Chiral Differential Operators

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Abstract

This is just a personal rewriting of some parts of [AM21], with some backgrounds taken from [Kle] and [Mal17]. Every mistake is due to me.

1 Background

Let $X = \operatorname{Spec} A$ be an affine algebraic variety over $\mathbb{C}$ (we will focus particularly on the case in which X is an algebraic group). Let $\mathcal{O}_X$ be the structure sheaf.

Definition 1.1. A global section $\theta \in \operatorname{End}_{\mathbb{C}}(\mathcal{O}_X)(X)$ is a *vector field* on X if for each open $U \subset X$, the section $\theta(U) := \theta\big|_U \in \operatorname{Der}_{\mathbb{C}}(\mathcal{O}_X(U), \mathcal{O}_X(U))$, i.e. it satisfies the Leibniz rule. Call $\Theta(X)$ the set of vector fields on X.
The *tangent sheaf* Θ_X is defined by

$$U \mapsto \Theta(U)$$

and one can verify it is a $\mathcal{O}_X$-module, where one identifies $f \in \mathcal{O}_X$ with $\mu_f \in \operatorname{End}_{\mathbb{C}}(\mathcal{O}_X)$ being the multiplication by f.

Compactly, $\Theta_X = \operatorname{Der}_{\mathbb{C}}(\mathcal{O}_X)$ and one can prove it is a coherent sheaf. In-fact, if

$$A = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_r),$$

then it is well known

$$\operatorname{Der}_{\mathbb{C}}(A, A) \simeq \operatorname{Hom}_A(\Omega^1_{A/\mathbb{C}}, A), \qquad \Omega^1_{A/\mathbb{C}} = \frac{\bigoplus_{i=0}^n A dx_i}{(df_1, \dots, df_r)}$$

so that we see $\operatorname{Der}_{\mathbb{C}}(A)$ is finitely generated as an A-module.

Let's now talk about cotangent sheaf. One local construction of $\Omega^1_{A/\mathbb{C}}$ is given by considering the A-module I/I^2 where I is the kernel of the multiplication map $\mu \colon A \otimes_{\mathbb{C}} A \to A$, and one can prove it is generated by elements $df = f \otimes 1 - 1 \otimes f \mod I^2$. Globalizing this construction we obtain the following.

Definition 1.2. The *cotangent sheaf* of X is defined by

$$\Omega^1_X := \delta^{-1}(\mathcal{I}/\mathcal{I}^2)$$

where $\delta \colon X \to X \times X$ is the diagonal embedding and $\mathcal{I}$ is the ideal sheaf of $\delta(X)$ in $X \times X$ (X is affine so automatically separated, i.e. δ is a closed immersion). Sections of Ω^1_X are called *differential forms*.

The cotanget sheaf is an $\mathcal{O}_X$-module in a natural way and it has a natural derivation $d \colon \mathcal{O}_X \to \Omega^1_X$ given by $df = f \otimes 1 - 1 \otimes f \mod \delta^{-1}(\mathcal{I}^2)$. Analogue to the affine case, we have

Lemma 1.3. As an $\mathcal{O}_X$-module, Ω^1_X is generated by df, for $f \in \mathcal{O}_X$.

And, of course, the same universal property holds.

Proposition 1.4. We have an isomorphism in $\mathcal{O}_X - Mod$:

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \cong \Theta_X.$$

We have an analogue situation of what happens in a manifold with charts, where we can always choose a local neighborhood trivializing the tangent bundle.

Theorem 1.5. Let X be smooth. For each $x \in X$ there exists an affine open neighborhood $V \ni x$, regular functions $x_i \in \mathcal{O}_X(V)$ and verctor fields $\partial_i \in \Theta_X(V)$ satisfying

$$[\partial_i, \partial_j] = 0, \quad \partial_i(x_j) = \delta_{i,j}, \qquad \Theta_V = \bigoplus_i \mathcal{O}_V \partial_i.$$

Moreover one can choose $x_1, \dots, x_n$ so that they generate the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_{X,x}$.

Proof. By assumption the local ring $\mathcal{O}_{X,x}$ is regular, so there exist $n = \dim X$ functions $x_1, \dots, x_n$ generating the ideal $\mathfrak{m}_x$ (use definition of regular local ring and Nakayama's lemma). Then $dx_1, \dots, dx_n$ is a basis of the free $\mathcal{O}_{X,x}$-module $\Omega^1_{X,x} \simeq \Omega^1_{\mathcal{O}_{X,x}}$. This is a well-known result: since $\mathcal{O}_{X,x}$ is a finitely generated local $\mathbb{C}$-algebra, whose residue field is $\mathbb{C}$ (Nullstellensatz), then we have $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong \mathbb{C} \otimes_{\mathcal{O}_{X,x}} \Omega^1_{X,x}$ and, using again Nakayama, we get what we claimed.

Thus we can find an affine open neighborhood V of x such that $\Omega^1(V)$ is a free $\mathcal{O}_X(V)$-module with basis $dx_1, \dots, dx_n$. If we define $\partial_1, \dots, \partial_n \in \Theta_X(V)$ as the dual basis, we get $\partial_i(x_j) = \delta_{i,j}$. To obtain the desired commutation relations, write

$$[\partial_i, \partial_j] = \sum_{i=1}^{n} g^l_{i,j} \partial_l \in \mathcal{O}_X(V)$$

and observe that $g^l_{i,j} = [\partial_i, \partial_j] x_l = \partial_i \partial_j x_l - \partial_j \partial_i x_l = 0$. ■

The set $\{x_i, \partial_i \mid 1 \leq 1 \leq n\}$ over an affine open neighborhood of x, satisfying the above conditions, is called a *local coordinate system.*

1.1 Differential Operators

Let's now define a sheaf $\mathcal{D}_X$ on X as the sheaf of $\mathbb{C}$-subalgebras of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ (embedded, as before, as left multiplications) and Θ_X.

Definition 1.6. The sheaf $\mathcal{D}_X$ is called the *sheaf of differential operators* on X. The algebra $\mathcal{D}_X(X)$ is called the *algebra of differential operators* on X.

Remark. For now we should think only about $\mathcal{D}_A$ for A a commutative $\mathbb{C}$-algebra. We will see later that this definition, for finitely generated $\mathbb{C}$-algebras, behaves well with localization, hence it gives rise to a sheaf.

Observe that, on a trivializing neighborhood U of x, we have

$$[\partial, f] = \partial(f) \in \mathcal{O}_X(U) \qquad \forall\, f \in \mathcal{O}_X(U),\, \partial \in \Theta_X(U)$$

so that we have

$$\mathcal{D}(U) = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^\alpha, \qquad \partial^\alpha := \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}.$$

We have an obvious order filtration, which we can define locally by

$$F_l \mathcal{D}_U = \sum_{|\alpha| \leq l} \mathcal{O}_U \partial^\alpha$$

and then glue globally just by requiring all restrictions to trivial neighborhood to be in the corresponding degree. There is, though, a more natural way to define it:

$$F_k \mathcal{D}_X = \{\theta \in \mathcal{E}nd(\mathcal{O}_X) \mid [f_{k+1}, \dots, [f_2, [f_1, \theta]] \dots] = 0 \quad \forall\, f_1, \dots, f_{k+1} \in \mathcal{O}_X\}\,.$$

It is clear that $F_0 \mathcal{D}_X = \mathcal{O}_X$ and we have the split short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_1 \mathcal{D}_X \longrightarrow \Theta_X \longrightarrow 0$$

where $\mathcal{O}_X$ is embedded as multiplication, the map $F_1 \mathcal{D}_X \to \Theta_X$ is given by $P \mapsto [P, -]$. Here are some basic properties of differential operators.

Proposition 1.7. (i) $F_\bullet \mathcal{D}_X$ is an increasing filtration of $\mathcal{D}_X$ such that $\mathcal{D}_X = \bigcup_{l \geq 0} F_l \mathcal{D}_X$ and each $F_l \mathcal{D}_X$ is locally a free $\mathcal{O}_X$-module.

(ii) $F_0\mathcal{D}_X = \mathcal{O}_X$ and $F_l\mathcal{D}_X \circ F_m\mathcal{D}_X \subseteq F_{l+m}\mathcal{D}_X$.

(iii) $[F_l\mathcal{D}_X, F_m\mathcal{D}_X] \subseteq F_{l+m-1}\mathcal{D}_X$.

Observe that a corollary of this proposition is that the graded algebra

$$\operatorname{gr}\mathcal{D}_X = \bigoplus_{l\geq 0} F_l\mathcal{D}_X/F_{l-1}\mathcal{D}_X$$

is commutative. Assume now $A = \mathcal{O}_X(X)$ is smooth, in the sense that Ω^1_A is a finitely generated free A-module.

Proposition 1.8. There is a (sheaf of) algebra isomorphism

$$\operatorname{gr}\mathcal{D}_X \xrightarrow{\sim} \operatorname{Sym}\Theta_X \cong \pi_*\mathcal{O}_{T^*X}$$

where $\operatorname{Sym}\Theta_A$ is the symmetric algebra on Θ_A. Moreover, it is also a Poisson algebra isomorphism.

Proof. See [Mal17, p.75]. ■

1.2 Derivations and differential forms on a group

Now we focus on the case $X = G$, for G an affine algebraic group. Its Lie algebra $\mathfrak{g}$ can be defined in a lot of equivalent ways, for example as the T_eG (tangent space at the identity element). We prefer, though, to define it in another way.

Definition 1.9. The Lie algebra of G is the Lie algebra of *left invariant vector fields* on G, that is,

$$\mathfrak{g} = \operatorname{Lie}(G) \coloneqq \{\theta \in \Theta(G) \mid \Delta\circ\theta = (1\otimes\theta)\circ\Delta\}$$

where $\Delta\colon \mathcal{O}(G) \to \mathcal{O}(G)\otimes\mathcal{O}(G)$ is the comultiplication and $1\colon \mathcal{O}(G)\to\mathbb{C}$ is the co-unit (recall that any affine algebraic group is a Hopf algebra).

