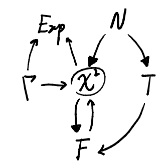
Note Side A

Note for TT1

		$P(X=x)$	$E(X)$	$Var(X)$	MGF
~ Bernoulli(p)	$x \in \{0,1\}$	$p^x(1-p)^{1-x}$	p	$p(1-p)$	$(1-p)+pe^t$
~ Binomial(n,p)	$x \in \{0,1,2,\dots,n\}$	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$	$((1-p)+pe^t)^n$
~ Geometric(p)	$x \in \{0,1,2,\dots\}$	$(1-p)^x p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)e^t}$
~ Negative Binomial(r,p)	$x \in \{0,1,2\dots\}$	$\binom{r-1+x}{r-1}p^r(1-p)^x$	$\frac{rp}{1-p}$	$\frac{rp}{(1-p)^2}$	$\left(\frac{1-p}{1-pe^t}\right)^r$
~ Poisson(λ)	$x \in \{0,1,2\dots\}$	$\frac{\lambda^x e^{-\lambda}}{x!}$	λ	λ	$e^{\lambda(e^t-1)}$
~ Hypergeometric(M,N,n)	$x \in \{0,1,2\dots\min(n,M)\}$	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$	$n\frac{M}{N}$	$n\frac{M}{N}\frac{N-M}{N}\frac{N-n}{N-1}$	
~ Exponential(λ)	$x \in [0,\infty)$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$
~ Normal(μ,σ^2)	$x \in \mathbb{R}$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	μ	σ^2	$\exp(\mu t+\sigma^2t^2/2)$
~ Gamma(α,β)	$x \in (0,\infty)$	$\frac{1}{\Gamma(\alpha)\beta^\alpha}\cdot x^{\alpha-1}\exp\left(\frac{-x}{\beta}\right)$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(1-\frac{t}{\beta}\right)^{-\alpha}$
~ Beta(α,β)	$x \in (0,1)$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$1+\sum_{k=1}^{\infty}\left(\frac{\alpha}{\alpha+\beta}\cdot\frac{\alpha-1}{\alpha+\beta-1}\cdots\frac{\alpha-(k-1)}{\alpha+\beta-(k-1)}\right)\cdot\frac{t^k}{k!}$
~ Multinomial($n,\theta_1,\theta_2,\theta_3$)	$x_i \in \{0,1,2\dots n\}$	$\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_1-x_2}{x_3}\theta_1^{x_1}\theta_2^{x_2}\theta_3^{x_3}$ or $\frac{n!}{x_1!x_2!x_3!}\theta_1^{x_1}\theta_2^{x_2}\theta_3^{x_3}$	$n\cdot\theta_i$	$n\theta_i(1-\theta_i)$	$\left(\sum_{i=1}^{3}\theta_i e^{t_i}\right)^n$
~ Chi-squared(k)	$x \in [0,\infty)$	$\frac{1}{2^{k/2}\Gamma(k/2)}\cdot x^{k/2-1}\cdot e^{-x/2}$	k	$2k$	$(1-2t)^{-k/2}$
~ F-distribution(v_1,v_2)	$x \in [0,\infty)$	/	$\frac{v_2}{v_2-2}$	$\frac{2v_2^2(v_1+v_2-2)}{v_1(v_2-2)^2(v_2-4)}$	DNE
~ T-distribution(v)	$x \in (-\infty,\infty)$	$\frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi}\,\Gamma(\frac{v}{2})}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}$	0 for $v>1$	$\frac{v}{v-2}$ for $v>2$, ∞ for $1<v\le2$	DNE

- $X_i \overset{i.i.d.}{\sim}$ Geometric(p) $\Rightarrow$ $X_1+X_2+\dots+X_r \sim$ Negative Binomial(r,p)
- $Z_1, Z_2 \sim$ Normal $\Rightarrow$ $\mu_1+\sigma_1Z_1,\ \mu_2+\sigma_2(\rho Z_1+(1-\rho^2)^{1/2}Z_2) \sim$ Bivariate Normal
- $X_i \sim$ Normal(0,1) $\Rightarrow$ $X_1^2+X_2^2+\dots+X_n^2 \sim$ Chi-squared(n)
  - $X \sim$ Chi-squared(n=2) $\Rightarrow$ $X \sim$ Exponential($\frac{1}{2}$)
- $X_1 \sim$ Chi-squared(v_1), $X_2 \sim$ Chi-squared(v_2) $\Rightarrow$ $\frac{X_1/v_1}{X_2/v_2} \sim$ F-distribution(v_1,v_2)
  - $X \sim$ F-distn(m,n) $\Rightarrow$ $1/X \sim$ F-distn(n,m)
  - $X_n \sim$ F-distn(m,n) $\Rightarrow$ $\lim_{n\to\infty} m\cdot X_n \sim$ Chi-squared(m)
- $X_i \sim$ Normal(μ,σ^2), σ^2 unknown $\Rightarrow$ $\frac{X}{\sqrt{(X_1^2+\dots+X_n^2)/n}} \sim$ T-distribution(n)
  - $X \sim$ T-distn(k) $\Rightarrow$ $X^2 \sim$ F-distn(1,k)
- $X \sim$ Gamma($\alpha=1,\beta$) $\Rightarrow$ $X \sim$ Exponential(β)
  - $X \sim$ Gamma($\alpha=\frac{n}{2},\beta=\frac{1}{2}$) $\Rightarrow$ $X \sim$ Chi-squared(n)
  - $X \sim$ Beta($\alpha=1,\beta=1$) $\Rightarrow$ $X \sim$ Uniform(0,1)



