Mead®

211
Linear Systems

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ME 211A: Modeling and Analysis of Dynamic Systems (Fall 2010)

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projects & exams graded by Kumar
~~home~~ assignments graded by TA

Office Hours Tuesday 4 – 5PM, Friday 11 – 2pm,

Important: I have an open door policy, so you are welcome to stop by my office and if you need appointment, please see me after the class or send me an email.

TA **TBD**
Lecture 10:00 – 10:50 MWF (McNutt Hall 212)
Prerequisite Math 14 (or 8), 15 (or 21), 22, and 204; Physics 24; C or better in ME/AE 160
Textbook *Modeling and Analysis of Dynamic Systems* 3rd Edition by Close, Frederick, and Newell. ISBN: 0471394424 Paperback ok $92.50 used

Course Objectives: The course is designed to introduce students to the basics of modeling and analyzing dynamic systems. A group course project is included for students to model and analyze a dynamic system. The Matlab/Simulink software package is used in this course. Topics covered include: modeling mechanical translational, mechanical rotational, electromechanical, and fluid systems, solutions of linear equations, frequency response of dynamic systems, linearization of nonlinear systems, transfer function formulation, block diagrams, dynamic performance analysis, and simulation.

Must use engineer paper or 8.5x11

Assignments: 6 - 8 homework problems will be assigned each week. These problems provide ample opportunity for learning the topics covered in class. HW's are due in class, a week after they are assigned. There will be several assignments during the semester. You are strongly encouraged to discuss the assignments with other students; however, you **must** turn in your own work. HW's may be turned in ahead of time if you know you will not be in class on a particular day. Late assignments will **not** be accepted for a grade. Work that is not neat will be returned. The lowest grade on HW will be dropped before calculating your final grade. 2pm in office Ok

Blackboard: This is not a web based course but Blackboard will be used as an alternative means of communication between the students and instructor/TA for posting lecture notes, homework solutions, grades, announcements and some useful information. You are advised to log in to your Blackboard account frequently.

Examinations: There will be three in-class midterm exams. The dates will be announced in the class atleast a week before the exam is scheduled. All examinations are closed book and notes. You are allowed the sheet of Laplace Transforms (to be posted on blackboard) and one 8.5 by 11 inch (both side) formula sheet for each examination. The formula sheets may not contain solved problems. Make up examinations will not be provided except in the case of a documented emergency. Permission to miss an examination must be obtained from the instructor prior to examination. Family emergency, University activity and health problems need to have written documentation. Those with unexcused absences will be given a zero grade for this portion of the course; the unexcused absence to the regular date of your exam will not give you the permission to come at the make up time scheduled for that particular test.

Attendance: Attendance is not mandatory. However, each lecture introduces significant new materials. If class is missed, you are responsible for obtaining announced information, handouts and notes from other students or from me. If you miss more than 2 or 3 class sessions and do not make them up, you will certainly get into difficulties.

Cheating: Cheating and/or plagiarism is unethical conduct and will not be tolerated. Anyone found to be cheating or assisting another student in cheating will automatically receive a failing grade for the course and the situation will be brought before the Office of Academic Affairs.

Grading: The following grading scale will be used unless class average requires some adjustment.
$A \geq 90\%$, $80\% \leq B < 90\%$, $70\% \leq C < 80\%$, $60\% \leq D < 70\%$, $F < 60\%$.
The percent credit for assignments, course projects, and examinations is:

Assignments	20%		
Course Project	10%		
In–Class Examinations (3)	50% (Total)	Top 2 Scores	20%
Final Examination	20%	last score	10%

Reference Books:

1). Introduction to Dynamic Systems Analysis by T. D. Burton, 1994.
2). Feedback control of Dynamic Systems by Franklin, Powell and Emami-Naeini, 2002.
3). MATLAB – The language of technical computing, Version 5, The Math Works, Inc.
4). Fundamentals of Vibrations by Leonard Meirovitch, 2000

Important: Try to understand the material as much as you can during class. Write and draw big. Plan to spend average of 3-5 hours to do each assignment. Learn how to study efficiently. Students are strongly encouraged to visit the instructor during office hours, if they have questions on the homework assignments or any related technical issues. You are welcome to stop by my office during non-office hours.

Notes 8-23

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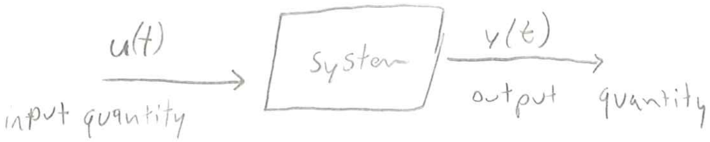
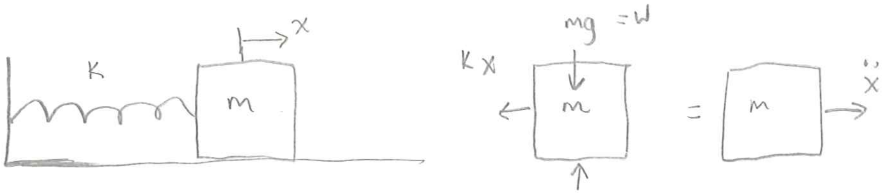
new sheet for each problem in Homework
have textbook by next friday



for this course, u(t) & y(t) are functions of time



$\xrightarrow{+}$ $\Sigma F_x = m\ddot{x}$

$-kx = m\ddot{x}$

$\boxed{m\ddot{x} + kx = 0}$ solve this diff eq., gives $x = f(t)$

(use $x = e^{\lambda t}$ where λ = const)

- review diff. eq. • method of undetermined coefficients, laplace transforms

non-linear systems require ~~a~~ numerical integration (not used in this class)

☆ copy slide "Easy way to identify ... linear or nonlinear d.e."