Let's try to unfold this definition: a vector field $\theta \in \operatorname{Der}_{\mathbb{C}}(\mathcal{O}(G))$ is in $\mathfrak{g}$ if and only if, for every $g\in G$, $[\lambda_g, \theta] = 0$ as endomorphism of $\mathcal{O}(G)$, where λ_g corresponds to the action of g (acting by left multiplication on G) on $\mathcal{O}(G)$, i.e. $\lambda_g(f) = f(g\cdot -)$. One way to see this is to consider the map $\phi\colon G\to G$ (so that $\theta = -\circ\phi$) and the multiplication $\mu\colon G\times G\to G$. The condition on θ translates to

$$\phi(g_1\cdot g_2) = g_1\cdot\phi(g_2)$$

and, using now $\theta = -\circ\phi$, one obtains that for every $f\in\mathcal{O}(G)$ and $y\in G$ we have

$$\lambda_g\theta(f)(y) = \theta(f)(gy) = \theta(f(gy)) = \theta\lambda_g(f)(y).$$

As the above reasoning suggests, the only "important" information is the value at the identity e, as formalized by the following lemma.

Lemma 1.10. We have an isomorphism of Lie algebras

$$\mathfrak{g} = \operatorname{Lie}(G) \to T_eG \stackrel{\text{def}}{=} \operatorname{Der}_{\mathbb{C}}(\mathcal{O}(G),\mathbb{C}), \qquad \theta\mapsto\epsilon\circ\theta$$

where $\epsilon\colon\mathcal{O}(G)\to\mathbb{C}$ is the co-unit.

Proof. The map is clearly well-defined. Its inverse is given by $\delta\mapsto(\operatorname{id}\otimes\delta)\circ\Delta$. ■

We have a dual definition of *right invariant vector fields*, requiring simply the symmetric condition $\Delta\circ\theta = (\theta\otimes 1)\circ\Delta$ for $\theta\in\operatorname{Der}_{\mathbb{C}}(\mathcal{O}(G))$. As before, this concretely means that θ commutes with all ρ_g for $g\in G$, the contragradient action induced by the right multiplication $(g,x)\mapsto xg$ on G (i.e. $\rho_g(f) = f(-\cdot g)$). The proof is exactly as before, just by observing that this condition translates to $\phi(g_1g_2) = \phi(g_1)g_2$. Also observe that this latter condition does not give us automatically commutativity with λ_g, as well as the former doesn't give commutativity with the ρ_g, so there is a real distinction among left and right invariant vector fields, although they are both determined by just their value at the identity e (indeed, what changes is "how" they are determined). They are canonically isomorphic though, so given $x\in T_eG$ we write x_L (resp. x_R) to mean the corresponding left (resp. right) invariant vector field.

Sometimes using the definition of $\mathfrak{g}$ as derivation of $\mathcal{O}(G)$ at e can be useful, so let's state the following, which is a more concrete reformulation of the above. Let's consider left and right translations

$$\begin{aligned}\lambda_g\colon G\to G,\quad x\mapsto gx \rightsquigarrow \lambda_g^*\colon \mathcal{O}(G)\to\mathcal{O}(G),\quad f\mapsto f\circ\lambda_g(x\mapsto f(gx)),\\ \rho_g\colon G\to G,\quad x\mapsto xg \rightsquigarrow \rho_g^*\colon \mathcal{O}(G)\to\mathcal{O}(G),\quad f\mapsto f\circ\rho_g(x\mapsto f(xg)).\end{aligned}$$

We have

Lemma 1.11. Let $x\in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G),\mathbb{C})$ an element of T_eG. Then the corresponding left and right invariant vector fields are given by

$$x_L(f)(g)=x(\lambda_g^*f)=x(f(g\cdot -)),\qquad x_R(f)(g)=x(\rho_x^*f)=x(f(-\cdot g))$$

where $f\in\mathcal{O}(G)$, $g\in G$.

We can prove that left invariant and right invariant vector fields commute, as the commutation between λ and ρ suggests.

Lemma 1.12. Given $x,y\in\mathfrak{g}$ we have

$$[x_L,y_R]=0\in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)).$$

Proof. This is classical in Lie groups. One algebraic way to prove is to use Definition 1.9 stating that we have

$$\Delta\circ x_L=(\mathrm{id}\otimes x_L)\circ\Delta,\qquad \Delta\circ y_R=(y_R\otimes\mathrm{id})\circ\Delta.$$

Then let's consider

$$\begin{aligned}\Delta\circ x_L\circ y_R=(\mathrm{id}\otimes x_L)\circ\Delta\circ y_R=(\mathrm{id}\otimes x_L)\circ(\mathrm{id}\otimes y_R)\circ\Delta=(y_R\otimes x_L)\circ\Delta=\\ =(y_R\otimes\mathrm{id})\circ(\mathrm{id}\otimes x_L)\circ\Delta=(y_R\otimes\mathrm{id})\circ\Delta\circ x_L=\Delta\circ y_R\circ x_L\end{aligned}$$

which basically says $\Delta\circ[x_L,y_R]=0$. Composing with the map $\epsilon\otimes\mathrm{id}\colon\mathcal{O}(G)\otimes\mathcal{O}(G)\to\mathcal{O}(G)$ and using Hopf algebra axioms we can conclude. ■

We also have

Lemma 1.13. Given $x,y\in\mathfrak{g}$, and $S\colon\mathcal{O}(G)\to\mathcal{O}(G)$ the antipode map, we have

$$x_R\circ S=S\circ x_L.$$

Proof. Given $f\in\mathcal{O}(G), g\in G$ observe that

$$x_R(S(f))(g)=x(\rho_g^*S(f))=x(f\circ\iota\circ\rho_g)=x(f\circ\lambda_{g^{-1}}\circ\iota)=S(x(\lambda_{g^{-1}}^*f))=(Sx_L(f))(g)$$

where $\iota\colon G\to G$ is the inverse map, so that $S=-\circ\iota$. ■

Recall also that any affine algebraic linear group is smooth (as a scheme) and it has trivial tangent and cotangent bundles, i.e.

$$TG\cong G\times T_eG\cong G\times\mathfrak{g},\qquad T^*G\cong G\times\mathfrak{g}^*.$$

Lemma 1.14. The embedding $\mathfrak{g}\hookrightarrow \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$ given by $x\mapsto x_L$ induces an isomorphism in $\mathcal{O}(G)-Mod$

$$\mathcal{O}(G)\otimes_{\mathbb{C}}\mathfrak{g}\xrightarrow{\sim}\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)).$$

Proof. Both sides are free $\mathcal{O}(G)$-modules of rank equal to $\dim_{\mathbb{C}}\mathfrak{g}=\dim G$ since G is smooth. ■

We have the canonical $\mathcal{O}(G)$-bilinear pairing

$$\langle,\rangle\colon\ \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))\times\Omega^1(G)\to\mathcal{O}(G).$$

Fix a $\mathbb{C}$-basis of $\mathfrak{g}$ given by $(x^1,\dots,x^d)$ (corresponding hence to an $\mathcal{O}(G)$-basis of $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$) and let $(\omega^1,\dots,\omega^d)$ be the dual $\mathcal{O}(G)$-basis of $\Omega^1(G)$. Let's introduce the structure coefficients writing

$$[x^i,x^j]=\sum_p c_p^{i,j}x^p,\qquad \text{for } i,j=1,\dots,d$$

with $c_p^{i,j} \in \mathbb{C}$. Having embedded $\mathfrak{g}$ into $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$ as left-invariant vector fields, we can consider also the dual embedding and write

$$x^i_R = \sum_p f^{i,p} x^p, \qquad \text{for } i = 1, \dots, d$$

for some invertible (the x^i_R are also a basis) matrix $(f^{i,p})_{1 \le i,p \le d}$ with coefficients in $\mathcal{O}(G)$ (observe that by x^i we mean the corresponding left-invariant vector field x^i_L in $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$).

Lemma 1.15. We have the following identities:

(i) For all $i, j, s = 1, \dots, d$,

$$x^i_L f^{j,s} + \sum_p c_s^{i,p} f^{j,p} = 0.$$

(ii) For all $i, j, s = 1, \dots, d$,

$$\sum_p f^{i,p} \cdot x^p_L f^{j,s} = \sum_q c_q^{i,j} f^{q,s}.$$

Proof. The first identity is equivalent to the commutation relation

$$[x^i_L, x^j_R] = 0$$

for all i, j (just substitute the expression of x^j_R and then put to zero every component multiplying the base elements x_s).

To prove the second, let's write the relation

$$[x^i_R, x^j_R] = [x^i, x^j]_R$$

which says nothing else than that also $(-)_R$ is a Lie algebra morphism (same reasoning of left one). Using coordinates we have

$$[x^i_R, x^j_R] = \sum_s [x^i_R, f^{j,s} x^s] = \sum_s (x^i_R f^{j,s}) x^s = \sum_{s,p} f^{i,p} (x^p_L f^{j,s}) x^s$$

by the previous commutation relation. Plugging it back, we obtain the searched identities (as usual insulating every component). ■

Definition 1.16. The Lie algebra $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$ acts on $\Omega^1(G)$ by *Lie derivative* as follows:

$$\Omega^1(G) \ni (\mathrm{Lie}\,\theta).\omega \colon \theta_1 \mapsto \theta(\langle \theta_1, \omega \rangle) - \langle [\theta, \theta_1], \omega \rangle,$$

where $\omega \in \Omega^1(G)$ and $\theta, \theta_1 \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$.

