An introduction to spectral sequences

Carlo Buccisano

January 16, 2020

Contents

1 Preliminaries

We'll denote with $\mathcal{C}$ a $\mathcal{U}$-small categories, for some Grothendieck universe $\mathcal{U}$. For our purpose, we'll need an abelian category $\mathcal{C}$ which respects the following Grothendieck axioms (and their dual versions, i.e. substitute coproduct with product and mono with epi):

AB3 For every (small) indexed family $(A_i)_{i\in I} \subset \mathrm{Ob}(\mathcal{C})$, the coproduct $\coprod_i A_i$ exists, i.e. $\mathcal{C}$ is cocomplete.

AB4 $\mathcal{C}$ satisfies AB3 and the coproduct of a family of mono is mono.

For the sake of simplicity, we'll use the Freyd-Mitchell embedding of $\mathcal{C}$ in a full subcategory of $\mathbf{Mod}(A)$ for some ring A. Similar to the notation used in [Wei94], we'll adopt left arrows for chain complexes: let $\Omega \in \mathrm{C}(\mathcal{C})$, then

$$\dots \longleftarrow \Omega_{n-1} \xleftarrow{d_n} \Omega_n \xleftarrow{d_{n+1}} \Omega_{n+1} \longleftarrow \dots$$

which is the same as adopting the right-arrows notation and working in $\mathcal{C}^{\mathrm{op}}$. For double chain complexes, we'll use left and downward arrows, i.e. let $E \in \mathrm{C}^2(\mathcal{C})$, then

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longleftarrow & E_{p-1,q+1} & \xleftarrow[d_{p,q+1}]{} & E_{p,q+1} & \xleftarrow[d_{p+1,q+1}]{} & E_{p+1,q+1} & \longleftarrow & \dots \\
 & & \downarrow {\scriptstyle d'_{p-1,q+1}} & & \downarrow {\scriptstyle d'_{p,q+1}} & & \downarrow {\scriptstyle d'_{p+1,q+1}} & & \\
\dots & \longleftarrow & E_{p-1,q} & \xleftarrow[d_{p,q}]{} & E_{p,q} & \xleftarrow[d_{p+1,q}]{} & E_{p+1,q} & \longleftarrow & \dots \\
 & & \downarrow {\scriptstyle d'_{p-1,q}} & & \downarrow {\scriptstyle d'_{p,q}} & & \downarrow {\scriptstyle d'_{p+1,q}} & & \\
\dots & \longleftarrow & E_{p-1,q-1} & \xleftarrow[d_{p,q-1}]{} & E_{p,q-1} & \xleftarrow[d_{p+1,q-1}]{} & E_{p+1,q-1} & \longleftarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We ask for the squares of a double chain complex to be anti-commutative, that is, we must have $d' \circ d + d \circ d' = 0$. We recall that there are two different definitions of total complexes, $\mathrm{Tot}^{\coprod}(E)$ and $\mathrm{Tot}^{\oplus}(E)$, defined by

$$\mathrm{Tot}^{\coprod}(E)_n = \coprod_{p+q=n} E_{p,q}, \quad \mathrm{Tot}^{\oplus}(E)_n = \bigoplus_{p+q=n} E_{p,q}$$
$$d_n^{\coprod}\Big|_{E_{p,q}} = d_{p,q} + d'_{p,q}, \qquad d_n^{\oplus}\big|_{E_{p,q}} = d_{p,q} + d'_{p,q}$$

which coincides if E is such that, for any n, we have only a finite number of non-zero terms along the anti-diagonal $p + q = n$.

2 Terminology

Definition 2.1. A *homology spectral sequence* in $\mathcal{C}$ is the following data:

1. A family $\{E^r_{p,q}\} \subset \mathcal{C}$ defined for all integers p, q and $r \geq a$, for some $a \in \mathbb{Z}$;
2. Differentials $d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $d^r \circ d^r = 0$, which corresponds to chain complexes of slope $-(r+1)/r$ in the lattice $E^r_{\bullet\bullet}$.
3. Isomorphisms between $E^{r+1}_{p,q}$ and the homology of $E^r_{\bullet\bullet}$ at (p, q), i.e. $E^{r+1}_{p,q} \cong \ker d^r_{p,q} / \operatorname{im} d^r_{p+r,q-r+1}$.

The *total degree* of $E^r_{p,q}$ is $p+q$ and each differential $d^r_{p,q}$ decreases the total degree by one. These objects form a category: a morphism $f \colon E \to E'$ is a family of maps $f^r_{p,q} \colon E^r_{p,q} \to E'^r_{p,q}$ in $\mathcal{C}$ with $d'^r \circ f^r = f^r \circ d^r$ and such that $f^{r+1}_{p,q}$ is the map induced by $f^r_{p,q}$ on homology.

