = 0$. Panel A shows shell power fractions. Panel B gives the fractional-growth statistic against peak mass. Panel C gives the distribution of absolute peak-neighbour alignments. These panels resolve the spectral and object records along with transfer records of the field realization in Section 4.2. The power-budget history is shown in Figure 2.

The boundary probe $\mathcal{L}_h(1, a)$ decreases from 8.51×10^{-7} to 7.46×10^{-11} . It measures residual spectral localization near the moving endpoint as described by the shell moments in Section 4.5. A bound on the resulting equation defect additionally uses the actual interaction and the reconstruction error as in (4.37).

The equation-defect estimate is evaluated independently of these power diagnostics. In Appendix B.2 a one-mode reduction of a three-mode matter system has uniform state error below 5×10^{-5} on the specified interval. Rational arithmetic bounds the omitted nonlinear feedback and verifies positive density and the continuation tube for the stated coefficients and initial data on the prescribed time grid.

5.4.3 Cosmological particle trajectories and spatial force

The cosmological trajectory example follows particle motion with cosmological damping. Its force law and time reconstruction determine the trajectory-error estimate. The cold sheet reconstructed from this same trajectory then permits comparison with a continuous matter force. These comparisons measure time-integration and particle-sampling errors separately. Spatial assignment contributes its own force discrepancy.

The particle-mesh calculation uses 4096 particles on a 16^3 mesh in a $32 h^{-1} \text{Mpc}$ box over $0.2 \leq a \leq 0.4$ with the background parameters of Section 4.1. The initial displacement

is Gaussian-smoothed with rms amplitude 0.002 times the particle spacing and growing-mode velocity. The rms position differences between 16 and 32 steps and between 32 and 64 steps are 7.680×10^{-8} and $1.914 \times 10^{-8} h^{-1} \text{Mpc}$. The mean density error is below 5×10^{-10} and the sampled density is positive. Independent Einstein–de Sitter growing-mode tests show second-order convergence. For the background considered here the linear growth solution agrees with an independent integration to within 3.72×10^{-6} . These convergence estimates apply to the specified mesh and time interval. The trajectory estimate controls the numerical trajectory against the continuous-time equations of that mesh.

Proposition 5.5 (Enclosure of the cosmological particle trajectory). Let $h_m = L/N_m$ be the mesh spacing with one initial particle in each cell and let $F_h(X)$ denote the periodic cloud-in-cell (CIC) force including deposition and the specified discrete Poisson operator followed by CIC interpolation. Write its grid convolution kernels as K_b for $b = 1, 2, 3$ and set

$$A_h = \max_b \sum_{d=1}^3 \frac{\|K_b - \mathsf{T}_d K_b\|_{\ell^1}}{h_m}, \quad C(e) = \left(1 + \frac{2e}{h_m}\right)^2, \quad D(e) = \left(1 + \frac{2e}{h_m}\right)^3 - 1. \quad (5.21)$$

Here T_d is a one-cell translation. In a convex configuration set for which each particle remains within distance $e < h_m/2$ in the maximum norm of its own cell centre

$$\|F_h(X) - F_h(Y)\|_\infty \leq A_h[C(e) + D(e)]\|X - Y\|_\infty. \quad (5.22)$$

For the canonical cosmological equations with $\Omega_{m0} > 0$ and $\Omega_{\Lambda 0} \geq 0$

$$X' = \alpha(a)P, \quad P' = \beta(a)F_h(X), \quad \alpha = \frac{1}{a^3 E(a)}, \quad \beta = \frac{3\Omega_{m0}}{2a^2 E(a)}, \quad (5.23)$$

use $\|(X, P)\|_w = \max\{\|X\|_\infty, w\|P\|_\infty\}$. On a step $I_j = [a_j, a_{j+1}]$ let $(\hat{X}, \hat{P})$ be the linear interpolation of the computed canonical states. Suppose its equation defect is bounded by d_j and the configuration set contains a radius- r tube about this interpolation. Then the continuous-time particle solution with the same initial state satisfies

$$\begin{aligned} B_0 &= 0, & B_{j+1} &= \exp(\ell_j \Delta a_j)(B_j + \Delta a_j d_j), \\ \ell_j &= \max \left\{ \frac{\alpha(a_j)}{w}, w\beta(a_j)A_h[C(e_j) + D(e_j)] \right\}, \end{aligned} \quad (5.24)$$

provided $B_{j+1} < r$ at every step. The same recurrence bounds the error throughout each step.

Proof. Every grid vertex receives eight nonnegative trilinear weights. The cell condition bounds the assigned density between $(1 - 2e/h_m)^3$ and $(1 + 2e/h_m)^3$. For a displacement V differentiation of the assignment gives

$$D\rho[X]V = - \sum_d \frac{J_d - \mathsf{T}_d J_d}{h_m}, \quad \|J_d\|_\infty \leq C(e)\|V\|_\infty.$$

Each face current contains four transverse bilinear weights. Discrete summation by parts therefore bounds the derivative of the deposited force by $A_h C(e)$. Differentiation of its CIC sampling contributes at most $A_h D(e)$ since the force is generated by the density contrast. This proves Eq. (5.22). Both α and β decrease when $\Omega_{m0} > 0$ and $\Omega_{\Lambda 0} \geq 0$. The weighted maximum norm consequently gives the stated ℓ_j . The integrating-factor argument in the proof of Theorem 3.6 used in this weighted norm with no reconstruction jumps gives Eq. (5.24) using $(e^{\ell h} - 1)/\ell \leq h e^{\ell h}$. The strict tube inequality supplies its continuation condition. $\square$

The step defect is evaluated from the stored endpoints. Put $V_j = (X_{j+1} - X_j)/\Delta a_j$ and $W_j = (P_{j+1} - P_j)/\Delta a_j$ and let g_q enclose $F_h(X_q)$ with error η_q for $q = j, j+1$. With $L_j = A_h[C(e_j) + D(e_j)]$ valid bounds are

$$r_{X,j} = \max_{\substack{t \in \{a_j, a_{j+1}\} \\ q \in \{j, j+1\}}} \|V_j - \alpha(t)P_q\|_\infty, \quad (5.25)$$

$$r_{P,j} \leq \min_{q \in \{j, j+1\}} \left[\max_{t \in \{a_j, a_{j+1}\}} \|W_j - \beta(t)g_q\|_\infty + \beta(a_j)(L_j \|X_{j+1} - X_j\|_\infty + \eta_q) \right]. \quad (5.26)$$

The calculation uses $d_j = \max\{r_{X,j}, wr_{P,j}\}$ including the roundoff allowance for these expressions. The positive cosmological coefficients are enclosed by directed arithmetic. CIC roundoff and the error of an explicit radix-two Fourier transform are propagated into η_q . The roots of unity are enclosed once by interval arithmetic. A positive Taylor series with a geometric remainder bounds the exponential in Eq. (5.24).

For the 4096-particle trajectory the mesh spacing is $h_m = 2 h^{-1} \text{Mpc}$. The norm and tube parameters are $w \simeq 0.934487$ and $r = 0.1 h^{-1} \text{Mpc}$. The computed bounds with $U = (X, P)$ give

$$A_h < 8.079631, \quad \max_j \ell_j < 22.463, \quad \sup_{0.2 \leq a \leq 0.4} \|\widehat{U} - U\|_w < 5.715 \times 10^{-3} h^{-1} \text{Mpc}.$$

The position enclosure is below 0.286% of a mesh cell. It controls the finite CIC equations at fixed spatial resolution. The mesh-cell normalization expresses how closely the reconstructed positions track that discrete dynamical system. It also makes explicit the spatial scale against which the time-integration error is measured. The finite reference problem uses the numerically represented initial state and Fourier multipliers. Roundoff incurred in subsequent evaluation is included in the defect bound.

The bound is evaluated from consecutive endpoint forces along the same trajectory used in the refinement comparison. The 64-step calculation uses 65 force evaluations and their interval enclosures. The enclosure work includes 263 three-dimensional complex Fourier transforms and bounds on roundoff and elementary functions. The full arithmetic counts accompany the numerical data [14].