Let's now consider the case $\omega = f\partial g \in \Omega^1(G)$, $\tau \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$ and try to give more explicit formulas. We have

$$\langle \tau, f\partial g \rangle = f\tau(g), \qquad (\mathrm{Lie}\,\tau).(f\partial g) = \tau(f)\partial g + f\partial(\tau(g))$$

as one can verify with few computations.

Proposition 1.17. We have the following identities:

(1) $(\mathrm{Lie}\,\tau).(f\omega) = \tau(f)\omega + f(\mathrm{Lie}\,\tau).\omega$.

(2) $(\mathrm{Lie}\,f\tau).\omega = f(\mathrm{Lie}\,\tau).\omega + \langle \tau, \omega \rangle \partial f$.

Proof. Easy computations. ■

Using the previously introduced $\mathcal{O}(G)$-basis $\{\omega_1, \dots, \omega_d\}$ of $\Omega^1(G)$ we can write

$$(\mathrm{Lie}\,x^i).\omega^j = \sum_s \alpha_s^{i,j} \omega^s$$

for $\alpha_s^{i,j} \in \mathbb{C}$ coefficients. As it happens in differential geometry, we will sometimes write

$$(\mathrm{Lie}\,\theta).f = \theta(f)$$

for $\theta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$ and $f \in \mathcal{O}(G)$ (i.e. the Lie derivative of a function along a vector field is exactly the corresponding directional derivative, seeing the vector field as a differential operator). We have another technical lemma.

Lemma 1.18. The following identities hold:

(i) For all $i, j = 1, \ldots, d$,
$$(\mathrm{Lie}\, x^i).\omega^j = \sum_s c_j^{s,i} \omega^s.$$

(ii) For all $i, j = 1, \ldots, d$,
$$(\mathrm{Lie}\, x_R^i).\omega^j = 0.$$

Proof. For all $i, j, s = 1, \ldots, d$ we have
$$\alpha_s^{i,j} = \langle x^s, (\mathrm{Lie}\, x^i).\omega^j \rangle = x_L^i(\langle x^s, \omega^j \rangle) + \langle [x^s, x^i], \omega^j \rangle = c_j^{s,i}$$
where we used the fact that ω^i is a $\mathcal{O}(G)$-dual basis of the x^j's, and that $x_L^i(\delta_{s,j}) = 0$ being the derivation of a constant. This clearly implies (i).

To prove (ii) let's observe first of all that
$$\langle x_R^i, \omega^j \rangle = \sum_s \langle f^{i,s} x^s, \omega^j \rangle = f^{j,s} \in \mathcal{O}(G).$$

To prove $(\mathrm{Lie}\, x_R^i).\omega^j = 0$ it suffices to show it is zero against the base of right-invariant vector fields x_R^s. We have
$$\langle x_R^s, (\mathrm{Lie}\, x_R^i).\omega^j \rangle = x_R^i(\langle x_R^s, \omega^j \rangle) + \langle [x^s, x^i]_R, \omega^j \rangle = x_R^i(f^{s,j}) + \sum_p c_p^{s,i} f^{p,j} =$$
$$= \sum_k f^{i,k} x_L^k(f^{s,j}) + \sum_p c_p^{s,i} f^{p,j} = \sum_q c_q^{i,s} f^{q,j} + \sum_p c_p^{s,i} f^{p,j} = 0$$
where we used the second identity of Lemma 1.15 and the fact that $c_p^{i,j} = -c_p^{j,i}$. ■

Proposition 1.19. The map
$$\Omega^1(G) \to \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathcal{O}(G)), \qquad dg \mapsto (x \mapsto x_L(g))$$
is an isomorphism in $\mathcal{O}(G) - Mod$.

Proof. By Frobenius reciprocity, using that $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)) \cong \mathcal{O}(G) \otimes_{\mathbb{C}} \mathfrak{g}$, we have
$$\mathrm{Hom}_{\mathcal{O}(G)}(\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)), \mathcal{O}(G)) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathcal{O}(G))$$
and thus, as $\mathbb{C}$-vector spaces, we obtain
$$\Omega^1(G) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathcal{O}(G)).$$
This holds since $\Omega^1(G)$ is a free $\mathcal{O}(G)$-module of finite rank, hence its bi-dual is (canonically) isomorphic to itself as an $\mathcal{O}(G)$-module, and hence also as a $\mathbb{C}$-vector space.

Let's write $\Omega^1(G) \ni \omega = \sum_i f_i dg_i$ with $f_i \in \mathcal{O}(G)$. In this isomorphism, the element ω gets sent to the map $\widetilde{\omega} \colon \mathfrak{g} \to \mathcal{O}(G)$ acting
$$\mathfrak{g} \ni x \mapsto \sum_i f_i \cdot x_L(g_i) \in \mathcal{O}(G).$$
■

2 Chiral differential operators

2.1 Definitions

Let G be an affine algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra (over $\mathbb{C}$) and κ be an invariant bilinear form. Let's recall the following.

Definition 2.1. The *Kac-Moody affinization* of $\mathfrak{g}$ (related to κ) is

$$\hat{\mathfrak{g}}_\kappa := \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}1$$

where the bracket is given by

$$[xt^n, yt^m] = [x,y]t^{n+m} + n\delta_{n,-m}\kappa(x,y)1$$

and 1 is central.

Let's set

$$\mathcal{A}_G := U(\hat{\mathfrak{g}}_\kappa) \otimes_{\mathbb{C}} \mathcal{O}(\mathscr{L}G)$$

where $\mathscr{L}G$ is the loop space of G. The algebra structure on $\mathcal{A}_G$ is such that

$$U(\hat{g}_\kappa) \hookrightarrow \mathcal{A}_G, \qquad \mathcal{O}(\mathscr{L}G) \hookrightarrow \mathcal{A}_G$$

are algebra embeddings (which, from now on, we'll implicitely use to identify for example $x \otimes 1$ with $x \in U(\hat{g}_\kappa)$) and bracket

$$[xt^m, f_{(n)}] = (x_L f)_{(m+n)} \qquad x \in \mathfrak{g},\ f \in \mathcal{O}(G),\ n,m \in \mathbb{Z}.$$

Let's define the subalgebra

$$\mathcal{A}_{G,+} := U(\mathfrak{g}[t] \oplus \mathbb{C}1) \otimes_{\mathbb{C}} \mathcal{O}(\mathscr{L}G)$$

and consider $\mathcal{O}(\mathscr{J}_\infty G)$ as an $\mathscr{A}_{G,+}$-module. To define this structure it suffices to say that $\mathcal{O}(\mathscr{L}G)$ acts by the natural surjection

$$\mathcal{O}(\mathscr{L}G) \to \mathcal{O}(\mathscr{J}_\infty G), \quad f_{(n)} \mapsto \chi_{\mathbb{Z}_{<0}}(n) \cdot f_{(n)} \quad f \in \mathcal{O}(G),$$

$\mathfrak{g}[t] \subset \mathfrak{g}[\![t]\!]$ acts by left invariant vector fields, (recall that $\mathrm{Lie}(\mathscr{J}_\infty G) \cong \mathfrak{g}[\![t]\!]$) and finally 1 acts as identity. We are finally ready to define our object of interest.

Definition 2.2. The algebra of global *chiral differential operators* on G is defined by

$$\mathcal{D}^{\mathrm{ch}}_{G,\kappa} := \mathcal{A}_G \otimes_{\mathcal{A}_{G,+}} \mathcal{O}(\mathscr{J}_\infty G).$$

Let's immediately observe that, as $\hat{f}g$-module, we have

$$\mathcal{D}^{\mathrm{ch}}_{G,\kappa} \cong U(\hat{g}_\kappa) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}1)} \mathcal{O}(\mathscr{J}_\infty G).$$

Let's define two families of fields on $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$:

$$x(z) = \sum_{n\in\mathbb{Z}} (xt^n) z^{-n-1}, \qquad f(z) = \sum_{n\in\mathbb{Z}} f_{(n)} z^{-n-1} \qquad x \in \mathfrak{g},\ f \in \mathcal{O}(G)$$

where both xt^n and $f_{(n)}$ are seen in $\mathrm{End}_{\mathbb{C}}(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})$ by left multiplication. Observe that, thanks to the commutation relation defined on $\mathcal{A}_G$, there can happen that the action of $f_{(n)}$ for $n \geq 0$ is non-trivial.

Proposition 2.3. The two fields above satisfy the following OPEs:

$$x(z)y(w) \sim \frac{[x,y](w)}{z-w} + \frac{\kappa(x,y)}{(z-w)^2}, \qquad f(z)g(w) \sim 0,$$
$$x(z)f(w) \sim \frac{(x_L f)(w)}{z-w}$$

for any $x, y \in \mathfrak{g}$, $f, g \in \mathcal{O}(G)$.