We can also define a cohomology spectral sequence, with differentials going "to the right", and we'll denote one by $\{E^{p,q}_r\}$. From now on, unless otherwise specified, we'll work with homology spectral sequences.

Definition 2.2 (E^∞ terms)**.** Let $\{E^r_{p,q}\}$ be a spectral sequence; each $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$ and with an induction on r we can prove there exists a nested family of subobjects in $E^a_{p,q}$

$$0 = B^a_{p,q} \subseteq \cdots \subseteq B^r_{p,q} \subseteq B^{r+1}_{p,q} \subseteq \cdots \subseteq Z^{r+1}_{p,q} \subseteq Z^r_{p,q} \subseteq \cdots \subseteq Z^a_{p,q} = E^a_{p,q}$$

such that $E^r_{p,q} \cong Z^r_{p,q}/B^r_{p,q}$. We define

$$B^\infty_{p,q} \coloneqq \bigcup_{r=a}^{\infty} B^r_{p,q}, \qquad Z^\infty_{p,q} \coloneqq \bigcap_{r=a}^{\infty} Z^r_{p,q}$$

and set $E^\infty_{p,q} \coloneqq Z^\infty_{p,q}/B^\infty_{p,q}$.

Definition 2.3. A spectral sequence is:

1. *Bounded* if for each $n \in \mathbb{Z}$ there are only finitely many nonzero terms of total degree n in $E^a_{\bullet\bullet}$ (and hence in each $E^r_{\bullet\bullet}$ for $r \geq a$). In this case, for each (p, q), there exists an r_0 such that $E^r_{p,q} = E^{r+1}_{p,q}$ for all $r \geq r_0$ and hence $E^\infty_{p,q}$ corresponds to this stable value.
2. *Bounded below* if for each $n \in \mathbb{Z}$ there exists $s(n) \in \mathbb{Z}$ such that $E^a_{p,n-p} = 0$ for every $p < s(n)$.
3. *Regular* if for each (p, q) the differentials $d^r_{p,q}$ (i.e. leaving $E^r_{p,q}$) are zero for all large r. Equivalently, there exists $r = r(p, q) \in \mathbb{Z}$ such that $Z^\infty_{p,q} = Z^r_{p,q}$.

It is easily seen that every property implies the ones below it.

Example 2.4. A first quadrant spectral sequence is one where $E^r_{p,q} = 0$ unless (p, q) belongs to the first quadrant. Clearly it is bounded. A right half-plane spectral sequence, instead, is bounded below but not bounded.

Definition 2.5 (Convergence)**.** The spectral sequence $\{E^r_{p,q}\}$ *weakly converges* to $H_\star = \{H_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ if for each H_n we have a filtration (indexed in $\mathbb{Z}$)

$$\cdots \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq F_{p+1}H_n \subseteq \cdots \subseteq H_n$$

and isomorphisms $\beta_{p,q}\colon E^\infty_{p,q} \xrightarrow{\sim} F_pH_{p+q}/F_{p-1}H_{p+q}$ for every (p,q).

We say that E *approaches* $H_\star$ if it weakly converges to it and every filtration is Hausdorff and exhaustive, i.e. $\cap F_pH_n = 0$ and $\cup F_pH_n = H_n$.

Finally, we say that E *converges* to $H_\star$ if E is regular, it approaches $H_\star$ and $H_\star$ is also complete, i.e. $H_n = \varprojlim(H_n/F_pH_n)$. We'll denote convergence by

$$E^a_{p,q} \Longrightarrow H_{p+q}.$$

Example 2.6. If a first quadrant spectral sequence converges to $H_\star$ then every H_n has a finite filtration (sometimes called "canonical filtration")

$$0 = F_{-1}H_n \subseteq F_0H_n \subseteq \cdots \subseteq F_nH_n = H_n$$

such that the bottom piece $F_0H_n = E^\infty_{0,n}$ is on the y-axis and the top piece $H_n/F_{n-1}H_n \cong E^\infty_{n,0}$ is on the x-axis. Since this is a first quadrant sequence, each $E^\infty_{0,n}$ is a quotient of $E^a_{0,n}$ and each $E^\infty_{n,0}$ is a subobject of $E^a_{n,0}$. Hence we have the edge morphisms

$$E^a_{0,n} \to E^\infty_{0,n} \subseteq H_n, \qquad H_n \to E^\infty_{n,0} \subseteq E^a_{n,0}.$$

Definition 2.7. A spectral sequence E *collapses* at E^r ($r \geq 2$) if there is exactly one nonzero row/column in the lattice $E^r_{\bullet\bullet}$. If E converges to $H_\star$ then H_n is the unique nonzero $E^r_{p,n-p}$. (The majority of applications of spectral sequences involve collapsing spectral sequences at E^2).

