A 44-vertex triangulation of real projective 6-space

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September 6, 2026

Abstract

We give an explicit triangulation of the six-dimensional real projective space on 44 vertices. It is the antipodal quotient of the boundary of a centrally symmetric simplicial 7-polytope with 88 integer vertices. The construction improves the 45-vertex bound of Guyer, Steinerberger, and Yang and answers their Question 3.2 affirmatively. The quotient has f-vector $(44, 938, 7024, 22555, 34936, 25914, 7404)$. A finite certificate consists of the coordinates and 7404 representatives of antipodal facet pairs. Exact supporting-hyperplane tests certify each facet, ridge-incidence tests certify completeness of the facet list, and a graph test certifies the antipodal quotient. Two verification programs using only the Python standard library are supplied. No vertex-minimality claim is made.

Keywords. Real projective space; simplicial triangulation; centrally symmetric polytope; exact computation.
Mathematics Subject Classification (2020). 57Q15, 52B11, 52B55.

1 Introduction and main result

Let $\nu(M)$ denote the smallest number of vertices in a finite simplicial complex whose geometric realization is homeomorphic to a manifold M. Arnoux and Marin [AM91] proved that

$$\nu(\mathbb{R}P^d) \geq \binom{d+2}{2} + 1 \qquad (d \geq 3); \tag{1.1}$$

see also [VZ21, Theorem 1.1]. For $d = 6$ this gives 29 vertices. Venturello and Zheng [VZ21] constructed a triangulation of $\mathbb{R}P^6$ with 53 vertices. Guyer, Steinerberger, and Yang [GSY26] subsequently obtained one with 45 vertices and asked whether fewer than 45 suffice. We give an explicit affirmative answer.

Theorem 1.1. *Let $a_1, \ldots, a_{44} \in \mathbb{Z}^7$ be the vectors in Appendix A, and set*

$$P = \operatorname{conv}\{a_1, \ldots, a_{44}, -a_1, \ldots, -a_{44}\}.$$

Then P is a centrally symmetric simplicial 7-polytope with exactly 88 vertices. The antipodal quotient $K = \partial P/(a_i \sim -a_i)$ is a simplicial triangulation of $\mathbb{R}P^6$, with

$$f(K) = (44,\ 938,\ 7024,\ 22555,\ 34936,\ 25914,\ 7404). \tag{1.2}$$

Here $f_j(K)$ counts the j-dimensional faces; the empty face is not included in the f-vector. Combining Theorem 1.1 with (1.1) yields

$$29 \le \nu(\mathbb{R}P^6) \le 44. \tag{1.3}$$

The upper inequality improves the construction in [GSY26]; it does not assert that 44 is optimal.

The proof is computer-assisted, but its input is entirely discrete. Neither an approximate convex hull nor the output of a numerical optimizer is accepted as a certificate. We specify exact finite tests and explain why their success proves the theorem. The coordinate search and these certificate checks have different roles: only the latter enter the proof.

2 Antipodal quotients of polytopal spheres

For a simplicial complex Δ, write $N_\Delta(v)$ for the set of vertices joined to v by an edge. The closed star $\mathrm{st}_\Delta(v)$ is the subcomplex generated by the simplices containing v. We use the following version of the disjoint-star criterion in [GSY26, Section 2.1], stated specifically for the geometric antipodal action.

Lemma 2.1. *Let $P \subset \mathbb{R}^{d+1}$ be a full-dimensional centrally symmetric simplicial polytope, and put $\Delta = \partial P$. Suppose that, for every vertex v,*

$$-v \notin N_\Delta(v), \qquad N_\Delta(v) \cap N_\Delta(-v) = \varnothing. \tag{2.1}$$

Let $\pi(v) = \{v, -v\}$ and let K consist of the sets $\pi(\sigma)$ for $\sigma \in \Delta$. Then K triangulates $\mathbb{R}P^d$. Every nonempty face of K has exactly two lifts in Δ, exchanged by the antipodal map. Moreover, $\mathrm{lk}_K(\pi(v))$ is isomorphic to $\mathrm{lk}_\Delta(v)$.

Proof. No simplex contains both v and $-v$, so π is injective on each simplex. Thus the vertex map defines a simplicial map $q : |\Delta| \to |K|$. We first prove uniqueness of a lifted simplex after fixing the lift of one vertex.

Suppose $\sigma, \sigma' \in \Delta$ have the same nonempty image. After replacing σ' by $-\sigma'$ if necessary, choose $v \in \sigma \cap \sigma'$. For any other vertex $w \in \sigma$, the representative of $\pi(w)$ in σ' is either w or $-w$. In the latter case v is adjacent to both w and $-w$, contrary to (2.1) applied to w. Hence $\sigma' = \sigma$. It follows that the two lifts of each nonempty face are precisely σ and $-\sigma$.