Notes 8-25

System Analysis
↓
modeling
↓
solve the model

for all linear ODE's

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} \dots a_1(t)\frac{dx}{dt} + a_0(t)x = g(t)$$

if $g(t) = 0$ ∴ homogeneous equation

reread ch 1 when I get textbook

chapter 2 - translational mechanical systems

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mechanical systems → translational → rectilinear motion, curvilinear motion

mechanical systems → rotational

translational + rotational → general plane motion

- variables used to describe a translational system
1) $x \to$ displacement, always measured from a fixed reference
2) $v \to$ velocity m/s, ft/s, $= \frac{dx}{dt} = \dot{x}$
3) $a \to$ accel m/s², ft/s² $= \frac{d^2x}{dt^2} = \ddot{x}$
4) $f \to$ force (N, lb) $\Sigma f_x = m\ddot{x}$
   $\Sigma f_y = m\ddot{y}$

- elements of a translational system
1) mass - represented by a particle. For eg. $\boxed{M} \to f$ $\xrightarrow{+} \Sigma F_x = \frac{d}{dt}(MV)$, $\vdash \to V$
   or $f = \frac{d}{dt}(MV)$
2) damper or friction - many types, static, coulomb, viscous
   <u>viscous friction</u>: friction between moving surfaces separated by viscous fluid
   this force is linearly proportional to velocity

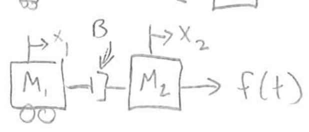
$\boxed{f_v = B(V_2 - V_1)}$ b = damping coefficient

$B = \left(\frac{NS}{m}\right)$ units

in textbook:

3) damping devices (eg shock absorbers) symbol (|]) 

 $BV \leftarrow \boxed{M} \equiv \boxed{M} \to M\frac{dv}{dt}$

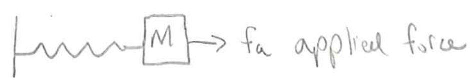


$B(\dot{x}_2 - \dot{x}_1)$ or $(V_2 - V_1)B$

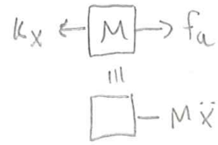
|||

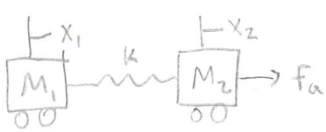
$\boxed{M_1} \to M_1\ddot{x}_1$

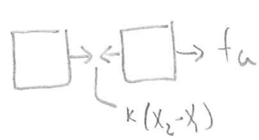
4) Spring force kx or stiffness * elongation



$\xrightarrow{+} \Sigma F_x = M\ddot{x}$ $\quad f_a - kx = M\ddot{x} \Rightarrow M\ddot{x} + kx = f_a(t)$







D-Alembert's law, a re-statement of Newton's law

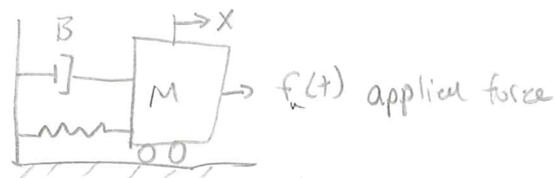
$$\Sigma f_{ext} = M \frac{dv}{dt}$$

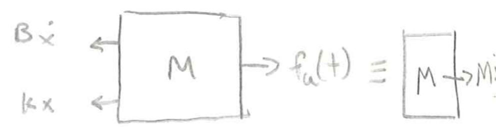
obtaining the system model

a) draw FBD

b) apply Alembert's law/Newton's 2nd law to generate math model

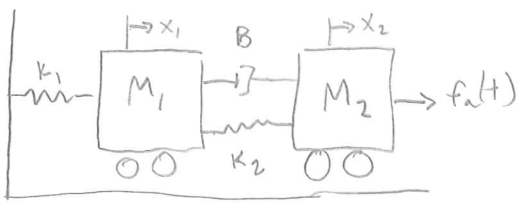
Ex-1

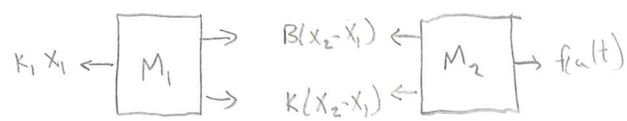




$\overset{+}{\rightarrow} \Sigma F_x = M\ddot{x}$ $\quad -B\dot{x} - kx + f_a(t) = M\ddot{x}$ $\quad M\ddot{x} + B\dot{x} + kx = f_a(t)$

Ex-2

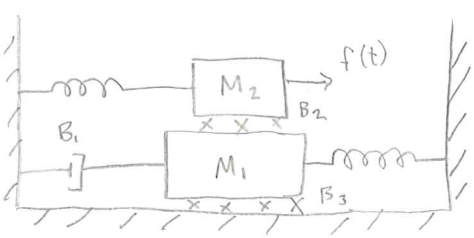




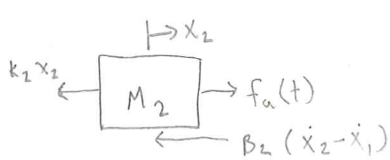
M_1: $\overset{+}{\rightarrow} \Sigma F_x = M_1\ddot{x}_1$ $\quad -k_1 x_1 + B(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) = M_1\ddot{x}_1$

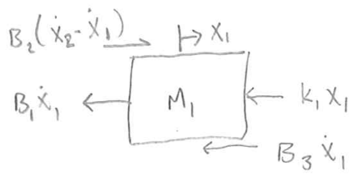
M_2 $\quad -B(\dot{x}_2 - \dot{x}_1) - k(x_2 - x_1) + f_a(t) = M_2\ddot{x}_2$

Notes 8-27



FBD of masses





Newton's law: for mass M_2 $\quad \overset{+}{\rightarrow} \Sigma F_x = M_2\ddot{x}_2$

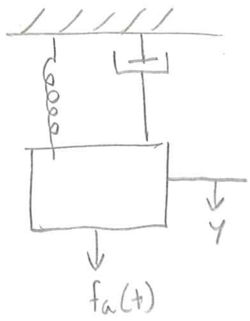
$-k_2 x_2 - B_2(\dot{x}_2 - \dot{x}_1) - M_2\ddot{x}_2 = -f_a(t) \Rightarrow f_a(t) = (M_2\ddot{x}_2 + B_2\dot{x}_2 + k_2 x_2 - B_2\dot{x}_1)$

then flip left & right sides of eq'n for proper math model

for mass M_1 $\quad \overset{+}{\rightarrow} \Sigma F_x = M_1\ddot{x}_1 \Rightarrow -B_1\dot{x}_1 + B_2(\dot{x}_2 - \dot{x}_1) - B_3\dot{x}_1 - k_1 x_1$

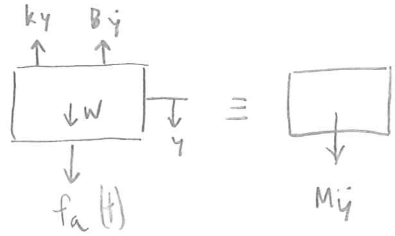
$M_1\ddot{x}_1 + B_1\dot{x}_1 - B_2(\dot{x}_2 - \dot{x}_1) + B_3(\dot{x}_1) + k_1 x_1 = 0 \Rightarrow M_1\ddot{x}_1 + (B_1 + B_2 + B_3)\dot{x}_1 + k_1 x_1 - B_2\dot{x}_2 = 0$