Proof. It suffices to check brackets, and then to use the "locality" proposition. For the first case we get

$$[x(z), y(w)] = \sum_{n,m} [xt^n, yt^m] z^{-n-1} w^{-m-1} \overset{\mathrm{def}}{=} \sum_{n,m} [x,y]t^{n+m} z^{-n-1} w^{-m-1} + \sum_n n\kappa(x,y)1 z^{-n-1} w^{n-1} =$$
$$= \sum_k [x,y]t^k w^{-k-1} \cdot \left(\sum_n z^{-n-1} w^n \right) + \kappa(x,y)1 \cdot \sum_n n w^{n-1} z^{-n-1} =$$
$$= [x,y](w) \cdot \delta(z-w) + \kappa(x,y)1 \cdot \partial_w \delta(z-w)$$

and thus we conclude. For the second case it is immediate since $\mathcal{O}(\mathscr{L}G)$ is abelian. Finally the third case is proven analogously by

$$[x(z), f(w)] = \sum_{n,m} [x_{(n)}, f_{(m)}] z^{-n-1} w^{-m-1} = \sum_{n,m} (x_L f)_{(n+m)} z^{-n-1} w^{-m-1} =$$
$$= \sum_k (x_L f)_{(k)} w^{-k-1} \cdot \left(\sum_n z^{-n-1} w^n \right) = (x_L f)(w) \cdot \delta(z-w). \qquad \blacksquare$$

Let $(x^1, \dots, x^n)$ be an ordered basis of $\mathfrak{g}$ and let $\mathcal{O}(G)$ be generated by coordinates $\xi^1, \dots, \xi^r$. Using PBW theorem we see that we get a "PBW" basis of $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ by tensoring the respective two bases. Namely we obtain that $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ is spanned by vectors of the form

$$x^{i_1}_{(n_1)} \cdots x^{i_s}_{(n_r)} \otimes \xi^{j_1}_{(m_1)} \cdots \xi^{j_t}_{(m_t)} |0\rangle$$

where $|0\rangle = \overline{1 \otimes 1}$, $n_i < 0$. We can use Reconstruction Theorem to endow $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ with a vertex algebra structure: indeed we just declare to associate xt^{-1} to field $x(z)$ for $x \in \mathfrak{g}$ and $f_{(-1)}$ to the field $f(z)$ for $f \in \mathcal{O}(G)$, since we already know they are mutually local and their coefficients span the whole space.

Theorem 2.4. There is a unique vertex algebra structure on $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ such that the embeddings

$$\pi_L \colon V^\kappa(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}_{G,\kappa}, \qquad u\,|0\rangle \mapsto u \otimes 1,$$
$$j \colon \mathcal{O}(\mathscr{J}_\infty G) \hookrightarrow \mathcal{D}^{\mathrm{ch}}_{G,\kappa}, \qquad f \mapsto 1 \otimes f$$

are homomorphisms of vertex algebras and

$$x(z) f(w) \sim \frac{(x_L f)(w)}{z-w}, \qquad x \in \mathfrak{g},\ f \in \mathcal{O}(G).$$

The vertex algebra $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ is also $\mathbb{Z}_{\geq 0}$-graded by setting $\deg x_{(n)} = -n$ and $\deg f_{(-1-j)} = j$, where $x \in \mathfrak{g}$, $f \in \mathcal{O}(G)$, $n < 0$ and $j \geq 0$. To declare this it suffices to ask for $x \in \mathfrak{g}$ (embedded through π_L) to have weight 1 and for $f \in \mathcal{O}(G)$ (embedded through j) to have weight 0, since then translation just adds 1 to the weight.

Consider now the subspace of $(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})_1$ spanned by vectors $f\partial g$ with $f, g \in \mathcal{O}(G)$ (we are using j again, this is the same as writing $f_{(-1)} g_{(-2)}$), and call it Ω. Recalling that

$$\mathcal{O}(G) \otimes_{\mathbb{C}} \mathfrak{g} \xrightarrow{\simeq} \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$$

we see that

$$(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})_1 = \Omega \oplus \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$$

since they represent the only two possible ways for a vector to have weight 1, i.e. being either $xt^{-1} \otimes f_{(-1)}$ (this is the explicit writing, we will just write xf from now on) or $f\partial g$. Recall also that $\Omega^1(G)$, the space of 1-differential forms on G, generated by $d\mathcal{O}(G)$ as $\mathcal{O}(G)$-module, satisfies

$$\Omega^1(G) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathcal{O}(G))$$

by $df \mapsto (x \mapsto x_L f)$.

Lemma 2.5. The $\mathbb{C}$-linear map

$$\Gamma \colon \Omega \to \mathrm{Hom}_{\mathcal{O}(G)}(\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)), \mathcal{O}(G)), \qquad f\partial g \mapsto \big(\mathcal{O}(G) \otimes \mathfrak{g} \ni h \otimes x \mapsto (hx)_{(1)}(f\partial g)\big)$$

is an isomorphism of $\mathbb{C}$-vector spaces. Therefore $\Omega \cong \Omega^1(G)$ as $\mathbb{C}$-vector spaces.

Proof. Observe that $hx \in \mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ (it can be re-written in a different order using commutation relation, but here it does not matter), so our first problem is to understand its associated field. Thanks to reconstruction theorem, and to the tautological $hx = h_{(-1)} x_{(-1)} |0\rangle$, we obtain

$$Y(hx, \zeta) = {:}Y(h,\zeta) Y(x,\zeta){:}$$

so that

$$(hx)_{(1)} = \sum_{n \leq -1} h_{(n)} x_{(-n)} + \sum_{n \geq 0} x_{(-n)} h_{(n)}.$$

Applying this to $f\partial g = f_{(-1)}g_{(-2)}$ we see that all terms in the second series give 0, since $h_{(n)}$ acts by zero having $n \geq 0$ (i.e. $h_{(n)}(1 \otimes *) = 0$). Using

$$h_{(n)}x_{(-n)}(f\partial g) = h_{(n)} \cdot \left[(x_L f)_{(-1-n)}g_{(-2)} + f_{(-1)}(x_L g)_{(-2-n)}\right]$$

we see that the only nonzero term comes from $n = -1$, so that

$$(hx)_{(1)}(f\partial g) = hf(x_L g).$$

The map $h \otimes x \mapsto (hx)_{(1)}(f\partial g)$ is thus a clear morphism of $\mathcal{O}(G)$-modules and so the map Γ is well defined. Observe that by Frobenius reciprocity we have

$$\mathrm{Hom}_{\mathcal{O}(G)}(\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)), \mathcal{O}(G)) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathcal{O}(G))$$

and, using this correspondence, the map $\Gamma(f\partial g)$ is just given by $\mathfrak{g} \ni x \mapsto f(x_L g)$. This means that it suffices to prove that the map sending ∂g to $(x \mapsto x_L g)$ is an isomorphism of $\mathcal{O}(G)$-modules. Using Lemma 1.15 this is the same as saying that $\partial g \mapsto dg \in \Omega^1(G)$ is an isomorphism of $\mathcal{O}(G)$-modules. Finally, $\partial g = g_{(-2)}\,|0\rangle$ is a regular function on $\mathscr{J}_1 G \cong TG$ (tangent bundle of G) and it corresponds to dg. ■

We can now consider the canonical $\mathcal{O}(G)$-bilinear pairing $\langle , \rangle \colon \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)) \times \Omega \to \mathcal{O}(G)$ and the action of $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$ on Ω by Lie derivative.

Lemma 2.6. Let $x \in \mathfrak{g}$ and $\omega \in \Omega$. Then $x_{(1)}\omega = \langle x, \omega\rangle$ and $x_{(0)}\omega = (\mathrm{Lie}\, x).\omega$.

Proof. The first identity holds by Lemma 2.5 (put $\omega = f\partial g$ and then extend by linearity). We can use it to prove the second one; we have

$$y_{(1)}((\mathrm{Lie}\, x).\omega) = x_L(y_{(1)}\omega) - [x, y]_{(1)}\omega = x_{(0)}y_{(1)}\omega - [x, y]_{(1)}\omega$$

for all $y \in \mathfrak{g}$. Since

$$y_{(1)}(x_{(0)}\omega) = x_{(0)}y_{(1)}\omega - [x, y]_{(1)}\omega$$

for all $y \in \mathfrak{g}$, we conclude $x_{(0)}\omega = (\mathrm{Lie}\, x).\omega$. ■

3 Main results

Theorem 3.1. (i) There is a vertex algebra embedding

$$\pi_R \colon V^{\kappa^*}(\mathfrak{g}) \hookrightarrow \mathrm{Com}(V^{\kappa}(\mathfrak{g}), \mathcal{D}^{\mathrm{ch}}_{G,\kappa}) \subset \mathcal{D}^{\mathrm{ch}}_{G,\kappa}$$

such that

$$[\pi_R(x)_{(m)}, f_{(n)}] = (x_R f)_{(m+n)} \qquad \text{for } f \in \mathcal{O}(G),\ m, n \in \mathbb{Z},$$

where x_R is the right invariant vector field corresponding to $x \in \mathfrak{g}$.

(ii) There is a vertex algebra isomorphism

$$\mathcal{D}^{\mathrm{ch}}_{G,\kappa} \cong \mathcal{D}^{\mathrm{ch}}_{G,\kappa^*}$$

that sends $\mathcal{O}(G) \ni f$ to $S(f) \in \mathcal{O}(G)$, where $S \colon \mathcal{O}(G) \to \mathcal{O}(G)$ is the antipode.

Proof. For all this proof we will identify $x \in \mathfrak{g}$ (corresponding to $xt^{-1} \in V^{\kappa}(\mathfrak{g})$) with its image in $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ through π_L. We will give a formula for the map π_R and we will prove, in order, that the images commute with $V^{\kappa}(\mathfrak{g})$, that it is injective and that it defines a vertex algebra homomorphism.

Let as before $(x^1, \ldots, x^d)$ be a basis of $\mathfrak{g}$, with $(\omega^1, \ldots, \omega^d)$ the dual $\mathcal{O}(G)$-basis of $\Omega \cong \Omega^1(G)$. Then, we obtain that $(x^1, \ldots, x^d)$ is also an $\mathcal{O}(G)$-basis of $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$ (identifying $x \in \mathfrak{g}$ with x_L). Thus, the corresponding right-invariant vector fields can be expressed by

$$\mathcal{D}^{\mathrm{ch}}_{G,\kappa} \ni x^i_R = \sum_p f^{i,p} x^p$$

where $(f^{i,p})_{1\le i,p\le d}$ is an invertible $\mathcal{O}(G)$-matrix. To define π_R, since it must be a morphism of vertex algebras, it suffices to define it only on the basis $x^i \cong x^i t^{-1}$ of $V^{\kappa^*}(\mathfrak{g})$, so let's set

$$\pi_R(x^i) := x^i_R + \sum_{q,p} \kappa^*(x^p, x^q) f^{i,p}\omega^q \in (\mathcal{D}^{\mathrm{ch}}_{G,\kappa})_1 = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)) \oplus \Omega.$$

To verify it commutes with all elements of $V^\kappa(\mathfrak{g})$ it suffices to prove (classical argument using Borcherds identities) that

$$v_{(n)}\pi_R(s) = 0 \qquad \forall\, v \in V^\kappa(\mathfrak{g}),\ s \in V^{\kappa^*}(\mathfrak{g}),\ n \ge 0$$

and, more specifically, we can just verify this relation for $v = x^i$ and $s = x^j$ for all i, j (i.e. we just check it on generators, thanks to Borcherds identities). So our first step will be to prove

$$(x^i)_{(n)}\pi_R(x^j) = 0 \tag{1}$$

for all i, j and $n \ge 0$.

