Online Appendix: The Effects of Monetary Policy: Theory with Measured Expectations

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In this Appendix, we introduce notation (Section 1.1), formally state the decision problem of a household (Section 1.2) and derive equations (9) and (12) in the paper (Section 1.3).

1 Appendix A: Proofs

In this Appendix, we introduce notation (Section 1.1), formally state the decision problem of a household (Section 1.2) and derive equations (9) and (12) in the paper (Section 1.3).

1.1 Notation

Let s_t denote the vector of exogenous shocks that nature draws in period t. Let x_{-1} denote the vector of initial conditions that nature drew before period zero. Let $s^t = \{x_{-1}, s_0, s_1, \ldots, s_t\}$ denote the exogenous history of the economy up to and including period t. Let z_i^t denote the information set of household i in period t. We assume that s^t is drawn from a finite set S^t and z_i^t is drawn from a finite set Z_i^t. Let $p_{i,\tau}\left(s^t, z_i^t\right)$ denote household i's period-τ subjective probability of the realization $\left(s^t, z_i^t\right)$. Finally, let $X_{i,\tau}\left(s^t\right)$ denote household i's subjective belief in period τ about the value of the endogenous variable X_t at history s^t.

This setup is extremely general. It imposes no restrictions on the exogenous histories, s^t, the information sets, z_i^t, the subjective probabilities, $p_{i,\tau}\left(s^t, z_i^t\right)$, and the subjective models, $X_{i,\tau}\left(s^t\right)$, apart from the finiteness of the set of possible realizations of $\left(s^t, z_i^t\right)$.

The standard procedure in DSGE models is to impose additional restrictions on these objects by making assumptions. The assumption of rational expectations imposes two restrictions: (i) the subjective probability $p_{i,\tau}\left(s^t, z_i^t\right)$ equals the *objective* probability of the realization $\left(s^t, z_i^t\right)$ given information z_i^τ, and (ii) the subjective model $X_{i,\tau}\left(s^t\right)$ equals the equilibrium outcome of variable X_t at history s^t. The assumption of full information, rational expectations imposes the restriction $z_i^t = s^t$ and restrictions (i)-(ii). These three restrictions may or may not be satisfied in reality. The literatures on overconfidence, diagnostic expectations, and adaptive learning impose restrictions on the subjective probabilities, $p_{i,\tau}\left(s^t, z_i^t\right)$, and the subjective models, $X_{i,\tau}\left(s^t\right)$, that deviate from restriction (i) or (ii). The literatures on exogenous incomplete information and rational inattention impose restrictions on z_i^t that deviate from the restriction $z_i^t = s^t$. In this paper, we follow a different strategy. In the following subsections 1.2-1.3, we impose no restrictions on the exogenous histories, s^t, the information sets, z_i^t, the subjective probabilities, $p_{i,\tau}\left(s^t, z_i^t\right)$, and the subjective models, $X_{i,\tau}\left(s^t\right)$, apart from the finiteness of the set of possible realizations of $\left(s^t, z_i^t\right)$.

1.2 Statement of the decision problem

Before we formally state the decision problem of a household, we introduce three concepts: planned consumption, highest feasible consumption, and actual consumption.

Let $C\left(z_i^t\right)$ and $N\left(z_i^t\right)$ denote household i's period-zero plan for consumption and hours worked in period t at information set z_i^t. The two brackets indicate that household's actions in period t have to be measurable with respect to the household's information in period t

The highest feasible consumption of household i in period t at history s^t and information set z_i^t, denoted $\bar{C}\left(s^t, z_i^t\right)$, is given by the flow budget constraint and the borrowing limit:

$$\bar{C}\left(s^t, z_i^t\right) = \frac{1}{P\left(s^t\right)}\left[W\left(s^t\right)N\left(z_i^t\right) + D\left(s^t\right) + R\left(s^{t-1}\right)B\left(s^{t-1}, z_i^{t-1}\right) - T\left(s^t\right) + L\left(s^t\right)\right]. \tag{1}$$

Here $W\left(s^t\right)$, $D\left(s^t\right)$, $T\left(s^t\right)$, and $L\left(s^t\right)$ denote the household's individual nominal wage rate, dividend income, tax payment, and borrowing limit at history s^t, $R\left(s^{t-1}\right)$ denotes the gross nominal interest rate on bond holdings between periods $t-1$ and t at history s^{t-1}, and $P\left(s^t\right)$ denotes the price level at history s^t. Furthermore, $N\left(z_i^t\right)$ are household i's hours worked in period t at information set z_i^t and $B\left(s^{t-1}, z_i^{t-1}\right)$ denotes household i's nominal bond holdings between periods $t-1$ and t at history s^{t-1} and information set z_i^{t-1}. The highest feasible consumption of household i in period t depends on the history s^t and the information set z_i^t, because all variables that the household takes as given are a function of the history s^t, the household's hours worked are a function of the information set z_i^t, and the household's nominal bond holdings between periods $t-1$ and t are a function of s^{t-1} and z_i^{t-1}.