Notes 8-27-10

fig a) 

Translational system in vertical motion

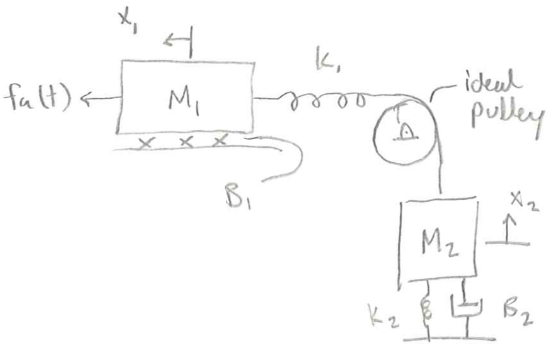
FBD of block



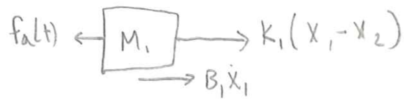
$W = mg$

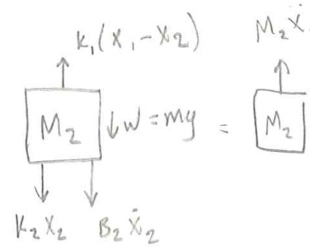
$\Sigma F_y = M\ddot{y}$ $\quad -ky - B\dot{y} + W + f_a(t) = M\ddot{y}$ $\quad \boxed{f_a(t) + W = M\ddot{y} + B\dot{y} + ky}$

at $t=0$, $f_a(t)=0$ $\quad y \to y_0$ $\quad$ at $t=0$ $\quad Mg = \cancel{M\ddot{y}}^{0} + \cancel{B\dot{y}} + ky_0 \rightarrow y_0 = \frac{Mg}{K}$ $\quad$ static eql^{m} deflection



FBD M_1 & M_2





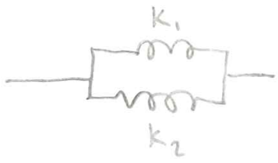
$\overset{+}{\longleftarrow}$ $\Sigma F_x = M_1\ddot{x}_1 \Rightarrow f_a(t) - K_1(x_1 - x_2) - B_1\dot{x}_1 = M_1\ddot{x}_1 \ldots$ ~~$f_a(t) = M_1\ddot{x}_1 + B_1\dot{x}_1$~~

$\ldots \boxed{M_1\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2 = f_a(t)}$

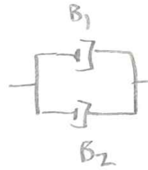
$+\uparrow \Sigma F_y = M_2\ddot{x}_2 \quad K_1(x_1 - x_2) - W - K_2x_2 - B_2\dot{x}_2 = M_2\ddot{x}_2$

$M_2\ddot{x}_2 + B_2\dot{x}_2 + x_2(K_2 + K_1) - K_1x_1 + W = 0$

Parallel combination of springs & damping devices

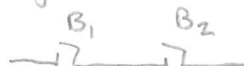


$K_{eq} = K_1 + K_2$



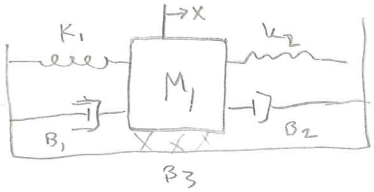
$B_{eq.} = B_1 + B_2$

Series combination of springs & damping devices



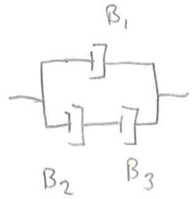
$K_{eq} = \frac{K_1K_2}{K_1 + K_2}$ $\qquad B_{eq.} = \frac{B_1B_2}{B_1 + B_2}$

these springs/dampers not in series



$M\ddot{x} + \dot{x}(B_1+B_2+B_3) + x(k_1+k_2) \equiv 0$
↳ $M\ddot{x} + B_{eq}\dot{x} + k_{eq}x = 0$

simplify



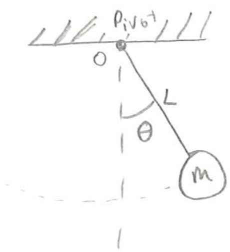
$B_{eq} = B_1 + B_4$

$B_4 = \dfrac{B_2 B_3}{B_2 + B_3}$

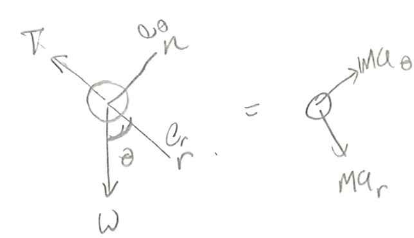
Notes 8-30-10

Translational mech. systems → curvilinear

Ex: find math model for mass system



$\Sigma F_n = ma_n$
$\Sigma F_t = ma_t$
$\left.\begin{array}{l}\Sigma F_r = ma_r \\ \Sigma F_\theta = ma_\theta\end{array}\right\}$ polar coords
θ called "transverse component"
$\Sigma F_x = ma_x$
$\Sigma F_y = ma_y$



Eq'n of motion:

$+\searrow \ \Sigma F_r = ma_r \qquad mg\cos\theta - T = ma_r$
$+\nearrow \ \Sigma F_\theta = ma_\theta \qquad mg\sin\theta = ma_\theta$ ⇒ $-mg\sin\theta = mL\ddot{\theta}$
$mL\ddot{\theta} + mg\sin\theta = 0$ dividing by mL
$\boxed{\ddot{\theta} + \frac{g}{L}\sin\theta = 0}$

e_θ & e_r are unit vectors

recall from dynamics: $V_r = \dot{r} \quad V_\theta = r\dot{\theta} \quad a_r = \ddot{r} - r\dot{\theta}^2 \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

alternative method: $M_o = \dot{H}_o = \dfrac{dH_o}{dt} = I\alpha \qquad M_o = \vec{r} \times F = Le_r \times (-Te_r + W\cos\theta\, e_r - W\sin\theta\, e_\theta)$