Lemma 2.8 (Mapping Lemma)**.** *Let $f\colon E \to E'$ be a morphism of spectral sequences such that, for a certain r, $f^r\colon E^r \xrightarrow{\sim} E'^r$ is an iso. Then f^s is an iso as well for every $s \geq r$. Also, the natural morphism $f^\infty\colon E^\infty \xrightarrow{\sim} E'^\infty$ is an iso.*

Proof. The fact that f^s is an iso can be proved by induction on s. It easily derives from the fact that if $f^{s-1}\colon E^{s-1} \xrightarrow{\sim} E'^{s-1}$ then it also the induces an iso between homologies, which then implies that f^s is also an iso. To conclude that also f^∞ is an iso we observe that

$$Z^\infty/B^\infty = \varprojlim Z^n/B^m = \varprojlim Z^\infty/B^m = \varprojlim Z^n/B^\infty$$

and that we have a natural induced map $Z^\infty/B^\infty \to Z'^\infty/B'^\infty$ and we conclude using $AB4$. ☐

Definition 2.9 (Compatible maps)**.** Let E, E' be two spectral sequences weakly convergent to $H_\star$ and $H'_\star$ respectively. A map $h\colon H_\star \to H'_\star$ is *compatible* with a morphism $f\colon E \to E'$ if h maps F_pH_n to $F_pH'_n$ and the following diagram commutes

$$\begin{array}{ccc} F_pH_n/F_{p-1}H_n & \xrightarrow{h} & F_pH'_n/F_{p-1}H'_n \\ \uparrow{\scriptstyle \wr\,\beta_{p,q}} & & \uparrow{\scriptstyle \wr\,\beta'_{p,q}} \\ E^\infty_{p,q} & \xrightarrow{f^\infty_{p,q}} & E'^\infty_{p,q} \end{array}$$

Theorem 2.10 (Comparison theorem)**.** *Let E, E' converge to $H_\star$ and $H'_\star$ and let $h\colon H_\star \to H'_\star$ be a compatible map with a morphism $f\colon E \to E'$. If $f^r\colon E^r \to E'^r$ is an iso for some r then also h is an isomorphism.*

Proof. By the Mapping Lemma we know that also f^r and f^∞ are iso. Weak convergence gives us the following exact sequences

$$\begin{array}{ccccccccc} 0 & \to & F_{p-1}H_n/F_sH_n & \to & F_pH_n/F_sH_n & \to & E^\infty_{p,n-p} & \to & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \wr} & & \\ 0 & \to & F_{p-1}H'_n/F_sH'_n & \to & F_pH'_n/F_sH'_n & \to & E'^\infty_{p,n-p} & \to & 0 \end{array}$$

Fixed s, an easy induction shows us that $F_pH_n/F_sH_n \cong F_pH'_n/F_sH'_n$ (use the 5-lemma) for any p. Since this filtration is exhaustive (by def of convergence) then we have $H_n/F_sH_n \cong H'_n/F_sH'_n$ for any s, and we conclude using completeness taking the projective limit of both members. $\square$

Remark 2.10.1. The same spectral sequence can converge to many different $H_\star$. For example consider, in **Ab**, a first quadrant spectral sequence defined by

$$E^0_{p,q} := \begin{cases} 0, & \text{if } p < 0 \text{ or } q < 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise} \end{cases}$$

where the differentials are the zero morphisms. Then $E^\infty = E^0$ and a H_3 can be $\mathbb{Z}/16\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^4$ or even $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. The comparison theorem allows us to reconstruct $H_\star$ in a different way, having the right maps.

3 Spectral sequence of a filtration

Definition 3.1. Denote by $C \in \mathrm{C}(\mathcal{C})$ a chain complex. A filtration on C is a family of chain subcomplexes

$$\cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq \cdots \subseteq C.$$

Our goal is to show that we can associate a spectral sequence to any filtered complex.

Definition 3.2. A filtration on C is called *bounded* if for any n there are integers $s < t$ such that $F_sC_n = 0$ and $F_tC_n = C_n$. If $s = -1$ and $t = n$ then the filtration if canonically bounded. If $s = -\infty$ $(t = +\infty)$ then the filtration is bounded above (below).

Theorem 3.3 (Construction of the spectral sequence)**.** *A filtration F on a chain complex C naturally determines a spectral sequence starting with $E^0_{p,q} = F_pC_{p+q}/F_{p-1}C_{p+q}$ and $E^1_{p,q} = H_q(E^0_{p,\bullet})$.*