The actual consumption of household i in period t at history s^t and information set z_i^t, denoted $C\left(s^t, z_i^t\right)$, is given by

$$C\left(s^t, z_i^t\right) = min\left\{C\left(z_i^t\right); \bar{C}\left(s^t, z_i^t\right)\right\}. \tag{2}$$

If planned consumption exceeds the highest feasible consumption, the credit card payment does not go through and actual consumption equals the highest feasible consumption. By contrast, if planned consumption does not exceed the highest feasible consumption, the credit card payment goes through and actual consumption equals planned consumption.

The expected utility of household i in period zero can now be written as:

$$E_{i,0}\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{C_{i,t}^{1-\gamma}}{1-\gamma} - v_i\left(N_{i,t}\right)\right)\right] = \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\sum_{z_i^t\in Z_i^t} p_{i,0}\left(s^t, z_i^t\right)\beta^t\left(\frac{C_{i,0}\left(s^t, z_i^t\right)^{1-\gamma}}{1-\gamma} - v_i\left(N\left(z_i^t\right)\right)\right), \tag{3}$$

with

$$C_{i,0}\left(s^t, z_i^t\right) = min\left\{C\left(z_i^t\right); \bar{C}_{i,0}\left(s^t, z_i^t\right)\right\}. \tag{4}$$

Here $p_{i,0}\left(s^t, z_i^t\right)$ denotes household i's period-zero subjective probability of the realization $\left(s^t, z_i^t\right)$, $C_{i,0}\left(s^t, z_i^t\right)$ denotes household i's period-zero belief about its actual consumption at history s^t and information set z_i^t, $C\left(z_i^t\right)$ denotes planned consumption of household i in period t at information set z_i^t, and $\bar{C}_{i,0}\left(s^t, z_i^t\right)$ denotes household i's period-zero belief about its highest feasible consumption at history s^t and information set z_i^t. Equation (4) states that the household is aware of the fact that, if it attempts to spend more than it can, its actual consumption will equal its highest feasible consumption. Finally, $N\left(z_i^t\right)$ denotes planned hours worked of household i in period t at information set z_i^t.

We endow households with knowledge of the *structure* of the flow budget constraint. Each household understands that bond holdings between periods t and $t+1$ equal the difference between total after-tax income (including gross interest income) and consumption expenditure in period t. Formally, household i's period-zero subjective belief about the value of its nominal bond holdings between periods t and $t+1$ at history s^t and information set z_i^t, denoted $B_{i,0}\left(s^t, z_i^t\right)$, is given by:

$$B_{i,0}\left(s^t, z_i^t\right) = W_{i,0}\left(s^t\right) N\left(z_i^t\right) + D_{i,0}\left(s^t\right) + R_{i,0}\left(s^{t-1}\right) B_{i,0}\left(s^{t-1}, z_i^{t-1}\right) - T_{i,0}\left(s^t\right) - P_{i,0}\left(s^t\right) C_{i,0}\left(s^t, z_i^t\right). \tag{5}$$

Here $W_{i,0}\left(s^t\right)$, $D_{i,0}\left(s^t\right)$, and $T_{i,0}\left(s^t\right)$ denote household i's period-zero belief about its nominal wage rate, its dividend income, and its tax payment at history s^t; $R_{i,0}\left(s^{t-1}\right)$ denotes household i's period-zero belief about the gross nominal interest rate at history s^{t-1} ; and $P_{i,0}\left(s^t\right)$ denotes household i's period-zero belief about the price level at history s^t. Furthermore, $N\left(z_i^t\right)$ denotes household i's period-zero plan for hours worked at information set z_i^t and $C_{i,0}\left(s^t, z_i^t\right)$ denotes household i's period-zero belief about its actual consumption at history s^t and information set z_i^t.

It is important to emphasize that equation (5) is extremely general. To see this, consider the consequences of imposing additional assumptions. For example, imposing the assumption that households have correct beliefs about the outcomes at history s^t and information set z_i^t, equation (5) reduces to

$$B\left(s^t, z_i^t\right) = W\left(s^t\right) N\left(z_i^t\right) + D\left(s^t\right) + R\left(s^{t-1}\right) B\left(s^{t-1}, z_i^{t-1}\right) - T\left(s^t\right) - P\left(s^t\right) C\left(s^t, z_i^t\right). \tag{6}$$

Furthermore, imposing the additional assumption that households have complete information (i.e.,

$z_i^t = s^t$), equation (5) reduces to

$$B\left(s^t\right) = W\left(s^t\right) N\left(s^t\right) + D\left(s^t\right) + R\left(s^{t-1}\right) B\left(s^{t-1}\right) - T\left(s^t\right) - P\left(s^t\right) C\left(s^t\right). \tag{7}$$

The last equation is the usual formulation of the flow budget constraint with history notation. Going from equation (7) to equation (6) one allows for the possibility that households have incomplete information. Going from equation (6) to equation (5) one, in addition, allows for the possibility that households have non-rational expectations.