- **Expectation**

Definition of expectation	$E(X)$	$\sum_x x_i\cdot p_X(x)$	$\int x\cdot f(x)\,dx$
Expectation of a function	$E(g(x))$	$\sum_x g(x)\cdot p_X(x)$	$\int g(x)\cdot f(x)\,dx$
	$E(h(x,y))$	$\sum_y\sum_x h(x,y)\cdot p_{XY}(x,y)$	$\iint h(x,y)\cdot f(x,y)\,dx\,dy$
Conditional expectation	$E(X\mid Y=y)$	$\sum_x x\cdot\frac{p_{XY}(x,y)}{p_Y(y)}$	$\int x\cdot\frac{f_{XY}(x,y)}{f_Y(y)}\,dx$

Properties:
$E(X+Y)=E(X)+E(Y)$.
$E(aX+b)=aE(X)+b$
independent $\Rightarrow$ $E(XY)=E(X)E(Y)$

Law of total expectation: $E(E(X\mid Y))=E(X)$

- **Variance**

Variance: $\sum(x_i-\mu)^2\cdot p_i$ / $\int(x-\mu)^2f(x)\,dx$
$Var(X)=E(X^2)-E(X)^2$

$Var(aX+b)=a^2Var(X)$
$Var(aX+bY)=a^2Var(X)+b^2Var(Y)+2ab\,Cov(X,Y)$
independent $\Rightarrow$ $Var(X+Y)=Var(X)+Var(Y)$

Covariance: $E[(X-\mu_x)(Y-\mu_y)]$
$Cov(X,Y)=E(XY)-E(X)E(Y)$

$Cov(aX+b, cY+d)=ac\,Cov(X,Y)$
$Cov(X,Y+Z)=Cov(X,Y)+Cov(X,Z)$
independent $\Rightarrow$ $Cov(X,Y)=0$

Law of total variance: $V(X)=V(E(X\mid Y))+E(V(X\mid Y))$

Conditional Variance: $V(X\mid Y)=E(X^2\mid Y)-E(X\mid Y)^2$

- **PMF, PDF & CDF**

Definition of PDF	$f_X(x)=P(X=x)$	$f_{XY}(x,y)=P(X=x,Y=y)$
Marginal PMF/PDF	$f_X(x)=\int f_{XY}(x,y)\,dy$	$p_X(x)=\sum_y p_{XY}(x,y)$
Conditional PMF/PDF	$f_{X\mid Y}(x\mid y)=\frac{f_{XY}(x,y)}{f_X(y)}$	$p_{X\mid Y}(x\mid y)=\frac{p_{XY}(x,y)}{p_X(x)}$
Independence	$f_{XY}(x,y)=f_X(x)\cdot f_Y(y)$	$p_{XY}(x,y)=p_X(x)\cdot p_Y(y)$
Definition of CDF	$F_X(x)=\int_{-\infty}^{x}f_X(x)\,dx$	$F_X(x)=\sum_{y\le x}p_X(y)$
Marginal CDF	$F_X(x)=\int_y F_{XY}(x,y)\,dy$	$F_X(x)=\sum_y F_{XY}(x,y)$

$F_{XY}(x,y)=P(X\le x, Y\le y)$ | $\frac{d}{dx}F_X(x)=f_X(x)$

$P(a\le X\le b, c\le y\le d)=F_{XY}(b,d)-F_{XY}(a,d)-F_{XY}(b,c)+F_{XY}(a,c)$

- **MGF**: $m_X(s)=E(e^{sX})$

$m_X'(0)=E(X)$
$m_X''(0)=E(X^2)$

$m_{X+Y}(t)=m_X(t)\cdot m_Y(t)$
$m_Y(t)=e^{at}m_X(bt)$ if $Y=a+bX$

$M_Y(t)=E(e^{t_1Y_1+t_2Y_2})$ if $Y=Y_1+Y_2$

- **Inequalities**

Markov	$P(X\ge a)\le\frac{E(X)}{a}$
Chebychev's	$P(Y-\mu_Y\ge a)\le\frac{V(Y)}{a^2}$
Chernoff's	$P(X>a)\le e^{-at}m_X(t)$
Cauchy-Schwartz	$\lvert Cov(X,Y)\rvert\le\sqrt{V(X)V(Y)}$
Jensen	f is convex, $E(f(x))$ is finite $\Rightarrow f(E(X))\le E(f(x))$; f is linear $\Rightarrow f(E(X))=E(f(x))$

