III IIR Digital filter

→ Basically a digital filter is a LTI discrete time system.

→ The terms FIR, IIR are used to distinguish filter types.

→ The FIR filter are of non-recursive type, where by the present o/p sampled depends on the present i/p sample & previous i/p samples.

→ IIR filters are of recursive type, where the present o/p sample depends on present i/p, past i/p samples and o/p samples.

The impulse response h(n) for a realizable filter is

$$h(n) = 0 \quad \text{for} \quad n \leq 0$$

and for stability, it must satisfy the condition

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

IIR digital filters have the Transfer f^n of the form

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

→ The design of an IIR filter for the given specifications is to find the filter coefficients a_k, b_k.

Analog lowpass filter Design

The most general form of analog filter transfer f^n is

$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{M} a_i s^i}{1 + \sum_{i=1}^{N} b_i s^i}$$

where $H(s)$ is the Laplace transform of the impulse response $h(t)$, i.e.,

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

and $N \geq M$ must be satisfied. For a stable analog filter, the poles of $H(s)$ lies in the left half of s-plane.

→ Now we study 2 types of analog filter design.

(i) Butterworth filter

(ii) Chebyshev filter

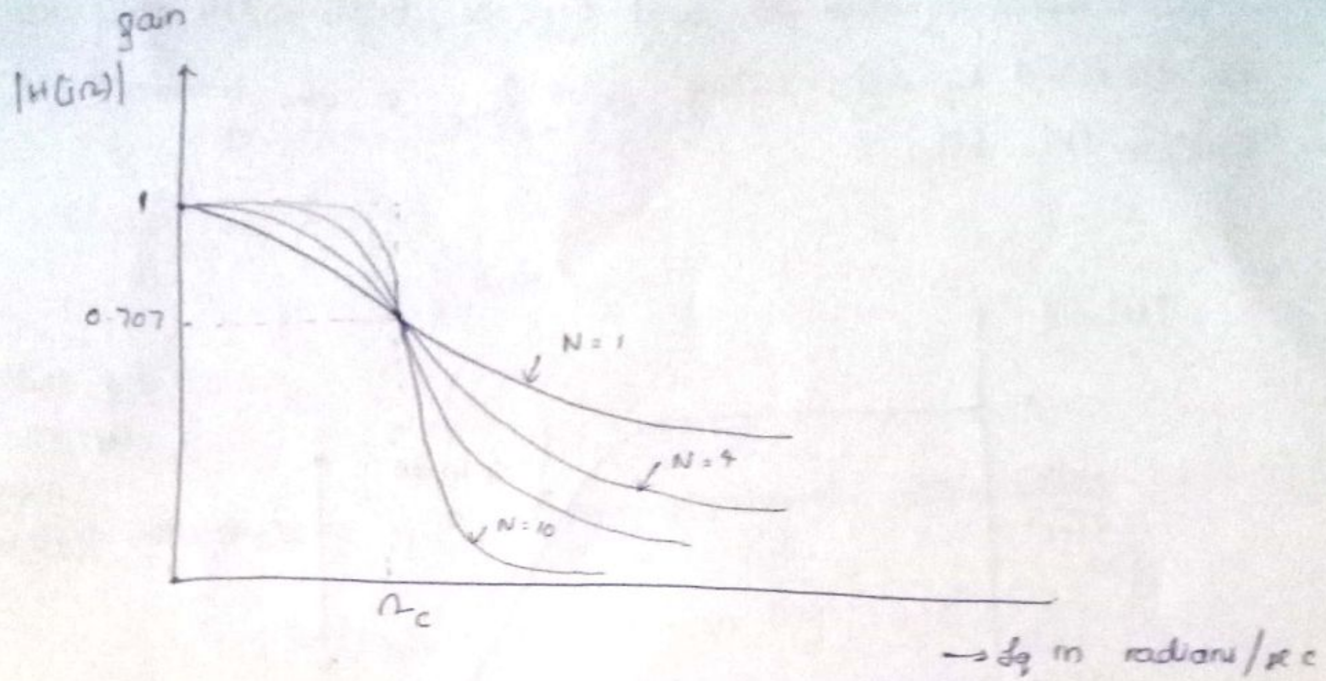
(i) Analog lowpass Butterworth filter

The magnitude f^n of Butterworth lowpass filter is given by

$$|H(j\Omega)| = \frac{1}{\left[1 + (\Omega/\Omega_c)^{2N}\right]^{1/2}} \qquad N = 1, 2, 3 \ldots$$

where $N \to$ order of the filter

$\Omega_c \to$ cut off fq.



Low pass Butterworth Magnitude response

→ The dashed line indicates the ideal response of LPF.

→ As the order N increases, the response approaches the ideal low pass characteristics.

List of Butterworth Polynomials

N	Denominator of H(s)
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.8477s+1)$

→ The above polynomials in the table are normalised poles

→ In general, the unnormalized poles are given by

$$s_k' = \Omega_c s_k$$

→ The transfer function of such type of Butterworth filter can be obtained by substituting $s \to s/\Omega_c$ in the transfer f^n of Butterworth filter.

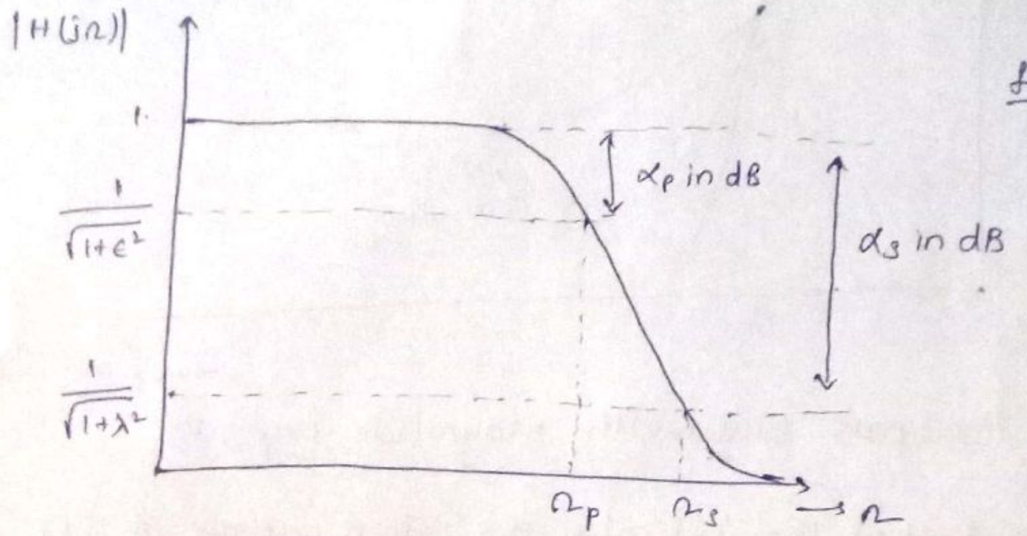


fig: Butterworth approxn of magnitude response

-3 dB attenuation → at Ω_c

α_p → max pass band attenuation in dB at → Ω_p

α_s → max stop band attenuation in dB at → Ω_s

Now the magnitude function can be written as

$$|H(j\Omega)| = \frac{1}{\left[1+\epsilon^2(\Omega/\Omega_p)^{2N}\right]^{1/2}}$$

$$\Rightarrow |H(j\Omega)|^2 = \frac{1}{1+\epsilon^2(\Omega/\Omega_p)^{2N}}$$

Taking log on both sides

$$20\log|H(j\Omega)| = 10\log 1 - 10\log\left[1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right] \quad - ①$$

$$\Rightarrow 20\log|H(j\Omega_p)| = -\alpha_p = -10\log(1+\epsilon^2)$$

$$\Rightarrow \quad \alpha_p = 10 \log(1+\epsilon^2)$$

$$0.1\alpha_p = \log(1+\epsilon^2)$$

Taking antilog

$$1+\epsilon^2 = 10^{0.1\alpha_p}$$

$$\epsilon = \left(10^{0.1\alpha_p} - 1\right)^{1/2}$$

Now at $\Omega = \Omega_s$, $\alpha_s \rightarrow$ min stopband attenuation, from (1)

$$\Rightarrow \quad 20 \log |H(j\Omega_s)| = 10\log 1 - 10 \log\left[1+\epsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

$$-\alpha_s = -10 \log\left[1+\epsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

$$0.1\alpha_s = \log\left[1+\epsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

After simplification, we get

$$\epsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = 10^{0.1\alpha_s} - 1$$

substituting the value of ϵ

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}$$

Since we are interested in N

$$\Rightarrow \quad N = \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log \dfrac{\Omega_s}{\Omega_p}}$$

$\because$ The above expression normally doesnot result in an integer value, we therefore, round off N to next higher integer.

ie, $$N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log \dfrac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log\left(\dfrac{\lambda}{\epsilon}\right)}{\log \dfrac{\Omega_s}{\Omega_p}}$$

where $\epsilon = \left(10^{0.1\alpha_p}-1\right)^{1/2}$

$\lambda = \left(10^{0.1\alpha_s}-1\right)^{1/2}$

for simplicity, we define A, k as

$$A = \frac{\lambda}{\epsilon} = \left(\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right)^{1/2}$$

$$k = \frac{\Omega_p}{\Omega_s}$$

where $k \rightarrow$ transition ratio.

The order eqn for lowpass Butterworth analog filter is

$$\boxed{N \geq \frac{\log A}{\log(1/k)}}$$

Also we can prove from the main transfer function $H(j\Omega)$

$$\boxed{\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{1/2N}} = \frac{\Omega_s}{\left(10^{0.1\alpha_s}-1\right)^{1/2N}}}$$

Steps to design an analog Butterworth lowpass filter

1. From the given specifications, find the order of the filter N
2. Round off it to the next higher integer.
3. Find the transfer f^n $H(s)$ for $\Omega_c = 1$ rad/sec for the value of N.
4. Calculate the value of cut off frequency Ω_c
5. Find the transfer f^n $H_a(s)$ for the above value of Ω_c by substituting $s \rightarrow \frac{s}{\Omega_c}$ in $H(s)$.

Ex: Design an analog Butterworth filter that has a -2dB passband attenuation at fq of 20 rad/sec and atleast -10dB stopband attenuation at 30 rad/sec

soln: Given $\alpha_p = 2$ dB $\quad \Omega_p = 20$ rad/sec

$\alpha_s = 10$dB $\quad \Omega_s = 30$ rad/sec

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \frac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log \sqrt{\frac{10 - 1}{10^{0.2} - 1}}}{\log \frac{30}{20}}$$

$$\geq 3.37$$

$$\Rightarrow \quad N = 4$$

$$\Rightarrow \quad H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.84775s + 1)}$$

also $\quad \Omega_c = \dfrac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \dfrac{20}{(10^{0.2}-1)^{1/8}} = 21.3868$

To get unnormalised transfer f^n

$$s \to \frac{s}{\Omega_c} = \frac{s}{21.3868} \quad \text{in } H(s)$$

ie.,
$$H(s) = \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 0.76537\left(\frac{s}{21.3868}\right) + 1}$$

$$\times \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 1.8477\left(\frac{s}{21.3868}\right) + 1}$$

$$H(s) = \frac{0.20921 \times 10^6}{(s^2 + 16.3686s + 457.394)(s^2 + 39.5176s + 457.394)}$$

HW : For the given specifications, design an analog Butterworth filter

$0.9 \le |H(j\Omega)| \le 1$ for $0 \le \Omega \le 0.2\pi$.

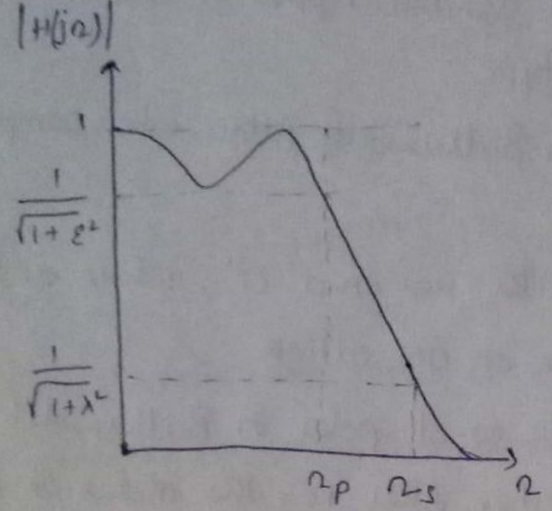
$|H(j\Omega)| \le 0.2$ for $0.4\pi \le \Omega \le \pi$

Analog lowpass Chebyshev filters :

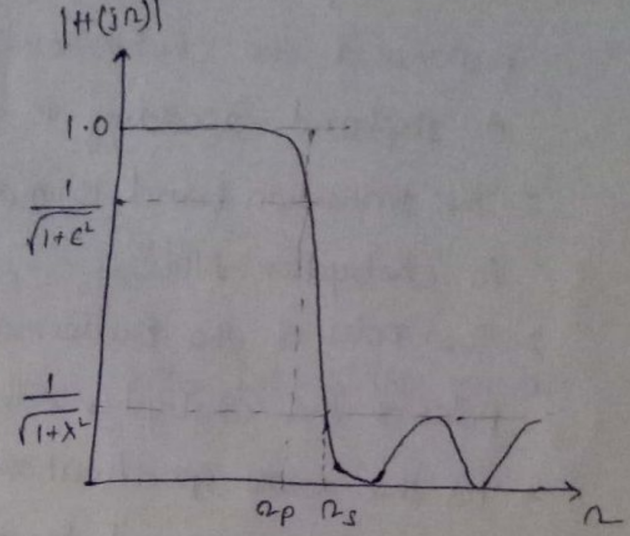
There are two types of Chebyshev filters.