If $q(x) = q(y)$, the minimal faces containing x and y have the same image, and the positive barycentric coordinates agree. The preceding argument gives $y = x$ or $y = -x$. Consequently q induces a continuous bijection

$$|\Delta|/(x \sim -x) \longrightarrow |K|.$$

It is a homeomorphism, since its source is compact and its target is Hausdorff. Central symmetry and full dimension imply $0 \in \mathrm{int}\, P$. The radial map $x \mapsto x/\|x\|$ is an antipodally equivariant homeomorphism $\partial P \to S^d$. Therefore

$$|K| \cong \partial P/(x \sim -x) \cong S^d/(x \sim -x) = \mathbb{R}P^d.$$

Finally, a face in the link of $\pi(v)$ has a unique lift that extends to a face containing v. This establishes the asserted link isomorphism. $\square$

The vertex sets of the closed stars in (2.1) are $\{v\} \cup N_\Delta(v)$ and $\{-v\} \cup N_\Delta(-v)$, respectively. Thus the graph test is exactly a disjointness test for these closed stars, interpreted geometrically. In particular, the vertex links in K are polytopal spheres, so the quotient is a combinatorial manifold. The argument uses the actual antipodal action on a polytopal sphere; it does not identify the quotient of an arbitrary free involution on an abstract sphere with projective space.

3 A finite certificate for the convex hull

For a $(d+1)$-polytope, a facet has dimension d and a ridge has dimension $d-1$. We use the following elementary completeness test.

Proposition 3.1. *Let $V \subset \mathbb{R}^{d+1}$ be a finite set of distinct points, spanning $\mathbb{R}^{d+1}$ linearly, with $V = -V$. Let $\mathcal{F}$ be a nonempty collection of distinct $(d+1)$-element subsets of V. Assume:*

(i) *for every $F \in \mathcal{F}$, the matrix with the points of F as rows is nonsingular, and there are $c_F \in \mathbb{R}^{d+1}$ and $b_F > 0$ such that*

$$\langle c_F, v\rangle = b_F \ (v \in F), \qquad \langle c_F, w\rangle < b_F \ (w \in V \setminus F); \tag{3.1}$$

(ii) *every point of V occurs in some member of $\mathcal{F}$;*

(iii) *each d-element subset of a member of $\mathcal{F}$ belongs to exactly two members of $\mathcal{F}$.*

Then $P = \operatorname{conv} V$ is a simplicial $(d+1)$-polytope, V is exactly its vertex set, and $\mathcal{F}$ is its full facet list.

Proof. Symmetry and spanning imply $0 \in \operatorname{int} P$. In particular P has dimension $d+1$. For $F \in \mathcal{F}$, nonsingularity implies that its points are affinely independent. The strict support tests show that

$$P \cap \{x : \langle c_F, x\rangle = b_F\} = \operatorname{conv} F.$$

This is a d-simplex, hence a genuine facet of P. Its vertices are vertices of P. Assumption (ii) now proves that every point in V is a vertex.

Every d-element subset of F spans a facet of the simplex $\operatorname{conv} F$. It is therefore a genuine $(d-1)$-face of P, that is, a ridge. A ridge of a convex polytope belongs to exactly two facets. Assumption (iii) implies that whenever a facet in $\mathcal{F}$ is crossed through a ridge, the other incident facet is also in $\mathcal{F}$.

The facet-adjacency graph of a convex polytope is connected; equivalently, the graph of its polar is connected. These standard polytope facts are discussed in [Zie95]. A nonempty set of facets closed under all ridge-adjacency steps must therefore contain every facet. Thus no unlisted facet remains, and all facets of P are simplices. □

Checking only that listed facets have supporting hyperplanes would not establish completeness. The ridge-incidence step is essential: it prevents a missing facet, including a possible nonsimplicial facet, from being silently omitted.

For integer data, the support test requires no rounding. If B_F is the matrix of a proposed facet, solve

$$B_F u_F = \mathbf{1} \tag{3.2}$$

over $\mathbb{Q}$, clear denominators, and obtain $c_F \in \mathbb{Z}^{d+1}$ and $b_F \in \mathbb{Z}_{>0}$ with $B_F c_F = b_F \mathbf{1}$. The verifier then checks all equalities and inequalities in (3.1) directly with integers. The antipodal facet has certificate $(-c_F, b_F)$.

4 The 88-vertex polytope and its quotient

4.1 Coordinates and indexing

The 44 rows of Appendix A define $a_1, \ldots, a_{44}$ without any normalization. All coordinates are integers; they are not rounded placeholders for unspecified real numbers. The certificate files use zero-based labels

$$v_i = a_{i+1} \quad (0 \le i < 44), \qquad v_{i+44} = -a_{i+1} \quad (0 \le i < 44), \tag{4.1}$$

with involution $\tau(i) = (i + 44) \bmod 88$.