The continuous initial distribution. The initialization of the cosmological trajectory example defines a Lagrangian cold sheet. Let D be the generated displacement grid with

common amplitude normalization s . Its trilinear interpolation is $I_h D$. For uniform Lagrangian coordinate q the continuous initial map is

$$X_0(q) = q + sI_h D(q), \quad P_0(q) = a_0^2 E(a_0) f(a_0) sI_h D(q). \quad (5.27)$$

The growth coefficient and common normalization are the values used in the initialization. The particle initial data sample this map at cell centres. If $L_\psi = \|\nabla(sI_h D)\|_{\infty \rightarrow \infty}$ coupling each Lagrangian cell to its centre gives

$$W_{\infty, w}(\mu_0, \mu_{N,0}) \leq \max \left\{ \frac{h_p}{2} (1 + L_\psi) + \eta_X, w \left[\frac{h_p}{2} |a_0^2 E(a_0) f(a_0)| L_\psi + \eta_P \right] \right\}. \quad (5.28)$$

Here h_p is the particle lattice spacing and η_X, η_P bound the initialization roundoff. This follows by applying the Lipschitz bound to the same cell coupling in position and momentum. The displacement arrays already used for initialization give $L_\psi < 0.006530$ and the bound $W_{\infty, w}(\mu_0, \mu_{N,0}) < 1.006534 h^{-1} \text{Mpc}$. The radius-0.1 tube bounds time integration for the particle initial-value problem. Equation (5.28) measures the separate initial sampling error of the cold sheet.

For a common regularized force kernel the characteristic coupling propagates such an initial transport error by the corresponding Lipschitz estimate. A change of spatial force operator additionally contributes $w\beta(a)\epsilon_h(a)$ to the defect where ϵ_h bounds the force consistency error on the transported state set. The initial-distribution error and spatial-force error are therefore separate terms in a continuum comparison. After shell crossing the state is represented by the phase-space measure.

Initial Newtonian force observables. The initial cold sheet permits a direct comparison with the continuous Newtonian force at specified spatial frequencies. Write $\psi = sI_h D$ and $k_m = 2\pi m/L$ and use the Fourier convention $\hat{f}_m = L^{-3} \int_{\mathbb{T}_L^3} f(x) e^{-ik_m \cdot x} dx$. Mass conservation under Eq. (5.27) gives for $m \neq 0$

$$\hat{\rho}_m = \frac{1}{L^3} \int_{\mathbb{T}_L^3} e^{-ik_m \cdot [q + \psi(q)]} dq.$$

The homogeneous contribution vanishes exactly.

Expanding the displaced density to second order separates its linear response from the quadratic correction and a bounded cubic remainder. The trilinear displacement supplies an exact interpolation multiplier while cellwise Gaussian quadrature encloses the quadratic term. Proposition C.1 in Appendix C gives the resulting coefficient bounds including the interpolation and quadrature constants. Applying the Poisson multiplier and Parseval's identity gives both upper and lower force discrepancies in the specified frequency bands.

Table 7 resolves the initial spatial discrepancy independently of time discretization. Its growth with the included frequency range reflects the finite assignment and force resolution. The bounds in the table concern the initial state and the two specified Fourier bands.

5.4.4 Continuous spatial force and equation defects

The cold-sheet reconstruction of the cosmological trajectory example associates its particle data with a material map at every Lagrangian position. Its Jacobian determines the density

Table 7. Initial spatial force discrepancy for the continuous cold sheet and the force field used by the CIC particle calculation. The norm is $L^{-3/2}\|P_M(F^{\text{CIC}} - F^N)\|_{L^2}$ where P_M retains $0 < \|m\|_\infty \leq M$. The relative error uses the Newtonian force in the same band. All displayed interval endpoints are rounded outwards.

M	Modes	Absolute error ($h^{-1}\text{Mpc}$)	Relative error
1	26	$[1.50260, 1.50351] \times 10^{-4}$	$[4.7747, 4.7778]\%$
2	124	$[4.40684, 4.41879] \times 10^{-4}$	$[11.4336, 11.4667]\%$

and that density determines a continuous Newtonian force. In the present weakly perturbed cold sheet the map remains invertible so both the reconstructed matter distribution and its force can be controlled throughout the interval.

The computed particle trajectory defines a continuous reconstruction anchored to the original cold sheet. Let I_n and I_c denote periodic Q1 interpolation on the original nodal grid and the grid of cell centres. Write $x(a), p(a)$ for the linear time interpolants of these states and $\psi_0 = sI_n D$ for the original displacement. Set

$$\begin{aligned}\hat{X}(q, a) &= q + \psi(q, a), & \psi(q, a) &= \psi_0(q) + I_c[x(a) - x(a_0)](q), \\ \hat{P}(q, a) &= f_0\psi_0(q) + I_c[p(a) - p(a_0)](q), & f_0 &= a_0^2 E(a_0)f(a_0),\end{aligned}\tag{5.29}$$

where $f(a_0)$ is the initial logarithmic growth rate. The continuous initial displacement is retained exactly in this definition. The density is $\hat{\rho} dx = \hat{X}_*(dq)$ and the force $F^N = -\nabla\Phi$ is determined by $\Delta\Phi = \hat{\rho} - 1$ with zero mean force. Norms below are normalized by the box volume.

Proposition 5.6 (Continuous cold-sheet force and equation defects). For the 4096-particle 16^3 -mesh trajectory on $a \in [0.2, 0.4]$ the continuous reconstruction satisfies $\|D\psi\|_{L^\infty, 2 \rightarrow 2} < 0.011204$ and $0.96713 < \hat{\rho} < 1.03438$.

(a) Spatial force. Its Newtonian force and the continuous CIC field of the saved particle configurations obey

$$\sup_{0.2 \leq a \leq 0.4} \|F^N[\hat{\rho}(a)] - G[x(a)]\|_{L_x^2} < 1.870 \times 10^{-3} h^{-1} \text{Mpc}.\tag{5.30}$$

The corresponding uniform relative bound is 37.50%.

(b) Equation defects. With $\alpha = (a^3 E)^{-1}$ and $\beta = 3\Omega_{m,0}/(2a^2 E)$ define

$$r_X = \partial_a \hat{X} - \alpha \hat{P}, \quad r_P = \partial_a \hat{P} - \beta F^N[\hat{\rho}] \circ \hat{X}.$$

For the same reconstruction and interval in the stated position and canonical momentum units

$$\begin{aligned}\sup_a \|r_X\|_{L_q^2} &< 3.148 \times 10^{-3}, & \sup_a \|r_P\|_{L_q^2} &< 5.440 \times 10^{-3}, \\ \int_{0.2}^{0.4} \|r_X\|_{L_q^2} da &< 3.732 \times 10^{-4}, & \int_{0.2}^{0.4} \|r_P\|_{L_q^2} da &< 1.079 \times 10^{-3}.\end{aligned}\tag{5.31}$$

The associated phase-space measure satisfies

$$|\mathcal{D}_\varphi| < 3.732 \times 10^{-4} \|\nabla_x \varphi\|_\infty + 1.079 \times 10^{-3} \|\nabla_p \varphi\|_\infty,$$

where $\mathcal{D}_\varphi$ is its weak Vlasov–Poisson defect (C.11).

Proof. For (a) Lemma C.2 bounds the nonlinear Poisson source map by its longitudinal displacement. The half-grid phase and complete spline norm give outward force bounds at nine anchors with the complementary Fourier energy included. Equation (C.6) covers all 64 time segments. The outward bounds on the displacement gradient and Jacobian give the stated density interval.

For (b) the pullback estimate (C.8) and transverse displacement bound combine with the Q1 inner products to give (C.9)–(C.10). Separate endpoint bounds for each norm together with intervals for α, β enclose every time segment. Integrating these upper bounds gives the second line of (5.31). The chain rule and Cauchy–Schwarz give the test-function estimate. The numerical and arithmetic details for both parts are given in Appendix C and the accompanying data. $\square$

The largest relative bound occurs on $a \in [0.209375, 0.2125]$. At its anchor $a = 0.2$ the relative bound is 22.75%. Time extension increases the error numerator and decreases the force-norm lower bound giving 37.50%. Appendix C separates this contribution from the Fourier complement and nonlinear homotopy remainder. The density and displacement bounds place this continuous certification in the weakly perturbed regime. Here the relative error is normalized by a lower bound on the continuous force norm. It includes the complete frequency complement and the extension between time anchors. The 0.286% trajectory enclosure uses a mesh cell as denominator and refers to a different observable. The two percentages quantify distinct sources of error in the same matter reconstruction.

The spatial-force comparison bounds two fields in Eulerian L^2 . The equation-defect estimate bounds the failure of the reconstructed cold sheet to satisfy the continuous kinetic equation. The finite-CIC trajectory enclosure instead compares the reconstruction with the continuous-time solution of the spatially discrete system. A transport comparison with an exact continuous solution also requires density bounds for that solution as in the bounded-density estimates and periodic formulation of Refs. [12, 13].

5.4.5 Object partition and growth budget

The large particle catalogue with 64^3 particles records material identity during structure formation. Its particle counts and recorded membership intersections constrain transfers between the diffuse population and detected objects. This information distinguishes mass transfer from a change in the number of objects selected at each epoch and gives a direct material interpretation of the representation balance.