Lemma 3.2. We have $(x^i)_{(n)}\pi_R(x^j) = 0$ for all $n \ge 2$.

Proof. One way to convince oneself about this is to use the OPEs in Proposition 2.3 and Wick's theorem. In this case we can do explicit computation, though, and we will do them to warm us up. First thing first $(x^i)_{(n)}x^j_R = \sum_p (x^i)_{(n)}(f^{j,p}x^p)$, so let's focus on terms of this type. Observe that by Borcherds 2 (commutators identity) and by the fact that $(x^i)_{(n)}x^j = 0$ for $n \ge 2$ (OPE) we have

$$(x^i)_{(n)}(f^{j,p}x^p) = [x^i_{(n)}, f^{j,p}_{(-1)}]x^p = \sum_{j\ge 0}\binom{n}{j}(x^i_{(j)}f^{j,p})_{(n-1-j)}x^p$$

and using the OPE of $x(z)f(z)$ we get

$$(x^i)_{(n)}(f^{j,p}x^p) = (x^i_{(0)}f^{j,p})_{(n-1)}x^p = (x^i_L f^{j,p})_{(n-1)}x^p$$

which is zero for $n \ge 2$.

The second part is just proving $(x^i)_{(n)}(f^{j,p}\omega^q) = 0$ for $n \ge 2$. Recalling that Ω is generated by $f\partial g = f_{(-1)}g_{(-2)}$ and that $x_L g = \langle x, \omega\rangle$ for $x \in \mathfrak{g}$, we obtain, recalling that $\mathfrak{g}[\![t]\!]$ acts by derivation, that

$$x^i_{(n)}(f^{j,p}\omega^q) = (x_L f)_{(n-1)}\omega^q - f^{j,p}\langle x^i, \omega^q\rangle_{(n-2)} = 0$$

for $n \ge 2$. Summing these two we obtain the statement. ■

By the lemma we just need to prove (1) for $n = 0, 1$. For the case $n = 1$ we can write:

$$(x^i_R)_{(1)}x^j = \sum_p (f^{i,p}_{(-1)}x^p)_{(1)}x^j \overset{Borcherds1}{=} \sum_p (f^{i,p}_{(-1)}x^p_{(1)}x^j + x^p_{(0)}f^{i,p}_{(0)}x^j) = \tag{2}$$

$$\overset{OPE}{=} \sum_p (f^{i,p}\kappa(x^p, x^j) - x^p_L(x^j_L f^{i,p})). \tag{3}$$

We used the fact that $f^{i,p}_{(0)}x^j = -x^j_{(0)}f^{i,p}$ (again skew symmetry or do the inverse OPE). Using Lemma 1.15 twice we obtain

$$-x^p_L(x^j_L f^{i,p}) = \sum_s x^p_L(c^{j,s}_p f^{i,s}) = -\sum_{s,u} c^{p,u}_s c^{j,s}_p f^{i,u} = \sum_{s,u} c^{u,p}_s c^{j,s}_p f^{i,u}.$$

Lemma 3.3. The Killing form can be expressed using structure coefficients as

$$\kappa_{\mathfrak{g}}(x^i, x^j) = \sum_{p,q} c^{i,q}_p c^{j,p}_q$$

for any i, j.

Proof. Easy computation. ■

We deduce that (writing explicitely and renaming the three indexes)

$$-\sum_p x_L^p(x_L^j f^{i,p}) = \sum_u \kappa_{\mathfrak{g}}(x^u, x^j) f^{i,u}. \tag{4}$$

Recalling the definition of κ^* we get

$$(x_R^i)_{(1)} x^j = -\sum_p \kappa^*(x^p, x^j) f^{i,p}.$$

Let's observe the following.

Lemma 3.4. We have $(f\omega)_{(1)}(x) = x_{(1)}(f\omega)$, with obvious notation.

Proof. Observe that

$$x_{(1)}(f\omega) \overset{derivation}{=} (x_L f)_{(0)}\omega + f\langle x, \omega\rangle_{(-1)} = f\langle x, \omega\rangle.$$

By skew-symmetry, i.e. $Y(f\omega, \zeta)x = e^{\zeta T} Y(x, -\zeta) f\omega$, we have

$$\sum_n (f\omega)_{(n)} x \zeta^{-n-1} = \sum_h \zeta^{-h-1} \cdot \sum_{k\geq 0} \frac{(-1)^{-k-h-1}}{k!} T^k(x_{(k+h)}(f\omega))$$

so that

$$(f\omega)_{(1)} x = \sum_{k\geq 0} \frac{(-1)^{-k-2}}{k!} T^k(x_{(k+1)}(f\omega)) \overset{derivation}{=} x_{(1)}(f\omega)$$

where only $k = 0$ survives. ■

By Lemma 2.6, for any p, q, we have

$$\left(\sum_{p,q} \kappa^*(x^p, x^q) f^{i,p}\omega^q\right)_{(1)} x^j \overset{above}{=} \sum_{p,q} \kappa^*(x^p, x^q) x^j_{(1)}(f^{i,p}\omega^q) =$$

$$= \sum_{p,q} \kappa^*(x^p, x^q) f^{i,p}\langle x^j, \omega^q\rangle = \sum_p \kappa^*(x^p, x^j) f^{i,p}$$

and therefore we can conclude that $\pi_R(x^i)_{(1)} x^j = 0$. Using skew-symmetry (check proof of above lemma) we can also conclude that $x^i_{(1)}(\pi_R(x^j)) = 0$.

Let's now focus on the case $n = 0$, i.e. we want to prove $(x^i)_{(0)}\pi_R(x^j) = 0$ for any i, j. Observe that using Lemma 1.15 we have

$$x^i_{(0)} x^j_R = \sum_q x^i_{(0)}(f^{j,q} x^q) = \sum_q x^i_{(0)} f^{j,q}_{(-1)} x^q \overset{OPE}{=} \sum_q ((x^i_L f^{j,q})_{(-1)} x^q + f^{j,q}_{(-1)}[x^i, x^q]).$$

This last sum is equal to zero because

$$x^i_L f^{j,q} = -\sum_p c^{i,p}_q f^{j,p}, \qquad \sum_q f^{j,q}_{(-1)}[x^i, x^q] = \sum_{q,s} c^{i,q}_s f^{j,q} x^s$$

and summing we get

$$\sum_q (x^i_L f^{j,q})_{(-1)} x^q = -\sum_{q,p} c^{i,p}_q f^{j,p} x^q$$

so that we just need to switch indexes. Thus $x^i_{(0)} x^j_R = 0$. On the other hand, using Lemma 1.15 and Lemma 2.6, we have

$$\sum_{p,q} x^i_{(0)}(\kappa^*(x^p, x^q) f^{j,p}\omega^q) = \sum_{p,q} \kappa^*(x^p, x^q)((x^i_L f^{j,p})\omega^q + f^{j,p}(\operatorname{Lie} x^i).\omega^q).$$

Writing $x^i_L f^{j,p} = -\sum_s c^{i,s}_p f^{j,s}$ and $(\operatorname{Lie} x^i).\omega^q = -\sum_s c^{i,s}_q \omega^s$ we obtain that the above term is equal to

$$-\sum_{p,q,r} \kappa^*(x^p, x^q) c^{i,r}_p f^{j,r}\omega^q + \kappa(x^p, x^q) c^{i,r}_q f^{j,p}\omega^r.$$

Observing that

$$\kappa^*([x^i, x^s], x^r) = \sum_k c_k^{i,s} \kappa^*(x^k, x^r)$$

we can write the first part as

$$\sum_{p,q,r} \kappa^*(x^p, x^q) c_p^{i,r} f^{j,r} \omega^q = \sum_{q,r} \kappa^*([x^i, x^r], x^q) f^{j,r} \omega^q$$

whereas the second part is equal to

$$\sum_{p,q,r} \kappa^*(x^p, x^q) c_q^{i,r} f^{j,p} \omega^r = \sum_{p,r} \kappa^*(x^p, [x^i, x^r]) f^{j,p} \omega^r$$

so that, by the $\mathfrak{g}$-invariance of κ^*, their sum is zero. This means that

$$x^i_{(0)} \left(\sum_{p,q} \kappa^*(x^p, x^q) f^{j,p} \omega^q \right) = 0$$

and therefore we can conclude that $x^i_{(0)} \pi_R(x^j) = 0$. In conclusion, we proved formula (1), which means that π_R is a map from $V^{\kappa^*}(\mathfrak{g})$ to $\mathrm{Com}(V^\kappa(\mathfrak{g}), \mathcal{D}^{\mathrm{ch}}_{G,\kappa})$.

Let's now prove injectivity. This is easy since $(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})_1 \cong \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)) \oplus \Omega$ and we can consider the projection of $\pi_R(x^i)$ onto $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$ which is equal to x^i_R. Since the map $\mathfrak{g} \ni x \mapsto x_R \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$ is injective, we conclude that also π_R is injective.