The points are distinct. Their span is seven-dimensional; for example, the determinant of the matrix with rows $a_1, \ldots, a_7$ is

$$-121850351295546734739 \neq 0.$$

The file `facets88_half.txt` contains 7404 sorted seven-element subsets of $\{0, \ldots, 87\}$, one from each antipodal facet pair. The full proposed facet set is obtained by adjoining their images under τ. No antipodal orbit is repeated.

4.2 Exact verification

For each of the 7404 representatives, the matrix in (3.2) is nonsingular. The exact support tests succeed at all 88 input points, with equality precisely at the seven listed vertices. Every input point occurs in a certified facet. After adding antipodes, every ridge occurs in exactly two facets. The graph of the listed facets is also checked to be connected, although the latter is redundant for Proposition 3.1.

For an additional numerical invariant that is still exact, define the scale-independent support slack

$$\eta = \min_{F \in \mathcal{F}} \min_{w \in V \setminus F} \frac{b_F - \langle c_F, w \rangle}{b_F}.$$

The computed value is

$$\eta = \frac{1233505615204901}{57134216104482583462} > 0. \tag{4.2}$$

Positivity is certified by integer comparisons, not by a tolerance. The full facet list therefore describes a simplicial polytope, by Proposition 3.1.

All edges are obtained by taking two-element subsets of its facets. For all 44 antipodal vertex pairs, the tests in (2.1) succeed. Lemma 2.1 identifies the quotient with $\mathbb{R}P^6$. The principal counts are summarized below.

Quantity	∂P	K
Vertices	88	44
Edges	1876	938
Ridges (dimension 5)	51828	25914
Facets (dimension 6)	14808	7404

Enumerating all nonempty subsets of the quotient facets yields (1.2). In particular,

$$f(\partial P) = (88, 1876, 14048, 45110, 69872, 51828, 14808),$$

and

$$\chi(K) = 44 - 938 + 7024 - 22555 + 34936 - 25914 + 7404 = 1.$$

The Euler characteristic is a consistency check, not a substitute for the quotient argument. This completes the computer-assisted proof of Theorem 1.1.

4.3 The graph of the quotient

The graph of K is missing only eight of the $\binom{44}{2} = 946$ possible edges. In the one-based labeling of the coordinate rows, these missing pairs are

$$\begin{gathered} \{1,6\},\ \{6,29\},\ \{7,27\},\ \{7,41\}, \\ \{8,20\},\ \{9,33\},\ \{9,44\},\ \{28,35\}. \end{gathered} \tag{4.3}$$

There are three vertices of degree 41, ten of degree 42, and 31 of degree 43. These counts also follow directly from (4.3). The graph information is descriptive; no assertion that K is 2-neighborly is intended.

5 Optimization viewpoint

The search began with the 45 antipodal pairs in the explicit construction of [GSY26, Section 2.3]. One pair was removed and the remaining coordinates were varied while preserving central symmetry. The supplied data reproduce the final certificate, rather than a complete stochastic search trajectory.

A useful sufficient condition for (2.1) is

$$\langle v, w\rangle > 0 \qquad \text{for every edge } \{v, w\}. \tag{5.1}$$

Indeed, a common neighbor of v and $-v$ would need to have both a positive and a negative inner product with v. However, the final construction does not satisfy (5.1). Exactly eight of its 1876 edges have negative inner product. For instance, $\{a_{10}, -a_{34}\}$ is an edge and

$$\langle a_{10}, -a_{34}\rangle = -127509.$$

Thus insisting on (5.1) would exclude this particular realization, despite its valid antipodal quotient.

For a full-dimensional centrally symmetric vertex configuration and a non-antipodal pair v_i, v_j, an edge can be monitored through the linear program

$$\begin{aligned} \rho_{ij} = \max_{c,t} \quad & t, \\ \text{subject to} \quad & \langle c, v_i\rangle = \langle c, v_j\rangle = 1, \\ & \langle c, v_k\rangle + t \le 1 \quad (k \notin \{i, j\}). \end{aligned} \tag{5.2}$$

Here $\rho_{ij} > 0$ if and only if $\{v_i, v_j\}$ is an edge. In one direction, positive slack exposes exactly those two vertices. In the other, an exposing functional for an edge has positive support value because 0 is interior; scaling that value to 1 gives a positive minimum slack over the other vertices.

This formulation targets the existence of conflicting edges directly, without imposing positivity of their inner products. In a numerical search the edge set changes when the combinatorial type changes, so it must be recomputed rather than kept fixed throughout the optimization. For certification, the final coordinates were rounded to thousandths and multiplied by 1000. All support and graph tests were then repeated on the resulting integers. The theorem concerns only these integer coordinates, and does not depend on convergence claims or floating-point decisions made during exploration.