The PM construction assigns particles to disjoint detected components. The mass budget evaluates Proposition 5.2 directly from particle counts checking $M_h = m_p n_h$ for every object and the fixed total $N = 262144$. The assigned object population starts at zero and reaches 42 particles at $z = 2$. It contains 2638 particles at $z = 0$ and 17050 at $a = 3$.

The remaining particles form the diffuse carrier. The two counts sum to N at each epoch and their net changes cancel exactly in integer arithmetic. Stored best-parent intersections give the stated intervals for the unresolved gross transfers. Thus the aggregate mass balance is exact with interval bounds on transfers not determined by the selected parent links.

The 837 entries count objects across epochs. The final population contains 96 selected objects under the imposed catalogue limit. Neighbour strengths and alignments describe the instantaneous geometry of this population. The particle intersections determine mass transfer whereas momentum and angular-momentum transfer additionally depend on the particle velocities and forces on each link.

The snapshot mean-velocity norm r_n supplies the further global bound

$$\frac{\|P_{n+1} - P_n\|}{Nm_p} \leq r_{n+1} + r_n.$$

The maximum adjacent-epoch bound read from the table is 3.595×10^{-9} per unit total mass. This global estimate follows from the triangle inequality. Individual momentum and angular-momentum transfers instead require particle and force records for each link. Proposition 5.4 supplies the error estimate for replacing their contributions by object moments.

6 Conclusions

Spatial concentration and material exchange change which details of matter control its external action. A finite-window description is selected according to the interaction and observables whose evolution it must reproduce. Localized matter can be represented by its mass and moments when they reproduce the required force while unresolved modes continue to act through residual variables and their history.

The common matter–source applications retain the generated source relations in the coupled variation of matter and geometry. The phase of the same matter state supplies the clock reading. In the homogeneous example a source-residual estimate controls response reuse and its effect on the matter state and scale factor. The same estimate bounds the radius and phase-clock readings. The nonuniform planar construction solves and propagates the Einstein constraints and retains positive principal forms near the occupied background. On the occupied homogeneous background the scalar constraints give an explicit linear response, whose canonical evolution preserves the constraints and determines the material acceleration. The nonuniform numerical evolution is checked under spatial and temporal refinement. The RMS application gives an autonomous finite displacement evolution with its periodic potential solved at all spatial frequencies. The binding term couples material motion to the expansion constraint and averaged pressure. On the certified interval the flow remains noncrossing, the reduced physical momentum has positive kinetic curvature, and the weak-field material clock stays regular. The orthogonal force defect records the drive of the displacement components excluded from this material window.

The ordering of object variables follows from their contribution to the force. Total mass fixes the leading response. The mass centroid cancels the first spatial correction.

Higher moments measure the remaining influence of the internal distribution. Eight CIC cell moments preserve the trilinear mesh source with cell-crossing corrections for a change of assignment region. For the collective response a positive lower bound on the source derivative of the stationary constraint controls the accuracy of source elimination. The material partition preserves additive quantities during a change of representation. Force and trajectory bounds control the accuracy of the subsequent action.

For each finite reference evolution reconstruction in a common state space measures the combined effect of these changes. Integrable stability and defect bounds together with strict containment in the state tube control the accumulated error and reconstruction jumps over the prescribed finite interval. The observable modulus then determines sufficient resolution. The first accepted window is the least certified choice in the specified candidate family. Approximate readings add their own evaluation bound to this propagated state error. Quadratic observations retain the cross contribution of the active and residual fields including their relative phase.

The Galerkin calculations achieve positive density and uniform accuracy with less arithmetic work including verification. The particle example encloses the continuous-time CIC trajectory within 0.286 percent of a mesh cell at fixed spatial resolution. Its weakly perturbed cold-sheet reconstruction gives full-frequency Newtonian force bounds. It also controls the continuous position and momentum equation defects. These statements retain their respective reference dynamics and norms.

The spectral-field histories describe the relation between a moving window and peak statistics while particle membership determines ancestry and mass transfer. The cosmological CIC and cold-sheet enclosures use the specified merged pressureless background and force. The generated phase-current equations determine the source exchange and radial stress. Their acceleration terms enter the comparison with resolved matter. Bounds on these terms enter the same equation-defect estimate as spectral and object replacement.

The nonlinear scale locates a changing structural regime. Force and observable bounds determine which details of that structure can be represented by moments or residual history. Gaussian composition fixes the averaging law and the generated interactions determine how matter develops correlations and localized structure under that law. The regularity and reconstruction estimates follow these changes over the specified interval. When the influence of omitted details grows finer coordinates re-enter with the accumulated reconstruction error. An effective object description remains adequate for as long as its retained variables reproduce the object's action on the surrounding matter to the prescribed accuracy. The object's internal structure remains part of the same evolving material system.

A Observable stability and notation

The error estimate in Theorem 3.6 requires a common modulus of continuity for the chosen observables. For object statistics this modulus is local to a region with stable selection and ancestry. The catalogue stability lemma specifies that region.

Lemma A.1 (Stability of catalogue observables). On a finite grid suppose the density-threshold margin and particle distances from membership cell boundaries exceed their

certified perturbations. Assume the same for the catalogue-ranking gap and selected-parent advantage (twice the score error for comparisons of two scores). Then the selected cells and connected components retain their particle membership. The retained ranking and parent choices are unchanged. On this region their mass and coordinate readings are locally Lipschitz using fixed periodic coordinate charts. For angular-average centres the resultant length and coordinate branch margin also have positive lower bounds. Ratios and logarithmic growth additionally require positive lower bounds on masses and elapsed times as well as the relevant vector norms.

Proof. The margins preserve every sign and strict comparison in the finite selection algorithm hence its graph and membership. With those fixed the coordinate readings are finite sums and smooth functions with denominators bounded away from zero so their derivatives are bounded on a compact certified region. $\square$

The catalogue guarantee applies while the whole comparison tube remains in such a stable region. Nearby continuous trajectories may cross a selection threshold at different times. For the reading $O(s) = \mathbf{1}_{\{s>0\}}$ the states $s = \pm\eta$ have distance 2η but readings differing by one for every $\eta > 0$. Both reconstructions may have zero jumps. Consequently the jump terms in (3.61) control reconstructed-state changes while intervals containing uncertified selection crossings require separate event-time enclosures and sets of admissible discrete labels. The admissible object counts and parent labels are recorded in those event enclosures. Alignment statistics are evaluated where spin and torque norms have positive lower bounds.

For a spectral ratio $O(P) = A(P)/T(P)$ where $A(P) = \int KP$ and $T(P) = \int P$ with $|K| \leq K_0$ suppose both compared denominators are at least $T_0 > 0$. The identity

$$O(\tilde{P}) - O(P) = \frac{\int [K - O(P)](\tilde{P} - P)}{T(\tilde{P})}$$

gives $|O(\tilde{P}) - O(P)| \leq (K_0 + |O(P)|)\|\tilde{P} - P\|_{L^1}/T_0$. To apply this estimate to a state error one also needs the denominator bound and a bound on the induced power error. On a Fourier L^2 ball of radius R $\||x|^2 - |y|^2\|_{\ell^1} \leq 2R\|x - y\|_{\ell^2}$. For catalogue observables the required local modulus follows from Lemma A.1.

Table 8 summarizes the scales and time variables together with the object quantities used in these estimates. In the common response units $e = \mu E$ and $P = M_P/\mu$. Also the physical matter mass coefficient is μm_0 . The initial occupied expansion H_0 in Proposition 3.2 is determined by its Einstein constraint. The cosmological examples use H_0 for the present Hubble scale. The internal source ζ is the variable denoted by η in the inherited action while η here denotes conformal time.

Table 8. Frequently used symbols.