Correlation: $\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

Median: M s.t. $F(M)=0.5$

Mode: x s.t. $P(x)\ge P(x_i)$

Note Side B

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- Law of total probability $P(A) = \sum_i P(A|B_i) \cdot P(B_i)$

  Conditional and Bayes

  $P(A|B) = \frac{P(A \cap B)}{P(B)}$ $\quad \frac{P(A)}{P(B)} \cdot P(B|A) = \frac{P(A) \cdot P(B|A)}{\sum_i P(A_i) \cdot P(B|A_i)}$

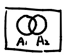
- Countably additive $A = A_1 \cup A_2 \cup A_3 \cdots$
  $P(A) = P(A_1) + P(A_2) + P(A_3) \cdots$

  Finitely additive $A = A_1 \cup A_2 \cdots \cup A_n$
  $P(A) = P(A_1) + P(A_2) \cdots + (A_n)$

  $\{A_n\} \nearrow A$ if $A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq A$ and $\bigcup_{n=1}^{\infty} A_n = A$
  $\{A_n\} \searrow A$ if $A_1 \supseteq A_2 \supseteq A_3 \cdots \supseteq A$ and $\bigcap_{n=1}^{\infty} A_n = A$

  Theorem 1.6.1 Suppose $\{A_n\} \nearrow A$ or $\{A_n\} \searrow A$, then $\lim_{n \to \infty} P(A_n) = P(A)$

  Boole's inequality: $P(\bigcup_k A_k) \le \sum_k P(A_k)$

  $P(A_1 \cup A_2) \le P(A_1) + P(A_2)$

  Bonferroni's inequality: $P(\bigcap_k A_k) \geq 1 - \sum_k P(A_k)$ $\quad P(A_1 \cap A_2) \geq 1 - P(A_1) - P(A_2)$

- Binomial Theorem $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

  Multinomial Theorem $(a_1 + a_2 + \cdots + a_m)^n = \sum \frac{n!}{k_1! k_2! \cdots k_m!} \cdot a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}$

  $\frac{n!}{x_1! x_2! \cdots x_m!} = \binom{n}{x_1} \binom{n - x_1}{x_2} \cdots \binom{n - x_1 - x_2 - \cdots - x_{m-1}}{x_m}$

  $a \sum_{k=1}^{n} r^k = a\left(\frac{1 - r^n}{1 - r}\right)$ for $r < 1$ | $a \sum_{k=1}^{\infty} r^k = \frac{a}{1 - r}$ for $|r| < 1$

  $\sum n = \frac{n(n+1)}{2}$ | $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$ | $\sum n^3 = \frac{n^2 (n+1)^2}{4}$

- Gamma properties:

  $\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx$ | $\Gamma(a+1) = a \cdot \Gamma(a)$ | $n \in \mathbb{N} \Rightarrow \Gamma(n) = (n-1)!$ | $\Gamma(1/2) = \sqrt{\pi}$

- Absolutely continuous : if a density function f satisfy $P(a \le x \le b) = \int_a^b f(x) dx$

- $f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(x_1(y_1, y_2), x_2(y_1, y_2)) \cdot \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$

  $E(Y_i) = \frac{\partial M_Y(t)}{\partial (t_i)}$ $\quad E(Y_i^2) = \frac{\partial^2 M_Y(t)}{\partial t_i^2}$ $\quad E(Y_i, Y_j) = \frac{\partial^2 M_Y(t)}{\partial t_i \partial t_j}$

- Convergeance in probability: $X_n \xrightarrow{P} Y$ $\quad \lim_{n \to \infty} P(|X_n - Y| > \varepsilon) = 0$ for all $\varepsilon > 0$. (or $\lim_{n \to \infty} P(|X_n - Y| \le \varepsilon) = 1$)

  Convergeance in distribution: $X_n \xrightarrow{D} Y$ $\quad P(Y = x) = 0 \Rightarrow \lim_{n \to \infty} P(X_n \le x) = P(Y \le x)$ ($\lim_{n \to \infty} F_n(x) = F(y)$)