We also endow households with the knowledge that there *exists* a borrowing limit in period t. Let $L_{i,0}\left(s^t\right)$ denote household i's period-zero subjective belief about its borrowing limit in period t at history s^t. We impose no restrictions on $L_{i,0}\left(s^t\right)$. This formulation nests the possibility that the household has correct beliefs about the borrowing limit at history s^t ($L_{i,0}\left(s^t\right) = L\left(s^t\right)$) as well as the possibility that the household has incorrect beliefs about the borrowing limit at history s^t ($L_{i,0}\left(s^t\right) \neq L\left(s^t\right)$). Finally, we assume that households are capable of combining their beliefs about the borrowing limit in period t with their knowledge of the structure of the flow budget constraint to arrive at beliefs about the highest feasible consumption in period t:

$$\bar{C}_{i,0}\left(s^t, z_i^t\right) = \frac{1}{P_{i,0}\left(s^t\right)}\left[W_{i,0}\left(s^t\right) N\left(z_i^t\right) + D_{i,0}\left(s^t\right) + R_{i,0}\left(s^{t-1}\right) B_{i,0}\left(s^{t-1}, z_i^{t-1}\right) - T_{i,0}\left(s^t\right) + L_{i,0}\left(s^t\right)\right]. \tag{8}$$

Here $\bar{C}_{i,0}\left(s^t, z_i^t\right)$ denotes household i's period-zero belief about its highest feasible consumption in period t at history s^t and information set z_i^t, which is the object that appears on the right-hand side of equation (4). Equation (8) follows from setting $B_{i,0}\left(s^t, z_i^t\right)$ in equation (5) equal to $-L_{i,0}\left(s^t\right)$ and solving for consumption.

Statement of the decision problem: In period zero, each household i makes a plan for consumption and hours worked $\left\{C\left(z_i^t\right), N\left(z_i^t\right)\right\}_{t=0}^{\infty}$, with $C\left(z_i^t\right) \in \mathbb{R}_{++}$ and $N\left(z_i^t\right) \in \mathbb{R}_{+}$, so as to maximize the expected utility (3) subject to equations (4), (5) and (8).

Equation (3) is the equation for the expected utility of household i in period zero. Equation (4) states that the household believes that its credit card payment does not go through if attempted consumption exceeds the highest feasible consumption. Equation (5) characterizes the household's period-zero belief about the evolution of bond holdings over time. Equation (8) characterizes the household's period-zero belief about the highest feasible consumption in period t.

1.3 Consumption functions of different types of households

Definition: We say household i is "ex-ante non-HTM in period zero," if the solution to the household's decision problem, $\left\{C^*\left(z_i^t\right), N^*\left(z_i^t\right)\right\}_{t=0}^{\infty}$, has the property that $C^*\left(z_i^0\right) < \bar{C}_{i,0}\left(s^0, z_i^0\right)$ for all $s^0 \in S^0$ with $p_{i,0}\left(s^0, z_i^0\right) > 0$.

In other words, a household is called ex-ante non-HTM in period zero, if the household believes that a marginal increase in current consumption would go through with probability one.

Proposition 1: Consider any household who is ex-ante non-HTM in period zero and believes that it will be HTM in period one with probability one.

- The plan of the household has to satisfy the usual Euler equation for period-zero consumption:

$$C\left(z_i^0\right)^{-\gamma} = E_{i,0}\left[\beta \frac{R_0}{\Pi_1} C_{i,1}^{-\gamma}\right],$$

which using the more precise notation reads

$$C\left(z_i^0\right)^{-\gamma} = \sum_{s^1 \in S^1} \sum_{z_i^1 \in Z_i^1} p_{i,0}\left(s^1, z_i^1\right) \beta \frac{R_{i,0}\left(s^0\right)}{P_{i,0}\left(s^1\right)/P_{i,0}\left(s^0\right)} C_{i,0}\left(s^1, z_i^1\right)^{-\gamma}. \tag{9}$$

- The planned consumption of the household in period zero, $C\left(z_i^0\right)$, is given by this Euler equation, the household's belief that it will be HTM in period one with probability one (i.e., $C_{i,0}\left(s^1, z_i^1\right) = \bar{C}_{i,0}\left(s^1, z_i^1\right)$ for all $\left(s^1, z_i^1\right)$ with $p_{i,0}\left(s^1, z_i^1\right) > 0$), and the household's beliefs about the highest feasible consumption in period one, given by equations (8) and (5). Formally, $C\left(z_i^0\right)$ is given by

$$1 = \sum_{s^1 \in S^1} \sum_{z_i^1 \in Z_i^1} p_{i,0}\left(s^1, z_i^1\right) \beta \frac{R_{i,0}\left(s^0\right) P_{i,0}\left(s^0\right)}{P_{i,0}\left(s^1\right)} \left(\frac{\tilde{X}_{i,0}\left(s^1, z_i^1\right)}{C\left(z_i^0\right)} - \tilde{R}_{i,0}\left(s^1\right)\right)^{-\gamma}, \tag{10}$$

where

$$\tilde{X}_{i,0}\left(s^1, z_i^1\right) = \tilde{Y}_{i,0}\left(s^1, z_i^1\right) + \tilde{L}_{i,0}\left(s^1\right) + \tilde{R}_{i,0}\left(s^1\right)\left[\tilde{Y}_{i,0}\left(s^0, z_i^0\right) + \tilde{R}_{i,0}\left(s^0\right) \tilde{B}_{i,0}\left(s^{-1}, z_i^{-1}\right)\right].$$