Proof. Let $\eta_{p,q} \colon F_pC_{p+q} \to F_pC_{p+q}/F_{p-1}C_{p+q} = E^0_{p,q}$ be the natural surjection. Let's define the sets

$$A^r_{p,q} := \{c \in F_pC_{p+q} : d(c) \in F_{p-r}C_{p+q-1}\}$$

i.e. the elements of F_pC_{p+q} that are cycles "mod $F_{p-r}C_{p+q-1}$". Let's immediately observe that we have the inclusions

$$\begin{array}{ccccccccc} \cdots & \hookrightarrow & A^{r+1}_{p,q} & \hookrightarrow & A^r_{p,q} & \hookrightarrow & A^{r-1}_{p,q} & \hookrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \hookrightarrow & A^{r+2}_{p+1,q-1} & \hookrightarrow & A^{r+1}_{p+1,q-1} & \hookrightarrow & A^r_{p+1,q-1} & \hookrightarrow & \cdots \end{array} \tag{1}$$

$$d(A^r_{p,q}) \subseteq A^s_{p-r,q+r-1} \quad \text{for any } s. \tag{2}$$

Now let's define

$$Z^r_{p,q} := \eta_{p,q}(A^r_{p,q}) \subseteq E^0_{p,q}, \qquad B^r_{p,q} := \eta_{p,q}(d(A^{r-1}_{p+r-1,q-r+2})) \subseteq E^0_{p,q}$$
$$Z^\infty_{p,q} := \bigcap_{r=1}^{\infty} Z^r_{p,q}, \qquad B^\infty_{p,q} := \bigcup_{r=1}^{\infty} B^r_{p,q}$$

so that we have the following inclusions in $E^0_{p,q}$

$$0 = B^0_{p,q} \subseteq B^1_{p,q} \subseteq \cdots \subseteq B^r_{p,q} \subseteq B^\infty_{p,q} \subseteq Z^\infty_{p,q} \subseteq \cdots \subseteq Z^r_{p,q} \subseteq \cdots \subseteq Z^1_{p,q} \subseteq Z^0_{p,q} = E^0_{p,q}.$$

The inclusions $B^r_{p,q} \subseteq B^{r+1}_{p,q}$ and $Z^{r+1}_{p,q} \subseteq Z^r_{p,q}$ immediately comes from (1) while $B^s_{p,q} \subseteq Z^r_{p,q}$ comes from (2). Let's now observe some other "rules" we'll use in all the following isomorphisms (together with the classical theorem $S + T/T \cong S/S \cap T$) and some clever tricks):

$$A^{r-1}_{p-1,q+1} = A^r_{p,q} \cap F_{p-1}C_{p+q} \tag{3}$$
$$A^r_{p-1,q+1} = A^{r+1}_{p,q} \cap A^{r-1}_{p-1,q+1}. \tag{4}$$

Let's now define the terms of the spectral sequence:

$$
\begin{aligned}
E^r_{p,q} := \frac{Z^r_{p,q}}{B^r_{p,q}} &= \frac{A^r_{p,q}/A^r_{p,q} \cap F_{p-1}C_{p+q}}{d(A^{r-1}_{p+r-1,q-r+1})/d(A^{r-1}_{p+r-1,q-r+1}) \cap F_{p-1}C_{p+q}} \cong \\
&\cong \frac{A^r_{p,q} + d(A^{r-1}_{p+r-1,q-r+2}) + F_{p-1}C_{p+q}}{d(A^{r-1}_{p+r-1,q-r+2}) + F_{p-1}C_{p+q}} \cong \\
&\cong \frac{A^r_{p,q}}{A^{r-1}_{p-1,q+1} + d(A^{r-1}_{p+r-1,q-r+2})}.
\end{aligned}
$$

The differentials are the natural maps induced by the differential of the complex

$$
\begin{array}{ccc}
E^r_{p,q} & \xrightarrow{\quad d^r_{p,q} \quad} & E^r_{p-r,q+r-1} \\
\uparrow \wr & & \uparrow \wr \\
\frac{A^r_{p,q}}{A^{r-1}_{p-1,q+1}+d(A^{r-1}_{p+r-1,q-r+2})} & \xrightarrow{\quad d \quad} & \frac{A^r_{p-r,q+r-1}}{A^{r-1}_{p-r-1,q+r}+d(A^{r-1}_{p-1,q+1})}
\end{array}
$$

which are well defined since $d(A^r_{p,q}) \subseteq A^r_{p-r,q+r-1}$. To conclude the proof we only need to give the isomorphisms between E^{r+1} and $H_\star(E^r)$. First of all, let's prove that $d^r_{p,q}$ induces an iso

$$
Z^r_{p,q}/Z^{r+1}_{p,q} \xrightarrow{\sim} B^{r+1}_{p-r,q+r-1}/B^r_{p-r,q+r-1}.
$$