We now have to prove that π_R is indeed a vertex algebra homomorphism, which amounts to say that it respects OPE, i.e. that we have

$$(\pi_R(x))(z)(\pi_R(y))(w) \sim \frac{1}{z-w} \pi_R([x,y])(w) + \frac{\kappa^*(x,y)}{(z-w)^2}$$

for all $x, y \in \mathfrak{g}$. This basically means that we have to prove

$$\begin{aligned} \pi_R(x)_{(n)} \pi_R(y) &= 0 \qquad \forall\, n \geq 2, \\ \pi_R(x)_{(1)} \pi_R(y) &= \kappa^*(x,y), \\ \pi_R(x)_{(0)} \pi_R(y) &= \pi_R([x,y]) \end{aligned}$$

for all $x, y \in V^{\kappa^*}(\mathfrak{g})$. As usual, we can just assume $x = x^i$ and $y = x^j$ to be in the basis of $\mathfrak{g}$.

Proposition 3.5. We have $\pi_R(x^i)_{(n)} \pi_R(x^j) = 0$ for every $n \geq 2$.

Proof. Expanding both sides we find terms like $(fx)_{(n)}(gy)$, $(fx)_{(n)}(g\omega)$ and $(f\omega)_{(n)}(gy)$. Let's focus on the first kind, the other are similar. We can use Reconstruction theorem to obtain the relative fields

$$Y(fx, \zeta) =: Y(f, \zeta) Y(x, \zeta):$$

and then Wick's theorem to get OPEs. We see that have terms like $\langle f, g \rangle = 0$, $\langle f, y \rangle = -\frac{(y_L f)(w)}{z-w}$ and products of at most two of them, so no denominators with powers bigger than $(z-w)^2$. Thus, for $n \geq 2$ we have zero product of fields. ∎

Let's now compute

$$\pi_R(x^i)_{(1)} \pi_R(x^j) = \left(x^i_R + \sum_{q,p} \kappa^*(x^p, x^q) f^{i,p} \omega^q \right)_{(1)} \left(x^j_R + \sum_{q,p} \kappa^*(x^p, x^q) f^{j,p} \omega^q \right)$$

so that we see there are 4 terms. We have

$$(x^i_R)_{(1)} x^j_R = \sum_{p,s} (f^{i,p} x^p)_{(1)} (f^{j,s} x^s).$$

Small computation proposition time.

Proposition 3.6. With obvious notation we have

$$(fx)_{(1)}(gy) = fg\kappa(x,y) - fy_L(x_Lg) - gx_L(y_Lf) - (x_Lf)(y_Lg).$$

Proof. By Borcherds 1 we have

$$(fx)_{(1)} = \sum_{l\geq 0}(-1)^l((-1)^1 f_{(-1-l)}x_{(1+l)} + x_{(-l)}f_{(l)}).$$

Observe that

$$\begin{aligned} x_{(1+l)}(gy) = g_{(-1)}x_{(1+l)}y + [x_{(1+l)}, g_{(-1)}]y = g_{(-1)}x_{(1+l)}y + (x_Lg)_{(l)}y = \\ = g\cdot x_{(1+l)}y + y(x_Lg)_{(l)} - (y_Lx_Lg)_{(l-1)} \end{aligned}$$

so the only surviving l is 0, for which we have $x_{(1)}(gy) = g\kappa(x,y) - y_L(x_Lg)$, using OPEs. Instead we have

$$\begin{aligned} f_{(l)}(gy) \overset{abelian}{=} g_{(-1)}f_{(l)}y = gyf_{(l)} + g[f_{(l)}, y] = \\ = yg_{(-1)}f_{(l)} - (y_Lg)_{(-2)}f_{(l)} - g(y_Lf)_{(l-1)} \overset{l\geq 0}{=} -g(y_Lf)_{(l-1)} \end{aligned}$$

so that also here the only survival is $l = 0$ with $-g(y_Lf)$. Hence we have

$$(fx)_{(1)}(gy) = f\cdot x_{(1)}(gy) + x_{(0)}f_{(0)}(gy) = fg\kappa(x,y) - fy_L(x_Lg) - x_{(0)}(gy_Lf)$$

and recalling the action of $x_{(0)}$ we conclude. ■

Using the above computation we can write

$$(x_R^i)_{(1)}x_R^j = \sum_{p,s}\left(f^{i,p}f^{j,s}\kappa(x^p,x^s) - f^{i,p}x_L^s(x_L^pf^{j,s}) - f^{j,s}x_L^p(x_L^sf^{i,p}) - (x_L^pf^{j,s})(x_L^sf^{i,p})\right).$$

Let's now observe that using Lemma 1.15

$$\begin{aligned} -f^{i,p}x_L^s(x_L^pf^{j,s}) = \sum_k c_s^{p,k}f^{i,p}(x_L^sf^{j,k}) = -\sum_{l,k}c_s^{p,k}c_k^{s,l}f^{i,p}f^{j,l}, \\ -f^{j,s}x_L^p(x_L^sf^{i,p}) = -\sum_{k,l}c_p^{s,k}c_k^{p,l}f^{j,s}f^{i,l}, \\ -(x_L^pf^{j,s})(x_L^sf^{i,p}) = -\sum_{k,l}c_s^{p,k}c_p^{s,l}f^{j,k}f^{i,l}. \end{aligned}$$

Summing over p and s and summing those three terms above, using the expression of Killing form in coordinates, we obtain

$$\sum_{a,b}f^{i,a}f^{j,b}\kappa_{\mathfrak{g}}(x^a,x^b)$$

so reinserting into the initial expression we get

$$(x_R^i)_{(1)}x_R^j = -\sum_{p,s}\kappa^*(x^p,x^s)f^{i,p}f^{j,s}.$$

Proposition 3.7. We have

$$(f^{i,p}\omega^q)_{(1)}x_R^j = f^{j,q}f^{i,p}.$$

Proof. Expanding x_R^j we see that we just need to study terms like $(f^{i,p}\omega^q)_{(1)}(f^{j,l}x^l)$. We will use the skew-symmetry formula

$$(f^{i,p}\omega^q)_{(1)}(f^{j,l}x^l) = \sum_{k\geq 0}\frac{(-1)^{-k-2}}{k!}T^k\left((f^{j,l}x^l)_{(k+1)}(f^{i,p}\omega^q)\right).$$

By Borcherds 1 we have

$$(f_{(-1)}^{j,l}x^l)_{(1+k)}(f^{i,p}\omega^q) = \sum_{t\geq 0}(-1)^t\left((-1)^t f_{(-1-t)}^{j,l}x_{(1+k+t)}^l(f^{i,p}\omega^q) + x_{(k-l)}^l f_{(t)}^{j,l}(f^{i,p}\omega^q)\right).$$

We have
$$x^l_{(1+k+t)}(f^{i,p}\omega^q) = (x^l_L f^{i,p})_{(k+t)}\omega^q + f^{i,p}\langle x^l, \omega^q\rangle_{(k+t-1)}$$
so that the first term dies since $k+t \geq 0$, as well as the second part in Borcherds identity. Then we obtain
$$(f^{j,l}x^l)_{(k+1)}(f^{i,p}\omega^q) = \sum_{t\geq 0} f^{j,l}_{(-1-t)} f^{i,p}\delta^{l,q}_{(k+t-1)}.$$
Then the only nonzero term is $t=-k$, so that $t=k=0$ and we obtain
$$(f^{j,l}x^l)_{(k+1)}(f^{i,p}\omega^q) = \delta_{l,q}\cdot\delta_{k,0}\cdot f^{j,q}f^{i,p}.$$
Finally, putting back into the skew symmetry, we get
$$(f^{i,p}\omega^q)_{(1)}(f^{j,l}x^l) = \delta_{q,l}\cdot f^{j,q}f^{i,p}$$
and summing over l we conclude $(f^{i,p}\omega^q)_{(1)}x^j_R = f^{j,q}f^{i,p}$. ■

Using the above proposition we obtain
$$\left(\sum_{p,q}\kappa^*(x^p,x^q)f^{i,p}\omega^q\right)_{(1)} x^j_R = \sum_{p,q}\kappa^*(x^p,x^q)f^{j,q}f^{i,p}.$$
The remaining part is
$$(x^i_R)_{(1)}\left(\sum_{u,s}\kappa^*(x^s,x^u)f^{j,s}\omega^u\right) = \sum_{s,u}\kappa^*(x^s,x^u)(x^i_R)_{(1)}(f^{j,s}\omega^u).$$
Expanding x^i_R we see that we need to study terms like $(f^{i,l}x^l)_{(1)}(f^{j,s}\omega^u)$ and this is the usual reasoning with Borcherds 1. We have
$$(f^{i,l}x^l)_{(1)}(f^{j,s}\omega^u) = \sum_{t\geq 0}(-1)^t\left((-1)^t f^{i,l}_{(-1-t)}x^l_{(1+t)}(f^{j,s}\omega^u) + x^l_{(-t)}f^{i,l}_{(t)}(f^{j,s}\omega^u)\right),$$
$$x^l_{(1+t)}(f^{j,s}\omega^u) \overset{derivation}{=} (x^l_L f^{j,s})_{(t)}\omega^u + f^{j,s}\langle x^l,\omega^u\rangle_{(t-1)} \overset{t\geq 0}{\equiv} f^{j,s}\delta^{l,u}_{(t-1)}$$
so that the only surviving term is for $t=0$ and $l=u$, in which case we obtain $f^{i,u}f^{j,s}$. Summing over l we obtain
$$(x^i_R)_{(1)}\left(\sum_{u,s}\kappa^*(x^s,x^u)f^{j,s}\omega^u\right) = \sum_{u,s}\kappa^*(x^s,x^u)f^{i,u}f^{j,s}.$$
Finally the fourth term is zero since $\mathcal{O}(\mathscr{J}_\infty G)$ is commutative, i.e. we have
$$\left(\sum_{p,q}\kappa^*(x^p,x^q)f^{i,p}\omega^q\right)_{(1)}\left(\sum_{u,s}\kappa^*(x^s,x^u)f^{j,s}\omega^u\right) = 0.$$
Summing over these terms we obtain
$$\pi_R(x^i)_{(1)}\pi_R(x^j) = \sum_{p,q}\kappa^*(x^p,x^q)f^{i,p}f^{j,q}.$$
A priori this is an element of $\mathcal{O}(\mathscr{J}_\infty G)$, let's prove it is actually a constant. We just need to show it gets annihilated by all left-invariant vector fields, and specifically we just need to test x^s_L for all s. Using identities of Lemma 1.15 we have
$$x^s_L\left(\sum_{p,q}\kappa^*(x^p,x^q)f^{i,p}f^{j,q}\right) = \sum_{p,q}\kappa^*(x^p,x^q)\left[(x^s_L f^{i,p})f^{j,q} + f^{i,p}(x^s_L f^{j,q})\right] =$$
$$= -\sum_{p,q,u}\kappa^*(x^p,x^q)c^{s,u}_p f^{i,u}f^{j,q} - \sum_{p,q,v}\kappa^*(x^p,x^q)c^{s,v}_q f^{j,v}f^{i,p} =$$
$$\overset{linearity}{=} -\sum_{u,q}\kappa^*([x^s,x^u],x^q)f^{i,u}f^{j,q} - \sum_{v,p}\kappa^*(x^p,[x^s,x^v])f^{j,v}f^{i,p} =$$
$$= \sum_{n,m}\left[\kappa^*([x^n,x^s],x^m) - \kappa^*(x^n,[x^s,x^m])\right]f^{i,n}f^{j,m} = 0$$