- Log-linearizing the consumption function (10) at the point, where all variables are constant over time and $R/\Pi = 1/\beta$, yields the consumption function

$$c\left(z_i^0\right) = \begin{array}{c} \frac{1}{\frac{C_i}{\tilde{Y}_i}} \frac{1}{1+\beta} E_{i,0}\left[\sum_{t=0}^{1} \beta^t \left(\frac{1}{\beta} \frac{\tilde{B}_i}{\tilde{Y}_i}\left(r_{t-1} - \pi_t\right) + \tilde{y}_{i,t}\right)\right] \\ -\frac{1}{\gamma} \frac{\beta}{1+\beta} E_{i,0}\left[r_0 - \pi_1\right] \\ + \frac{1}{\frac{C_i}{\tilde{Y}_i}} \frac{1}{1+\beta} E_{i,0}\left[\frac{1}{\beta} \frac{\tilde{B}_i}{\tilde{Y}_i} \tilde{b}_{i,-1} + \beta \frac{\tilde{L}_i}{\tilde{Y}_i} \tilde{l}_{i,1}\right] \end{array}. \tag{11}$$

Proof: Consider any household who is ex-ante non-HTM in period zero and believes that it will be HTM in period one with probability one; under the optimal plan. We show that the optimal plan of this household has to satisfy the Euler equation (9) and the consumption function (10).

Consider the following deviation from an original plan: marginally *decrease* $C\left(z_i^0\right)$ without changing $N\left(z_i^0\right)$, $N\left(z_i^1\right)$ and $\left\{C\left(z_i^t\right), N\left(z_i^t\right)\right\}_{t=2}^{\infty}$. For a household who is ex-ante non-HTM in period zero and believes that it will be HTM in period one with probability one, the effect of this deviation from the original plan on expected utility is

$$-C\left(z_i^0\right)^{-\gamma}+\sum_{s^1\in S^1}\sum_{z_i^1\in Z_i^1} p_{i,0}\left(s^1,z_i^1\right)\beta\frac{R_{i,0}\left(s^0\right)P_{i,0}\left(s^0\right)}{P_{i,0}\left(s^1\right)}\bar{C}_{i,0}\left(s^1,z_i^1\right)^{-\gamma}. \tag{12}$$

For the deviation to be non-profitable, the original plan has to satisfy

$$-C\left(z_i^0\right)^{-\gamma}+\sum_{s^1\in S^1}\sum_{z_i^1\in Z_i^1} p_{i,0}\left(s^1,z_i^1\right)\beta\frac{R_{i,0}\left(s^0\right)P_{i,0}\left(s^0\right)}{P_{i,0}\left(s^1\right)}\bar{C}_{i,0}\left(s^1,z_i^1\right)^{-\gamma}\leq 0. \tag{13}$$

Consider another deviation from the original plan: marginally *increase* $C\left(z_i^0\right)$ without changing $N\left(z_i^0\right)$, $N\left(z_i^1\right)$ and $\left\{C\left(z_i^t\right), N\left(z_i^t\right)\right\}_{t=2}^{\infty}$. For a household who is ex-ante non-HTM in period zero and believes that it will be HTM in period one with probability one, the effect of this deviation from the original plan on expected utility is

$$C\left(z_i^0\right)^{-\gamma}-\sum_{s^1\in S^1}\sum_{z_i^1\in Z_i^1} p_{i,0}\left(s^1,z_i^1\right)\beta\frac{R_{i,0}\left(s^0\right)P_{i,0}\left(s^0\right)}{P_{i,0}\left(s^1\right)}\bar{C}_{i,0}\left(s^1,z_i^1\right)^{-\gamma}. \tag{14}$$

For the deviation to be non-profitable, the original plan has to satisfy

$$C\left(z_i^0\right)^{-\gamma}-\sum_{s^1\in S^1}\sum_{z_i^1\in Z_i^1} p_{i,0}\left(s^1,z_i^1\right)\beta\frac{R_{i,0}\left(s^0\right)P_{i,0}\left(s^0\right)}{P_{i,0}\left(s^1\right)}\bar{C}_{i,0}\left(s^1,z_i^1\right)^{-\gamma}\leq 0. \tag{15}$$