Let's note that $d(A^r_{p,q}) \cap F_{p-r-1}C_{p+q-1} = d(A^{r+1}_{p,q})$ and that $d(A^r_{p-1,q+1}) = d(A^{r+1}_{p,q}) \cap d(A^{r-1}_{p-1,q+1})$ (apply d to (4) and it's easy to show that it commutes with $\cap$). Using these facts together with classical isomorphism theorems and linearity of d we can prove that

$$
\begin{aligned}
B^r_{p-r,q+r-1} = \frac{d(A^{r-1}_{p-1,q+1})}{d(A^r_{p-1,q+1})} &= \frac{d(A^{r-1}_{p-1,q+1})}{d(A^{r+1}_{p,q}) \cap d(A^{r-1}_{p-1,q+1})} \cong \frac{d(A^{r-1}_{p-1,q+1} + A^{r+1}_{p,q})}{d(A^{r+1}_{p,q})} \\
\implies \frac{B^{r+1}_{p-r,q+r-1}}{B^r_{p-r,q+r-1}} &\cong \frac{d(A^r_{p,q})}{d(A^{r-1}_{p-1,q+1} + A^{r+1}_{p,q})}.
\end{aligned}
$$

In a similar way we obtain

$$
\frac{Z^r_{p,q}}{Z^{r+1}_{p,q}} \cong \frac{A^r_{p,q}}{A^{r+1}_{p,q} + A^{r-1}_{p-1,q+1}}
$$

so there is a natural map induced by $d^r_{p,q} \colon A^r_{p,q} \to d(A^r_{p,q})$ and it's an isomorphism because its kernel is contained in $A^{r+1}_{p,q}$. Now we have that

$$
\ker d^r_{p,q} = \frac{\{z \in A^r_{p,q} : d(z) \in d(A^{r-1}_{p-1,q+1}) + A^{r-1}_{p-r-1,q+r}\}}{d(A^{r-1}_{p+r-1,q-r+2}) + A^{r-1}_{p-1,q+1}} = \frac{A^{r-1}_{p-1,q+1} + A^{r+1}_{p,q}}{d(A^{r-1}_{p+r-1,q-r+2}) + A^{r-1}_{p-1,q+1}} \cong \frac{Z^{r+1}_{p,q}}{B^r_{p,q}}
$$

where the last isomorphism derives from the fact that left and right member are both isomorphic to $A^{r+1}_{p,q}/d(A^{r-1}_{p+r-1,q-r+2}) + A^r_{p-1,q+1}$. In-fact we have

$$
\begin{aligned}
\ker d^r_{p,q} &= \frac{d(A^{r-1}_{p+r-1,q-r+2}) + A^{r-1}_{p-1,q+1} + A^{r+1}_{p,q}}{d(A^{r-1}_{p+r-1,q-r+2}) + A^{r-1}_{p-1,q+1}} \cong \frac{A^{r+1}_{p,q}}{d(A^{r-1}_{p+r-1,q-r+2}) + A^r_{p-1,q+1}} \\
\frac{Z^{r+1}_{p,q}}{B^r_{p,q}} &\cong \frac{A^{r+1}_{p,q} + d(A^{r-1}_{p+r-1,q-r+2}) + F_{p-1}C_{p+q}}{d(A^{r-1}_{p+r-1,q-r+2}) + F_{p-1}C_{p+q}} \cong \frac{A^{r+1}_{p,q}}{(d(A^{r-1}_{p+r-1,q-r+2}) + F_{p-1}C_{p+q}) \cap A^{r+1}_{p,q}} = \\
&= \frac{A^{r+1}_{p,q}}{d(A^{r-1}_{p+r-1,q-r+2}) + A^r_{p-1,q+1}}
\end{aligned}
$$

The map $d^r_{p,q}$ factors as

$$
E^r_{p,q} \overset{\text{def}}{=} \frac{Z^r_{p,q}}{B^r_{p,q}} \twoheadrightarrow \frac{Z^r_{p,q}}{Z^{r+1}_{p,q}} \xrightarrow[d]{\sim} \frac{B^{r+1}_{p-r,q+r-1}}{B^r_{p-r,q+r-1}} \hookrightarrow \frac{Z^r_{p-r,q+r-1}}{B^r_{p-r,q+r-1}} \overset{\text{def}}{=} E^r_{p-r,q+r-1}
$$

from which we see that $\operatorname{im} d^r_{p,q} = B^{r+1}_{p-r,q+r-1}/B^r_{p-r,q+r-1}$. This, finally, implies

$$E^{r+1}_{p,q} = \frac{Z^{r+1}_{p,q}}{B^{r+1}_{p,q}} \cong \frac{\ker d^r_{p,q}}{\operatorname{im} d^r_{p+r,q-r+1}},$$

which concludes the proof (quotient both by $B^r_{p,q}$). □

Let C be a filtered complex; then we have an induced filtration on homology:

$$F_p H_n(C) \coloneqq \operatorname{im}(H_n(F_p C) \to H_n(C)).$$

If F is exhaustive on C then it is also exhaustive on H (any element of $H_n(C)$ is represented by some $c \in F_p C_n$ s.t. $d(c) = 0$). If F is bounded below on C then it is bounded below also on H, since $F_p C = 0$ implies $F_p H_n(C) = 0$.