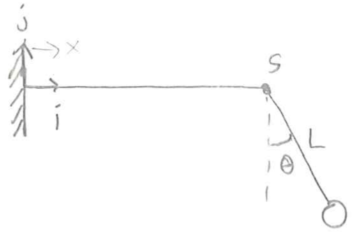
$= -TL\underbrace{(e_r \times e_r)}_{0} + WL\cos\theta\underbrace{(e_r \times e_r)}_{0} - WL\sin\theta\,(e_r \times e_\theta) = \boxed{M_o = -mgL\sin\theta\,\hat{k}}$

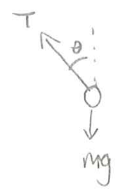
H_o = angular momentum of particle about o $\quad H_o = \vec{r} \times \vec{p} \quad p$ = linear momentum
$H_o = (\vec{r} \times m\vec{v}) \qquad H_o = (Le_r) \times m(V_\theta e_\theta + \overset{0}{V_r} e_r) = (Le_r) \times m(L\dot{\theta}e_\theta) = L^2 m\,(e_r \times e_\theta)(\dot{\theta})$
$H_o = mL^2\dot{\theta} \qquad \dot{H}_o = \dfrac{d}{dt}(H_o) = \boxed{mL^2\ddot{\theta}}$

$H_o = I\alpha: \quad I = mr^2 \quad \alpha = \ddot{\theta} \quad \Rightarrow \boxed{H_o = mr^2\ddot{\theta}}$

Notes 8-30-10

Pendulum with a moving support





$\xrightarrow{+} \Sigma F_x = m\ddot{x} \qquad -T\sin\theta = m\ddot{x} \quad (1)$

$+\uparrow \Sigma F_y = m\ddot{y} \qquad T\cos\theta - W = m\ddot{y} \quad (2)$

$\vec{r} = xi + (-L\cos\theta j + L\sin\theta i)$

$\vec{r} = (x + L\sin\theta)i + (-L\cos\theta)j$

$\dot{r} = (\dot{x} + L\dot{\theta}\cos\theta)i + (\dot{\theta}L\sin\theta)j$

$\ddot{r}\ \underbrace{(\ddot{x} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta)}_{a_x}i + L\ddot{\theta}\sin\theta \ldots$

$-T\sin\theta = m(\ddot{x} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) \ (1)$

$T\cos\theta - W = m(L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta) \quad (2)$

$\frac{(1)}{(2)} : -\frac{\cancel{T}\sin\theta}{\cancel{T}\cos\theta} \Rightarrow$ cross multiply last eq'n

$\ldots \ \ddot{x}\cos\theta + L\ddot{\theta}\cos^2\theta + L\ddot{\theta}\sin^2\theta + g\sin\theta = 0$

$\boxed{\ddot{x}\cos\theta + L\ddot{\theta} + g\sin\theta = 0}$

Notes 9-1-10

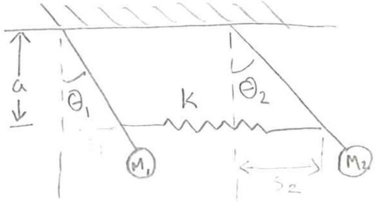
Blackboard will be up soon (today or tomorrow)

$a_1 \frac{d^2x}{dt^2} + a_2\frac{dx}{dt} + a_3 x = f(t)$

$a_2 \frac{d^2x_1}{dt^2} + \frac{a_2\, dx_1}{dt} + a_3 x_1 = f(t)$

starting to show how to prove superposition

Example



$\theta_2 > \theta_1$ at the instant shown

Consider small angles

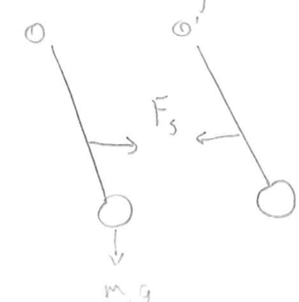
Small angles: $\sin\theta \approx 0 \qquad \sin 1^\circ \approx 0$

$\sin 1^\circ \equiv 1$

$180^\circ \equiv \pi$ radian $\qquad 1^\circ \equiv \frac{\pi}{180}$ radii $= \frac{3.14}{180}$

$\cos\theta \approx 1$

$\Delta x \approx L\theta$



$M_o = \overset{o}{\dot{H}_o}$

$= F\, d_{,\text{perpend.}}$

$S_1 = a\theta_1$

$S_2 = a\theta_2$

$\dot{H}_o = m_1 L^2\ddot{\theta}_1$ } for mass M_1

$M_o = -(m_1 g)L\sin\theta_1 + Ka(\theta_2 - \theta_1)a$

$M_o = -m_1 g L\sin\theta_1 + Ka^2(\theta_2 - \theta_1)$

$m_1 L^2\ddot{\theta} = -m_1 g L\sin\theta + ka^2(\theta_2 - \theta_1)$

$\boxed{m_1 L^2\theta + m_1 g L\sin\theta - ka^2(\theta_2 - \theta_1)}$

Notes 9-1-10

Chapter 3 — Standard forms for system model

so far: drawn FBD
↓
used Newton's / D'Alembert's law to gen. math model

in chap 3: how to convert diff eq's in standard forms (state-variable forms / state-space fm)
state var. forms converting to an n^{th} order ode to a set of 1^{st} order ODE

Formulation of state-variable form - consider an n^{th} order ode

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + a_2 \frac{d^{n-2} x}{dt^{n-2}} + \dots a_{n-1}\frac{dx}{dt} + a_n x = F(t)$$

to reduce the above D.E. into state-variable form:

(1) $x = x_1$

(2) $\frac{dx}{dt} = x_2$ — $\dot{x}_1 = x_2$ from (1) & (2)

(3) $\frac{d^2x}{dt^2} = x_3$ — $\dot{x}_2 = x_3$ from 2 & 3

(4) $\frac{d^3x}{dt^3} = x_4$ — $\dot{x}_3 = x_4$ from 3 & 4

(n) $\frac{d^{n-1}x}{dt^{n-1}} = x_n$

by following $\dot{x}_{n-1} = x_n$

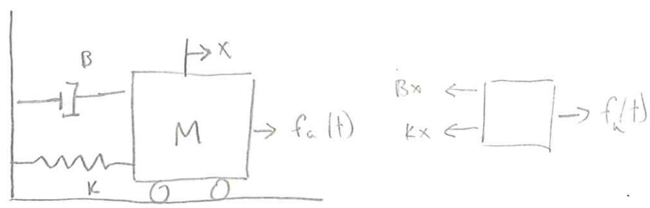
substituting these relations in the D.E.

$$\frac{d}{dt}\left(\frac{d^{n-1}x}{dt}\right) + a_1 x_n + a_2 x_{n-1} + \dots a_{n-1}x_2 + a_n x_1 = F(t)$$

$$\frac{d}{dt}(x_n) = (F(t) - a_1 x_n - a_2 x_{n-1} \dots a_{n-1}x_2 - a_n x_1)$$

$$\dot{x}_n = F(t) - a_1 x_n - a_2 x_{n-1} \dots - a_{n-1}x_2 - a_n x_1$$

Ex.