(i) Type I : These filters are all pole filters that exhibit equiripple behaviour in the passband and a monotonic characteristics in the stop band.

(ii) Type II : This filter contains both poles and zeros and exhibits a monotonic behaviour in the passband and an equiripple behaviour in the stopband.



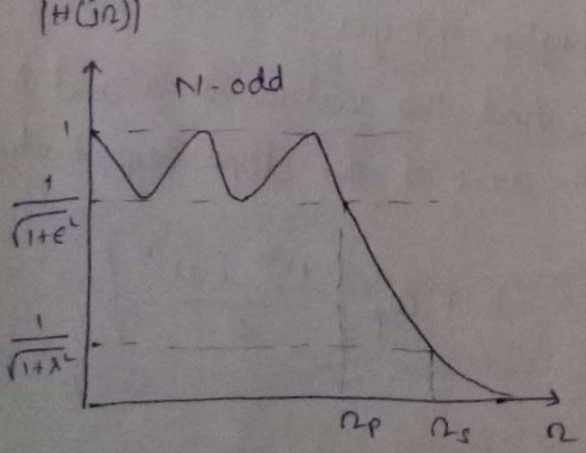
Type I

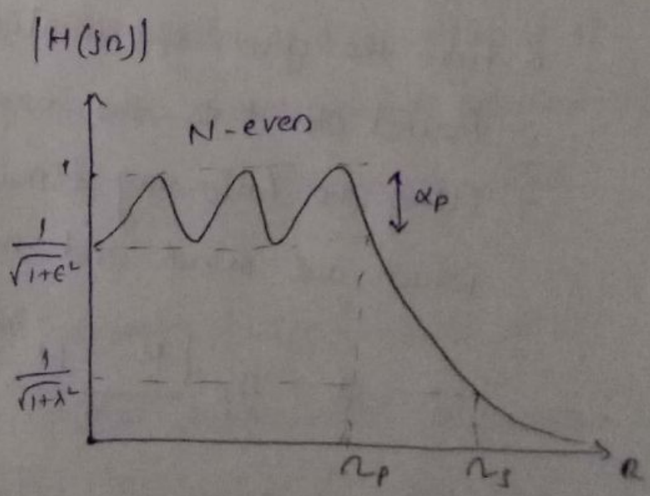


Type II

Characteristics of Chebyshev filters

equiripple characteristics of Chebyshev filter :





→ for odd values of N, the oscillatory curve starts from unity and for even values of N, the oscillatory curve starts from $\frac{1}{\sqrt{1+\epsilon^2}}$

Compairison b/w Butterworth & Chebyshev filter.

1. The magnitude response of Butterworth filter decreases monotonically as the fq Ω increases from 0 to ∞, where as the magnitude response of the chebyshev filter exhibits ripples in the passband or stopband according to the type.
2. The transition band is more in Butterworth filter when compared to chebyshev filter.
3. The poles of the Butterworth filter lie on a O^{le}, where as the poles of the chebyshev filter lie on an ellipse.
4. For the same specifications, the no. of poles in Butterworth are more when compared to chebyshev filter. ie, the order of the Chebyshev filter is less than that of Butterworth. This is a great advantage because, less no. of discrete components will be necessary to construct the filter.

Steps to design an analog Chebyshev lowpass filter.

1. From the given specifications find the order of the filter N.
2. Round off it to the next higher integer.
3. Using the following formulas, find the values of a and b, which are minor and major axis of the ellipse respectively.

$$a = \Omega_p \frac{\left[\mu^{1/N} - \mu^{-1/N}\right]}{2} \qquad b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

where

$$\mu = \epsilon^{-1} + \sqrt{\epsilon^{-2} + 1}$$

$$\epsilon = \sqrt{10^{0.1\alpha_p} - 1}$$

$\Omega_p \rightarrow$ pass band fq

$\alpha_p \rightarrow$ max attenuation in the passband.

N can be calculated from the formula

$$N = \frac{\cosh^{-1}\left(\frac{\lambda}{\epsilon}\right)}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1} A}{\cosh^{-1} 1/k}$$

$$A = \frac{\lambda}{\epsilon} \qquad K = \frac{\Omega_p}{\Omega_s}$$

$$\epsilon = \left(10^{0.1\alpha_p} - 1\right)^{1/2} \qquad \lambda = \left(10^{0.1\alpha_s} - 1\right)^{1/2}$$

4. Calculate the poles of Chebyshev filter which lie on an ellipse by using the formula.

$$S_k = a\cos\phi_k + jb\sin\phi_k \qquad k = 1, 2, \ldots N$$

where $\phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi \qquad k = 1, 2, \ldots N$

5. find the denominator polynomial of the transfer function using the above poles.

6. The numerator of the transfer f^n depends on the value of N.

(a) For N odd substitute $s = 0$ in the denominator polynomial and find the value. This value is equal to the numerator of the transfer f^n.

(∴ for N odd, the magnitude response $|H(j\Omega)|$ starts at 1).

(b) For N even, substitute $s = 0$ in the denominator polynomial and devide the result by $\sqrt{1+\epsilon^2}$. This value is equal to numerator.

Ex: Given the specifications $\alpha_p = 3db$, $\alpha_s = 16dB$, $f_p = 1KHz$, $f_s = 2KHz$. Determine the order of the filter using chebyshev approximation find H(s)

soln : Given

$$\Omega_p = 2\pi \times 1000\,Hz = 2000\pi \text{ rad/s}$$

$$\Omega_s = 2\pi \times 2000\,Hz = 4000\pi \text{ rad/s}$$

$$\alpha_p = 3\,dB \qquad \alpha_s = 16\,dB$$

step 1 :

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1}\sqrt{\frac{10^{1.6}-1}{10^{0.3}-1}}}{\cosh^{-1}\frac{4000\pi}{2000\pi}}$$

$$\geq 1.91$$

step 2 : Rounding N to next higher value we get $N = 2$

for N even, the oscillatory curve starts at $\frac{1}{\sqrt{1+\epsilon^2}}$

step 3 : The values of minor axis and major axis can be found as below.

$$\epsilon = (10^{0.1\alpha_p}-1)^{1/2} = (10^{0.3}-1)^{1/2} = 1$$

$$\mu = \epsilon^{-1} + \sqrt{1+\epsilon^{-2}} = 2.414$$

$$a = \Omega_p \frac{(\mu^{1/N} - \mu^{-1/N})}{2} = 2000\pi \frac{[(2.414)^{1/2} - (2.414)^{-1/2}]}{2} = 910\pi$$

$$b = \Omega_p \frac{(\mu^{1/N} + \mu^{-1/N})}{2} = 2000\pi \frac{[(2.414)^{1/2} + (2.414)^{-1/2}]}{2} = 2197\pi$$

step 4 : The poles are given by

$$s_k = a\cos\phi_k + jb\sin\phi_k \qquad k = 1, 2$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = 135^\circ$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = 225$$

$$S_1 = a\cos\phi_1 + jb\sin\phi_1 = -643.46\pi + j1554\pi$$

$$S_2 = a\cos\phi_2 + jb\sin\phi_2 = -643.46\pi - j1554\pi$$

Step 5: The denominator of $H(s) = (s+643.46\pi)^2 + (1554\pi)^2$

Step 6: The numerator of $H(s)$

$$= \frac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+\epsilon^2}} = (1414.38)^2\pi^2$$

The transfer f^n

$$H(s) = \frac{(1414.38)^2\pi^2}{s^2 + 1287\pi s + (1682)^2\pi^2}$$

HW:
Ex: Obtain an analog Chebyshev filter transfer f^n that satisfies the constraints $\frac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1$; $0 \le \Omega \le 2$

$|H(j\Omega)| < 0.1$; $\Omega \ge 4$

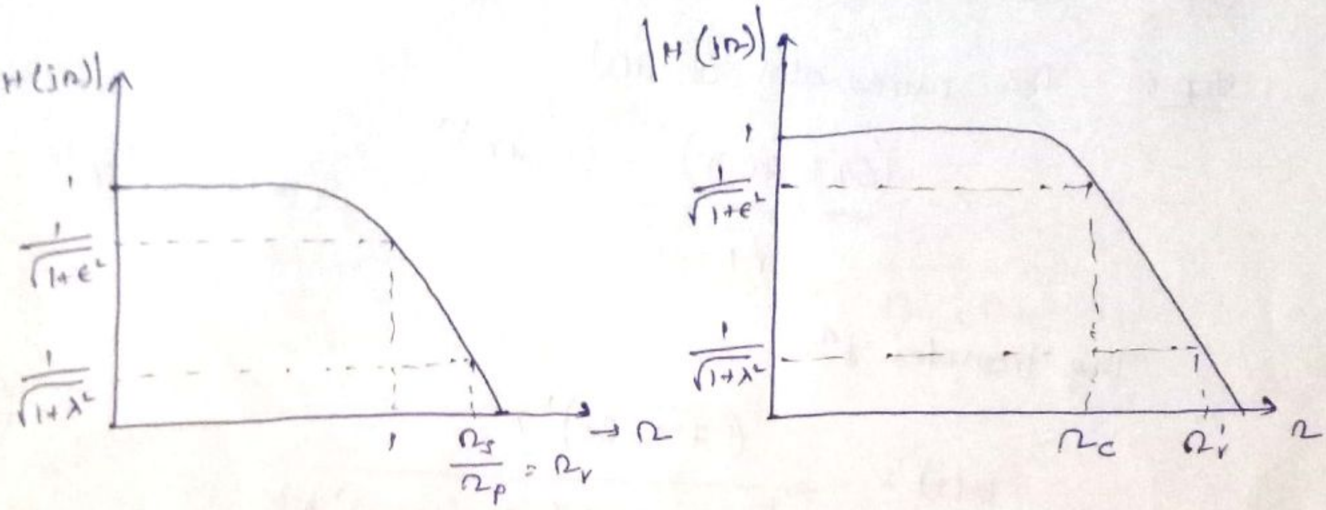
Frequency Transformation in Analog Domain:

So far we concentrated on designing a low pass filter for the given specifications. In this section we discuss the fq transformations that can be used to design LPF with diff pass band fqs, highpass filters, bandpass filters and bandstop filters from a normalized Lowpass analog filter ($\Omega_c = 1$rad/sec).

1) Lowpass to lowpass filter

Given a normalized LPF, it is desirable to have a LPF with different cut off fq Ω_c (or passband fq Ω_p). This can be accomplished by the transformation given below

$$s \rightarrow \frac{s}{\Omega_c}$$

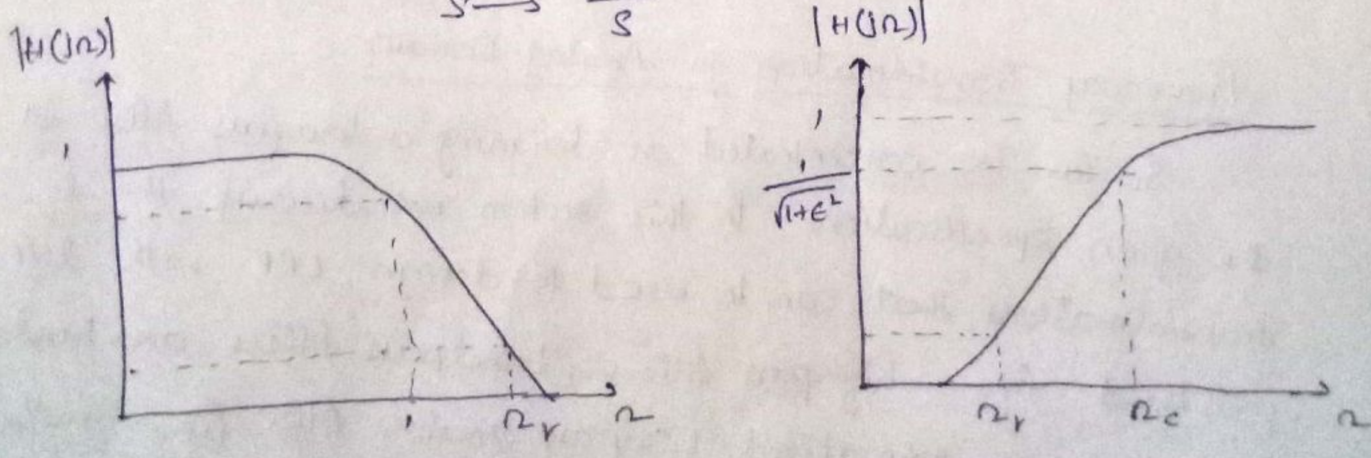


lowpass to Lowpass transformation

2) Lowpass to Highpass

Given a normalized LPF, it is desirable to have a HPF with cut off fq Ω_c, Then the transformation is