Theorem 3.4 (Classical convergence theorem)**.** *Let C be a filtered complex.*

1. *Suppose the filtration on C is bounded. Then the spectral sequence is bounded and converges to $H_\star(C)$, that is*
$$E^0_{p,q} = F_p C_{p+q}/F_{p-1}C_{p+q} \Longrightarrow H_{p+q}(C).$$
2. *Suppose the filtration on C is bounded below and exhaustive. Then the spectral sequence is bounded below and converges to $H_\star(C)$. Moreover, if $f \colon C \to C'$ is a map of filtered complexes (i.e. it respects the filtrations) then the induced map $f_\star \colon H_\star(C) \to H_\star(C')$ is compatible with the corresponding map induced on the spectral sequences.*

Proof. As already said above exhaustiveness and below-boundedness are inherited by the filtration on $H_\star(C)$. Then $H_\star(C)$ is Hausdorff, regular (both implied by bounded below) and complete (implied by bounded below and exhaustive) hence, recalling the definition Convergence, we just need to prove weak convergence. First of all, observe that, since the filtration on C is bounded below, fixed (p,q), we have that the $A^r_{p,q} = \{c \in F_p C_{p+q} : d(c) \in F_{p-r}C_{p+q-1}\}$ (see Construction of the spectral sequence) stabilize for a large enough r_0: we'll then define $A^\infty_{p,q} \coloneqq A^{r_0}_{p,q}$. Then we observe the following facts:

$$Z^\infty_{p,q} = \eta_{p,q}(A^\infty_{p,q}), \qquad A^\infty_{p,q} = \ker(F_p C_{p+q} \xrightarrow{d} F_p C_{p+q-1}).$$

Let's observe now that, since the filtration is exhaustive, we have

$$d(C_{p+q}) \cap F_p C_{p+q-1} = \bigcup_r d(A^r_{p+r,q-r}) = d(\cup A^r_{p+r,q-r}) \subseteq A^\infty_{p,q}$$

and that $A^\infty_{p-1,q+1} = \ker(A^\infty_{p,q} \xrightarrow{\eta_{p,q}} E^0_{p,q})$, since $A^\infty_{p-1,q+1} \subseteq F_{p-1}C_{p+q}$. We easily see that

$$B^r_{p,q} \overset{\text{def}}{=} \eta_{p,q}(d(A^{r-1}_{p+r-1,q-r+2})) \implies B^\infty_{p,q} \overset{\text{def}}{=} \bigcup_r B^r_{p,q} = \eta_{p,q}(d(\cup A^r_{p+r,q-r+1})).$$

Putting all together, recalling that $F_p H_{p+q}(C) = \operatorname{im}(H_{p+q}(F_p C) \to H_{p+q}(C))$, we have

$$\frac{F_p H_{p+q}(C)}{F_{p-1}H_{p+q}(C)} = \frac{A^\infty_{p,q}/d(C_{p+q+1}) \cap A^\infty_{p,q}}{A^\infty_{p-1,q+1}/d(C_{p+q+1}) \cap A^\infty_{p-1,q+1}} = \frac{A^\infty_{p,q}/d(\cup A^r_{p+r,q-r+1}) \cap A^\infty_{p,q}}{A^\infty_{p-1,q+1}/d(\cup A^r_{p+r,q-r+1}) \cap A^\infty_{p-1,q+1}} \cong$$
$$\cong \frac{A^\infty_{p,q}}{A^\infty_{p-1,q+1} + d(\cup A^r_{p+r,q-r+1})} \cong \frac{\eta_{p,q}(A^\infty_{p,q})}{\eta_{p,q}(d(\cup A^r_{p+r,q-r+1}))} = \frac{Z^\infty_{p,q}}{B^\infty_{p,q}} = E^\infty_{p,q}.$$

which concludes the proof of convergence. □

Example 3.5 (First quadrant spectral sequence)**.** Suppose that the filtration of C is canonically bounded, i.e. $F_{-1}C_n = 0$ and $F_n C_n = C_n$, so that the spectral sequence lies in the first quadrant. Then it converges to $H_\star(C)$.

We only cite a more powerful result,

Theorem 3.6 (Complete convergence theorem)**.** *Suppose the filtration on C is complete and exhaustive and the spectral sequence is regular. Then*

1. *the spectral sequence weakly converges to $H_\star(C)$;*
2. *if the spectral sequence is bounded above then it converges to $H_\star(C)$.*

Proof. See [Wei94, p. 140]. □

4 Spectral sequence of a double complex

One important application of spectral sequences is to compute the total homology of a double complex. Given a double complex $C \in \mathrm{C}^2(\mathcal{C})$ we have two filtrations for $\mathrm{Tot}(C)$, hence two different spectral sequences. We'll then be able to play them off against each other to prove some properties (e.g. 5-lemma, snake lemma).