$M\ddot{x} + B\dot{x} + kx = f_a(t)$ — $x = x_1$, $\frac{dx}{dt} = x_2$

$M\dot{x}_2 + Bx_2 + kx_1 = f_a(t)$

$M\dot{x}_2 = -Bx_2 - kx_1 + f_a(t)$ ⇒ (1) $\dot{x}_1 = x_2$

(2) $\dot{x}_2 = \frac{1}{M}(f_a(t) - Bx_2 - kx_1)$

x_1 & x_2 → state variable

☆ can express answer in terms of x & v instead of x_1 & x_2 ☆

Notes 9-3-10

Math Model

$$M_1\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2 = f_a(t) \quad (1)$$
$$M_2\ddot{x}_2 + B_2\dot{x}_2 + (K_1+K_2)x_2 - K_1x_1 + W = 0 \quad (2)$$

- put these eq'ns in state variable form

$$\begin{cases} x_1 \\ v_1 = \dfrac{dx_1}{dt} \\ x_2 \\ v_2 = \dfrac{dx_2}{dt} \end{cases}$$

$\dot{x}_1 = v_1$ (1)

$\dot{x}_2 = v_2$ (2)

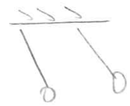
1st eqn above yields

$$M_1\dot{v}_1 + B_1v_1 + K_1x_1 - K_1x_2 = f_a(t)$$
(3) $$\dot{v}_1 = \frac{1}{M_1}\left[f_a(t) - B_1v_1 - K_1x_1 + K_1x_2\right]$$

2nd eqn above yields

$$M_2\dot{v}_2 + B_2v_2 + (k_1+k_2)x_2 - k_1x_1 + W = 0$$
(4) $$\dot{v}_2 = \frac{1}{M}\left[-B_2v_2 - (k_1+k_2)x_2 + k_1x_1 - W\right]$$

Set (1)(2)(3)(4) gives state variable model / state space eq'n's

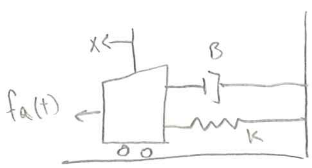
Using  example 2 pages back

$$\begin{rcases} \theta_1 \\ \omega_1 = \dfrac{d\theta_1}{dt} \\ \theta_2 \\ \omega_2 = \dfrac{d\theta_2}{dt} \end{rcases} \qquad \begin{rcases} \dot{\theta}_1 = \omega_1 \\ \dot{\theta}_2 = \omega_2 \end{rcases}$$

$$\begin{rcases} \dot{\omega}_1 = \dfrac{1}{m_1L^2}\left(a^2k(\theta_2-\theta_1) - m_1gL\theta_1\right) \\ \dot{\omega}_2 = \dfrac{1}{m_2L^2}\left(a^2k(\theta_1-\theta_2) - m_2gL\theta_2\right) \end{rcases}$$

Matrix formulation of state variable eqn - we can represent the state-var. eq'n by a single matrix D.E. (in compact form) let the state var. eq'n of an n'th order D.E. be

$$\dot{q}_1 = a_{11}q_1 + a_{12}q_2 + \dots a_{1n}q_n + b_{11}u_1 + b_{12}u_2 \dots b_{1m}u_m \quad (1)$$
$$\dot{q}_2 = a_{21}q_1 + a_{22}q_2 \dots \quad (2)$$
$$\vdots$$
$$\dot{q}_n = a_{n1}q_1 \quad (n)$$



where $q_1, q_2, q_3 \dots$ are the state variables

a_{ij} & b_{ij} are constants

u_1, u_2, u_m are inputs

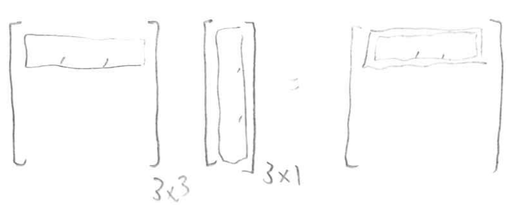
Matrix review $[A]_{n\times m}[B]_{p\times q} \equiv [C]_{n\times q}$ $\quad m = p$

$n\times m$ = rows x columns

$[A]_{n\times n}$ square matrix

$A = \begin{bmatrix} a_1 \\ a_2 \\ a_n \end{bmatrix}$ = column matrix

$A = [a_1 \ a_2 \dots]$ = row matrix

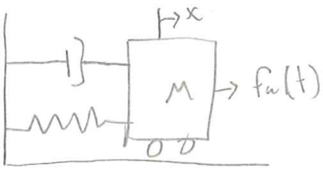


continued:

$$\underset{n\times 1}{\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \\ \vdots \end{bmatrix}} = \underset{n\times n}{\begin{bmatrix} a_{11} & a_{12} & a & \cdots \\ & & & \\ & & & \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}} + \underset{n\times m}{\begin{bmatrix} b & \cdots \\ & \\ & \end{bmatrix}} \underset{m\times 1}{\begin{bmatrix} u_1 \\ \vdots \end{bmatrix}}$$

$$[\dot q] = [A]\cdot[q] + [B][u]$$

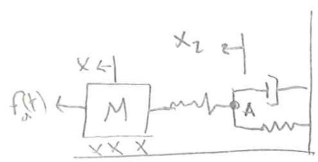
$\dot q = Aq + Bu$ where A & B → coefficient matrix



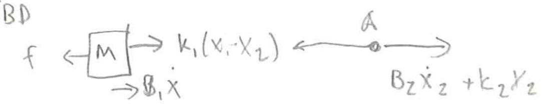
state var. model $\dot x = v$ $\quad \dot v = \frac{1}{m}\left(f_a(t) - Bv - kx\right)$

$q = \begin{Bmatrix} x \\ v \end{Bmatrix}$ $\quad \dot q = \begin{Bmatrix} \dot x \\ \dot v \end{Bmatrix}$ $\quad u = f_a(t)$

$$\begin{bmatrix} \dot x \\ \dot v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/M & -B/M \end{bmatrix}}_{A} \begin{Bmatrix} x \\ v \end{Bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/M \end{bmatrix}}_{B} f_a(t)$$