Symbol	Meaning
a	scale factor
η	conformal time
ζ	internal conformal source determined by its stationary constraint
s, ϑ	proper time and retained matter-phase reading
σ	physical Gaussian coarse-graining scale
Λ	active boundary selected for the prescribed observables and tolerance
k_{NL}	nonlinear reference wavenumber defined by $\Delta^2 = C_{\text{NL}}$
k_{box}	lowest nonzero wavenumber of the specified periodic volume
k_{cut}	maximum wavenumber of the numerical representation
k_{ref}	conservative reference cutoff for cost fractions
v_{Λ}	logarithmic rate of change of the nonlinear reference endpoint
M_{NL}	mass scale associated with k_{NL} under the convention $R = c_M/k_{\text{NL}}$
v_{vir}	virial velocity at the selected nonlinear mass scale
$R_{\text{vir},h}$	virial radius of object h under the mass convention
t	evolution parameter in the stated clock chart: background cosmic time in the matter–source equations, $\ln a$ in the window estimates, and $H_0 T$ in the RMS evolution at $N = 1$
T	cosmic proper time
g_h	logarithmic halo mass-growth rate $d \ln M_h / dT$
B_{frac}	fractional-mass growth statistic with the stated extent-based time
$r_{\text{sup},h}$	support radius used in the object force-error bound
$T_{ij}^{(p)}$	traceless tidal tensor of source channel p
$I_{ij}^{(q)}$	inertia tensor of target object q
J_q	angular momentum of the target object
W_h	object selection or weighting function in R_h
$\sigma_{\log}, Q$	dimensionless log-width and scale of the residual-floor diagnostic
$\mathcal{M}_T$	residual-tail moments and feedback variables
$\mathcal{O}$	generated object layer included in the coarse-grained state

B Source response, memory estimates and finite evolution

We derive the memory-truncation defect used in Theorem 3.6 and evaluate the theorem for the finite Galerkin examples. The projection identities are those of Section 3.4.

For a trajectory enclosure the verification data specify the reference equation and clock chart with the encoded initial state. They also specify the reconstruction and its norm. The source bracket and operator bounds are recorded with the spectral-tail bounds. The state region on which stability holds completes the mathematical inputs. The residual is evaluated from the reconstructed path and that reference equation. These data allow the error test to be checked independently of the algorithm that proposed the path [20]. In the source calculation shared spectral enclosures supply the finite-evaluation terms of Proposition 3.7. Their construction is counted once and subsequent use is included in the verification work.

B.1 Finite memory and hard-window events

On a fixed retained set write the same finite evolution as $y' = F_A(t, y, v)$ and $v' = L_T(t)v + g_T(t, y, v)$. Let $U_T(t, s)$ solve $\partial_t U_T = L_T(t)U_T$ with $U_T(s, s) = I$. Variation of constants gives the exact nonlinear memory relation

$$v(t) = U_T(t, t_0)v(t_0) + \int_{t_0}^t U_T(t, s)g_T(s, y(s), v(s)) ds.$$

Indeed differentiating the right side recovers the tail equation and initial data. Local uniqueness identifies it with v . Removing history older than $t - T$ incurs the norm bound

$$\|U_T(t, t_0)v(t_0)\| + \int_{t_0}^{t-T} \|U_T(t, s)\| \|g_T(s, y(s), v(s))\| ds.$$

This formula applies when $t - T \geq t_0$. Otherwise no history is discarded. The bound $\|U_T(t, s)\| \leq \exp(\int_s^t \|L_T(r)\| dr)$ follows from the integral equation and Gronwall. This finite-interval bound retains both the initial tail state and the forcing accumulated along the material history. Any sharper decay estimate concerns the actual L_T in the chosen clock and norm. Substituting approximate recent histories contributes a further defect controlled by the Lipschitz bound in Theorem 3.6.

At a hard-window event t_i the underlying state is continuous and the exact active jump is

$$y(t_i^+) - y(t_i^-) = (D_{t_i^+} - D_{t_i^-})x(t_i).$$

The complement receives the opposite transfer in the smoothed field. This follows by subtracting the two definitions of the active state. For an approximate reconstruction the norm of its jump enters as j_i in (3.61).

B.2 A finite Galerkin example with a uniform error bound

Consider a matter-dominated background with $\mathcal{H} \propto a^{-1/2}$. Put $t = \ln a$ and $w = v_1/\mathcal{H}$. On the 2π -periodic zero-mean subspace depending only on x_1 retain modes $0 < |k_1| \leq 3$. The single-stream equations give the finite Galerkin system

$$\delta' = -\partial_x w - P_3 \partial_x (\delta w), \tag{B.1}$$

$$w' = -\frac{1}{2}w - P_3(w \partial_x w) + \frac{3}{2} \partial_x (-\partial_x^2)^{-1} \delta. \tag{B.2}$$

Here P_3 discards all nonretained modes so products are exact finite convolutions without aliasing. The coefficient $1/2$ follows by differentiating $v_1 = \mathcal{H}w$ and the $3/2$ term follows from the Poisson coefficient divided by $\mathcal{H}^2$. The background and single-stream matter equations are the physical inputs to this finite-dimensional reduction.

In the joint Fourier ℓ^1 norm the linear column sums and the convolution inequality give

$$\begin{aligned} \|L\| &\leq \frac{7}{2}, & \|B(U, V)\| &\leq 3\|U\|\|V\|, & R &= \frac{3}{100}, \\ L_R &= \frac{7}{2} + 6R = 3.68, & M_R &= \frac{7}{2}R + 3R^2 = 0.1077. \end{aligned}$$

For example the linear column sums are $|k| + 1/2$ and $3/(2|k|)$. Differentiating a retained product costs at most three. Take initial data

$$\delta_0 = 10^{-2} \cos x + 10^{-5} \cos 3x, \quad w_0 = -10^{-2} \sin x - \frac{10^{-5}}{3} \sin 3x.$$

Advance only modes ± 1 with Euler step $h = 10^{-4}$ until $T = 10^{-2}$ reconstructing the other modes as zero. In this compressed system the quadratic projection vanishes and its growing-mode amplitude is exactly $10^{-2}(1+h)^n$. The initial compression error is $E_0 = 1/75000$. On the radius- R ball the omitted-mode defect is at most $3R^2$. The integrated defect of the piecewise linear Euler interpolant is at most $L_R M_R h T / 2$ since within a step $\|F(\widehat{U}(t)) - F(\widehat{U}(t_n))\| \leq L_R M_R (t - t_n)$. The resulting rational error bound is

$$E(T) \leq \frac{E_0 + 3R^2 T + L_R M_R h T / 2}{1 - L_R T} = \frac{15199313}{36120000000} < 5 \times 10^{-5}. \quad (\text{B.3})$$

We used $e^{L_R T} \leq (1 - L_R T)^{-1}$. The interpolant has norm at most $2/99$. $2/99 + 5 \times 10^{-5} < R$ proves the tube condition by first exit. The finite reference solution therefore exists on the whole interval and has $1 + \delta > 0.97$. Every normalized linear reading of norm at most one has error below 5×10^{-5} . Thus the one-mode compression is ε -sufficient with $\varepsilon = 10^{-4}$ and margin $\varepsilon/2$ for this three-mode reference. The bound also applies after Gaussian smoothing. The arithmetic evaluation uses rational coefficients throughout.

For the longer evolution with $N = 8$ retain $0 < |k| \leq 8$ and take $\delta_0 = 10^{-2} \cos x + 10^{-8} \cos 8x$ and $w_0 = -10^{-2} \sin x - (10^{-8}/8) \sin 8x$. The reduced initial state is the projection onto the first positive mode. For the numerically represented initial datum U_0 the initial bound is the upward-rounded value of $\|U_0 - P_1 U_0\|_1$ namely $E_0 \simeq 1.125 \times 10^{-8}$. The full Galerkin evolution has $E_0 = 0$. With $R = 0.03$ the same convolution bounds give $L_R = N + 1/2 + 2NR = 8.98$. Let y_n denote the embedded reduced state and $s_n = (y_{n+1} - y_n)/h$ the slope of its piecewise linear interpolant. The computed binary floating-point coefficients are treated as exact dyadic rationals when evaluating the polynomial F_N . The defect $d_n = \|s_n - F_N(y_n)\|_1$ therefore includes the rounding error incurred in the time step. The piecewise linear reconstruction has defect at most $d_n + L_R \|s_n\|_1 (t - t_n)$ on the radius- R ball. Hence

$$E_{n+1} \leq \frac{E_n + h d_n + \frac{1}{2} L_R \|s_n\|_1 h^2}{1 - h L_R}.$$

Each rational bound is rounded upwards before the next step. The same first-exit argument checks $\max(\|y_n\|_1, \|y_{n+1}\|_1) + E_{n+1} < R$. Writing $\Delta_n = y_{n+1} - y_n$ gives the equivalent expression

$$E_{n+1} \leq \frac{E_n + \|\Delta_n - h F_N(y_n)\|_1 + \frac{1}{2} L_R h \|\Delta_n\|_1}{1 - h L_R}.$$

A common binary denominator for the state coefficients and a common integer denominator for the polynomial coefficients reduce the evaluation of the integrated defect to integer arithmetic. Combining identical monomials before evaluation avoids repeated rational normalization. Only the scalar recurrence requires rational operations. This algebraic rearrangement preserves the outward-rounded error bound.