  Converges almost surely: $X_n \xrightarrow{a.s.} Y$ $\quad P(\lim_{n \to \infty} X_n = Y) = 1$ (or $P(|X_n - Y| \geq \varepsilon \text{ i.o.}) = 0$)

  $X_n \xrightarrow{a.s.} Y \Rightarrow X_n \xrightarrow{P} Y \Rightarrow X_n \xrightarrow{D} Y$

  If $\sum_1^{\infty} P(|X_n - X| > \varepsilon)$ is finite for all $\varepsilon > 0$, then $X_n \xrightarrow{a.s.} X$

- WLLN: $X_1, X_2 \cdots X_n \overset{iid}{\sim}$ Any distn, then $\lim_{n \to \infty} P(|M_n - \mu| > \varepsilon) = 0$ for all $\varepsilon > 0$ or $M_n \xrightarrow{P} \mu$

  SLLN: $X_1, X_2 \cdots X_n \overset{iid}{\sim}$ Any dist'n, then $P(\lim_{n \to \infty} M_n = \mu) = 1$ or $M_n \xrightarrow{a.s.} \mu$

  CLT: Average of any iid sequence is normally distributed.

  $Z_n = \sqrt{n} \cdot \frac{M_n - \mu}{\sigma} \Rightarrow Z_n \xrightarrow{D} N(0,1)$

  Slutsky's: $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} Y \Rightarrow X_n + Y_n \xrightarrow{P} X + Y$, $X_n Y_n \xrightarrow{P} XY$

  Cts. mapping Thm: $X_n \xrightarrow{P} X$, g absolutely cts. $\Rightarrow g(X_n) \xrightarrow{P} g(X)$

- Limit Theorems

  BCT: If $X_n \xrightarrow{a.s.} X$ and X_n is uniformly bounded ($\exists M < \infty$, $|X_n| < M$ for all n) $\Rightarrow \lim_{n \to \infty} E(X_n) = E(X)$

  MCT: If $X_n \xrightarrow{a.s.} X$ and $0 \le X_1 \le X_2 \cdots$ $\Rightarrow \lim_{n \to \infty} E(X_n) = E(X)$

  DCT: If $X_n \xrightarrow{a.s.} X$ and $E(|X|) < \infty$ and $|X_n| < \backslash$ for all n. $\Rightarrow \lim_{n \to \infty} E(X_n) = E(X)$

- Normal Distribution Theory.

  $X_i \overset{iid}{\sim} N(\mu_i, \sigma_i^2)$ $\quad Y = \sum a_i X_i + b.$ $\quad Y \sim N(\sum a_i \mu_i + b, \sum a_i^2 \sigma_i^2)$
  $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ $\quad \bar{X} = \sum x_i / n$ $\quad \bar{X} \sim N(\mu, \sigma^2 / n)$

  Compound dist'n: $S = X_1 + X_2 + \cdots + S_n$. $\quad E(S) = E(N) E(X_1)$
  $m_S(t) = r_N(m_{X_1}(t))$

  Mixture dist'n: $G(x) = p_1 F_1(x) + p_2 F_2(x) \cdots p_n F_n(x)$ is a CDF.
  $* \sum p_i = 1$

- Simple Random Walk:

  starting fund: \$a
  each round : +\$1 with chance p or
  -\$1 with chance 1-p
  fund after n rounds : X_n

  $X_n = a + Z_1 + Z_2 \cdots + Z_n$

  $P(X_n = a + k) = \begin{cases} 0 & \text{if } n+k \text{ is odd} \\ \binom{n}{\frac{n+k}{2}} \cdot p^{(n+k)/2} \cdot q^{(n-k)/2} \end{cases}$

  $E(X_n) = a + n \cdot (2p - 1)$

- Tail Events

  $\inf_{k \geq n} A_k = \bigcap_n^{\infty} A_k$ $\quad \lim_{n \to \infty} (\inf A_k) = \bigcup_1^{\infty} (\bigcap_n^{\infty} A_k) = \{A_n \text{ almost always}\}$
  After the nth trial, it's always heads

  $\sup_{k \geq n} A_k = \bigcup_n^{\infty} A_k$ $\quad \lim_{n \to \infty} (\sup A_k) = \bigcap_1^{\infty} (\bigcup_n^{\infty} A_k) = \{A_n \text{ infinitely often}\}$
  For any n, there's a m>n s.t. mth trial is head.

  $P(A_n \text{ i.o.}) = 1 - P(A_n^c \text{ a.a.})$

  Borel-Cantelli : if $\sum_1^{\infty} P(A_n)$ is finite, then $P(A_n. \text{ i.o.}) = 0$
  if $\sum_1^{\infty} P(A_n)$ is infinite, then $P(A_n. \text{ i.o.}) = 1$