where the last equality is due to the invariance of κ^*.

We conclude that $\sum_{p,q}\kappa^*(x^p,x^q)f^{i,p}f^{j,q}$ is constant. Observing that $f^{i,j}(e)=\delta_{i,j}$, where e is the identity of G, we see that we have

$$\pi_R(x^i)_{(1)}\pi_R(x^j)=\kappa^*(x^i,x^j)$$

as we wanted.

Let's now compute $\pi_R(x^i)_{(0)}\pi_R(x^j)$ and, as usual, let's start by expanding $\pi_R(x^j)$. Let's recall first a basic lemma of vertex algebras.

Lemma 3.8. Suppose $v_{(n)}w=0$ for each $n\geq 0$. Then $w_{(n)}v=0$ for each $n\geq 0$.

Proof. Write

$$w_{(n)}v=w_{(n)}v_{(-1)}\left|0\right\rangle=v_{(-1)}w_{(n)}\left|0\right\rangle+[w_{(n)},v_{(-1)}]\left|0\right\rangle\overset{n\geq 0}{=}-[v_{(-1)},w_{(n)}]\left|0\right\rangle$$

and using Borcherds 2

$$w_{(n)}v=-\sum_{j\geq 0}(-1)^j(v_{(j)}w)_{(n-1-j)}\left|0\right\rangle.$$

This is all equal to zero since $v_{(j)}w=0$ by assumption. ∎

Let's now focus on terms like $\pi_R(x^i)_{(0)}(f^{j,q}x^q)$. We proved before that $(x^j)_{(n)}\pi_R(x^i)=0$ for all non negative n and therefore, using the lemma, we also have $\pi_R(x^i)_{(n)}x^j=0$. Since in any vertex algebra the element $v_{(0)}$ is a "derivation", we have

$$\pi_R(x^i)_{(0)}(f^{j,q}_{(-1)}x^q)=(\pi_R(x^i)_{(0)}f^{j,q})_{(-1)}x^q.$$

Expanding $\pi_R(x^i)$ we need to understand terms like $(f^{i,l}x^l)_{(0)}f^{j,q}$. This is the usual Borcherds trick, for which we obtain $f^{i,l}(x^l_Lf^{j,q})$. Hence we have

$$\pi_R(x^i)_{(0)}f^{j,q}=\sum_l f^{i,l}(x^l_Lf^{j,q})$$

and summing over q we get

$$\pi_R(x^i)_{(0)}x^j_R=\pi_R(x^i)_{(0)}\left(\sum_q f^{j,q}x^q\right)=\sum_{q,l}f^{i,l}(x^l_Lf^{j,q})x^q=[x^i_R,x^j_R]=[x^i,x^j]_R$$

where the last equalities come from the proof of Lemma 1.15. We now need to study terms like $\pi_R(x^i)_{(0)}(f^{j,s}\omega^u)$, which we can already reduce to $(x^i_R)_{(0)}(f^{j,s}\omega^u)$ by the commutativity of the vertex algebra $\mathcal{O}(\mathscr{J}_\infty G)$. As before, using $v_{(0)}$ derivation, we have

$$(x^i_R)_{(0)}(f^{j,s}\omega^u)=((x^i_R)_{(0)}f^{j,s})_{(-1)}\omega^u+f^{j,s}(x^i_R)_{(0)}\omega^u$$

and now let's expand $x^i_R=\sum_l f^{i,l}x^l$. By Borcherds 1 we have

$$(f^{i,l}x^l)_{(0)}=\sum_{t\geq 0}(-1)^t\left((-1)^tf^{i,l}_{(-1-t)}x^l_{(t)}+x^l_{(-1-t)}f^{i,l}_{(t)}\right)$$

and observe that

$$x^l_{(t)}f^{j,s}=(x^l_Lf^{j,s})_{(t-1)}=\delta_{t,0}\cdot x^l_Lf^{j,s}$$

so that

$$(x^i_R)_{(0)}f^{j,s}=\sum_l(f^{i,l}x^l)_{(0)}f^{j,s}=\sum_l f^{i,l}(x^l_Lf^{j,s}).$$

Observe now, using Lemma 2.6, that

$$x^l_{(t)}\omega^u=\delta_{t,0}\cdot(\operatorname{Lie}x^l).\omega^u+\delta_{t,1}\cdot\langle x^l,\omega^u\rangle.$$

Plugging it in Borcherds 1 we get

$$(x^i_R)_{(0)}\omega^u = \sum_l (f^{i,l}x^l)_{(0)}\omega^u = \sum_l f^{i,l}(\mathrm{Lie}\, x^l).\omega^u + \sum_l f^{i,l}_{(-2)}\langle x^l, \omega^u\rangle = \sum_l (\mathrm{Lie}\, f^{i,l}x^l).\omega^u =$$
$$= (\mathrm{Lie}(\sum_l f^{i,l}x^l)).\omega^u = (\mathrm{Lie}\, x^i_R).\omega^u = 0$$

where the last equality come from Lemma 1.18 and identities on Lie derivatives in Proposition 1.17.
Thus, using Lemma 1.15, we can write

$$(x^i_R)_{(0)}\left(\sum_{s,u}\kappa^*(x^s,x^u)f^{j,s}\omega^u\right) = \sum_{s,u}\kappa^*(x^s,x^u)\left[\sum_l f^{i,l}(x^l_L f^{j,s})\omega^u\right] =$$
$$= \sum_{s,u,q}\kappa^*(x^s,x^u)c^{i,j}_q f^{q,s}\omega^u.$$

Observe that we have

$$\pi_R([x^i,x^j]) = \sum_q c^{i,j}_q \pi_R(x^q) = \sum_q c^{i,j}_q x^q_R + \sum_q c^{i,j}_q\left(\sum_{s,u}\kappa^*(x^s,x^u)f^{q,s}\omega^u\right).$$

Adding everything up we see that

$$\pi_R(x^i)_{(0)}\pi_R(x^j) = \pi_R([x^i,x^j])$$

so that we have proved that π_R is indeed a vertex algebra morphism.

Action by right invariant vector fields

Let's prove that we have the following OPE

$$(\pi_R(x)(z))(f(w)) \sim \frac{1}{z-w}(x_R f)(w) \tag{5}$$

and, as usual, assume $x = x^i$ is in the fixed basis. It is equivalent to prove that, for $n \geq 0$, we have

$$\pi_R(x^i)_{(n)}f = \delta_{n,0}\cdot(x^i_R f).$$

By the commutativity of $\mathcal{O}(\mathscr{J}_\infty G)$ we can write

$$(\pi_R(x^i))_{(n)}f = (x^i_R)_{(n)}f = \sum_l (f^{i,l}x^l)_{(n)}f$$

and hence we see that we just need to concentrate on terms of this kind. By Borcherds 1 we have

$$(f^{i,l}_{(-1)}x^l)_{(n)} = \sum_{j\geq 0}\left((-1)^j f^{i,l}_{(-1-j)}x^l_{(n+j)} + x^l_{(n-1-j)}f^{i,l}_{(j)}\right)$$

and we observe that

$$x^l_{(n+j)}f = (x^l_L f)_{(n+j-1)}, \qquad f^{i,l}_{(j)}f = 0.$$

The first term is also zero whenever $j \geq 1-n$ and, if $n \geq 1$, this always happens, so we conclude that

$$n > 0 \implies (\pi_R(x))_{(n)}f = \sum_l (f^{i,l}x^l)_{(n)}f = 0.$$

For $n = 0$ we obtain instead

$$(\pi_R(x^i))_{(0)}f = \sum_l (f^{i,l}x^l)_{(0)}f = \sum_l f^{i,l}(x^l_L f) = (x^i_R f).$$

This proves the OPE (5). Observe now that this implies, using Borcherds 2:

$$[\pi_R(x)_{(m)}, f_{(n)}] = \sum_{j\geq 0}\binom{m}{j}(\pi_R(x)_{(j)}f)_{(m+n-j)} = (\pi_R(x)_{(0)}f)_{(m+n)} = (x_R f)_{(m+n)}.$$

Second part

For the second point let's consider the vertex algebra map

$$\Phi\colon \mathcal{D}^{\mathrm{ch}}_{G,\kappa} \to \mathcal{D}^{\mathrm{ch}}_{G,\kappa^*}$$

whose restriction to $\mathcal{O}(G)$ is the antipode S and whose restriction to $\mathfrak{g}$ is π_R. To verify it is indeed a vertex algebra homomorphism we just need to check the "mixed" OPE

$$(\Phi(x))(z)(\Phi(f))(w) \sim \frac{1}{z-w}(\Phi(x_L f))(w)$$

for any $x \in \mathfrak{g}$, $f \in \mathcal{O}(G)$. This is true thanks to Lemma 1.13 because

$$\Phi(x_L f) = S(x_L f) = x_R(S(f))$$

and we know from the first point that

$$(\pi_R(x))(z)(S(f))(w) \sim \frac{1}{z-w}(x_R(S(f)))(w).$$

Finally, to show that Φ is an isomorphism we can just consider the map Ψ from $\mathcal{D}^{\mathrm{ch}}_{G,\kappa^*}$ to $\mathcal{D}^{\mathrm{ch}}_{G,(\kappa^*)^*} = \mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ induced by antipode on $\mathcal{O}(G)$ and by $\pi_R(x) \mapsto \pi_L(x)$ on $V^{\kappa^*}(\mathfrak{g})$. Similarly, also Ψ is a vertex algebra morphism and one can verify it is inverse to Φ. ∎

Let's do another theorem.