The retained dimension is chosen by comparing this bound with an allowance $A_j = qA_{j-1} + b$ where

$$q = (1 - hL_R)^{-1}, \quad A_0 = E_0, \quad b = \frac{(\varepsilon - q^J E_0)(q - 1)}{q^J - 1}, \quad J = T/h.$$

Provided $q^J E_0 < \varepsilon$ the recurrence assigns a constant error increment and satisfies $A_J = \varepsilon$. The allowance is evaluated in floating-point arithmetic and bounded above by ε . Every accepted step separately satisfies the exact error and continuation conditions. For $T = 0.2$ with $h = 1/640$ and $\varepsilon = 10^{-4}$ the first admissible state retains two positive modes. Additional coefficients are initialized to zero when the retained space is enlarged so the reconstruction remains continuous and its error is still bounded by E_n . Products of retained coefficients determine the defect in the omitted sector with support at most $2M$ for retained endpoint M . Thus the error estimate is evaluated from the reduced coefficients and the reference vector field.

B.3 Arithmetic work of finite-window evolution

The support of the retained nonlinear product determines the work needed to evaluate both the evolution and its defect. The complete comparison also includes the arithmetic used to construct and verify each step.

Proposition B.1 (Arithmetic work of a certified finite-window evolution). Consider the alias-free quadratic Galerkin system above. Let $\mathcal{A}$ contain every attempted step including rejected trials and let M_a be the retained positive-mode count in attempt a . Count setup and window selection separately from reconstruction. The floating-point evolution and defect-evaluation work satisfies

$$\mathcal{W}_{\text{fl}} \leq \mathcal{W}_{\text{setup,fl}} + \mathcal{W}_{\text{select,fl}} + \mathcal{W}_{\text{reconstruct,fl}} + C \sum_{a \in \mathcal{A}} M_a^2, \quad (\text{B.4})$$

with C independent of N and the retained counts. For each exact arithmetic phase p let B_p bound the intermediate integer bit lengths and let $a_p, m_p, d_p, f_p, k_p, g_p$ record the operation counts. The first three count elementary integer operations followed by multiplication and division. The last three count rational operations followed by least-common-multiple and greatest-common-divisor calls. A schoolbook implementation admits the conservative bit-work bound

$$\mathcal{W}_{\text{bit}} = O\left(\sum_p [a_p B_p + (m_p + d_p + f_p + k_p) B_p^2 + g_p B_p^3]\right). \quad (\text{B.5})$$

The phases include coefficient construction and coordinate conversion along with window selection and every attempted error update.

Proof. The quadratic product of the retained state has support at most $\min(2M_a, N)$. Both its retained response and its omitted-output defect therefore require at most a fixed multiple of M_a^2 coefficient operations. Summing over all attempts gives Eq. (B.4). The separately

displayed terms account for operations outside this convolution. Construction of each coefficient expression is counted once. On B_p -bit operands elementary integer operations require $O(B_p)$ work. Schoolbook multiplication and division require $O(B_p^2)$ work and the elementary Euclidean algorithm has the conservative bound $O(B_p^3)$. Rational arithmetic consists of these integer operations and reductions. Summing their phase counts proves Eq. (B.5). $\square$

The retained coefficient count and the complete encoded length specify the representation used in this work estimate. Every explicitly stored coefficient and every intermediate integer enters the corresponding count. The polynomial construction theorem of Ref. [20] concerns its fixed evolution problem and input bounds. Here (B.4) and (B.5) express the work for the stated Galerkin evolution and verification phases with the measured counts given in Section 5.3.

Common-source response costs. For the two evolutions in Section 5.2.1 the shared setup requires 367 interval additions and 684 multiplications. It also uses 248 divisions and 77 negations. The remaining work includes 718 point conversions and 36 absolute bounds. One square root is evaluated. The shared setup evaluates one spectral endpoint (65 shell pairs) and 65 tail-moment terms. The inherited invariant-box certificate is reused once. Its spectral work is eight endpoints and 520 shell pairs. All interval operations use the outward integer grid 10^{-40} . Stored endpoints have at most 161 bits. Each interval primitive therefore has a bounded integer-arithmetic cost while the analytic root proposals carry their separately counted transcendental evaluations. The counts in Table 1 include unsuccessful reuse decisions followed by refreshes. Common initialization and the invariant-box construction are shared by both methods.

B.4 A uniform enclosure for autonomous RMS expansion

For the canonical system in Section 5.2.3 an outward rational calculation establishes existence and physical kinetic positivity on $0 \leq t \leq 0.05$. The same exit box bounds the material density and weak-field clock readings. In the box (5.18) write the upper endpoints for a, D, p_D as a_+, D_+, p_+ and set

$$f_+ = \frac{mD_+}{2k}, \quad g_+ = \sqrt{f_+^2 + \alpha_0 f_+}.$$

The inverse flux gives $|g| \leq g_+$. Convexity and $0 \leq \Psi(g) \leq g^2/2$ imply

$$-\frac{ma_+D_+g_+}{k} \leq W \leq 0, \quad 0 \leq W_a \leq 3a_+^2g_+^2, \quad 0 < -W_D \leq \frac{ma_+g_+}{k}.$$

Here $3\Psi - g\mathcal{J} \geq 0$ follows by differentiation from $2\mathcal{J} - g\mathcal{J}' \geq 0$. These inequalities bound every component of (3.29). They give $\dot{p}_a < 0$ and the integrated upper displacements on the four faces leave strictly positive margins. The smallest margins for a, D, p_D are respectively larger than 0.00499 and 1.53×10^{-4} and 1.40×10^{-7} . The lower p_a face retains a margin greater than 0.0721. Thus the local solution remains in (5.18) until $t = 0.05$. Evaluation of R and the physical Hamiltonian Hessian together with $|r_X| \leq 3g_+$ on this same box gives (5.19).

Uniform material readings. The same box bounds the observables in (3.33). In particular

$$-\frac{1}{501} \leq \delta \leq \frac{1}{499}, \quad |u| \leq \frac{2kp_+}{m} < 1.70 \times 10^{-4}, \quad \frac{H_{\text{phys}}}{H_0} \geq \frac{5.9}{6(1.06)^2} > 0.875.$$

For a mean-zero periodic potential of period $\ell = 2\pi/k$,

$$|\Phi(x)| = \left| \ell^{-1} \int [\Phi(x) - \Phi(y)] dy \right| \leq \ell \|\partial_x \Phi\|_\infty \leq \frac{2\pi a_+ g_+}{k} < 9 \times 10^{-5}.$$

The final decimal bounds follow by rational comparison, using $\pi < 22/7$. Thus $|2\Phi - u^2| < 1/5000$ throughout the certified interval. The weak-field material clock in (3.31) obeys

$$\sqrt{1 - \frac{1}{5000}} < \frac{ds_C}{dT} < \sqrt{1 + \frac{1}{5000}}, \quad |R_C| < 5.01 \times 10^{-9}. \quad (\text{B.6})$$

The remainder is the Taylor remainder of this clock reading, obtained from (3.32). These inequalities use the existing exit and force bounds; they require no additional trajectory integration or spatial reconstruction.

B.5 Material force and joint scalar response

Exact velocity equation. Use the phase frequency q_ϑ and logarithm χ of Section 3.1. Antisymmetrizing $\partial_\mu \vartheta = -q_\vartheta n_\mu$ and contracting with n^ν gives $n^\nu \nabla_\nu n_\mu = -h_\mu{}^\nu \partial_\nu \chi$. For coordinate velocity $w^i = dx^i/dt$ set $V^\alpha = (1, w)$ and $W = -g_{\alpha\beta} V^\alpha V^\beta = (n^0)^{-2} > 0$. Eliminating the proper-time derivative yields

$$\frac{dw^i}{dt} = -\Gamma_{\alpha\beta}^i V^\alpha V^\beta + w^i \Gamma_{\alpha\beta}^0 V^\alpha V^\beta - W(g^{i\nu} - w^i g^{0\nu}) \partial_\nu \chi. \quad (\text{B.7})$$

In the FLRW metric the physical peculiar velocity is $v = aw$. With $D_t = \partial_t + (v/a) \cdot \nabla$ its exact equation is

$$D_t v + H v = H |v|^2 v - (1 - |v|^2) \left(\frac{\nabla \chi}{a} + v \partial_t \chi \right). \quad (\text{B.8})$$

The difference from its slow-motion force is bounded pointwise by

$$|H| |v|^3 + |v|^2 \left(\frac{|\nabla \chi|}{a} + |v| |\partial_t \chi| \right).$$

For the scalar metric $-N^2 dt^2 + A^2 dx^2$ and orthonormal velocity $v_o = Aw/N$, direct substitution into (B.7) gives

$$\begin{aligned} \mathfrak{D}_t v_o = & - (1 - |v_o|^2) (\partial_t \log A + \partial_t \chi) v_o - \frac{N}{A} \nabla \log N \\ & + \frac{N}{A} |v_o|^2 \nabla \log A + \frac{N}{A} v_o [v_o \cdot \nabla \log(N/A)] - \frac{N}{A} (1 - |v_o|^2) \nabla \chi. \end{aligned} \quad (\text{B.9})$$