$$s \rightarrow \frac{\Omega_c}{s}$$



Lowpass to high pass transformation

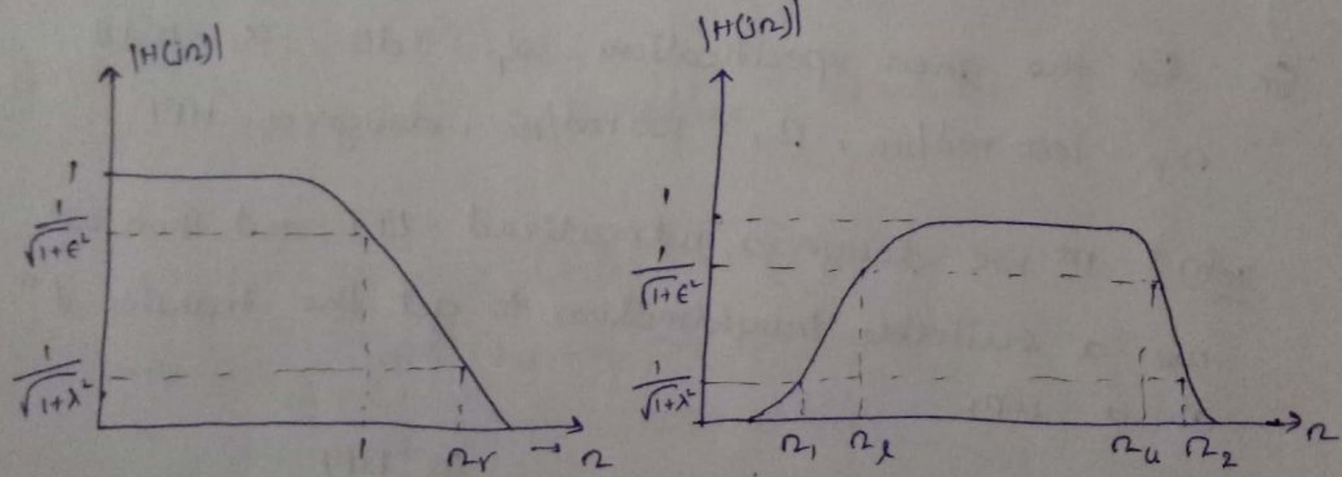
3) Lowpass to Bandpass.

The transformation for converting a normalized LPF to a bandpass filter with cut off fqs Ω_l, Ω_u can be accomplished by

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

$$\Omega_r = \min\{|A|, |B|\}.$$

$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)} \qquad B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$



Low pass to bandpass transformation

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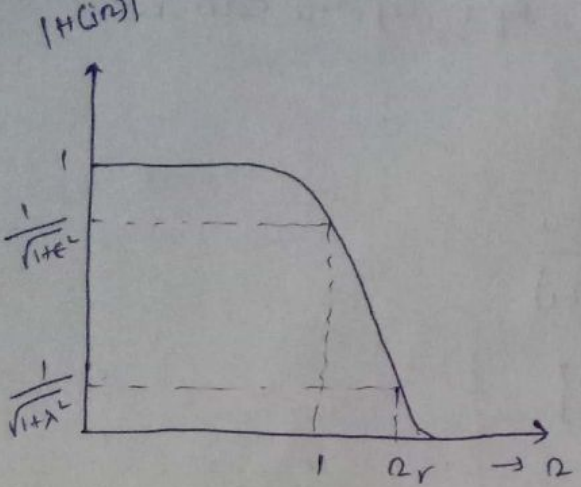
4) Lowpass to bandstop

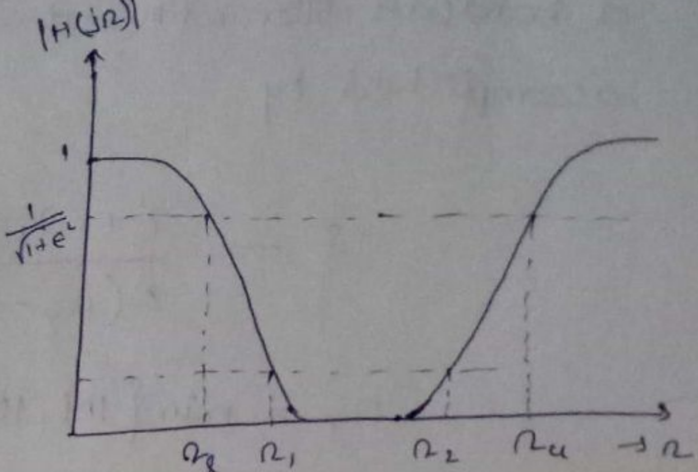
The transformation to convert a normalized LPF to BSF is

$$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$$

$$\Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l \Omega_u} \qquad B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l \Omega_u}$$





Lowpass to bandstop transformation

Ex: For the given specifications $\alpha_p = 3\,dB$, $\alpha_s = 15\,dB$, $\Omega_p = 1000$ rad/sec, $\Omega_s = 500$ rad/sec, design a HPF.

soln: 1st we design a normalized LPF and then we use a suitable transformation to get the transfer f^n of a HPF.

for a LPF

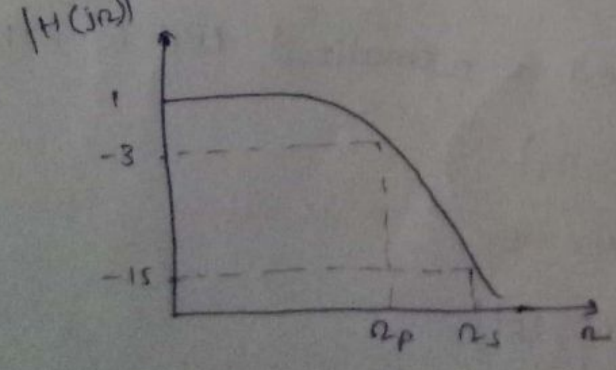
$\Omega_c = \Omega_p = 500$ rad/sec

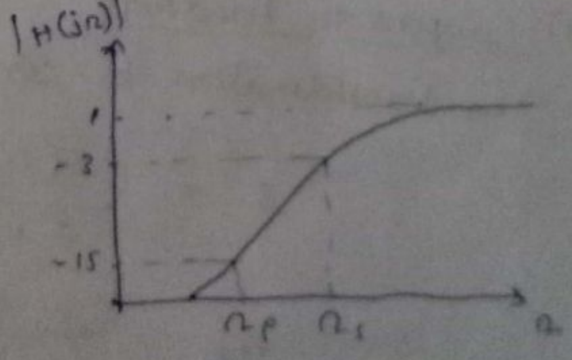
$\Omega_s = 1000$ rad/sec

for HPF

$\Omega_c = \Omega_p = 1000$ rad/sec

$\Omega_s = 500$ rad/sec





Lowpass to high pass transformation.

Lowpass filter specifications

$\Omega_c = \Omega_p = 500$ rad/sec $\quad \alpha_p = 3$ dB

$\Omega_s = 1000$ rad/sec $\quad \alpha_s = 15$ dB

We have

$$N = \frac{\log \lambda/\epsilon}{\log 1/k}$$

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = 5.533$$

$$\epsilon = \sqrt{10^{0.1\alpha_p} - 1} = 1$$

$$k = \frac{\Omega_p}{\Omega_s} = 0.5$$

$$\therefore \quad N = \frac{\log 5.533}{\log 2} = 2.468 \approx 3$$

$H(s)$ for $\Omega_c = 1$ rad/sec, $N = 3$ is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

to get a HPF having cut off fq

$\Omega_c = \Omega_p = 1000$ rad/sec

Substitute $\quad s \rightarrow \frac{1000}{s}$

$$H_a(s) = H(s) \Big|_{s \rightarrow \frac{1000}{s}}$$

$$= \frac{1}{(s+1)(s^2+s+1)} \Big|_{s \rightarrow \frac{1000}{s}}$$

$$= \frac{s^3}{(s+1000)\left[s^2 + 1000s + 1000^2\right]}$$

Design of IIR filters from analog filters

There are several methods that can be used to design digital filter having an infinite duration unit sample response. The techniques described are all based on converting an analog filter into a digital filter. If the conversion technique is to be effective, it should posses the following desirable properties.

1. The $j\Omega$-axis in the s-plane should map into the unit circle in the z-plane. Thus there will be a direct relationship between the two fq variables in the two domains.

2. The left-half plane of s-plane should map into the inside of the unit ckt in the z-plane. Thus a stable analog filter will be converted to a stable digital filter.

The 4 most widely used methods for digitizing the analog filter into a digital filter include

1. Approximation of derivatives
2. The impulse invariant transformation
3. Bilinear Transformation.
4. The matched z-Transformation technique.

1) Approximation of Derivatives

One of the simplest methods of digitizing an analog filter into a digital filter is to approximate the differential eqn by an equivalent difference eqn.

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT-T)}{T}$$

$$= \frac{y(n) - y(n-1)}{T}$$

where T represents the sampling interval & $y(n) = y(nT)$

L-T of $\frac{dy(t)}{dt} = sY(s)$ and is represented as

$$y(t) \longrightarrow \boxed{H(s) = s} \longrightarrow \frac{dy(t)}{dt}$$

The z-T of $\frac{y(n) - y(n-1)}{T}$ is $\frac{(1-z^{-1})Y(z)}{T}$, which can be represented as

$$y(t) \longrightarrow \boxed{\frac{1-z^{-1}}{T}} \longrightarrow \frac{y(n) - y(n-1)}{T}$$

On comparing analog and digital domains

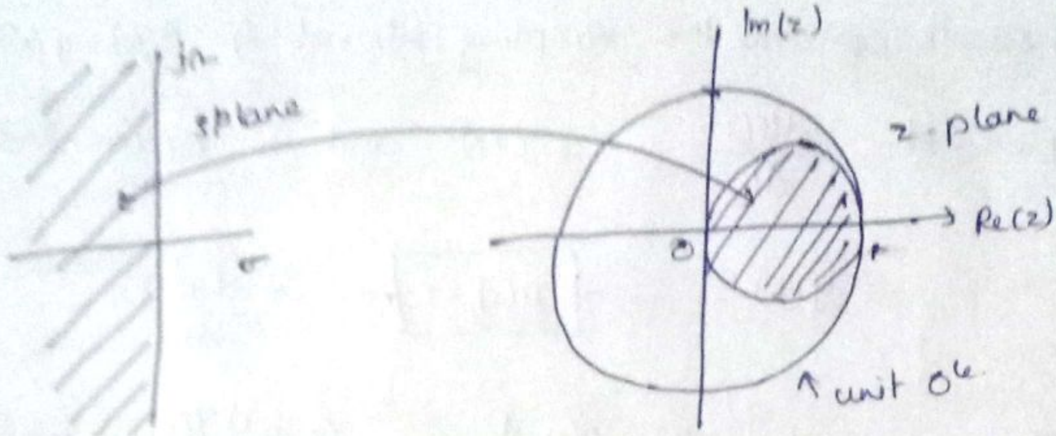
$$s = \frac{1-z^{-1}}{T}$$

Consequently the sys f^n for digital IIR filter obtained as a result of the approximation of the derivatives by finite difference is

$$\boxed{H(z) = H(s)\Big|_{s = \frac{1-z^{-1}}{T}}}$$

Note : → This transformation technique does map a stable analog filter into a stable digital filter, but the $j\Omega$-axis doesn't map on to the $z = e^{j\omega}$ O^{le}

→ Thus this mapping is restricted to the design of LPF and BFs having relatively small resonant fqs. It is not possible to transform analog HPF into corresponding digital HPF.



Mapping of the s-plane to z-plane using approximation of derivatives

2) Design of IIR filter using Impulse Invariance Technique:

In this method the IIR filter is designed such that the unit impulse response h(n) of digital filter is a sampled version of impulse response of analog filter.

if $H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - P_k}$ then, $H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$

for high sampling rates (for small T), the digital filter gain is high. Therefore, instead of above eqn, we can use

$$H(z) = \sum_{k=1}^{N} \frac{T C_k}{1 - e^{P_k T} z^{-1}}$$

Note: Due to the presence of aliasing, the impulse invariant method is appropriate for the design of LPF, BPF only. This method is unsuccessful in implementing digital HPFs.

Steps to design a digital filter using Impulse Invariance method:

1. For the given specifications, find $H_a(s)$, the transfer f^n of an analog filter.
2. Select the sampling rate of the digital filter, T sec/sample.
3. Express the analog filter transfer f^n as the sum of single-pole filters.