Hence, the requirement that there should not be a profitable deviation implies that an optimal plan has to satisfy

$$C\left(z_i^0\right)^{-\gamma}-\sum_{s^1\in S^1}\sum_{z_i^1\in Z_i^1} p_{i,0}\left(s^1,z_i^1\right)\beta\frac{R_{i,0}\left(s^0\right)P_{i,0}\left(s^0\right)}{P_{i,0}\left(s^1\right)}\bar{C}_{i,0}\left(s^1,z_i^1\right)^{-\gamma}= 0. \tag{16}$$

Finally, for a household who believes that it will be HTM in period one with probability one, we have, for all $\left(s^1,z_i^1\right)$ with $p_{i,0}\left(s^1,z_i^1\right)>0$,

$$C_{i,0}\left(s^1,z_i^1\right)=\bar{C}_{i,0}\left(s^1,z_i^1\right). \tag{17}$$

Substituting the last equation into the necessary condition (16) yields the necessary condition (9).

Next, household i's period-zero belief about its highest feasible consumption in period one at history s^1 and information set z_i^1 is given by equation (8) with $t = 1$. Furthermore, household i's period-zero belief about its bond holdings between periods zero and one at history s^0 and information set z_i^0 is given by equation (5) with $t = 0$. In addition, the fact that the household is ex-ante non-HTM in period zero implies that: $C_{i,0}\left(s^0, z_i^0\right) = C\left(z_i^0\right)$. Combining these three equations yields that household i's period-zero belief about its highest feasible consumption in period one at history s^1 and information set z_i^1 equals

$$\bar{C}_{i,0}\left(s^1, z_i^1\right) = \tilde{Y}_{i,0}\left(s^1, z_i^1\right) + \tilde{L}_{i,0}\left(s^1\right) + \tilde{R}_{i,0}\left(s^1\right)\left[\tilde{Y}_{i,0}\left(s^0, z_i^0\right) + \tilde{R}_{i,0}\left(s^0\right)\tilde{B}_{i,0}\left(s^{-1}, z_i^{-1}\right) - C\left(z_i^0\right)\right] \tag{18}$$

where $\tilde{Y}_{i,0}\left(s^1, z_i^1\right)$, $\tilde{Y}_{i,0}\left(s^0, z_i^0\right)$, $\tilde{L}_{i,0}\left(s^1\right)$, $\tilde{R}_{i,0}\left(s^1\right)$, $\tilde{R}_{i,0}\left(s^0\right)$, and $\tilde{B}_{i,0}\left(s^{-1}, z_i^{-1}\right)$ denote household i's period-zero belief about: the real non-interest income in period one, the real non-interest income in period zero, the real borrowing limit in period one, the real interest rate between periods zero and one, the real interest rate between periods minus one and zero, and the real bond holdings between periods minus one and zero. Formally,

$$\begin{aligned}
\tilde{Y}_{i,0}\left(s^1, z_i^1\right) &= \tfrac{W_{i,0}\left(s^1\right)N\left(z_i^1\right)+D_{i,0}\left(s^1\right)-T_{i,0}\left(s^1\right)}{P_{i,0}\left(s^1\right)} \\
\tilde{Y}_{i,0}\left(s^0, z_i^0\right) &= \tfrac{W_{i,0}\left(s^0\right)N\left(z_i^0\right)+D_{i,0}\left(s^0\right)-T_{i,0}\left(s^0\right)}{P_{i,0}\left(s^0\right)} \\
\tilde{L}_{i,0}\left(s^1\right) &= \tfrac{L_{i,0}\left(s^1\right)}{P_{i,0}\left(s^1\right)} \\
\tilde{R}_{i,0}\left(s^1\right) &= \tfrac{R_{i,0}\left(s^0\right)P_{i,0}\left(s^0\right)}{P_{i,0}\left(s^1\right)} \\
\tilde{R}_{i,0}\left(s^0\right) &= \tfrac{R_{i,0}\left(s^{-1}\right)P_{i,0}\left(s^{-1}\right)}{P_{i,0}\left(s^0\right)} \\
\tilde{B}_{i,0}\left(s^{-1}, z_i^{-1}\right) &= \tfrac{B_{i,0}\left(s^{-1}, z_i^{-1}\right)}{P_{i,0}\left(s^{-1}\right)}.
\end{aligned} \tag{19}$$

Substituting equation (18) into the Euler equation (16) and dividing by $C\left(z_i^0\right)^{-\gamma}$ on both sides of the equation yields equation (10), characterizing the planned consumption of household i in period zero, $C\left(z_i^0\right)$.

