Final Review

2023年4月9日 23:11

Finite Dimension (*All SONC contain FONC, $f\,g\,h$ C^1 in FONC, C^2 in SONC)

- Unconstrained – $\min f(x) : (\Omega, \mathbb{R})$

FONC: local min $\Rightarrow$ $\nabla f(x_0)\cdot v \geq 0$ for all feasible v.
Ω open $\Rightarrow$ all v feasible, $\nabla f(x_0) = 0$
(feasible: $x_0 + \varepsilon v \in \Omega$ for small $\varepsilon > 0$)

SONC: local min $\Rightarrow$ $\nabla f(x_0)\cdot v = 0 \Rightarrow v^T \nabla^2 f(x_0) v \geq 0$ for all feasible v.
Ω open $\Rightarrow$ $\nabla^2 f(x_0) \geq 0$

SOSC: $\nabla f(x_0) = 0, \nabla^2 f(x_0) > 0$ $\Rightarrow$ strict local min.

- Equality Constraint – $\min f(x)$ with $h_i(x) = 0$ (Ω open, x_0 regular)
Regular: $\{\nabla h_i(x_0)\}$ linearly independent, interior points are regular.

FONC: local min $\Rightarrow \exists \lambda_i \in \mathbb{R}$ s.t. $\nabla f + \lambda_i \nabla h_i = 0$ (Lagrange Multipliers)

SONC: local min $\Rightarrow \exists \lambda_i \in \mathbb{R}$ s.t. $\nabla^2 f + \lambda_i \nabla^2 h_i \geq 0$ on $T_{x_0}M$
$\mathcal{M} = \{x \in \mathbb{R}^n \mid h_i(x) = 0\}$ $\quad T_{x_0} = \{v \in \mathbb{R}^n \mid \nabla h_i(x_0)\cdot v = 0\}$
$Q \geq 0$ on $T_{x_0}M \iff v^T Q v \geq 0\ \forall v \in T_{x_0}\mathcal{M}$ ($T_{x_0}\mathcal{M} = T_{x_0}$ for x_0 regular)

SOSC: $\exists \lambda_i \in \mathbb{R}$ s.t. $\nabla f + \lambda_i \nabla h_i = 0, \nabla^2 f + \lambda_i \nabla^2 h_i > 0$ on $T_{x_0}\mathcal{M}$ $\Rightarrow$ strict local min

- Inequality Constraint – $\min f(x)$ with $h_i(x) = 0, g_j(x) \leq 0$ (Ω open, x_0 regular)
x_0 is regular if $\{\nabla h_i(x_0), \nabla g_j(x_0)$ active$\}$ linearly independent.
(active: $g_j = 0$, inactive: $g_j < 0$)

KT-condition: local min $\Rightarrow \exists \lambda_i \in \mathbb{R}, \mu_j \in \mathbb{R}$ s.t. $\begin{cases} \nabla f + \lambda_i \nabla h_i + \mu_j \nabla g_j = 0 \\ \mu_j \cdot g_j = 0,\ \mu_j \geq 0 \end{cases}$
(if g_j inactive, then $\mu_j = 0$)

SONC: local min $\Rightarrow \exists \lambda_i \in \mathbb{R}, \mu_j \in \mathbb{R}$ s.t. $L = \nabla^2 f + \lambda_i \nabla^2 h_i + \mu_j \nabla^2 g_j \geq 0$ on $\tilde{T}_{x_0}\mathcal{M}$
$\tilde{T}_{x_0} = \{v \in \mathbb{R}^n \mid \nabla h_i(x_0)\cdot v = 0, \nabla g_j(x_0)\cdot v = 0$ with g_j active$\}$

SOSC: x_0 feasible, $\exists \lambda_i \in \mathbb{R}, \mu_j \in \mathbb{R}$ s.t. $\begin{cases} \text{satisfy KT} \\ L = \nabla^2 f + \lambda_i \nabla^2 h_i + \mu_j \nabla^2 g_j > 0 \text{ on } \tilde{\tilde{T}}_{x_0}\mathcal{M} \end{cases}$
$\tilde{\tilde{T}}_{x_0} = \{v \in \mathbb{R}^n \mid \nabla h_i(x_0)\cdot v = 0, \nabla g_j(x_0)\cdot v = 0$ with g_j active, $\mu_j > 0\}$

Alogarithms ($\frac{1}{2}x^TQx - b^Tx$)

- Method of gradient descend

$x_{k+1} = x_k - a_k \cdot \nabla f(x_k)$ $\quad a_k = \dfrac{\|\nabla f(x_k)\|^2}{\nabla f(x_k)^T Q \nabla f(x_k)}$
Remark: $f(x_{k+1}) < f(x_k)$
$(x_{k+2} - x_{k+1})\cdot(x_{k+1} - x_k) = 0$

$f : \mathbb{R}^n \to \mathbb{R}^n = \frac{1}{2}x^TQx - b^Tx$
$\nabla f = Qx - b$, $\nabla^2 f = Q$, $A^{-1}b$ is global min.