FBD



For mass ... $\boxed{M\ddot x_1 + B_1\dot x_1 + k_1x_1 - k_1x_2 = f_a(t)}$

For pt. $\boxed{B_2\dot x_2 + (k_1 + k_2)x_2 - k_1x_1 = 0}$

* $M_1\dot v_1 + B_1v_1 + K_1X_1 - K_1X_2 = f_a(t) \rightarrow \dot v_1 = \ldots$ (use this form to do matrix)

*

x_1 $\quad \dot x_1 = v_1$

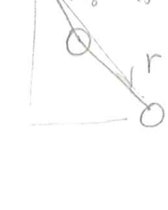
$v_1 = \frac{dx_1}{dt}$

x_2 $\quad \dot v_1 = \frac{1}{M}\left[f_a(t) - B_1v_1 - k_1x_1 + k_1x_2\right]$

$$\begin{bmatrix} \dot x_1 \\ \dot v_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k_1}{M} & -\frac{B_1}{m} & \frac{k_1}{M} \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \\ 0 \end{bmatrix} f_a(t)$$

$\dot q = Aq + Bu$

* Homework hint: Outputs cannot have any terms in their eq's that are not state variables



Notes 9-8-10

last topic Ch 3

$\dot{q} = Aq + Bu$

$\downarrow$ state vars; $Bu \to$ inputs

Output eq'ns: the state var. model in matrix form is usually written along with the output eq's. These outputs are separated. The output eq's have the form:

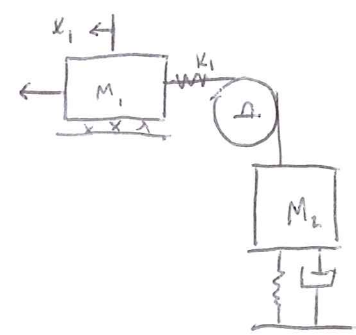
$$y_1 = c_{11}q_1 + c_{12}q_2 + \dots c_{1n}q_n + d_{11}u_1 + d_{12}u_2 \dots d_{1m}u_m$$
$$y_2 = c_{21}q_1 + c_{22}q_2 + \dots \dots d_{22}u_2 \dots d_{2m}u_m$$
$$\vdots$$
$$y_p = c_{p1}q_1 + \dots c_{pn}q_n + \dots d_{pm}u_m$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pn} \end{bmatrix}}_{\substack{p \times n \\ C}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_p \end{bmatrix}}_{\substack{n \times 1 \\ q}} + \underbrace{\begin{bmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ d_{21} & d_{22} & \dots & d_{2m} \\ \vdots & & & \vdots \\ d_{p1} & d_{p2} & \dots & d_{pm} \end{bmatrix}}_{\substack{p \times m \\ D}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}}_{u}$$

$y = Cq + Du$

$$\boxed{\begin{aligned} \dot{q} &= Aq + Bu \\ y &= Cq + Du \end{aligned}}$$

Example: Eq'ns
$M_1\ddot{x}_1 + B_1\dot{x}_1 + k_1x_1 - k_1x_2 = fa(t)$
$M_2\ddot{x}_2 + B_2\dot{x}_2 + (k_1 + k_2)x_2 - k_1x_1 + W = 0$



rewrite state var's in
a) matrix form (fa & g are inputs)
b) output eq'ns taking deformation in spring k_1 & viscous force in B_2 as outputs

$\dot{x}_1 = v_1$
$\dot{v}_1 = \frac{1}{M_1}[fa(t) - B_1v_1 - k_1x_1 + k_2x_2]$
$\dot{x}_2 = v_2$
$\dot{v}_2 = \frac{1}{M_2}[-B_2v_2 + k_1x_1 - (k_1 + k_2)x_2 - W]$

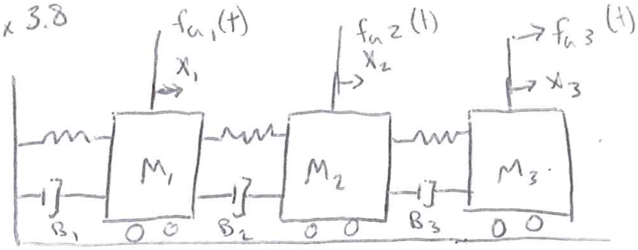
$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{M_1} & -\frac{B_1}{M_1} & \frac{k_1}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{M_2} & 0 & \frac{-(k_1+k_2)}{M_2} & -\frac{B_2}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{M_1} & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} fa(t) \\ g \end{bmatrix}$$

b) output eq'ns, taking deformation

$y_1 = x_1 - x_2 \qquad y_2 = B_2\dot{x}_2$ or B_2v_2

$$\underset{2 \times 1}{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & B_2 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} fa(t) \\ g \end{bmatrix}$$

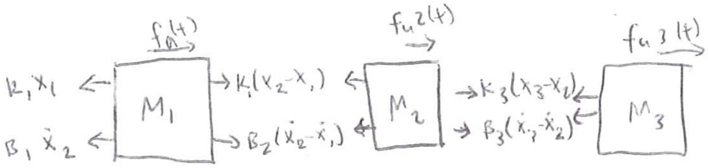
Ex 3.8



Given: output is total momentum of the system

Find a) state var. model

b) output eqns



x_1

$V_1 = \dot{x}_1$

x_2

$V_2 = \dot{x}_2$

x_3

$V_3 = \dot{x}_3$

$$M_1 \ddot{x}_1 + (B_1 + B_2)\dot{x}_1 + (k_1 + k_2)x_1 - B_2\dot{x}_2 - k_2 x_2 = f_{a1}(t)$$

$$M_2 \ddot{x}_2 + (B_2 + B_3)\dot{x}_2 + (k_2 + k_3)x_2 - B_2\dot{x}_1 - k_2 x_1 - B_3\dot{x}_3 - k_3 x_3 = f_{a2}(t)$$

$$M_2 \ddot{x}_3$$

$$\dot{x}_1 = V_1$$

$$\dot{V}_1 = \frac{1}{m_1}[f_{a1}(t) - (k_1 + k_2)x_1 - (B_1 + B_2)V_1 + k_2 x_2 + B_2 V_2$$

$$\dot{x}_2 = V_2$$

$$\dot{V}_2 = \frac{1}{m_2}[f_{a2}(t) - k_2 x_1 + B_2 V_1 - (k_2 + k_3)x_2 - (B_2 + B_3)V_2 + k_3 x_3 + B_3 V_3$$