All gradients here are comoving spatial gradients. Suppose $|2\Phi|, |2\Psi| \leq \eta_m < 1$, and put $R_m = \sqrt{(1 + \eta_m)/(1 - \eta_m)}$. The remainder in (3.3) obeys

$$\begin{aligned} |R_o| \leq & \frac{|\partial_t \Psi|}{1 - \eta_m} |v_o| + \frac{\eta_m}{a(1 - \eta_m)} |\nabla \Phi| + \frac{R_m - 1}{a} |\nabla \chi| \\ & + |v_o|^3 \left(|H| + \frac{|\partial_t \Psi|}{1 - \eta_m} + |\partial_t \chi| \right) \\ & + \frac{R_m}{a} |v_o|^2 \left(|\nabla \chi| + \frac{|\nabla \Phi| + 2|\nabla \Psi|}{1 - \eta_m} \right). \end{aligned} \quad (\text{B.10})$$

Indeed $\partial_t \log A = H - \partial_t \Psi / (1 - 2\Psi)$, aN/A lies between R_m^{-1} and R_m , and

$$\left| \frac{1}{\sqrt{(1 + 2\Phi)(1 - 2\Psi)}} - 1 \right| \leq \frac{\eta_m}{1 - \eta_m}.$$

These bounds applied term by term to (B.9) prove (B.10). Taking its norm or integrated bound on the chosen state region gives the corresponding force defect. The coordinate and orthonormal peculiar velocities are related by $a dx/dt = (aN/A)v_o$. The phase-clock reading $ds = d\vartheta/q_\vartheta$ follows the same trajectory.

Canonical perturbations of the common background. Use the variables of Section 3.2, with $P = M_P/\mu$, $\kappa = |k|/a > 0$ and $\dot{\kappa} = -H\kappa$. The homogeneous source satisfies $C_H = U_{,\zeta} + 3\mathcal{T} - \mathcal{V} = 0$. In canonical variables its first-order variation is

$$\delta C_H = U_{,E\zeta} e_1 + h_H \delta \zeta + \frac{3\dot{\varphi}^\top \pi_\varphi}{a^3} + 9\mathcal{A}_\varphi \Psi - B\varphi^\top f.$$

At fixed canonical data the energy density satisfies $\partial_\zeta \rho = C_H$, hence $\delta \rho = \mathcal{R}_0 + 3\mathcal{A}\Psi$. Eliminating $\dot{\Psi}$ between the Einstein scalar constraints gives $2P^2\kappa^2\Psi + 3H\mathcal{L} + \delta\rho = 0$. Together with $\delta C_H = 0$ this gives (3.16).

The kinetic Hamiltonian density is $N(p_E^2/Z_E + |p|^2/M)/(2v_\gamma)$, while the potential density is $Nv_\gamma(\mathcal{V} + U)$. To first order $N/v_\gamma = a^{-3}(1 + \Phi + 3\Psi)$ and $Nv_\gamma = a^3(1 + \Phi - 3\Psi)$. Varying these terms and the spatial-gradient terms, then using $\Phi = \Psi$, yields the six real canonical equations

$$\begin{aligned} \dot{e}_1 &= \frac{\pi_E}{a^3 Z_E} + 4\Psi \dot{E}, \\ \dot{f} &= \frac{\pi_\varphi}{a^3 M} + (4\Psi + 3\delta\zeta) \dot{\varphi}, \\ \dot{\pi}_E &= -a^3 \left[(Z_E \kappa^2 + U_{,EE}) e_1 + U_{,E\zeta} \delta\zeta - 2\Psi U_{,E} \right], \\ \dot{\pi}_\varphi &= -a^3 \left[(M \kappa^2 + B) f - B\varphi \delta\zeta - 2\Psi B\varphi \right]. \end{aligned} \quad (\text{B.11})$$

The metric-volume and lapse variations account for the 4Ψ velocity terms and the -2Ψ potential terms. The derivative $B_{,\zeta} = -B$ fixes the sign of the source term in the last equation. The background coefficients retain their full time dependence.

Propagation of the scalar constraints. The isotropic pressure perturbation is

$$\delta p_{\text{iso}} = \frac{\dot{E}\pi_E + \dot{\varphi}^\top \pi_\varphi}{a^3} - U_{,E}e_1 - B\varphi^\top f + 3\mathcal{A}\Psi + 3\mathcal{A}_\varphi\delta\zeta.$$

The source coefficient follows from $U_{,\zeta} = \mathcal{V} - 3\mathcal{T}$, so that $3\mathcal{T} + \mathcal{V} - U_{,\zeta} = 3\mathcal{A}_\varphi$. Differentiating the energy and momentum perturbations with (B.11) gives

$$\begin{aligned}\delta\dot{\rho} + 3H(\delta\rho + \delta p_{\text{iso}}) &= 3\mathcal{A}\dot{\Psi} - \kappa^2\mathcal{L}, \\ \dot{\mathcal{L}} + 3H\mathcal{L} &= \delta p_{\text{iso}} + \mathcal{A}\Psi.\end{aligned}\tag{B.12}$$

The energy identity uses the background source equation, its time derivative and $\delta C_H = 0$. To verify constraint propagation, set

$$C = 2P^2\kappa^2\Psi + 3H\mathcal{L} + \delta\rho, \quad Q = \dot{\Psi} + H\Psi - \frac{\mathcal{L}}{2P^2}.$$

Let E_ρ and $E_\mathcal{L}$ be the left sides minus the right sides of the two identities in (B.12). Using $\dot{H} = -\mathcal{A}/(2P^2)$ gives exactly

$$\dot{C} + 3HC = (2P^2\kappa^2 + 3\mathcal{A})Q + E_\rho + 3HE_\mathcal{L}.\tag{B.13}$$

The algebraic solution imposes $C = 0$, and the matter and source equations give $E_\rho = E_\mathcal{L} = 0$. Positivity of $2P^2\kappa^2 + 3\mathcal{A}$ therefore gives $Q = 0$, the momentum constraint. The Einstein spatial trace equation then follows from the contracted Bianchi identity. The canonical evolution and the algebraic response therefore satisfy the scalar Einstein and source equations for the retained action. On a compact regular background interval the reduced linear coefficients are continuous and their denominators have positive lower bounds. For each fixed nonzero Fourier mode, linear ODE theory gives a unique solution for every prescribed initial canonical perturbation throughout that interval. The kinetic and spatial principal matrices are given by (3.15); their positivity is evaluated on the same background interval.

Phase and number-density readings. For the occupied background with $r > 0$ and $\mathcal{Q} = \varphi_1 p_2 - \varphi_2 p_1 > 0$, number density is $\nu = \mathcal{Q}/a^3$. Its linear readings are

$$\begin{aligned}\delta\mathcal{Q} &= f_1 p_2 + \varphi_1(\pi_\varphi)_2 - f_2 p_1 - \varphi_2(\pi_\varphi)_1, \\ \frac{\delta\nu}{\nu} &= \frac{\delta\mathcal{Q}}{\mathcal{Q}} + 3\Psi, \\ \delta\chi &= \frac{\delta\nu}{\nu} + 3\delta\zeta - 2\frac{\delta r}{r}.\end{aligned}$$

Differentiation using the canonical equations gives

$$\delta\dot{\vartheta} = q_\vartheta(\Psi + \delta\chi), \quad \partial_t(\delta\nu/\nu) = 3\dot{\Psi} - \frac{\kappa^2}{q_\vartheta}\delta\vartheta.\tag{B.14}$$

The number-density relation is the linearized conserved phase current. Differentiating $v = -\nabla\delta\vartheta/(aq_\vartheta)$ now gives (3.17). Thus the constrained source and metric response determines material acceleration and clock frequency on the same evolving background.

C Continuous reconstruction and Newtonian force estimates

The initial force comparison resolves the discrepancy in specified Fourier bands. The subsequent estimates establish the full-frequency spatial-force bound and continuous equation defects in parts (a) and (b) of Proposition 5.6. They compare the Newtonian force with the longitudinal displacement then control the CIC field and the intervals between evaluated configurations.

Let the periodic box have volume $V = B^3$. Its side B is the length denoted by L in Section 5.4.3. Here L denotes a bound on the displacement gradient. All vector norms are Euclidean and L^2 norms include the factor $V^{-1/2}$. Lagrangian and Eulerian coordinates are denoted by q and x . For $X(q) = q + \psi(q)$ set $\rho dx = X_*(dq)$ and

$$F^N = -\nabla\Phi, \quad \Delta\Phi = \rho - 1, \quad \langle F^N \rangle = 0.$$

Let $P_{||}$ have Fourier symbol $kk^T/|k|^2$ at nonzero k and zero symbol at $k = 0$.