Remark 4.0.1. Let C be a double complex; we'll denote by H^h (H^v) the homology related to the horizontal (vertical) differentials. We'll use the generic notation $\mathrm{Tot}(C)$ meaning that we can define/do the same things for both total complexes.

Definition 4.1 (Filtration by columns)**.** Let $C = C_{\bullet\bullet}$ be a double complex and let ${}^{I}F_n\mathrm{Tot}(C)$ be the total complex of

$$\left({}^{I}F_nC\right)_{p,q} := \begin{cases} C_{p,q} & \text{if } p \leq n \\ 0 & \text{otherwise} \end{cases} \qquad \begin{array}{cc|cc} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{array}$$

We have defined a filtration of $\mathrm{Tot}(C)$, called filtration by columns.

It is easy to see that this filtration on $\mathrm{Tot}(C)$ gives rise to a spectral sequence $\{{}^{I}E^r_{p,q}\}$ starting with ${}^{I}E^0_{p,q} = C_{p,q}$, where the maps d^0 are exactly the vertical differentials of C so that ${}^{I}E^1_{p,q} = H^v_q(C_{p,\bullet})$. The maps $d^1 \colon H^v_q(C_{p,\bullet}) \to H^v_q(C_{p-1,\bullet})$ are clearly the ones induced on the homology by the horizontal differentials of C.

Definition 4.2 (Filtration by rows)**.** Let $C = C_{\bullet\bullet}$ be a double complex and let ${}^{II}F_n\mathrm{Tot}(C)$ be the total complex of

$$\left({}^{II}F_nC\right)_{p,q} := \begin{cases} C_{p,q} & \text{if } q \leq n \\ 0 & \text{otherwise} \end{cases} \qquad \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline * & * & * & * & * \\ * & * & * & * & * \end{array}$$

We have defined a filtration of $\mathrm{Tot}(C)$, called filtration by rows.

Since ${}^{II}F_p\mathrm{Tot}(C)/{}^{II}F_{p-1}\mathrm{Tot}(C)$ is the *row* $C_{\bullet,p}$ we have that the corresponding spectral sequence $\{{}^{II}E^r_{p,q}\}$ starts with ${}^{II}E^0_{p,q} = C_{q,p}$ (vertical morphisms d^0 of E^0 are exactly the horizontal morphisms of C) and ${}^{II}E^1_{p,q} = H^h_q(C_{\bullet,p})$ and, as one imagines, the differentials d^1 are induced by the vertical differentials of C. Let's now study the convergence of these sequences in some special cases.

First quadrant Let C be a first quadrant double complex then both the filtration of $\mathrm{Tot}(C)$ (here we have only one kind of total complex) are canonically bounded and so, by the Classical convergence theorem, both the spectral sequences converge to $H_\star(\mathrm{Tot}(C))$:

$${}^{I}E^0_{p,q},\ {}^{II}E^0_{p,q} \Longrightarrow H_{p+q}(\mathrm{Tot}(C)).$$

Zeroes in second quadrant Let C be such that $C_{p,q} = 0$ in the second quadrant (e.g. a fourth quadrant complex).

- **Columns** The filtration on $\mathrm{Tot}^{\prod}(C)$ by columns is bounded below but is not exhaustive (so we cannot apply our convergence theorems), in-fact we have

$$\bigcup_{p \geq p_0} \prod_{i=p_0}^{p} C_{i,p_0-i} \subsetneq \prod_{i=p_0}^{\infty} C_{i,p_0-i}$$

since the lhs contains only the "definitively zero terms". Instead, the column filtration on $\mathrm{Tot}^{\oplus}(C)$ is both bounded below and exhaustive so we can apply the Classical convergence theorem to obtain

$${}^{I}E^0_{p,q} \Longrightarrow H_{p+q}(\mathrm{Tot}^{\oplus}(C))$$

(hence here we must be careful because the two different total complexes are different and we have convergence only with $\mathrm{Tot}^{\oplus}(C)$).

- **Rows** The filtration on $\mathrm{Tot}^{\prod}(C)$ by rows is bounded above (hence exhaustive) and complete so we can apply Complete convergence theorem to obtain that ${}^{II}E^0_{p,q}$ weakly converges to $H_\star(\mathrm{Tot}^{\prod}(C))$. (We cannot say anything on $\mathrm{Tot}^{\oplus}(C)$ since its filtration by rows here it is not complete).

Zeroes in fourth quadrant Let C be such that $C_{p,q} = 0$ in the fourth quadrant (e.g. a second quadrant complex).