$$\dot{x}_3 = V_3$$

$$\dot{V}_3 = \frac{1}{m_3}(f_{a3} + k_3 x_2 + B_3 V_2 - k_3 x_3 - B_3 V_3)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{V}_1 \\ \dot{x}_2 \\ \dot{V}_2 \\ \dot{x}_3 \\ V_3 \end{bmatrix} = \underset{6\times 6}{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}} \underset{6\times 1}{\begin{bmatrix} x_1 \\ V_1 \\ x_2 \\ V_2 \\ x_3 \\ V_3 \end{bmatrix}} + \underset{6\times 3}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}} \begin{bmatrix} f_{a1} \\ f_{a2} \\ f_{a3}(t) \end{bmatrix}$$

$$Y(\text{output}) = m_1\dot{x}_1 + m_2\dot{x}_2 + m_3\dot{x}_3$$
$$= m_1 V_1 + m_2 V_2 + m_3 V_3$$

$$\underset{1\times 1}{[Y]} = \underset{1\times 6}{[0 \; M_1 \; 0 \; M_2 \; 0 \; M_3]} \underset{6\times 1}{\begin{bmatrix} x_1 \\ V_1 \\ x_2 \\ V_2 \\ x_3 \\ V_3 \end{bmatrix}} + [0 \;\; 0 \;\; 0] \begin{bmatrix} f_{a1} \\ f_{a2} \\ f_{a3} \end{bmatrix}$$

Casey Brennan

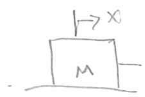
11-12 wednesdays   casey has "office" hours
will be in Toomey 2nd floor hallway

Notes 9-10-10

Ch. 5 rotational mechanical systems

Mechanical sys
- Translational (rectilinear & curvilinear
- Rotational - general plane motion

Variables

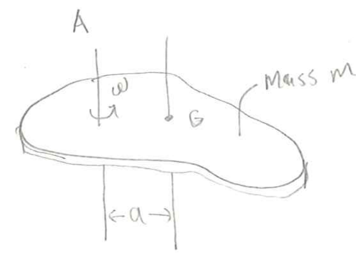


x - displacement
$\dot{x}$ - velocity
$\ddot{x}$ - accel
$\Sigma F = ma$

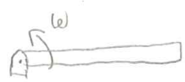
θ - angular displacement
$\dot{\theta} = \frac{d\theta}{dt} = \omega$ - angular velocity
$\ddot{\theta} = \alpha$ = angular accel
$\Sigma \tau = J\alpha$ sum of moments, J = moment of inertia (can use "I")

Parallel axis theorem

$J_{parallel\ axis}\ (J_{AA'}) = J_0 + Ma^2$



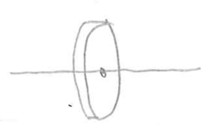
G: centroid



$J_0 = \frac{ML^2}{12}$

$= J_0 + Ma = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}$

uniform disk

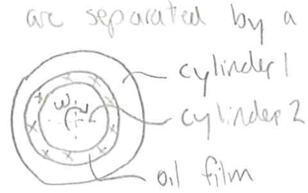


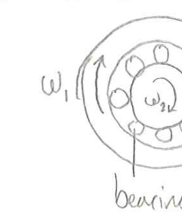
$J_0 = \frac{1}{2}MR^2$

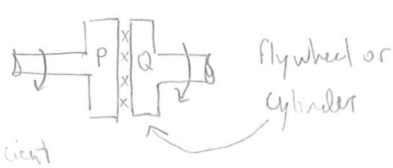
Elements of a rotational system

translational	rotation
1) mass	1) Moment of inertia (J)
2) stiffness	2) rotational friction
3) damping	3) rotational stiffness
	4) levers
	5) gears

2) rotational friction: when 2 rotating bodies are separated by a thin oil film

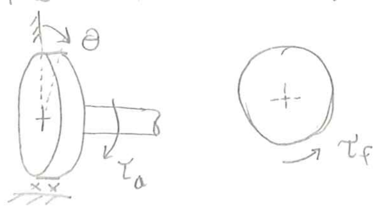




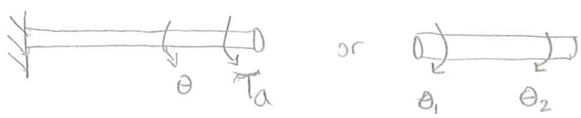


The friction torque, $\tau_f = B(\omega_2 - \omega_1)$
$\tau_f = B(\omega_2 - \omega_1) = B\Delta\omega$ where $B \Rightarrow$ friction coefficient
B given in N·M·S   Direction of τ_f: always opposite to system's rotation

Torque Friction continued: Notes 9-10-16

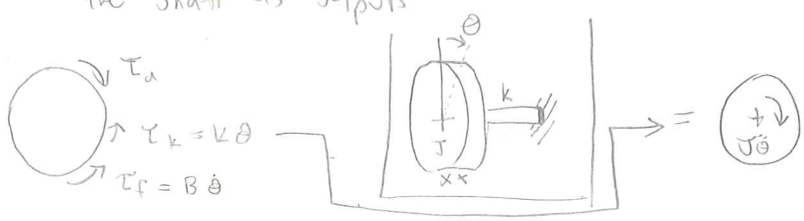


3) rotational stiffness a flexible/elastic spring that works by torsion or twisting. a straight bar of rubber or a thin shaft is also considered a torsion spring.



$\tau_k = k\,\Delta\theta = k(\theta_2 - \theta_1)$ rotational torque

Example: derive the math model for the system shown. write state variable form write output equations, taking angular accelleration α and torque exerted by the shaft as outputs



θ
$\omega = \dot\theta$

$\dot\theta = \omega$

$+\circlearrowright\ \Sigma\tau = J\ddot\theta \qquad \tau_a - B\dot\theta - k\theta = J\ddot\theta \qquad J\ddot\theta + B\dot\theta + k\theta = \tau_a$

$\dot\omega = \frac{1}{J}\left[\tau_a - B\omega - k\theta\right)$

$$\underbrace{\begin{bmatrix}\dot\theta\\ \dot\omega\end{bmatrix}}_{\dot q} = \underbrace{\begin{bmatrix}0 & 1\\ -\frac{k}{J} & -\frac{B}{J}\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\theta\\ \omega\end{bmatrix}}_{q} + \underbrace{\begin{bmatrix}0\\ 1/J\end{bmatrix}}_{B}\underbrace{\tau_a}_{u}$$