6 Reproducibility and scope

The ancillary directory contains the following principal files.

File	Contents
`coordinates44.txt`	44 rows of seven integer coordinates.
`facets88_half.txt`	7404 representatives of facet orbits, with labels 0–87 as in (4.1).
`facets_RP6.txt`	All 7404 quotient facets, with labels 1–44.
`verify.py`	Exact integer support tests and combinatorial checks.
`verify_rational.py`	A second implementation using rational Gaussian elimination.
`diagnostics.py`	The determinant, missing edges and inner-product statistics.
`SHA256SUMS`	Checksums for the data and verification programs.

From the ancillary directory, run

```
python verify.py
python verify_rational.py
python diagnostics.py
```

Python 3.9 or newer suffices. No third-party package or network access is required. The first verifier uses fraction-free Bareiss elimination [Bar68], with every integer division checked for zero remainder. The second uses ordinary elimination over `fractions.Fraction` and does not import the first verifier. Both recompute support hyperplanes, check ridge incidences, recover the graph, test antipodal neighborhoods, and enumerate quotient faces. Both also compare the separately supplied quotient facet file with the quotient they compute.

The mathematical decisions use arbitrary-precision integers or exact rational numbers. Floating-point time measurements in the reports are not used in any test. The reports are convenient summaries; the proof depends on rerunning the checks, not on trusting a stored `true` value. The certificate does not require an implementation of convex-hull enumeration, and the only topological identification used is Lemma 2.1.

For reference, the SHA-256 digests of the two main inputs are

```
coordinates44.txt
48c2601bdb435bfc00b5945863e74e428251234e99c2ee45f228685e7be46a4a
facets88_half.txt
264570209fd1e9ed3a6deb4e31376d8eceeda5575586c2694f7fa0a46a85cad1
```

No lower bound stronger than (1.1) is proved here. In particular, the existence of a 43-vertex triangulation is not decided, and the antipodal-polytopal search class need not contain every triangulation of $\mathbb{R}P^6$.

Computational assistance

OpenAI's ChatGPT was used in the computational exploration, in the development of the verification programs, and in preparing this manuscript. The two implementations were developed within the same AI-assisted workflow; their agreement is not external independent validation. The coordinates, finite checks and mathematical arguments are provided to enable such validation without using an AI system.

References

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A Integer coordinates

Each row below is $a_i \in \mathbb{Z}^7$. The vertex set of P consists of these 44 vectors and their negatives. No rescaling or normalization is required. The machine-readable file is `coordinates44.txt`.

i	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$a_{i,4}$	$a_{i,5}$	$a_{i,6}$	$a_{i,7}$
1	766	−8	2	−24	2	−363	530
2	10	−295	765	99	−26	138	−545
3	−11	478	186	−595	344	243	454
4	335	−499	6	6	−589	70	536
5	−716	−259	1	420	−159	7	467
6	373	−53	32	−65	383	−464	−700
7	−88	272	679	−213	−508	−61	388
8	−3	−560	199	−535	549	−201	−137
9	−2	−114	−239	6	418	504	−708
10	420	632	20	99	−92	48	635
11	−17	295	564	−590	369	−331	42
12	−289	−5	−581	485	16	−245	−532
13	−685	−41	518	−509	−3	−17	−31
14	42	−159	−235	−27	−591	569	−493
15	140	50	397	519	480	−278	−493
16	206	818	−312	−131	379	−133	−117
17	506	−5	267	11	−768	289	−8
18	72	−429	−144	−7	368	272	762
19	−615	662	322	111	−73	13	−248
20	−275	−112	−583	−9	523	545	8
21	−215	−640	−735	−42	43	−7	9
22	360	1	−669	−10	−507	−396	−95
23	620	−482	−34	−59	4	61	−612
24	−583	−455	−16	−328	68	25	584
25	33	7	−695	−425	−42	−12	−578
26	−417	−9	−302	−5	−748	−418	−2
27	−772	4	−265	−10	7	571	80
28	−6	−206	−12	−419	808	287	−217
29	−12	−165	−12	−246	−78	−742	596
30	140	−44	−298	−206	18	−863	−319
31	512	377	−196	−536	−511	−9	100
32	−17	−331	119	−759	−62	540	−69
33	−25	701	434	−148	−252	470	−115
34	−314	94	306	755	−433	170	112
35	−329	606	−631	156	−121	295	−7
36	−149	−790	78	25	−5	589	19
37	1	−715	32	384	12	−583	−12
38	129	−512	70	−647	−348	−321	−271
39	−129	−454	−485	415	28	−107	598
40	−689	185	−526	105	−92	−441	3
41	453	−182	318	709	−95	313	226
42	−507	−5	424	−6	11	−725	−195
43	244	−563	−82	657	421	75	−50
44	−521	−19	−19	−174	−763	183	−286