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - P_k}.$$

4. Compute the z-transform of the digital filter by using the formula

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

For high sampling rates, use

$$H(z) = \sum_{k=1}^{N} \frac{T C_k}{1 - e^{P_k T} z^{-1}}$$

Ex 1: For the analog transfer f^n $H(s) = \frac{2}{(s+1)(s+2)}$ determine $H(z)$ using impulse invarience method. Assume $T = 1$ sec

Soln:

$$H(s) = \frac{2}{(s+1)(s+2)}$$

$$= \frac{2}{s+1} - \frac{2}{s+2}$$

$$= \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

Using impulse invariance technique, we have,

$$H(s) = \sum_{K=1}^{N} \frac{C_K}{s - P_K} \Rightarrow H(z) = \sum_{K=1}^{N} \frac{C_K}{1 - e^{P_K T} z^{-1}}$$

→ There are two poles $P_1 = -1$ $P_2 = -2$. So

$$H(z) = \frac{2}{1 - e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} z^{-1}}$$

$$= \frac{2}{1 - 0.3678 z^{-1}} - \frac{2}{1 - 0.1353 z^{-1}}$$

$$= \frac{0.465 z^{-1}}{1 - 0.503 z^{-1} + 0.04976 z^{-2}}$$

Q: Repeat the above problem for $H(s) = \frac{10}{s^2 + 7s + 10}$, $T = 0.2$.

Soln: Given $H(s) = \frac{10}{s^2 + 7s + 10}$

$$= \frac{-3.33}{s+5} + \frac{3.33}{s+2}$$

$$\Rightarrow H(z) = T \left[\frac{-3.33}{1 - e^{-5T} z^{-1}} + \frac{3.33}{1 - e^{-2T} z^{-1}} \right]$$

$$= 0.2 \left[\frac{-3.33}{1 - e^{-1} z^{-1}} + \frac{3.33}{1 - e^{-0.4} z^{-1}} \right]$$

$$= \left[\frac{-0.666}{1 - 0.3678 z^{-1}} + \frac{0.666}{1 - 0.67 z^{-1}} \right]$$

$$= \frac{0.2012 z^{-1}}{1 - 1.0378 z^{-1} + 0.247 z^{-2}}$$

Ex

Ex: Apply Impulse invariant method and find $H(z)$ for

$$H(s) = \frac{s+a}{(s+a)^2 + b^2}$$

Soln: The inverse Laplace transform of given f^n is

$$h(t) = \begin{cases} e^{-at} \cos bt & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Sampling the f^n produces

$$h(nT) = \begin{cases} e^{-anT} \cos(bnT) & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H(z) = \sum_{n=0}^{\infty} e^{-anT} \cos(bnT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[e^{-anT} z^{-n} \left[\frac{e^{jbnT} + e^{-jbnT}}{2} \right] \right]$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left[\left(e^{-(a-jb)T} z^{-1}\right)^n + \left(e^{-(a+jb)T} z^{-1}\right)^n \right]$$

$$= \frac{1}{2} \left[\frac{1}{1 - e^{-(a-jb)T} z^{-1}} + \frac{1}{1 - e^{-(a+jb)T} z^{-1}} \right]$$

$$= \frac{1 - e^{-aT} \cos(bT) z^{-1}}{1 - 2e^{-aT} \cos(bT) z^{-1} + e^{-2aT} z^{-2}}$$

Ex: Use the above method with $T = 1$ sec, determine $H(z)$ if

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

soln:
$$H(z) = \frac{0.453z^{-1}}{1-0.7497z^{-1}+0.2432z^{-2}}$$

Ex: An analog filter has a T/F $H(s) = \frac{5}{s^3+6s^2+11s+6}$

Design a digital filter equivalent to this using impulse invariant method for T = 1 sec

3. Design of IIR filter using Bilinear Transformation:

Note: This transformation avoids aliasing of fq components.

Let us consider an analog linear filter with sys f^n.

$$H(s) = \frac{b}{s+a}$$ which can be written as

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

so $sY(s) + aY(s) = bX(s)$

This can be characterized by the differential eqn

$$\frac{dy(t)}{dt} + a\,y(t) = b\,x(t)$$

By using trapezoidal approximation formula and Z-transform of the difference eqn, we get

$$H(z) = \frac{b}{\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+a}$$

On comparing H(s) and H(z)

the mapping from s-plane to the z-plane can be obtained as

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

The relationship between s & z is known as <u>bilinear Transformation</u>.

Let $z = re^{j\omega}$ and $s = \sigma + j\Omega$

we have
$$S = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

$$= \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right] = \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]\left[\frac{r\cos\omega + 1 - jr\sin\omega}{r\cos\omega + 1 - jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{(r\cos\omega+1)^2 + r^2\sin^2\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{1 + r^2\cos^2\omega + 2r\cos\omega + r^2\sin^2\omega}\right]$$

Seperating imaginary and real parts, we have

$$s = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

$$\Rightarrow \sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right] \qquad \Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

when $r=1$, a then $\sigma = 0$ and

$$\Omega = \frac{2}{T}\frac{\sin\omega}{1+\cos\omega} = \frac{2}{T}\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\frac{\omega}{2}}$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

or $\omega = 2\tan^{-1}\frac{\Omega T}{2}$.

** ~~The~~ warping effect :

let Ω and ω represent the frequency variables in the analog filter and the derived digital filter respectively

we have

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

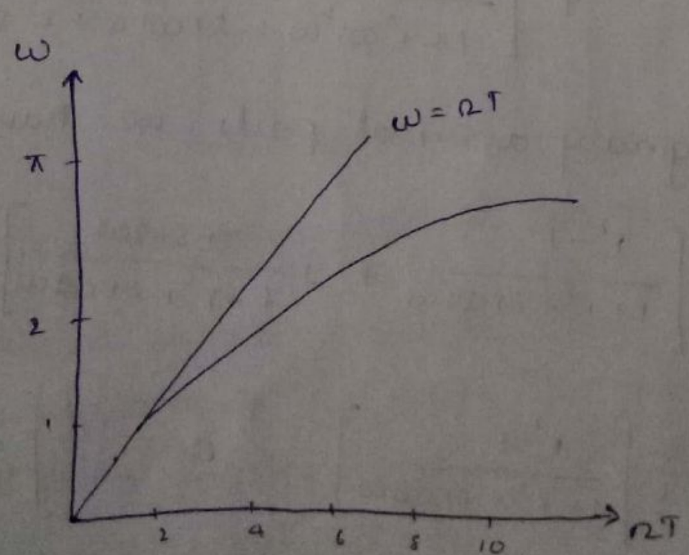
$\because$ for $\theta < 1$

$\tan\theta = \theta$

for small value of ω

$$\Omega = \frac{2}{T}\cdot\frac{\omega}{2} = \frac{\omega}{T}$$

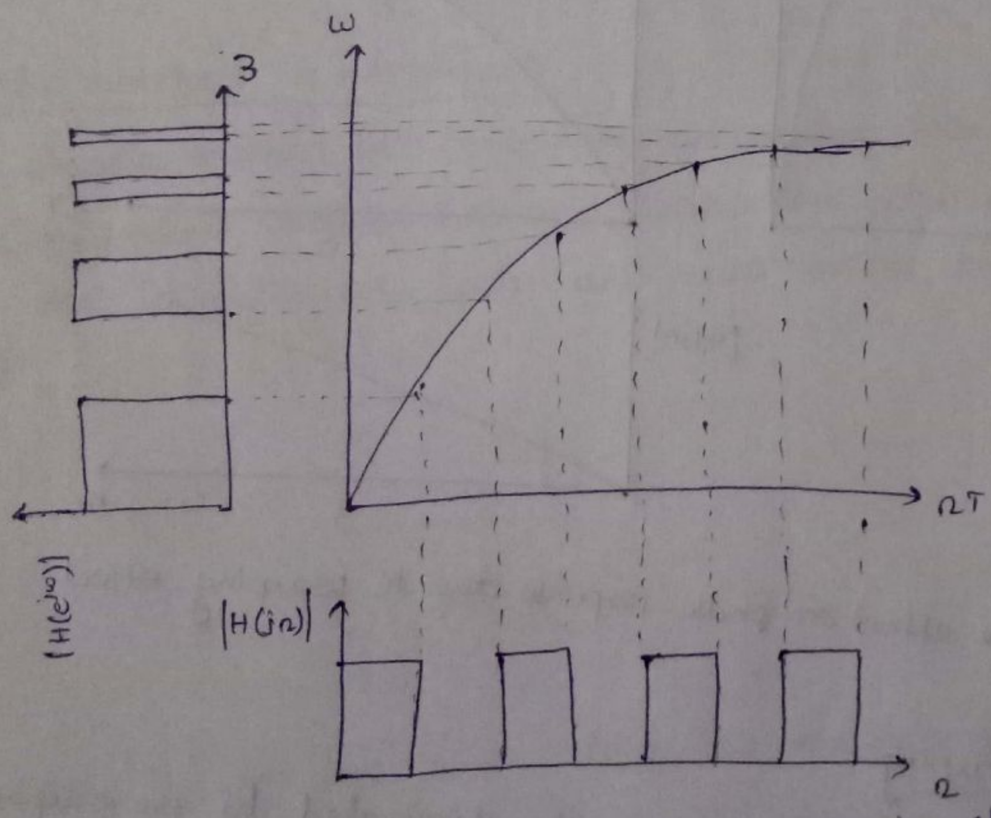
$$\omega = \Omega T$$



Relationship b/w Ω and ω

→ for low fqs the relationship b/w Ω and ω are linear, as a result, the digital filter have the same amplitude response as the analog filter.

→ for high fqs, the relationship b/w ω and Ω becomes non-linear and distortion is introduced in the fq scale of the digital filter to that of analog filter. This is known as <u>warping effect</u>.



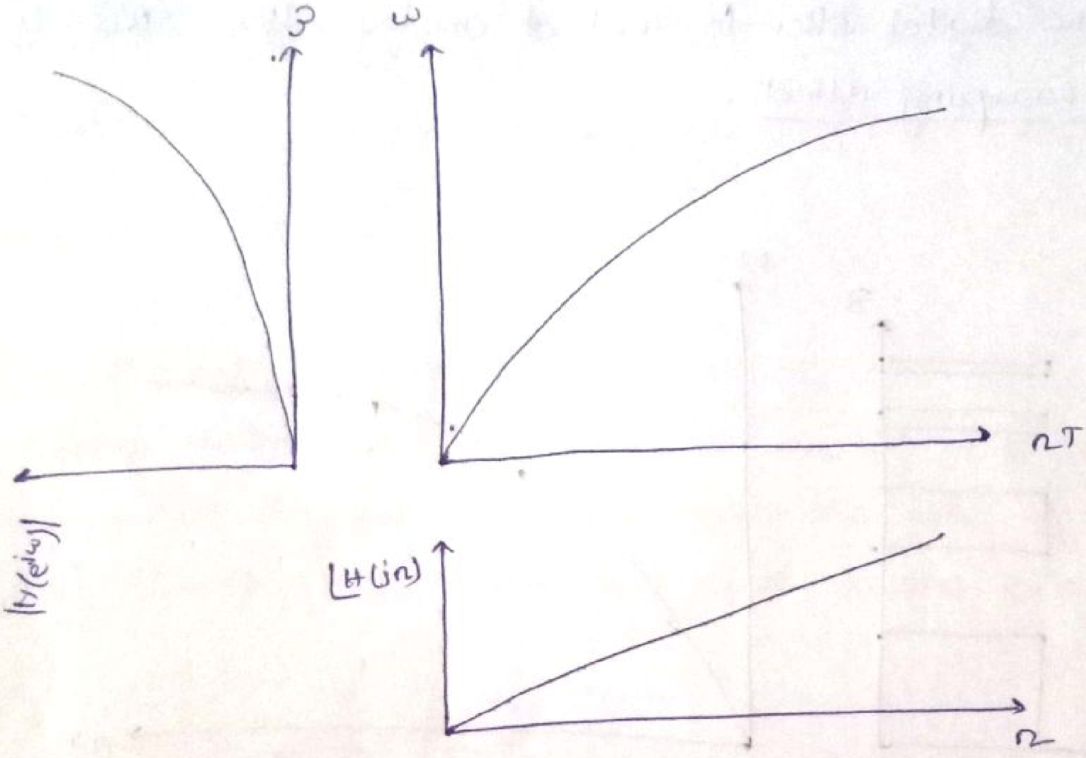
The effect on magnitude response due to warping effect.

→ The influence of ~~wapp~~ warping effect on the amplitude response is shown in the fig above, by considering an analog filter with a no. of passbands centered at regular intervals.

→ The desired digital filter will have same no of passbands. But the centre fqs and Bandwidth of higher fq passband will tend to reduce disproportionately.

→ The influence of the warping effect on the phase response is shown in the fig below

→ Considering an analog filter with linear phase response, the phase response of the desired digital filter will be non-linear.



The effect on phase response due to warping effect.

Prewarping

The warping effect can be eliminated by prewarping the analog filter. This can be done by finding prewarping analog fqs using the formula.

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

$\therefore$ we have $\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2}$

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2}$$

Steps to design digital filter using bilinear transform technique:

1. From the given specifications, find prewarping analog fq s using $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$
2. Using the analog fq s, find $H(s)$ of the analog filter.
3. Select the sampling rate of the digital filter, call it T sec/sample
4. Substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ into the transfer f^n found in step 2.

4. The matched z-transform

Another method for converting an analog filter into an equivalent digital filter is to map the poles & zeros of H(s) directly into poles and zeros in the z-plane.