C.1 The initial force in specified Fourier bands

For the initial map (5.27) write $\psi = sI_h D$ and $k_m = 2\pi m/B$. The Fourier convention is $\hat{f}_m = V^{-1} \int f(x) e^{-ik_m \cdot x} dx$. The mesh spacing is $h_m = B/N_m$. Mass transport gives $\hat{\rho}_m = V^{-1} \int e^{-ik_m \cdot [q + \psi(q)]} dq$ for $m \neq 0$. Its homogeneous term vanishes.

Proposition C.1 (Initial force comparison). Let $A \geq \|\psi\|_\infty$ in the Euclidean vector norm and $\mathcal{M}_2 \geq V^{-1} \int |\psi|^2 dq$. Let B_d bound the difference of neighbouring displacement-grid vectors in direction d . Set $K = |k_m|$ and $\theta_d = k_{m,d} h_m$ and define

$$Q_m = \frac{1}{V} \int e^{-ik_m \cdot q} (k_m \cdot \psi(q))^2 dq.$$

If $Q_m^{(2)}$ is evaluated by the tensor product of the two-point Gauss rule on each mesh cell then

$$\left| \hat{\rho}_m + ik_m \cdot \hat{\psi}_m + \frac{1}{2} Q_m^{(2)} \right| \leq \frac{K^3 A \mathcal{M}_2}{6} + \frac{\mathcal{G}_m}{2}, \quad (\text{C.1})$$

$$\mathcal{G}_m = \frac{1}{4320} \sum_{d=1}^3 [|\theta_d|^4 (KA)^2 + 8|\theta_d|^3 (KA)(KB_d) + 12|\theta_d|^2 (KB_d)^2]. \quad (\text{C.2})$$

Moreover

$$\hat{\psi}_m = W_m \hat{\psi}_m^{\text{grid}}, \quad W_m = \prod_{d=1}^3 \text{sinc}^2\left(\frac{\pi m_d}{N_m}\right), \quad \text{sinc}(z) = \frac{\sin z}{z}.$$

Thus Eq. (C.1) encloses the exact Newtonian coefficient $\hat{F}_m^N = ik_m \hat{\rho}_m / K^2$. The corresponding coefficient of the actual interpolated CIC force is $W_m \hat{K}_m \hat{\delta}_m^{\text{CIC}}$. For any finite set of nonzero modes these enclosures give upper and lower bounds on its volume-normalized L^2 force discrepancy.

Proof. Expand $e^{-ik_m \cdot \psi}$ to second order. Its pointwise third-order remainder is bounded by $|k_m \cdot \psi|^3/6$. Averaging gives the first term in Eq. (C.1). On one cell ψ is trilinear. Holding the other coordinates fixed put $z(t) = k_m \cdot \psi(t)$. Then $z'' = 0$ with $|z| \leq KA$ and $|z'| \leq KB_d$. The fourth derivative of $e^{-i\theta_d t} z(t)^2$ is bounded by the three terms in Eq. (C.2). The two-point Gauss remainder has a positive Peano kernel of integral $1/4320$. Applying it successively in the three directions proves the stated tensor-product bound. The Fourier transform of the periodic linear interpolation basis is the displayed sinc-squared factor. Finally the Poisson equation fixes each nonzero force coefficient and Parseval's identity gives the L^2 bounds. $\square$

In this calculation A is bounded by the largest displacement-grid norm and $\mathcal{M}_2$ by the mean squared grid norm using convexity of the interpolation weights. The three linear transforms and the 48 transforms of the six symmetric quadratic products at eight Gauss nodes are shared by all modes. The CIC density transform is reused from the initial trajectory-force evaluation. The reported bounds include interval phases and Fourier roundoff. Arithmetic enclosures are included in the same evaluation [14].

C.2 Mass transport and the full-frequency field

Lemma C.2 (A quantitative displacement estimate). If $\psi \in W^{1,\infty}$ is periodic and $\|D\psi\|_{L^\infty, 2 \rightarrow 2} \leq L < 1$ then

$$\begin{aligned} \|F^N - P_\parallel \psi\|_{2,x} &\leq C(L) \|\psi\|_2, \\ C(L) &= \frac{3L/2 + L^2 + L^3/4}{(1-L)^{3/2}} + \frac{L}{2}(1+L)^{3/2}. \end{aligned} \quad (\text{C.3})$$

For two such displacements whose straight interpolation has the same bound L

$$\|F^N[\psi_2] - F^N[\psi_1]\|_{2,x} \leq (1-L)^{-3/2} \|\psi_2 - \psi_1\|_{2,q}. \quad (\text{C.4})$$

Proof. For $T_t(q) = q + t\psi(q)$ the contraction equation $q = x - t\psi(q)$ gives its inverse on the periodic lift. Its positive Jacobian obeys $(1-tL)^3 \leq J_t \leq (1+tL)^3$ almost everywhere. The pushed-forward mass flux is

$$j_t(T_t q) = \psi(q)/J_t(q), \quad \partial_t \rho_t + \operatorname{div} j_t = 0, \quad F^N = P_\parallel \int_0^1 j_t \, dt.$$

Indeed $\widehat{\rho - 1} = -ik \cdot \int \widehat{j_t} \, dt$ and $\widehat{F^N} = ik\widehat{\rho}/|k|^2$. At $x = T_t q$

$$j_t(x) - \psi(x) = (J_t^{-1} - 1)\psi(q) + \psi(q) - \psi(T_t q).$$

Changing variables with $dx = J_t dq$ bounds the terms by

$$\frac{3tL + 3t^2L^2 + t^3L^3}{(1-tL)^{3/2}} \|\psi\|_2, \quad tL(1+tL)^{3/2} \|\psi\|_2.$$

Integration and the contractivity of $P_\parallel$ prove (C.3). The constant mode is removed by the projection so a mean displacement is permitted. For the second claim use the homotopy $\operatorname{id} + \psi_1 + t(\psi_2 - \psi_1)$. Its flux has squared norm $V^{-1} \int |\psi_2 - \psi_1|^2 / J_t \, dq$ bounded by $(1-L)^{-3} \|\psi_2 - \psi_1\|_2^2$. The weak continuity equation and the same projection give (C.4). $\square$

C.3 CIC continuity and interpolation between anchors

There is one equal-weight particle per original cell. Suppose every configuration under comparison remains within coordinate distance $e < h/2$ of its assigned cell centre including each configuration segment. For the vector force multiplier $H(m)$ define

$$S_h = \max_m \|H(m)\|_2 \left(\sum_d \left| \frac{1 - e^{-2\pi i m_d/N}}{h} \right|^2 \right)^{1/2}, \quad L_C = S_h(1 + 2e/h).$$

Then the continuously interpolated field satisfies

$$\|G[X] - G[Y]\|_{2,x} \leq L_C \|X - Y\|_{\text{rms}}. \quad (\text{C.5})$$

To prove this differentiate deposition as a discrete divergence of face currents. In each direction the current matrix is nonnegative with column sum one and row sum at most $(1 + 2e/h)^2$. Schur's bound is therefore $1 + 2e/h$. Discrete Parseval gives S_h . Jensen's inequality and $\int w_g dx = V/N_g$ show that continuous CIC interpolation contracts normalized grid ℓ^2 into L_x^2 . Here $N_p = N_g$. Integrating along the configuration segment proves (C.5) at a common Eulerian observation point.

For the reconstruction in (5.29) an anchor bound $E_b \geq \|F^N[\hat{\rho}(b)] - G[X(b)]\|_{2,x}$ consequently gives

$$\|F^N[\hat{\rho}(a)] - G[X(a)]\|_{2,x} \leq E_b + [(1 - L)^{-3/2} + L_C] \|X(a) - X(b)\|_{\text{rms}}. \quad (\text{C.6})$$

This follows from (C.4) (C.5) and the L^2 contraction of spatial interpolation. On each linear time segment the RMS distance to an anchor is convex hence bounded by its endpoint maximum. For a relative bound an anchor lower bound $N_b \leq \|F^N[\hat{\rho}(b)]\|_2$ becomes $N_b - (1 - L)^{-3/2}d$ for endpoint distances at most d . The reported denominator is strictly positive.