Finally, log-linearizing equation (10) at a point, where all variables are constant over time and $\tilde{R} = 1/\beta$, yields

$$c\left(z_i^0\right) = \frac{1}{1+\beta}E_{i,0}\left[\frac{\tilde{Y}}{C}\left(\tilde{y}_{i,0} + \beta\tilde{y}_{i,1}\right) + \frac{\tilde{B}}{C}\frac{1}{\beta}\left(\tilde{b}_{i,-1} + \tilde{r}_0 + \beta\tilde{r}_1\right) + \frac{\tilde{L}}{C}\beta\tilde{l}_{i,1}\right] - \frac{1}{\gamma}\frac{1}{1+\beta}E_{i,0}\left[\beta\tilde{r}_1\right]. \tag{20}$$

Here C, $\tilde{Y}$, $\tilde{B}$, and $\tilde{L}$ denote consumption, real non-interest income, real bond holdings, and real borrowing limit at the point around which we log-linearize, small roman letters denote log-deviations from this point

$$
\begin{aligned}
\tilde{y}_{i,0} &= ln\left(\tilde{Y}_{i,0}/\tilde{Y}\right)\\
\tilde{y}_{i,1} &= ln\left(\tilde{Y}_{i,1}/\tilde{Y}\right)\\
\tilde{b}_{i,-1} &= ln\left(\tilde{B}_{i,-1}/\tilde{B}\right)\\
\tilde{r}_{0} &= ln\left(\tilde{R}_{0}/\tilde{R}\right)\\
\tilde{r}_{1} &= ln\left(\tilde{R}_{1}/\tilde{R}\right)\\
\tilde{l}_{i,1} &= ln\left(\tilde{L}_{i,1}/\tilde{L}\right)\\
c\left(z_i^0\right) &= ln\left(C\left(z_i^0\right)/C\right),
\end{aligned}
\tag{21}
$$

and $\tilde{Y}_{i,0}$ denotes real non-interest income of household i in period zero, $\tilde{Y}_{i,1}$ denotes real non-interest income of household i in period one, $\tilde{B}_{i,-1}$ denotes real bond holdings of household i between periods minus one and zero, $\tilde{R}_0$ denotes the real interest rate between periods minus one and zero, $\tilde{R}_1$ denotes the real interest rate between periods zero and one, $\tilde{L}_{i,1}$ denotes the real borrowing limit of household i in period one, and $C\left(z_i^0\right)$ is the planned consumption of household i in period zero. Equation (20) states that the planned consumption of a household, who is ex-ante non-HTM in period zero and believes that it will be HTM in period one with probability one, depends on β, γ and two expectations.

Proposition 2: Consider any household who is ex-ante non-HTM for all periods $t = 0, 1, 2, \ldots$ (i.e., consider any household whose optimal plan has the property: $C^*\left(z_i^t\right) < \bar{C}_{i,0}\left(s^t, z_i^t\right)$ for all $s^t \in S^t$ and $z_i^t \in Z_i^t$ with $p_{i,0}\left(s^t, z_i^t\right) > 0$ and for all $t = 0, 1, 2, \ldots$) and who believes that it cannot run a Ponzi scheme.

- The planned consumption of the household in period zero, $C\left(z_i^0\right)$, is given by equations (28) and (29).
- Log-linearizing the consumption function at the point, where all variables are constant over time and $R/\Pi = 1/\beta$, yields the consumption function

$$
c\left(z_i^0\right) = \begin{array}{c}
(1-\beta)\,\frac{\tilde{Y}}{C}E_{i,0}\left[\sum_{t=0}^{\infty}\beta^t\tilde{y}_{i,t}\right]\\
-\left(\frac{1}{\gamma} - \frac{\left(\frac{1}{\beta}-1\right)\tilde{B}}{C}\right)E_{i,0}\left[\sum_{t=1}^{\infty}\beta^t\tilde{r}_t\right]\\
+(1-\beta)\,\frac{\frac{1}{\beta}\tilde{B}}{C}E_{i,0}\left[\tilde{r}_0 + \tilde{b}_{i,-1}\right]
\end{array}.
\tag{22}
$$

Proof: First, household i's period-zero belief about its nominal bond holdings between periods t and $t+1$ is given by equation (5). Dividing this equation by $P_{i,0}\left(s^t\right)$ yields household i's period-zero belief about its real bond holdings between periods t and $t+1$:

$$\tilde{B}_{i,0}\left(s^t, z_i^t\right) = \tilde{Y}_{i,0}\left(s^t, z_i^t\right) + \tilde{R}_{i,0}\left(s^t\right)\tilde{B}_{i,0}\left(s^{t-1}, z_i^{t-1}\right) - C_{i,0}\left(s^t, z_i^t\right), \tag{23}$$

where $\tilde{B}_{i,0}\left(s^t, z_i^t\right)$ denotes household i's period-zero belief about its real bond holdings between periods t and $t+1$, $\tilde{Y}_{i,0}\left(s^t, z_i^t\right)$ denotes household i's period-zero belief about its real non-interest income in period t, and $\tilde{R}_{i,0}\left(s^t\right)$ denotes household i's period-zero belief about the real interest rate between periods $t-1$ and t:

$$\begin{aligned} \tilde{B}_{i,0}\left(s^t, z_i^t\right) &= \frac{B_{i,0}\left(s^t, z_i^t\right)}{P_{i,0}\left(s^t\right)} \\ \tilde{Y}_{i,0}\left(s^t, z_i^t\right) &= \frac{W_{i,0}\left(s^t\right)N\left(z_i^t\right) + D_{i,0}\left(s^t\right) - T_{i,0}\left(s^t\right)}{P_{i,0}\left(s^t\right)} \\ \tilde{R}_{i,0}\left(s^t\right) &= \frac{R_{i,0}\left(s^{t-1}\right)P_{i,0}\left(s^{t-1}\right)}{P_{i,0}\left(s^t\right)}. \end{aligned} \tag{24}$$