Theorem 3.9. Suppose now that G is connected. The vertex algebras $V^\kappa(\mathfrak{g})$ and $V^{\kappa^*}(\mathfrak{g})$ form a *dual pair* in $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$, i.e.

$$V^\kappa(\mathfrak{g}) = (\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_R(\mathfrak{g}[\![t]\!])} := \{v \in \mathcal{D}^{\mathrm{ch}}_{G,\kappa} \mid \pi_R(xt^n)_{(m)}v = 0\,\forall\, m \geq 0,\, x \in \mathfrak{g},\}, \qquad V^{\kappa^*}(\mathfrak{g}) = (\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_L(\mathfrak{g}[\![t]\!])}.$$

Proof. By the preceding theorem we already know $V^\kappa(\mathfrak{g}) \subseteq (\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_R(\mathfrak{g}[\![t]\!])}$ and, using the isomorphism of the second part, $V^{\kappa^*}(\mathfrak{g}) \subseteq (\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_L(\mathfrak{g}[\![t]\!])}$, so we just need to prove the inverse inclusions. Observe that, since the image of π_R commutes with elements of $V^\kappa(\mathfrak{g})$ (embedded in $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$), we have

$$(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_R(\mathfrak{g}[\![t]\!])} = \left(U(\hat{g}_\kappa) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}1)} \mathcal{O}(\mathscr{J}_\infty G)\right)^{\pi_R(\mathfrak{g}[\![t]\!])} \cong U(\hat{g}_\kappa) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}1)} \mathcal{O}(\mathscr{J}_\infty G)^{\pi_R(\mathfrak{g}[\![t]\!])}.$$

Since G is connected we have

$$\mathbb{C} \cong \mathcal{O}(\mathscr{J}_\infty G)^{\mathscr{J}_\infty G} = \mathcal{O}(\mathscr{J}_\infty G)^{\mathfrak{g}[\![t]\!]} = \mathcal{O}(\mathscr{J}_\infty G)^{\pi_R(\mathfrak{g}[\![t]\!])}$$

and hence

$$(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_R(\mathfrak{g}[\![t]\!])} \cong V^\kappa(\mathfrak{g}).$$

The other claim comes for free using the isomorphism $\mathcal{D}^{\mathrm{ch}}_{G,\kappa} \cong \mathcal{D}^{\mathrm{ch}}_{G,\kappa^*}$. ∎

4 Other facts

Let's recall that for V a vertex algebra we have $R_V = V/F^1V$, where $F^1V = V_{(-2)}V$. If V has a PBW basis $(a^i)_i$, then one has

$$F^1V = \left\{a^i_{(-n-2)}v \mid n \geq 0,\, i \in I,\, v \in V\right\}.$$

Then the associated variety X_V is defined as the reduced scheme of $\operatorname{Spec} R_V$.

Proposition 4.1. We have $X_{\mathcal{D}^{\mathrm{ch}}_{G,\kappa}} \cong T^*G$.

Proof. We already mentioned that $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ has a PBW basis so we just need to prove $R_{\mathcal{D}^{\mathrm{ch}}_{G,\kappa}} \cong \mathbb{C}[T^*G]$. We have $T^*G \cong G \times \mathfrak{g}^*$ so

$$\mathcal{O}(T^*G) \cong \mathcal{O}(G) \otimes \mathcal{O}(\mathfrak{g}^*) \cong \mathcal{O}(G) \otimes \operatorname{Sym}(\mathfrak{g})$$

where $\operatorname{Sym}\mathfrak{g}$ is the symmetric algebra of $\mathfrak{g}$. Observe that given a generic vector

$$x^{i_1}_{(-n_1-1)} \cdots x^{i_m}_{(-n_m-1)} \xi^{j_1}_{(-1-t_1)} \cdots \xi^{j_r}_{(-1-t_r)} |0\rangle$$

if there exists a $n_j > 0$ then we can move, using commutators, $x^{i_j}_{(-(n_j-1)-2)}$ to the leftmost, so that this last term is in F^1V. Also all the other terms with commutators $[x^i_{(-n_i-1)}, x^j_{(-n_j-2)}] = [x^i, x^j]_{(-n_i-n_j-3)}$ will have "big" negative powers so we will be able to move to the leftmost position and prove they are in F^1V. Like this we see that the only surviving x part has only t^{-1} and commute, since $[xt^{-1}, yt^{-1}] = [x,y]t^{-2} \in F^1V$; thus it corresponds to $S(\mathfrak{g})$. More easily, since $\mathcal{O}(\mathscr{J}_\infty G)$ is abelian, we can move any $f_{(-1-j)}$ with $j > 0$ to the leftmost place (before the x's), and then we can use the relation $[x_{(-1)}, f_{(-1-j)}] = (x_L f)_{(-1-(j+1))}$ to continue as before. We obtain that only the nonderived functions survive, i.e. the $\mathcal{O}(G)$ part. Thus we proved

$$R_{\mathcal{D}^{\mathrm{ch}}_{G,\kappa}} \cong S(\mathfrak{g}) \otimes \mathcal{O}(G).$$

■

Recall now that given $a, b \in V$ homogeneous we can define

$$a \circ b = \sum_{i \geq 0} \binom{\Delta_a}{i} a_{(i-2)} b, a * b = \sum_{i \geq 0} \binom{\Delta_a}{i} a_{(i-1)} b.$$

It is then known that $\mathrm{Zhu}(V) = V/V \circ V$ is an associative unital almost-commutative algebra with product $*$.

Proposition 4.2. We have $\mathrm{Zhu}(\mathcal{D}^{\mathrm{ch}}_{G,\kappa}) \cong \mathcal{D}(G)$.

Proof. Since $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ has a PBW basis, we know that $R_{\mathcal{D}^{\mathrm{ch}}_{G,\kappa}} \cong \mathrm{gr}\,\mathrm{Zhu}(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})$. Since

$$R_{\mathcal{D}^{\mathrm{ch}}_{G,\kappa}} = S(\mathfrak{g}) \otimes \mathcal{O}(G) \cong \mathrm{gr}\, U(\mathfrak{g}) \otimes \mathcal{O}(G) \cong \mathrm{gr}(U(\mathfrak{g}) \otimes \mathcal{O}(G)) \cong \mathrm{gr}\,\mathcal{D}(G)$$

by PBW theorem and the isomorphism $U(\mathfrak{g}) \otimes \mathcal{O}(G) \cong \mathcal{D}(G)$. Let's consider the map of algebras

$$\mathcal{D}(G) \to \mathrm{Zhu}(\mathcal{D}^{\mathrm{ch}}_{G,\kappa}) = \frac{\mathcal{D}^{\mathrm{ch}}_{G,\kappa}}{\mathcal{D}^{\mathrm{ch}}_{G,\kappa} \circ \mathcal{D}^{\mathrm{ch}}_{G,\kappa}}, \quad \mathfrak{g} \ni x \mapsto xt^{-1} + \mathcal{D}^{\mathrm{ch}}_{G,\kappa} \circ \mathcal{D}^{\mathrm{ch}}_{G,\kappa}, \quad \mathcal{O}(G) \ni f \mapsto f_{(-1)} + \mathcal{D}^{\mathrm{ch}}_{G,\kappa} \circ \mathcal{D}^{\mathrm{ch}}_{G,\kappa}.$$

It is easy to verify that we have

$$xt^{-1} * yt^{-1} - yt^{-1} * xt^{-1} \overset{formula}{=} \sum_{j \geq 0} \binom{1-1}{j} x_{(j)} y = x_{(0)} y = [x,y]t^{-1}$$

so that our map is well defined. It clearly respects the filtration so it induces a map on the grading, which is the isomorphism of before, and thus we can conclude. ■

References

[AM21] Tomoyuki Arakawa and Anne Moreau. *Arc spaces and vertex algebras.* 2021. URL: `https://www.imo.universite-paris-saclay.fr/~moreau/CEMPI-arc_space-vertex_algebras.pdf`.

[Kle] Alexander Kleshchev. *Lectures on Algebraic Groups.* URL: `https://darkwing.uoregon.edu/~klesh/teaching/AGLN.pdf`.

[Mal17] Fyodor Malikov. "An Introduction to Algebras of Chiral Differential Operators". In: *Perspectives in Lie Theory.* Cham: Springer International Publishing, 2017, pp. 73–124. ISBN: 978-3-319-58971-8. DOI: `10.1007/978-3-319-58971-8_2`. URL: `https://doi.org/10.1007/978-3-319-58971-8_2`.