If,

$$H(s) = \frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - p_k)}$$

where $\{z_k\}$ are the zeros $\{p_k\}$ are the poles of the filter, then the sys f^n of the digital filter is

$$H(z) = \frac{\prod_{k=1}^{M}(1 - e^{z_k T} z^{-1})}{\prod_{k=1}^{M}(1 - e^{p_k T} z^{-1})}$$

where T - sampling interval. Thus each factor of the

$(s-a)$ in $H(s)$ is mapped into the factor $(1-e^{aT}z^{-1})$.
This mapping is called the matched z-T.

Ex: Apply bilinear Transformation to $H(s) = \dfrac{2}{(s+1)(s+2)}$ with $T = 1$ sec and find $H(z)$.

Soln: Given $H(s) = \dfrac{2}{(s+1)(s+2)}$

Substitute $s = \dfrac{2}{T}\left[\dfrac{1-z^{-1}}{1+z^{-1}}\right]$ in $H(s)$ to get $H(z)$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{2}{(s+1)(s+2)}\Bigg|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

Given $T = 1$ sec

$$H(z) = \frac{2}{\left\{2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})\,4}$$

$$= \frac{(1+z^{-1})^2}{6-2z^{-1}}$$

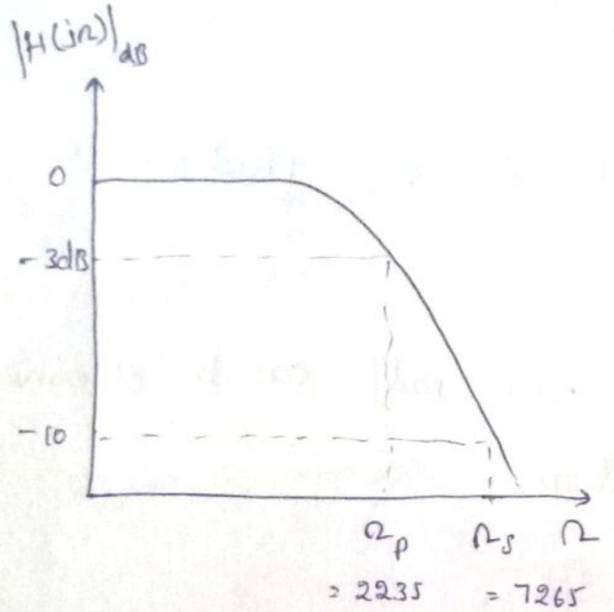
$$= \frac{0.166\,(1+z^{-1})^2}{(1-0.33\,z^{-1})}$$

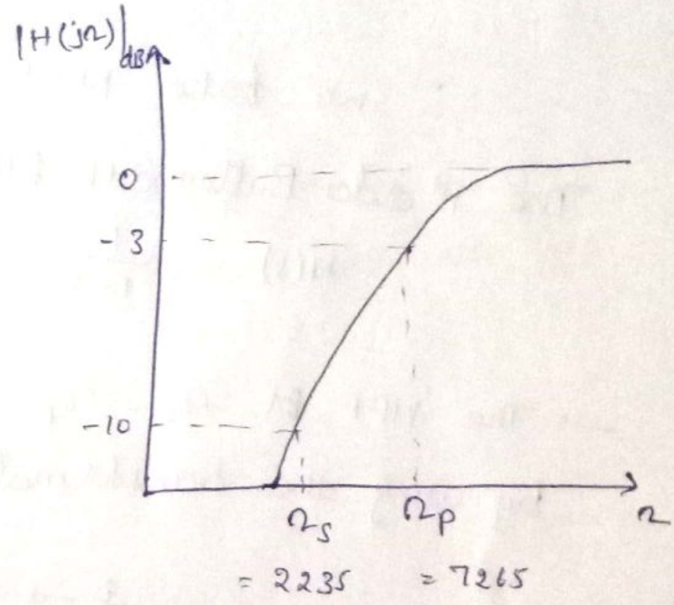
Ex Using the bilinear transform, design a HPF, monotonic in pass band & stop band with cut off fq of 1000 Hz and down 10dB at 350 Hz. The sampling fq is 5000 Hz.

Soln: Given $\alpha_p = 3dB$ $\omega_c = \omega_p = 2\times\pi\times1000 = 2000\pi$ rad/s.

$\alpha_s = 10dB$; $\omega_s = 2\times\pi\times350 = 700\pi$ rad/s

$T = \frac{1}{f} = \frac{1}{5000} = 2\times10^{-4}$ sec





The characteristics are monotonic in both passband and stopband. Therefore the filter is Butterworth filter.

Prewarping the digital fqs, we have

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p T}{2} = \frac{2}{2\times10^{-4}}\tan\frac{2000\pi\times2\times10^{-4}}{2}$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s T}{2} = \frac{2}{2\times10^{-4}}\tan\frac{700\pi\times2\times10^{-4}}{2}$$

$$= 10^4\tan(0.07\pi) = 2235 \text{ rad/s}.$$

→ first design a LPF for the given specifications and use suitable transformation to obtain Transfer function of HPF.

$$N = \frac{\log \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log \frac{\Omega_s}{\Omega_p}} = \frac{\log \sqrt{\frac{10^{0.1(10)}-1}{10^{0.1(3)}-1}}}{\log \frac{7265}{2235}} = \frac{\log 3}{\log 3.25}$$

$$= \frac{0.4771}{0.5118} = 0.932$$

$\therefore$ we take $N = 1$

The 1st order Butterworth filter for $\Omega_c = 1$ rad/sec is

$$H(s) = \frac{1}{1+s}$$

→ The HPF for $\Omega_c = \Omega_p = 7265$ rad/s can be obtained by using the transformation

$$s \to \frac{\Omega_c}{s}$$

$$\text{ie, } s \to \frac{7265}{s}$$

The transfer f^n of HPF is

$$H(s) = \left.\frac{1}{s+1}\right|_{s = \frac{7265}{s}}$$

$$= \frac{s}{s+7265}$$

Using bilinear Transformation

$$H(z) = \left. H(s) \right|_{s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \left. \frac{s}{s+7265} \right|_{s = \frac{2}{2\times10^{-4}}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{10000\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}{10000\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + 7265}$$

$$= \frac{0.5792\,(1-z^{-1})}{1 - 0.1584\,z^{-1}}$$

Frequency Transformation in Digital Domain:

A digital LPF can be converted into a digital HP, BP, BS filter.

1. Lowpass to lowpass

$$z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

where $\alpha = \dfrac{\sin[(\omega_p - \omega_p')/2]}{\sin[(\omega_p + \omega_p')/2]}$

$\omega_p \rightarrow$ passband fq of LPF

$\omega_p' \rightarrow$ " " of new LPF

2. Lowpass to highpass

$$z^{-1} = -\left[\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}\right]$$

where $\alpha = \dfrac{\cos[(\omega_p' + \omega_p)/2]}{\cos[(\omega_p' - \omega_p)/2]}$

ω_p = passband fq of LPF

$\omega_p^!$ = passband fq of HPF

3. Lowpass to Bandpass

$$z^{-1} \Rightarrow \frac{-\left[z^{-2} - \frac{2\alpha k}{1+k} z^{-1} + \frac{k-1}{k+1}\right]}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{1+k} z^{-1} + 1}$$

where $\alpha = \frac{\cos\left[(\omega_u+\omega_l)/2\right]}{\cos\left[(\omega_u-\omega_l)/2\right]}$

$$k = \cot\left[\frac{\omega_u-\omega_l}{2}\right] \tan\frac{\omega_p}{2}$$

$\omega_u \rightarrow$ upper cut off fq

$\omega_l \rightarrow$ lower " "

4) Low pass to Bandpass

$$z^{-1} \rightarrow \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$$

where $\alpha = \frac{\cos\left[(\omega_u+\omega_l)/2\right]}{\cos\left[(\omega_u-\omega_l)/2\right]}$

$$k = \tan\left(\frac{\omega_u-\omega_l}{2}\right) \tan\frac{\omega_p}{2}$$

Ex: Convert the single pole LPF with system f^n $H(z) = \frac{0.5(1+z^{-1})}{1-0.302z^{-2}}$

into BPF with upper & lower cut off fqs ω_u and ω_l respectively.
The LPF has 3 dB bandwidth $\omega_p = \frac{\pi}{6}$ and $\omega_u = \frac{3\pi}{4}$,
$\omega_l = \pi/4$

Soln: The digital-to-digital transformation from LPF to BPF is

$$z^{-1} \rightarrow \frac{-\left[z^{-2} - \frac{2\alpha k}{1+k} z^{-1} + \frac{k-1}{k+1}\right]}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} + 1}$$

where

$$k = \cot\left[\frac{\omega_u - \omega_l}{2}\right] \tan\frac{\omega_p}{2} = \cot\left[\frac{\frac{3\pi}{4} - \frac{\pi}{4}}{2}\right] \tan\frac{\pi}{12}$$

$$= \cot\left(\frac{\pi}{4}\right) \tan\frac{\pi}{12} = 0.268$$

$$\alpha = \frac{\cos\frac{\omega_u+\omega_l}{2}}{\cos\frac{\omega_u-\omega_l}{2}} = \frac{\cos\left[\frac{3\pi/4 + \pi/4}{2}\right]}{\cos\left[\frac{3\pi/4 - \pi/4}{2}\right]} = 0$$

substituting the values of α and k in the transformation

$$z^{-1} \rightarrow \frac{-\left[z^{2} + \frac{0.268-1}{0.268+1}\right]}{\frac{0.268-1}{0.268+1} z^{-2} + 1}$$

$$\rightarrow \frac{-(z^{-2} - 0.577)}{-0.577z^{-2}+1}$$

Now the transfer function of BPF can be obtained by substituting the above transformation in $H(z)$.

$$H(z) = \frac{0.5\left[1 + \frac{-z^{-2} + 0.577}{1 - 0.577z^{-2}}\right]}{1 - 0.302\left[\frac{-z^{-2} + 0.577}{1 - 0.577z^{-2}}\right]}$$

$$= 0.5\left[\frac{1.577(1 - z^{-2})}{0.82575 - 0.275z^{-2}}\right]$$

$$= \frac{0.955\left[1 - z^{-2}\right]}{1 - 0.333z^{-2}}$$

Realization of Digital filters:

A digital filter ~~rep~~ transfer function can be realized in a variety of ways. There are two types of realizations

1. Recursive 2. Nonrecursive

1. For recursive realization, the current o/p $y(n)$ is a f^n of past outputs, past and present inputs. This form corresponds to an Infinite Impulse Response (IIR) digital filter. In this section, we discuss this type of realization.

2. For non-recursive realization, current o/p sample $y(n)$ is a f^n of only past and present i/ps. This form corresponds to FIR digital filter.

IIR filter can be realized in many forms. They are

1. Direct form-I realization
2. Direct form-II realization
3. Transposed direct form realization
4. Cascade form realization.

Direct form- I realization :

let us consider an LTI recursive system described by difference eqn

$$y(n) = -\sum_{K=1}^{N} a_k\, y(n-k) + \sum_{K=0}^{M} b_k\, x(n-k)$$

$$= -a_1 y(n-1) - a_2 y(n-2) \cdots - a_{N-1} y(n-N+1)$$

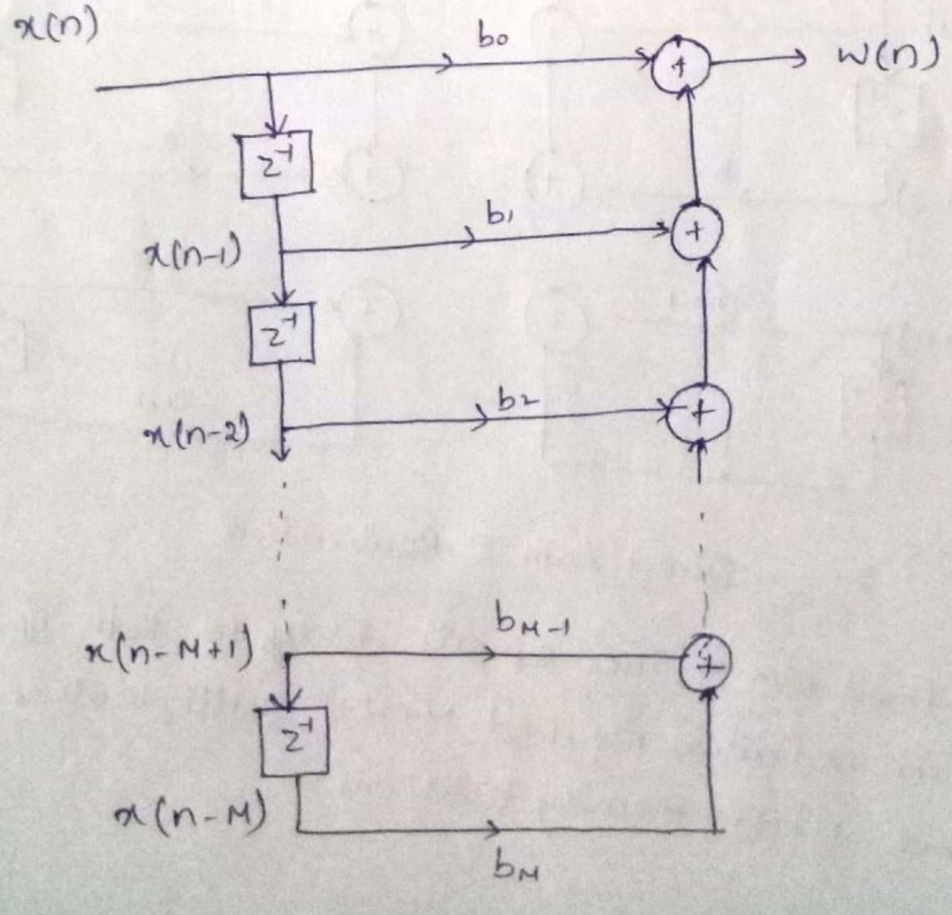
$$- a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$$

let

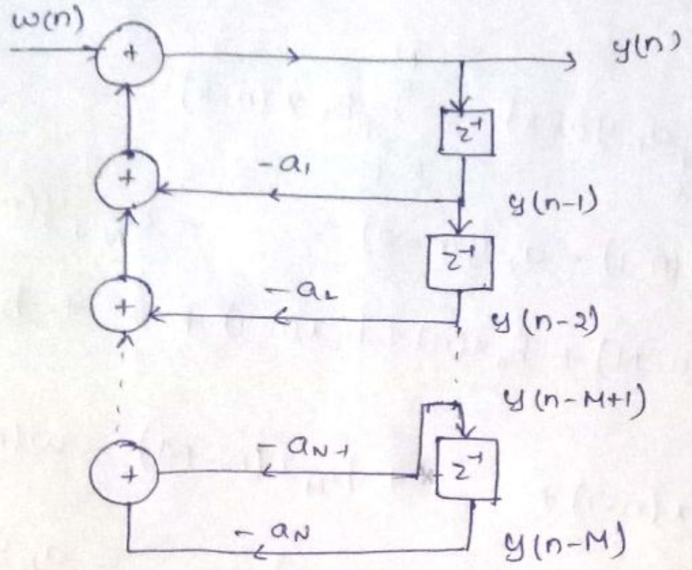
$$b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) = w(n) \quad -①$$