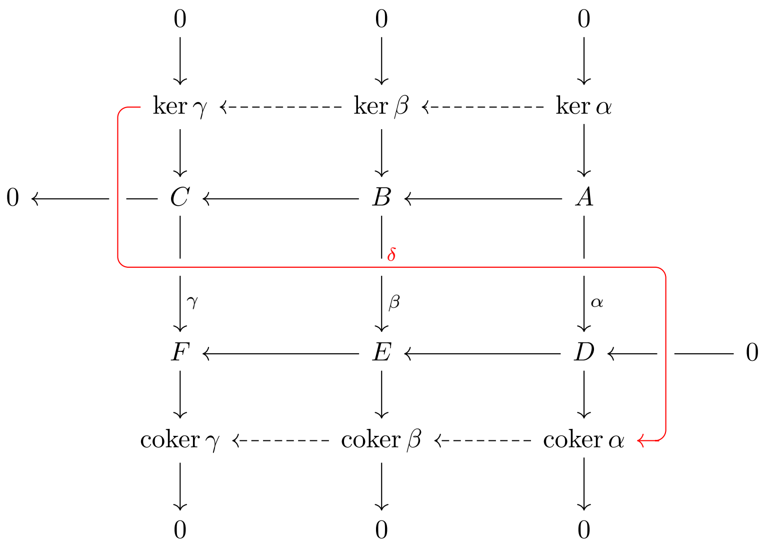
- **Columns** The column filtration of $\mathrm{Tot}^{\prod}(C)$ is bounded above, hence exhaustive, and complete so we can apply Complete convergence theorem to obtain that ${}^{I}E^0_{p,q}$ weakly converges to $H_\star(\mathrm{Tot}^{\prod}(C))$.(We cannot say anything on $\mathrm{Tot}^{\oplus}(C)$ since its filtration by columns here it is not complete).
- **Rows** The row filtration of $\mathrm{Tot}^{\prod}(C)$ is not exhaustive so we cannot apply the convergence theorems. Instead, the filtration on $\mathrm{Tot}^{\oplus}(C)$ is bounded below and exhaustive hence, by the Classical convergence theorem, we obtain

$$ {}^{II}E^0_{p,q} \Longrightarrow H_{p+q}(\mathrm{Tot}^{\oplus}(C)). $$

5 Snakes!

Finally, we prove the famous snake lemma using the whole machinery of spectral sequences, inspired by [Vak].

Proposition 5.1 (Snake Lemma). *Consider the following commutative diagram, whose rows are exact*



then there exists a morphism δ such that we have this exact sequence

$$ \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma. $$

Proof. Consider the double complex (fill with zeroes)

$$ \begin{array}{ccccccc} 0 & \longleftarrow & C & \longleftarrow & B & \longleftarrow & A & & \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \\ & & F & \longleftarrow & E & \longleftarrow & D & \longleftarrow & 0 \end{array} $$

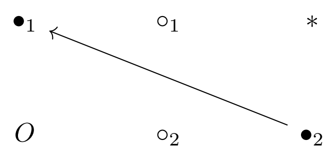
where rows and columns are numbered such that F is the origin and C is $(0,1)$. This is a first quadrant complex hence both filtrations converge to the total homology, call it $H_\star$. A little computation shows that ${}^{II}E^1$ (row filtration) is of this form

$$\begin{array}{ccc} 0 & \longleftarrow & * \\ \\ 0 & \longleftarrow & 0 \\ \\ * & \longleftarrow & 0 \end{array}$$

where the bottom left square is the origin. Hence we see that, having only two non-zero terms in different degrees, ${}^{II}E^{\infty} = {}^{II}E^{1}$ and we can already read that $H_1 = H_2 = 0$. We'll use this information with the spectral sequence corresponding to the column filtration. The first page is

$$\begin{array}{ccccc} \ker\gamma & \dashleftarrow & \ker\beta & \dashleftarrow & \ker\alpha \\ \\ \operatorname{coker}\gamma & \dashleftarrow & \operatorname{coker}\beta & \dashleftarrow & \operatorname{coker}\alpha \end{array}$$

and the second page is



where O is the origin and all the other non-written points are zero. Then we immediately see that, since $H_1 = H_2 = 0$ and the $\circ$'s are already stable, they must vanish. We also must have that the arrow between the $\bullet$'s is an iso, because at the next page its homologies must vanish. Let's translate all these information:

(i) $\circ_2 = 0$ means exactness at $\operatorname{coker}\beta$;

(ii) $\circ_1 = 0$ means exactness at $\ker\beta$;

(iii) the isomorphism between the $\bullet$'s is an iso

$$\delta \colon \bullet_1 = \operatorname{coker}(\ker\beta \to \ker\gamma) \xrightarrow{\sim} \ker(\operatorname{coker}\alpha \to \operatorname{coker}\beta) = \bullet_2$$

and so we can glue the two natural exact sequences by this iso to get the wanted exact sequence. □

References

[Vak] Ravi Vakil. *SPECTRAL SEQUENCES: FRIEND OR FOE?* URL: http://math.stanford.edu/~vakil/0708-216/216ss.pdf.

[Wei94] Charles A. Weibel. *An introduction to homological algebra.* Cambridge University Press, 1994.