Solving equation (23) for $\tilde{B}_{i,0}\left(s^{t-1}, z_i^{t-1}\right)$ yields

$$\tilde{B}_{i,0}\left(s^{t-1}, z_i^{t-1}\right) = \frac{1}{\tilde{R}_{i,0}\left(s^t\right)}\left[C_{i,0}\left(s^t, z_i^t\right) - \tilde{Y}_{i,0}\left(s^t, z_i^t\right)\right] + \frac{1}{\tilde{R}_{i,0}\left(s^t\right)}\tilde{B}_{i,0}\left(s^t, z_i^t\right). \tag{25}$$

Solving this equation forward from period zero onwards and using the fact that the household believes that it cannot run a Ponzi scheme along any path yields the present value budget constraint:

$$\tilde{B}_{i,0}\left(s^{-1}, z_i^{-1}\right) = \sum_{t=0}^{\infty}\frac{1}{\prod_{k=0}^{t}\tilde{R}_{i,0}\left(s^k\right)}\left[C_{i,0}\left(s^t, z_i^t\right) - \tilde{Y}_{i,0}\left(s^t, z_i^t\right)\right]. \tag{26}$$

Using the fact that the household is ex-ante non-HTM in all periods $t = 0, 1, 2, \ldots$, which implies $C_{i,0}\left(s^t, z_i^t\right) = C\left(z_i^t\right)$ in all periods $t = 0, 1, 2, \ldots$, and multiplying the last equation by $\tilde{R}_{i,0}\left(s^0\right)$ yields

$$\tilde{R}_{i,0}\left(s^0\right)\tilde{B}_{i,0}\left(s^{-1}, z_i^{-1}\right) = \sum_{t=0}^{\infty}\frac{1}{\prod_{k=1}^{t}\tilde{R}_{i,0}\left(s^k\right)}\left[C\left(z_i^t\right) - \tilde{Y}_{i,0}\left(s^t, z_i^t\right)\right]. \tag{27}$$

Since the last equation holds along any path, it also has to hold in expectations across paths, yielding

$$\sum_{s^0\in S^0} p_{i,0}\left(s^0\right)\tilde{R}_{i,0}\left(s^0\right)\tilde{B}_{i,0}\left(s^{-1}, z_i^{-1}\right) = \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\sum_{z_i^t\in Z_i^t} p_{i,0}\left(s^t, z_i^t\right)\frac{1}{\prod_{k=1}^{t}\tilde{R}_{i,0}\left(s^k\right)}\left[C\left(z_i^t\right) - \tilde{Y}_{i,0}\left(s^t, z_i^t\right)\right], \tag{28}$$

where $p_{i,0}\left(s^0\right)$ denotes household i's period-zero subjective probability of the realization s^0.

Second, an optimal plan has to satisfy the following Euler equation, for all $t = 1, 2, \ldots$:

$$C\left(z_i^0\right)^{-\gamma} = \sum_{s^t\in S^t}\sum_{z_i^t\in Z_i^t} p_{i,0}\left(s^t, z_i^t\right)\beta^t\left(\prod_{k=1}^{t}\tilde{R}_{i,0}\left(s^k\right)\right)C\left(z_i^t\right)^{-\gamma}. \tag{29}$$

Third, expressing equation (28) in terms of log-deviations from a point, where all variables are constant over time and $\tilde{R} = 1/\beta$, yields:

$$\begin{aligned} &\textstyle\sum_{s^0 \in S^0} p_{i,0}\left(s^0\right) \frac{1}{\beta} \tilde{B} e^{\tilde{r}_{i,0}\left(s^0\right)+\tilde{b}_{i,0}\left(s^{-1}, z_i^{-1}\right)} \\ &= \textstyle\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \sum_{z_i^t \in Z_i^t} p_{i,0}\left(s^t, z_i^t\right) \beta^t e^{-\sum_{k=1}^t \tilde{r}_{i,0}\left(s^k\right)} \left[C e^{c\left(z_i^t\right)} - \tilde{Y} e^{\tilde{y}_{i,0}\left(s^t, z_i^t\right)}\right]. \end{aligned} \tag{30}$$