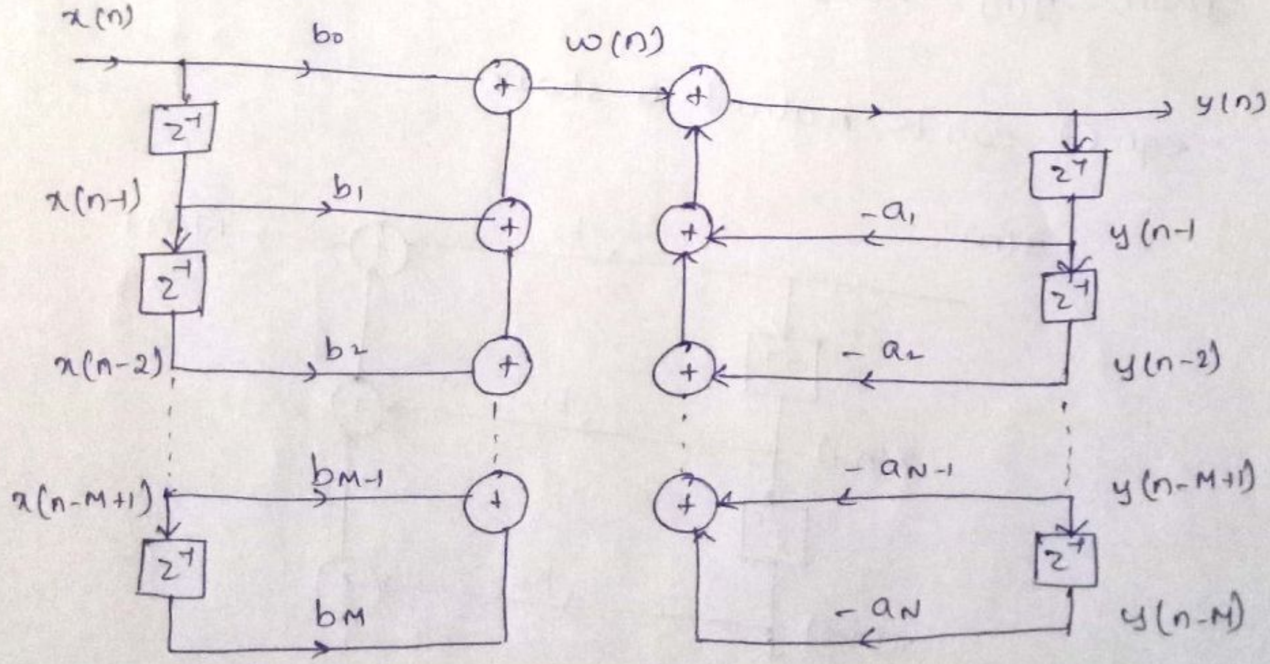
then $y(n) = -a_1 y(n-1) - a_2 y(n-2) + \cdots - a_N y(n-N) + w(n)$ $\quad -②$

eqn ① can be realized as shown



$||^{ly}$ the eqn ⑤ can be realized as shown below





Direct form I Realization

→ Direct form I uses seperate delays for both i/p & o/p.

→ This realization required M+N+1 multiplications, M+N additions and M+N+1 memory locations.

Ex: Realize the 2nd order digital filter

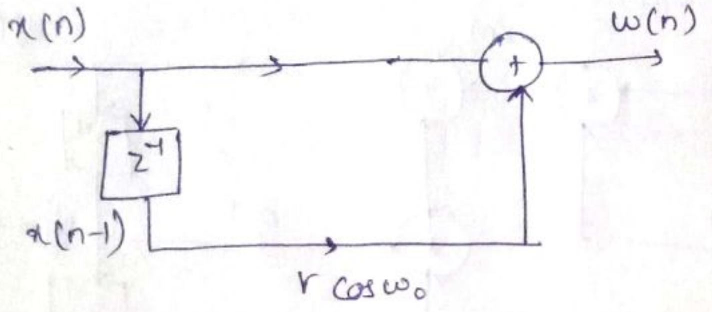
$$y(n) = 2r\cos(\omega_0)\, y(n-1) - r^2 y(n-2) + x(n) - r\cos\omega_0\, x(n-1)$$

Soln:

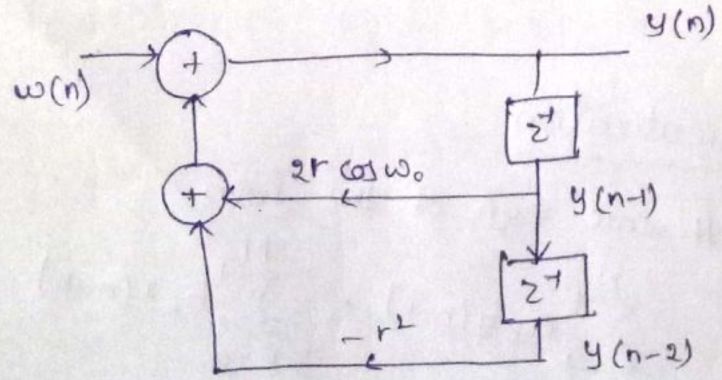
Let $x(n) - r\cos\omega_0\, x(n-1) = w(n)$ — ①

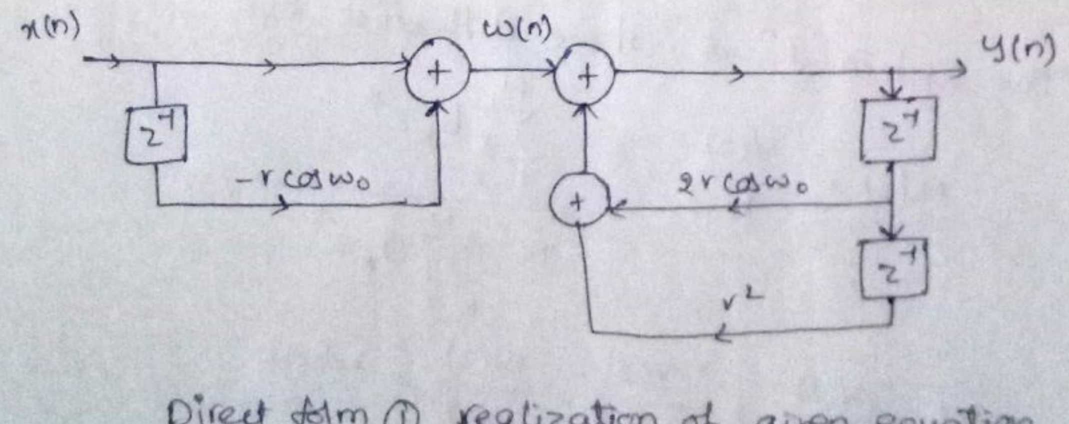
$$y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + w(n) \quad -②$$

Realization of eqn ①



realizing eqn ②





Direct form ① realization of given equation

Ex 2: Obtain the direct form-I realization for the system described by difference eqn

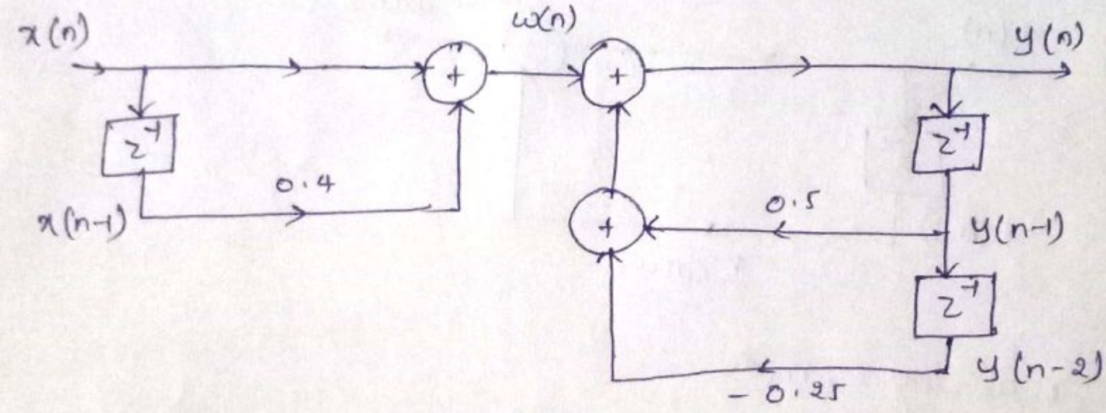
$$y(n) = 0.5\,y(n-1) - 0.25\,y(n-2) + x(n) + 0.4\,x(n-1)$$

Soln:

let $\quad x(n) + 0.4\,x(n-1) = w(n) \quad$ —①

then $\quad y(n) = 0.5\,y(n-1) - 0.25\,y(n-2) + w(n) \quad$ —②

Realizing eqn ① & ② and combining, we get



Direct form II realization.

Consider the difference eqn of the form

$$y(n) = -\sum_{k=1}^{N} a_k\,y(n-k) + \sum_{k=0}^{M} b_k\,x(n-k)$$

The system f^n of above difference eqn

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Let $\quad \dfrac{Y(z)}{X(z)} = \dfrac{Y(z)}{W(z)} \cdot \dfrac{W(z)}{X(z)} \quad$ where

$$\frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$\Rightarrow W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) \cdots - a_N z^{-N} W(z) \quad \text{—} (1)$$

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

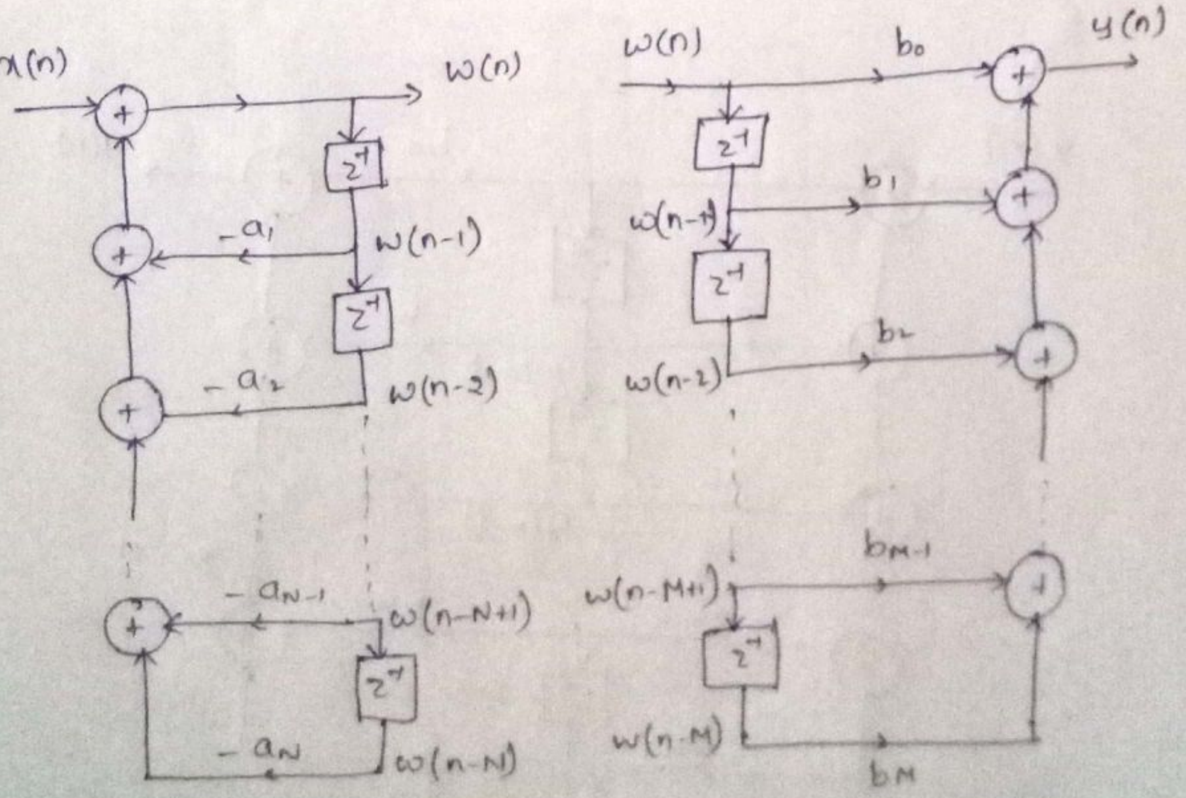
$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + \cdots + b_M z^{-M} W(z) \quad \text{—} (2)$$

eqn (1) & (2) can be expressed in difference eqn form ie,

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - \cdots - a_N w(n-N) \quad \text{—} (3)$$