- Method of conjugate directions

$x_{k+1} = x_k + a_k \cdot d_k$ $\quad a_k = -\dfrac{\nabla f(x_k)\cdot d_k}{d_k^T Q d_k}$
where $\{d_i\}$ is non-zero Q-orthogonal set.

Q-orthogonal set: $d_i^T Q d_j = 0\ \forall i \neq j$
Remark: Q symmetric $\Rightarrow$ exist $Q\perp$ eigenbasis.
eigenvectors of Q are $Q\perp$.
$Q\perp$ vectors are linearly independent.

> Local/Global min: small neighbourhood/for all x
> Non-strict/Strict: $f(x_0) \leq f(x)$ / $f(x_0) < f(x)$
>
> Pos-def: $A > 0 \iff x^TAx > 0\ \forall x \neq 0 \iff$ all eigenvalues > 0
> Pos semi-def: ≥ 0 instead of > 0

Infinite Dimension (Equality)

No constraint: $-\frac{d}{dx}L_{p_i} + L_{z_i} = 0$

- Type 1: Isoperimetric – $\min F(u) = \int_a^b (\ )dx$ with $G(u) = \int_a^b (\ )dx$

Write $L^F(x, z_i, p_i)$, $L^G(x, z_i, p_i)$
E.L. $-\frac{d}{dx}(L^F_{p_i} + \lambda L^G_{p_i}) + (L^F_{z_i} + \lambda L^F_{z_i})$

- Type 2: Holonomic – $\min F(u) = \int_a^b (\ )dx$ with $H(u) = c$

Write $L(x, z_i, p_i)$, L_{z_i}, L_{p_i}, H_{u_i}
E.L. $(-\frac{d}{dx}L_{p_i} + L_{z_i}) + \lambda(x)\cdot H_{u_i} = 0$

Testing function: all $C^1([0,1] \to \mathbb{R})$, $v(0) = v(1) = 0$
Theorem of CV: g cts. $[a,b]$, $\int_a^b g(x)v(x)dx = 0\ \forall v(x)$, then $v(x) = 0$.

- Basic ODEs

$u'(x) = 0 : u(x) = C$ $\qquad u''(x) = u(x) : u(x) = Ae^x + Be^{-x}$
$u''(x) = 0 : u(x) = Ax + B$ $\qquad u''(x) = -u(x) : u(x) = A\cos x + B\sin x$
$u'(x) = u(x) : u(x) = Ae^x$

Convexity & Subdifferentials

- Convex set: $sx_1 + (1-s)x_2 \in \Omega\ \forall x_1 x_2$
Convex func: $f(sx_1 + (1-s)x_2) \leq sf(x_1) + (1-s)f(x_2)\ \forall x_1, x_2, \forall s \in [0,1]$
Properties: $f_1 f_2$ convex, $a \geq 0 \Rightarrow f_1 + f_2$ convex, af convex
sublevel set $\{x \in \Omega \mid f(x) \leq c\}$ convex

In C^1, f convex $\iff f(y) \geq f(x) + \nabla f(x)\cdot(y - x)\ \forall x, y$
In C^2, f convex $\iff \nabla^2 f(x) \geq 0\ \forall x$

Ω compact, f convex attain max on $\Omega \Rightarrow \max_{\Omega} f$ on boundry

subdiff v at x_0: $f(x) \geq f(x_0) + v\cdot(x - x_0)\ \forall x$
subgrad $\partial f(x_0)$: set of all subdiff ($\partial f(x_0)$ is closed and convex)

$\partial(af) = a\cdot\partial f$, $a \geq 0$
$h(x) = f(Ax + b)$ convex $\Rightarrow \partial h(x_0) = A^T \partial f(Ax_0 + b)$
$f = \sum f_i$, f_i convex $\Rightarrow \partial f = \sum \partial f_i$ $(A + B = \{a + b, a \in A\ b \in B\})$
$f = \max\{f_i\}$, f_i convex $\Rightarrow \partial f(x_0) = \bigcup \partial f_i(x_0)$ s.t. $f_i = f$

- Constrained Optimization – $\min f(x)$, $x \in C$ convex

Let $g(x) = f(x) + \delta_C(x)$ (indicator func $= 0$ if $x \in C$, $= \infty$ otherwise)
x_0 minimize $f \iff x_0$ minimize $g \iff 0 \in \partial g(x_0)$

- Convex Programming – $\min f(x)$, $x \in C = \{g_i(x) \leq 0\}$, $f\,g$ convex

If C has interior points (Slater), then
$x_0 \in \mathbb{R}$ is optimal $\iff \exists y \in \partial f(x_0)$, $\mu_i \geq 0$ s.t. $\begin{cases} -y = \mu_i \partial g_i(x_0) \\ \mu_i g_i = 0 \end{cases}$