A first-order Taylor approximation of the last equation at zero yields

$$\begin{aligned} &\textstyle\sum_{s^0 \in S^0} p_{i,0}\left(s^0\right) \frac{1}{\beta} \tilde{B} \left(\tilde{r}_{i,0}\left(s^0\right) + \tilde{b}_{i,0}\left(s^{-1}, z_i^{-1}\right)\right) \\ &= \textstyle\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \sum_{z_i^t \in Z_i^t} p_{i,0}\left(s^t, z_i^t\right) \beta^t \left[C\left(-\sum_{k=1}^t \tilde{r}_{i,0}\left(s^k\right) + c\left(z_i^t\right)\right) - \tilde{Y}\left(-\sum_{k=1}^t \tilde{r}_{i,0}\left(s^k\right) + \tilde{y}_{i,0}\left(s^t, z_i^t\right)\right)\right], \end{aligned} \tag{31}$$

which can also be written as

$$c\left(z_i^0\right) + E_{i,0}\left[\sum_{t=1}^{\infty} \beta^t \left(-\sum_{k=1}^t \tilde{r}_k + c_{i,t}\right)\right] = \frac{\frac{1}{\beta}\tilde{B}}{C} E_{i,0}\left[\tilde{r}_0 + \tilde{b}_{i,-1}\right] + \frac{\tilde{Y}}{C} E_{i,0}\left[\tilde{y}_{i,0} + \sum_{t=1}^{\infty} \beta^t \left(-\sum_{k=1}^t \tilde{r}_k + \tilde{y}_{i,t}\right)\right]. \tag{32}$$

Equation (32) states that the expected present value of consumption equals the expected beginning-of-period-zero real financial wealth plus the expected present value of real non-interest income. Next, expressing the Euler equation (29) in terms of log-deviations from a point, where all variables are constant over time and $\tilde{R} = 1/\beta$, yields:

$$e^{-\gamma c\left(z_i^0\right)} = \sum_{s^t \in S^t} \sum_{z_i^t \in Z_i^t} p_{i,0}\left(s^t, z_i^t\right) e^{\sum_{k=1}^t \tilde{r}_{i,0}\left(s^k\right) - \gamma c\left(z_i^t\right)}. \tag{33}$$

A first-order Taylor approximation of the last equation at zero yields

$$-\gamma c\left(z_i^0\right) = \sum_{s^t \in S^t} \sum_{z_i^t \in Z_i^t} p_{i,0}\left(s^t, z_i^t\right) \left[\sum_{k=1}^t \tilde{r}_{i,0}\left(s^k\right) - \gamma c\left(z_i^t\right)\right], \tag{34}$$

which can also be written as

$$-\gamma c\left(z_i^0\right) = E_{i,0}\left[\sum_{k=1}^t \tilde{r}_k - \gamma c_{i,t}\right]. \tag{35}$$

Solving the last equation for household i's period-zero expectation of own consumption in period t yields

$$E_{i,0}\left[c_{i,t}\right] = \frac{1}{\gamma} E_{i,0}\left[\sum_{k=1}^t \tilde{r}_k\right] + c\left(z_i^0\right). \tag{36}$$

Using equation (36) to substitute for $E_{i,0}\left[c_{i,t}\right]$ in equation (32) yields

$$\frac{1}{1-\beta} c\left(z_i^0\right) + \left(\frac{1}{\gamma} - 1\right) \sum_{t=1}^{\infty} \beta^t E_{i,0}\left[\sum_{k=1}^t \tilde{r}_k\right] = \frac{\frac{1}{\beta}\tilde{B}}{C} E_{i,0}\left[\tilde{r}_0 + \tilde{b}_{i,-1}\right] + \frac{\tilde{Y}}{C} E_{i,0}\left[\tilde{y}_{i,0} + \sum_{t=1}^{\infty} \beta^t \left(-\sum_{k=1}^t \tilde{r}_k + \tilde{y}_{i,t}\right)\right]. \tag{37}$$

Finally, using the fact that

$$\sum_{t=1}^{\infty} \beta^t E_{i,0} \left[\sum_{k=1}^{t} \tilde{r}_k \right] = \frac{1}{1-\beta} \sum_{t=1}^{\infty} \beta^t E_{i,0} \left[\tilde{r}_t \right] \tag{38}$$

yields

$$\frac{1}{1-\beta} c\left(z_i^0\right) + \left(\frac{1}{\gamma} + \frac{\tilde{Y} - C}{C} \right) \frac{1}{1-\beta} \sum_{t=1}^{\infty} \beta^t E_{i,0} \left[\tilde{r}_t \right] = \frac{\frac{1}{\beta} \tilde{B}}{C} E_{i,0} \left[\tilde{r}_0 + \tilde{b}_{i,-1} \right] + \frac{\tilde{Y}}{C} E_{i,0} \left[\sum_{t=0}^{\infty} \beta^t \tilde{y}_{i,t} \right]. \tag{39}$$

Using the fact that $C = \tilde{Y} + \left(\frac{1}{\beta} - 1 \right) \tilde{B}$ yields equation (22).