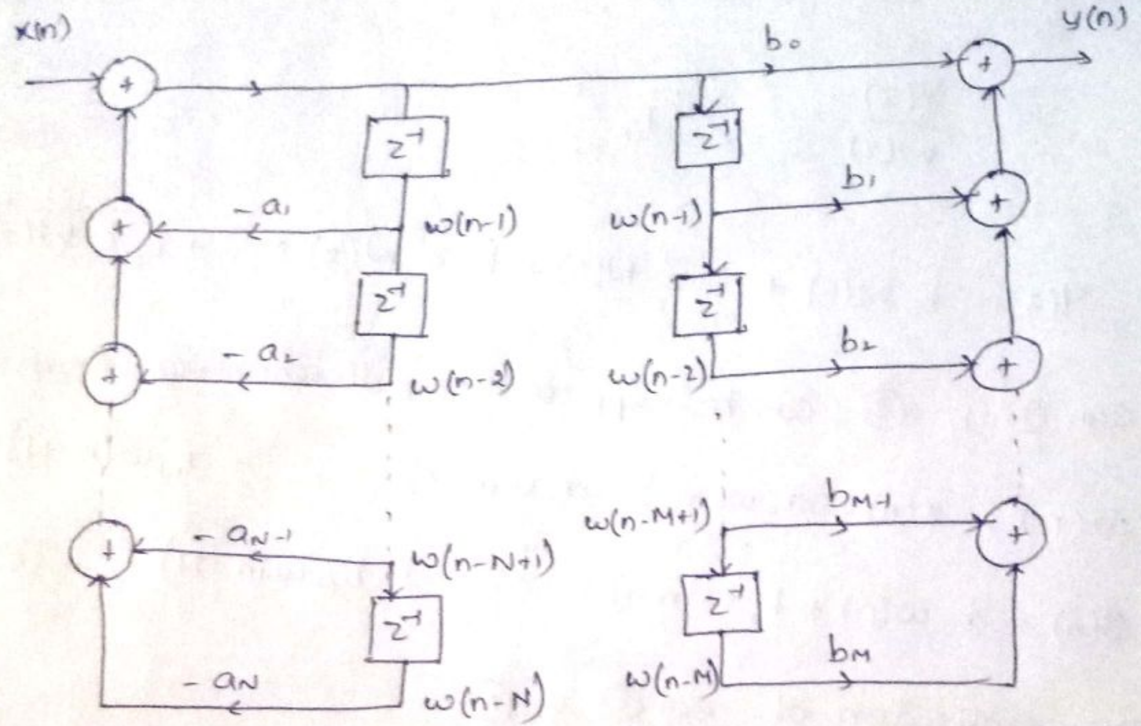
$$y(n) = b_0 w(n) + b_1 w(n-1) + \cdots + b_M w(n-M) \quad \text{—} (4)$$

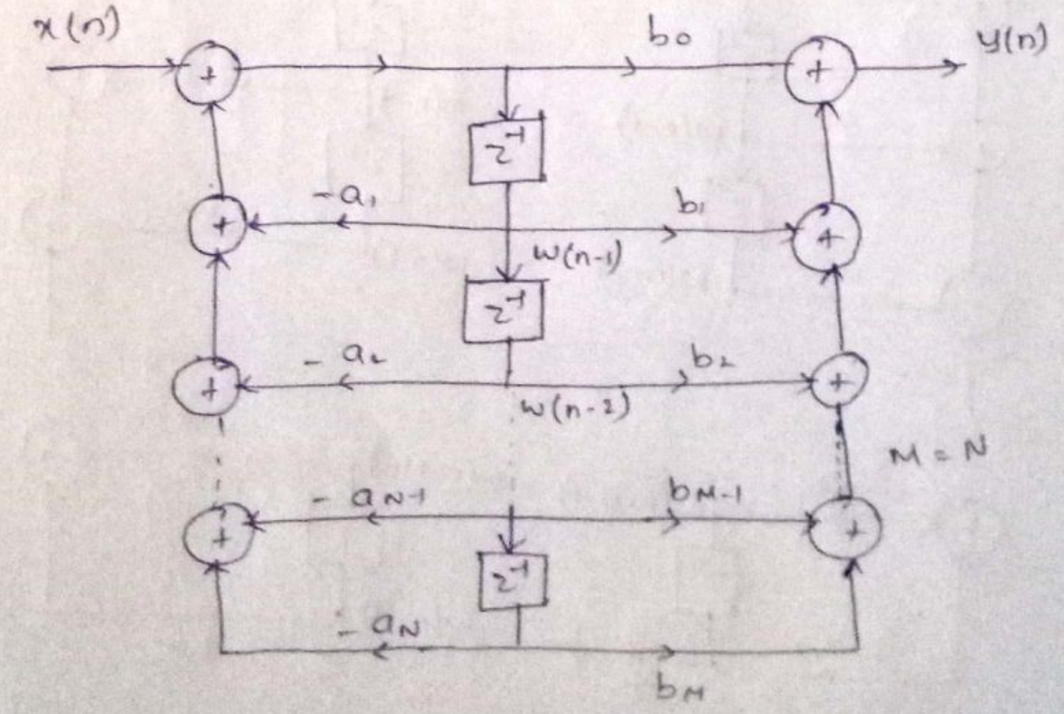
The realization of eqn (3) & (4)



from the above fig, it can be seen that the two delay elements contain the same i/p w(n) and hence the same o/p w(n-1).

→ So we can merge these delays into one delay





Direct form II realization

Ex: *

→ The realization structure shown above is called direct form II realization

→ This structure requires $M+N+1$ multiplications, $M+N$ additions and maximum of $\{M,N\}$ memory locations.

→ Since the direct form II realization minimizes the no. of memory locations, it is said to be canonic.

Ex: 1. Realize the 2nd order system

$$y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) - r\cos\omega_0\, x(n-1)$$ in direct form II.

Soln: Given

$$y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + x(n) - r\cos\omega_0\, x(n-1)$$

The system function

$$\frac{Y(z)}{X(z)} = \frac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$$

Let $$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

where $$\frac{Y(z)}{W(z)} = 1 - r\cos\omega_0 z^{-1}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$$

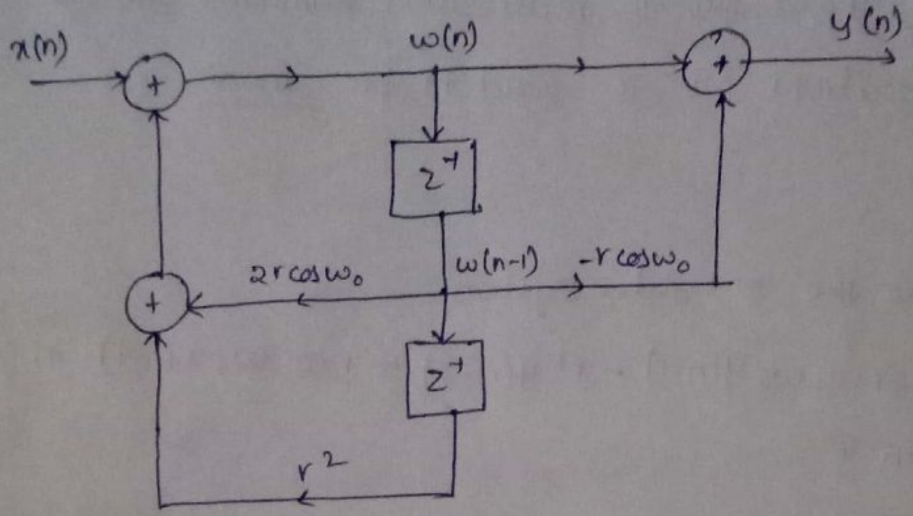
$$Y(z) = W(z) - r\cos\omega_0 z^{-1} W(z)$$

$$\Rightarrow \quad y(n) = w(n) - r\cos\omega_0\, w(n-1) \quad -①$$

$$W(z) = X(z) + 2r\cos w_0 z^{-1} W(z) - r^2 z^{-2} W(z)$$

$$\Rightarrow \quad w(n) = x(n) + 2r\cos w_0\, w(n-1) - r^2 w(n-2) \quad — ②$$

Realize eqn ① & ② and combine to get get direct form II.



Direct form II realization

Ex 2: Determine the direct form II realization for the following system

$$y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7x(n) - 0.252\,x(n-2)$$

soln:

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$\frac{Y(z)}{W(z)} = 0.7 - 0.252\,z^{-2}$$

$$Y(z) = 0.7\,W(z) - 0.252\,z^{-2}\,W(z)$$

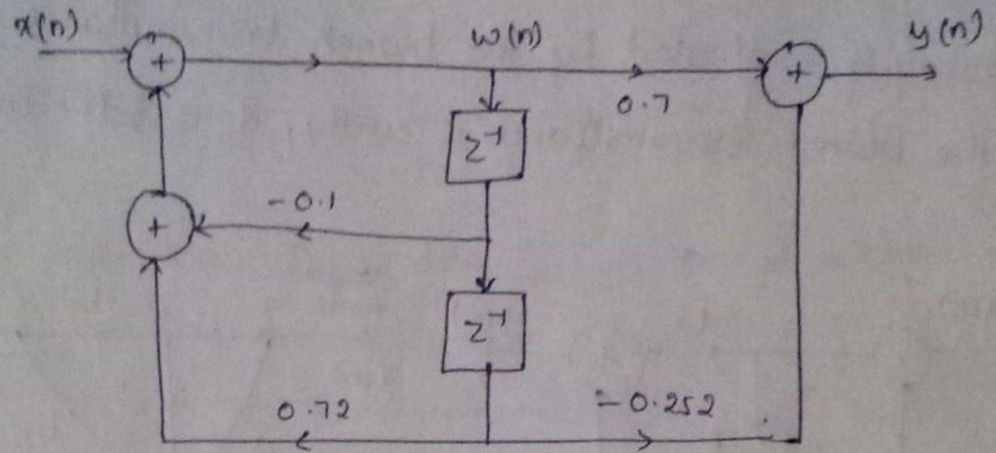
$$\Rightarrow \quad y(n) = 0.7\,w(n) - 0.252\,w(n-2)$$

$||^{ly}$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$W(z) = X(z) - 0.1z^{-1}W(z) + 0.72z^{-2}\,W(z)$$

$\Rightarrow w(n) = x(n) - 0.1\,w(n-1) + 0.72\,w(n-2)$



Direct form II realization.

Signal flowgraph

→ A signal flow graph is a graphical representation of the relationship between the variables of a set of linear difference equations.

→ The basic elements of a signal flow graph are branches and nodes.

→ The signal flow graph is basically a set of directed branches that connect at nodes.

→ A node represents a system variable, which is equal to the sum of incoming signals from all branches connecting to the node.

→ There are two types of nodes. Source nodes are nodes that have no entering branches.

→ Sink nodes are nodes that have only entering branches.

→ A signal travels along a branch from one node to another.

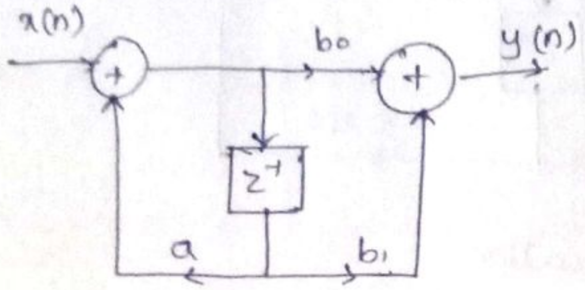
→ The signal out of a branch is equal to the branch gain times the signal into the branch.

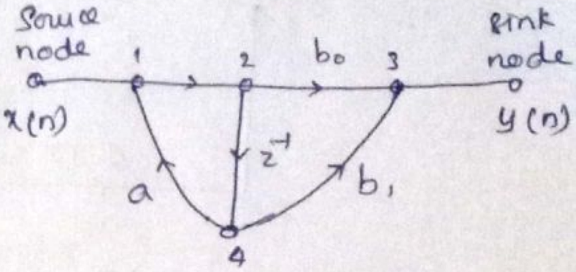
→ The arrow head shows the direction of the branch and the branch gain is indicated next to arrow head.