The anchor evaluation includes all spatial frequencies. With $W_m = \prod_d \text{sinc}^2(\pi m_d/N)$ the displacement coefficient at every integer frequency is

$$\hat{\psi}_m = W_m \left(\widehat{sD}_{m \bmod N} + e^{-i\pi \sum_d m_d/N} X(b) - \widehat{X}(a_0)_{m \bmod N} \right).$$

The half-grid phase distinguishes the original nodal source from the centred reconstruction. The CIC field coefficient is $W_m H(m \bmod N) \hat{\delta}_{m \bmod N}$. Complete spline norms follow from the one-dimensional mass stencil $(1, 4, 1)/6$ with eigenvalue $(2 + \cos(2\pi r/N))/3$. Subtracting enclosed retained-mode energies from complete norm bounds controls their complements including aliases and the constant mode. The triangle inequality and (C.3) then give E_b . Nine anchors at step indices $0, 8, \dots, 64$ use 728 nonzero modes with $|m|_\infty \leq 4$ and these full tail bounds.

Let D_b bound the retained-mode norm of $P_\parallel \psi - G$ and let $T_{\psi,b}$ and $T_{G,b}$ bound the complementary displacement and CIC-field norms. Orthogonality of the retained and complementary frequencies gives the anchor budget

$$E_b = [D_b^2 + (T_{\psi,b} + T_{G,b})^2]^{1/2} + H_b, \quad H_b = C(L) \|\psi(b)\|_2^{\text{up}}, \quad (\text{C.7})$$

evaluated with outward rounding. The complement enters in quadrature with the retained discrepancy. The nonlinear homotopy remainder H_b and the time-extension term in (C.6) enter additively.

For the segment giving the largest relative bound $a \in [0.209375, 0.2125]$ with anchor $b = 0.2$ the saved components give in $h^{-1}\text{Mpc}$

$$\begin{aligned} T_{\psi,b} &\simeq 3.1643 \times 10^{-4}, & T_{G,b} &\simeq 1.3664 \times 10^{-4}, & H_b &\simeq 9.4585 \times 10^{-5}, \\ E_b &\simeq 9.1471 \times 10^{-4}, & [(1-L)^{-3/2} + L_C]d &\simeq 4.9900 \times 10^{-4}. \end{aligned}$$

The displayed components are rounded summaries of interval bounds. The time-extended numerator is 1.4137056×10^{-3} or less and the force-norm lower bound decreases from approximately 4.020953×10^{-3} to more than 3.7706428×10^{-3} . Their ratio is below 0.3750. More closely spaced anchors reduce the time-extension contribution and the associated loss in the denominator. Enlarging the retained Fourier band targets the complement estimates at the same mesh resolution while sharper displacement reconstruction or homotopy estimates target H_b . Refining the force mesh also changes the retained discrepancy D_b which must be evaluated again.

C.4 The continuous position and momentum defects

Write $A(L) = (1-L)^{-3/2}$ and $M = \|\psi\|_\infty$. Changing variables in (C.3) followed by the Sobolev chain rule along T_t yields

$$\|F^N \circ X - P_\parallel \psi\|_{2,q} \leq A(L)\{C(L)\|\psi\|_2 + M\|D\psi\|_2\}. \quad (\text{C.8})$$

To justify the pullback for the nonsmooth Q1 field put $U = P_\parallel \psi \in H^1$. Smooth approximation gives $U(q + \psi(q)) - U(q) = \int_0^1 DU(T_t q)\psi(q) dt$. The Jacobian bounds imply $\|DU \circ T_t\|_2 \leq (1-tL)^{-3/2}\|D\psi\|_2$. The uniform boundedness of these pullbacks permits the H^1 limit proving (C.8) for the Q1 reconstruction. Parseval also gives with $k_{\min} = 2\pi/B$

$$\|(1 - P_\parallel)\psi\|_2 \leq |\langle \psi \rangle| + k_{\min}^{-1} \|\text{curl } \psi\|_2.$$

Thus for $d_p = (p_{j+1} - p_j)/(a_{j+1} - a_j)$

$$\begin{aligned} \|r_P\|_{2,q} &\leq \|I_c d_p - \beta \psi\|_2 + \beta[|\langle \psi \rangle| + k_{\min}^{-1} \|\text{curl } \psi\|_2 \\ &\quad + A(L)\{C(L)\|\psi\|_2 + \|\psi\|_\infty \|D\psi\|_2\}]. \end{aligned} \quad (\text{C.9})$$

Let b_0 sample ψ_0 on the centred grid and set $\varepsilon_0 = \|\psi_0 - I_c b_0\|_2$ and $\varepsilon_{P0} = \|f_0 \psi_0 - I_c p_0\|_2$. Positive interpolation weights give

$$\begin{aligned} \|r_X\|_2 &\leq \|d_x - \alpha p(a)\|_{\text{rms}} + \alpha \varepsilon_{P0}, \\ \|I_c d_p - \beta \psi\|_2 &\leq \|d_p - \beta[b_0 + x(a) - x(a_0)]\|_{\text{rms}} + \beta \varepsilon_0. \end{aligned} \quad (\text{C.10})$$

The initial half-grid difference is evaluated exactly as a Q1 field on the common 32^3 subdivision. The particle states remain on the 16^3 grid. Q1 mass and gradient inner products use the tensor mass stencil and directional differences. The periodic identity

$\|\text{curl } \psi\|_2^2 = \|D\psi\|_2^2 - \|\text{div } \psi\|_2^2$ includes the cross terms of the divergence inner product. Global continuity ensures that weak first derivatives have no surface delta terms.

For the positive matter–vacuum background α and β decrease on the interval. Coefficient intervals and endpoint bounds for each individual norm enclose every time segment. Products in (C.9) use the separate factor maxima. The initial growth integral is enclosed by its 20-term alternating binomial expansion with its remainder. The resulting initial mismatch bounds are $\varepsilon_0 < 5.462 \times 10^{-4}$ and $\varepsilon_{P0} < 1.370 \times 10^{-4}$.

Finally for $\hat{\mu}_a = (\hat{X}, \hat{P})_*(dq/V)$ the chain rule along each time segment gives the exact weak-defect identity

$$\begin{aligned} \mathcal{D}_\varphi &:= \left[\int \varphi d\hat{\mu}_a \right]_{a_0}^{a_1} - \int_{a_0}^{a_1} \int (\partial_a \varphi + \alpha p \cdot \nabla_x \varphi + \beta F^N \cdot \nabla_p \varphi) d\hat{\mu}_a da \\ &= \int_{a_0}^{a_1} \int (\nabla_x \varphi \cdot r_X + \nabla_p \varphi \cdot r_P)(a, q) \frac{dq}{V} da. \end{aligned} \quad (\text{C.11})$$

Gradients on the last line are evaluated at $(a, \hat{X}, \hat{P})$. The test function is C^1 periodic in position and compactly supported in momentum. Internal time-boundary terms cancel. Cauchy–Schwarz proves the bound in Proposition 5.6(b).

C.5 Evaluation of the bounds

The spatial-force estimate reuses the 65 trajectory states. Its 39 three-dimensional transforms consist of three for the initial source and four at each of the nine anchors. Complete spline norms control the Fourier complement. Forty-digit interval arithmetic encloses the trigonometric factors and roots of unity along with roundoff accumulated in the transforms and force evaluation.

The position and momentum defect bounds reuse the existing transforms and depositions without advancing the trajectory. They use the same states with exact spline inner products and interval bounds for the time-dependent coefficients. Independent rational cell integrations check the moment identities and shifted Fourier modes as well as grid-crossing terms. Numerical data and detailed arithmetic counts are supplied with Ref. [14].

The stored states determine the Q1 spatial reconstruction and the piecewise time interpolation. Their coefficients and background intervals determine the residual bounds with the force-kernel enclosures included. This evaluation is independent of the time-stepping recurrence. The initial mismatch and spatial force difference enter their respective terms in the displayed estimates. Time-segment defects account for the residual along the reconstruction. For a specified test function the last step is the exact weak-defect identity (C.11). This checks the continuous equation defect of the reconstructed material measure using the same trajectory data.

The arithmetic counts above apply to the specified reconstruction and mesh. Bit complexity also depends on intermediate integer sizes and retained precision. The treatment of these parameters for finite certificates is given in Ref. [20].

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Declarations

Conflict of interest. The author declares no competing interests.

Data and code availability. The numerical data and programs are available with the Coarse-Graining Genesis Framework [14]. The material includes initial conditions and cosmological parameters for the field and particle realizations. It also contains the finite Galerkin examples and common-source evolution with their error-bound calculations. The continuous-force and equation-defect estimates have separate reproduction procedures. Each calculation records its norm bounds and arithmetic conventions. For each enclosure the reference equation and encoded inputs identify the evolution being checked. The reconstruction and source enclosures together with force enclosures supply its residual tests. The reported work distinguishes construction and reuse from verification of these quantities.

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