→ The delay is indicated by the branch transmittance z^{-1}.

→ When the branch transmittance is unity, it is left unlabeled.

Ex





Block diagram representation

Signal flow graph representation.

__Transposition Theorem__ & __transposed structure__

The transpose of a structure is defined by following operations:

i) Reverse the direction of all branches in the signal flow graph.

ii) Interchange the inputs and outputs

iii) Reverse the roles of all nodes in the flowgraph.

iv) Summing points become branching points

v) Branching points become summing points.

→ According to transposition theorem, the system transfer function remain unchanged by transposition.

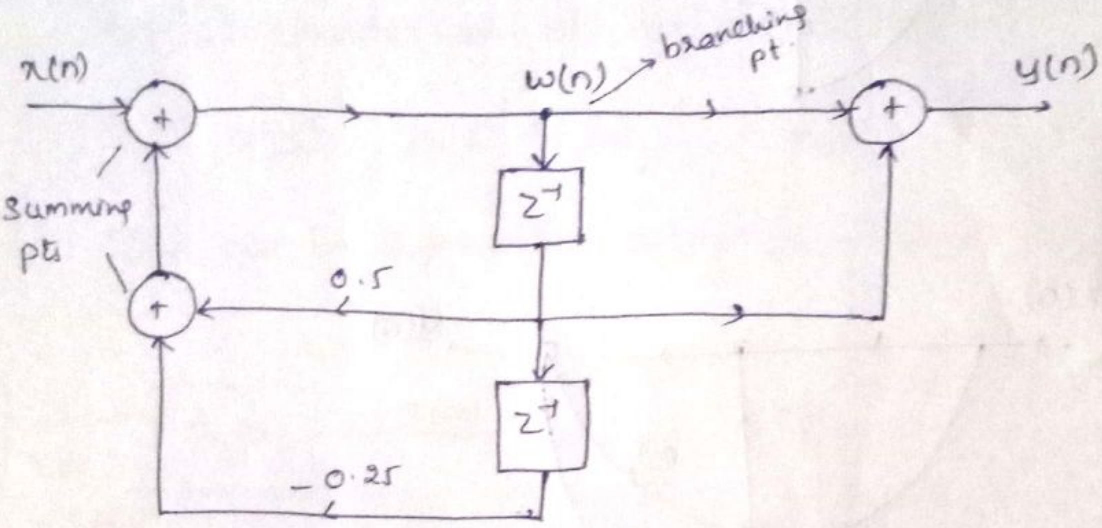
Ex: Det the direct form II and Transposed direct form II for the given system

$$y(n) = \frac{1}{2} y(n-1) - \frac{1}{4} y(n-2) + x(n) + x(n-1)$$

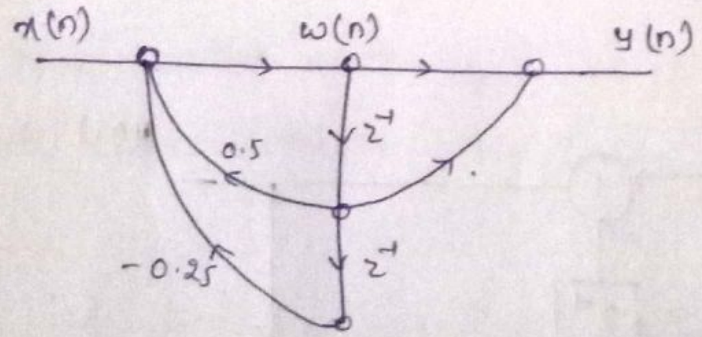
soln: The sys transfer function of the given difference eqn is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-0.5z^{-1}+0.25z^{-2}}$$

By inspection, Direct form II can be obtained



Signal flow graph representation



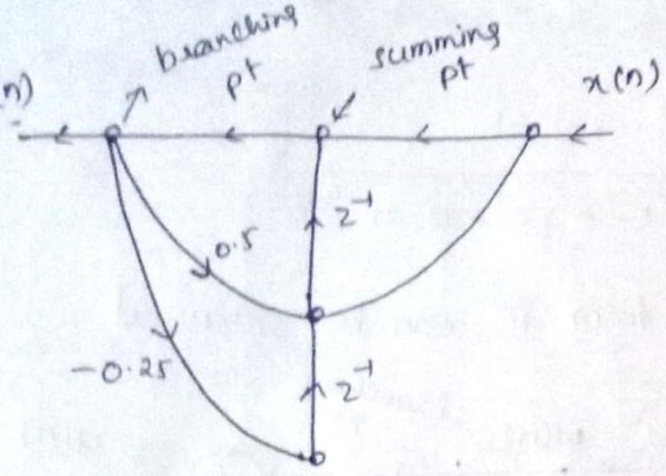
To get transposed direct form II do the following operations

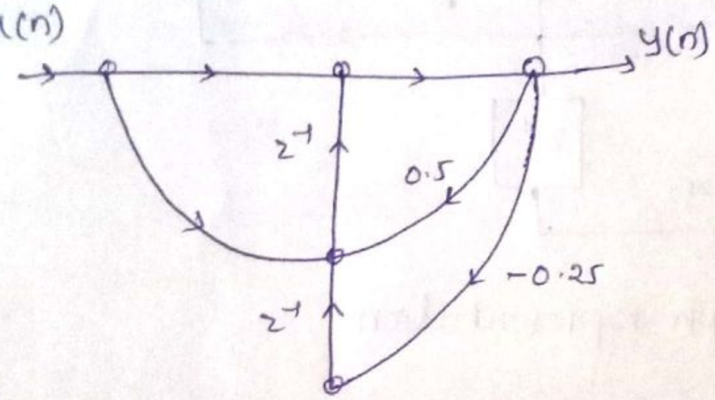
i) Change the direction of all branches.

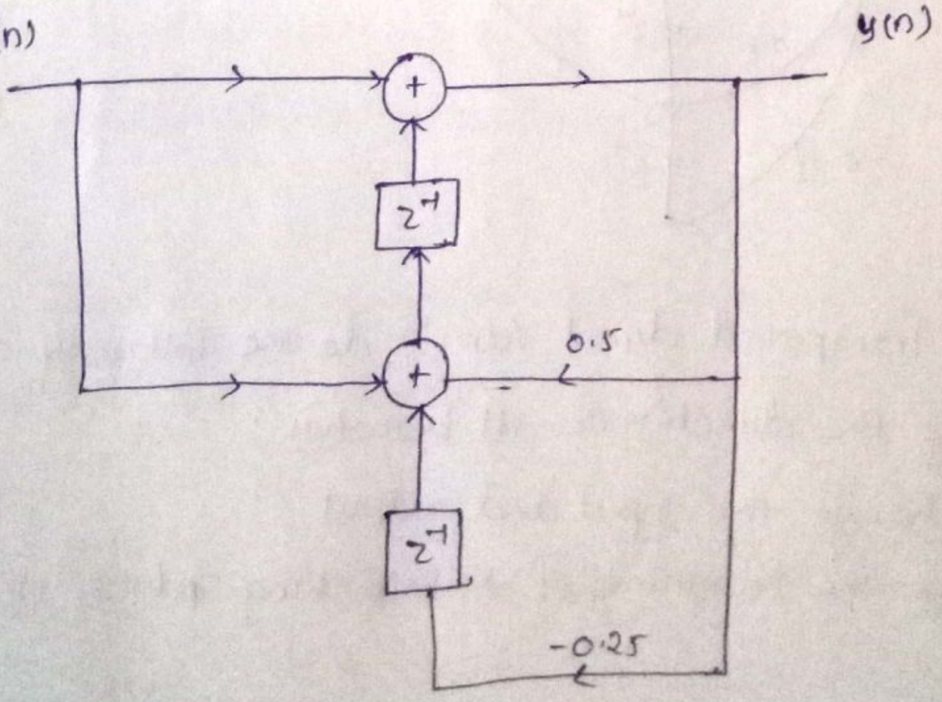
ii) Interchange the input and output.

iii) Change the summing pt to branching pt & vice versa.

Then we obtain







Transposition structure of given eqn.

En: Obtain the transposed direction form II structure of the following system

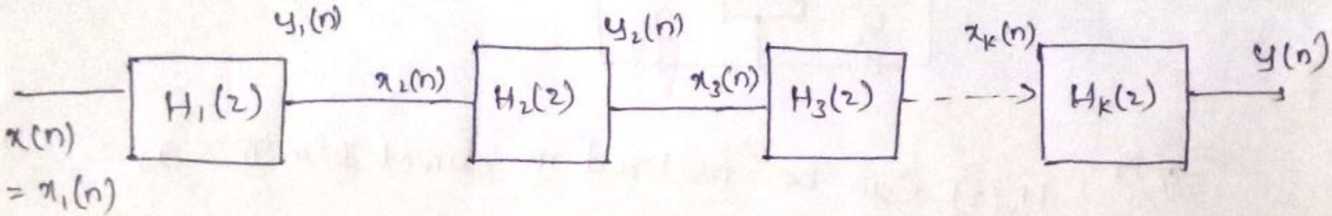
$$y(n) = \frac{1}{2} y(n-1) + \frac{1}{4} y(n-2) + x(n) + x(n-1)$$

Cascade form

Let us consider an IIR system with system f^n

$$H(z) = H_1(z) \cdot H_2(z) \cdot \cdots \; H_k(z)$$

This can be represented using the block diagram as shown



Block diagram representation

Now realize each $H_k(z)$ in direct form II and cascade all ~~structustures~~ structures

En: Realize the system with difference eqn

$y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n) + \frac{1}{3} x(n-1)$ in cascade form.

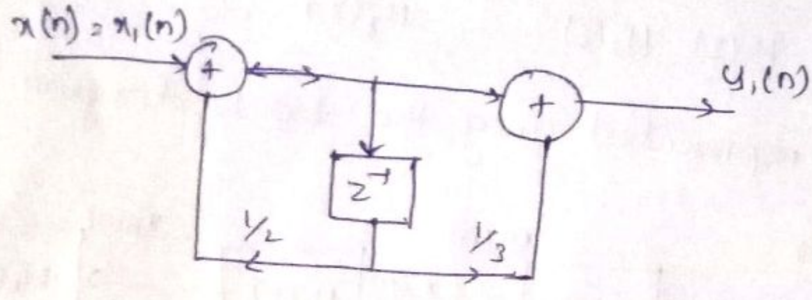
soln: from the difference eqn we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}$$

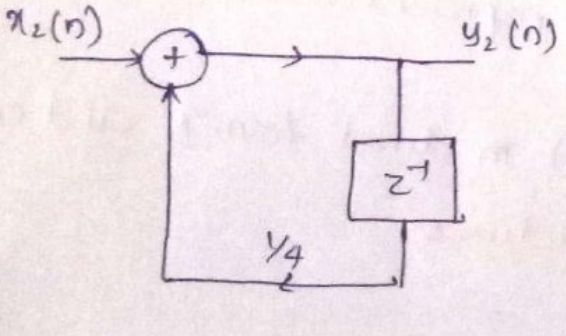
$$= \frac{1+\frac{1}{3}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} = H_1(z)\,H_2(z)$$

where $H_1(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$; $H_2(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$

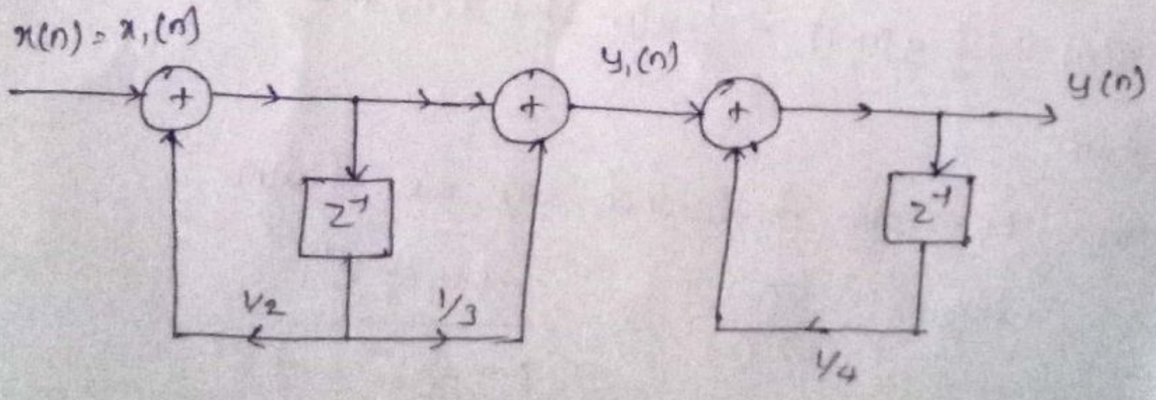
$H_1(z)$ can be realized in direct form II as



||ly $H_2(z)$ Can be realized in direct form II as



Cascading the realization of $H_1(z)$ and $H_2(z)$ we have



Ex: 2. For the system f^n given below, obtain cascade form

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$