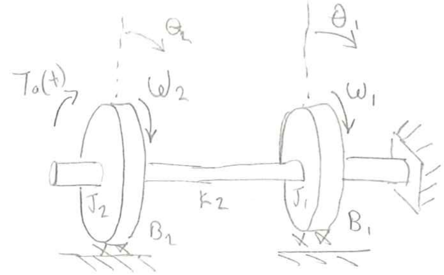
$J\alpha + B\omega + k\theta = \tau_a$

$\left.\begin{aligned} Y_1 &= \alpha = \tfrac{1}{J}\left[\tau_a - B\omega - k\theta\right] \\ Y_2 &= k\theta \end{aligned}\right\}$ outputs

$$\begin{bmatrix}Y_1\\ Y_2\end{bmatrix} = \underbrace{\begin{bmatrix}\alpha\\ \tau_k\end{bmatrix}}_{Y}\begin{bmatrix}-k/J & -B/J\\ k & 0\end{bmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix}$$

ex:

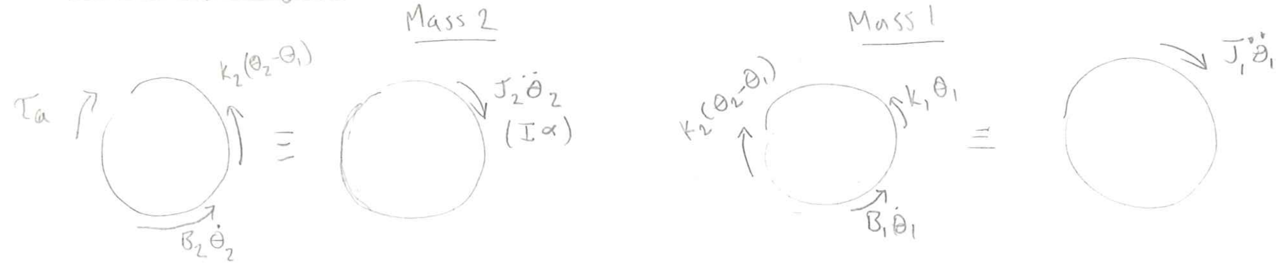
Notes 9-13-10



To find: a) D. E. for J_1 & J_2

b) output eq's when output is (τ_{k_2}) and total angular momentum

Free body diagram



$\circlearrowleft + \Sigma\tau = J_2\ddot{\theta}_2$ $\quad \tau_a - k_2(\theta_2-\theta_1) - B_2\dot{\theta}_2 = J_2\ddot{\theta}_2 \Rightarrow \boxed{J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + k_2\theta_2 - k_2\theta_1 = \tau_a}$ (1)

for disc 1 $\quad \circlearrowleft + \Sigma\tau = J_1\ddot{\theta}_1 \quad \ldots \quad \boxed{J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + (k_1+k_2)\theta_1 - k_2\theta_2 = 0}$ (2)

State vars

$\theta_1 \qquad \theta_2$

$\omega_1 = \dot{\theta}_1 \qquad \omega_2 \to \dot{\theta}_2$

$\dot{q} = aq + Bu$

$u = \tau_a(t)$

$\dot{\theta}_1 = \omega_1$

$\dot{\omega}_1 = \frac{1}{J_1}\left[-(k_1+k_2)\theta_1 - B_1\omega_1 + k_2\theta_2\right]$

$\dot{\theta}_2 = \omega_2$

$\dot{\omega}_2 = \frac{1}{J_2}\left[k_2\theta_1 - k_2\theta_2 - B_2\omega_2 + \tau_a\right]$

$$\begin{bmatrix}\dot{\theta}_1\\ \dot{\omega}_1\\ \dot{\theta}_2\\ \dot{\omega}_2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ \frac{-k_1+k_2}{J_1} & \frac{-B_1}{J_1} & \frac{k_2}{J_1} & 0\\ 0 & 0 & 0 & 1\\ \frac{k_2}{J_2} & 0 & \frac{-k_2}{J_2} & \frac{-B_2}{J_2}\end{bmatrix}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1/J_2\end{bmatrix}\tau_a(t)$$

$y_1 = \tau_{k_2} = k_2(\theta_2-\theta_1)$

$y_2 =$

$H_{total} = J_1\omega_1 + J_2\omega_2$

$$\begin{bmatrix}\tau_{k_2}\\ H_{total}\end{bmatrix} = \underbrace{\begin{bmatrix}-k_2 & 0 & k_2 & 0\\ 0 & J_1 & 0 & J_2\end{bmatrix}}_{C}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 0\end{bmatrix}}_{D}\tau_a$$

Note

$\dot{H}_0 = I\alpha$

$= I\frac{d}{dt}(\omega)$

$H = I\omega$ OR $J\omega$

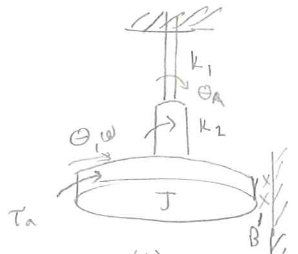
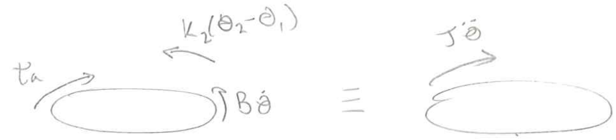


Fig (1)



$$J\ddot{\theta} + B\dot{\theta} + k_2(\theta - \theta_a) = \tau_a$$

To eliminate θ_a, use

$$J\ddot{\theta} + B\dot{\theta} + K_{eq}\theta = \tau_a \qquad \left(k_{eq} = \frac{k_1 k_2}{k_1 + k_2}\right)$$

Lever: A rigid bar pivoted at a point (fixed)

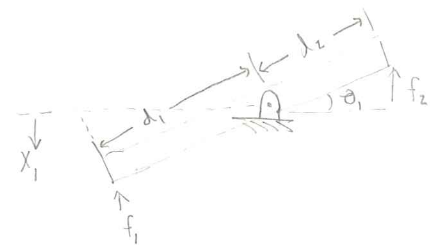
for this coursework: a) Pt. O (pivot) is fixed

b) angle of rotation is small (θ) $\Rightarrow \sin\theta \approx \theta$, $\cos\theta \approx 1$

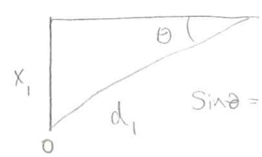
c) since θ is small, motion of the ends can be considered strictly translational

d) the bar is rigid

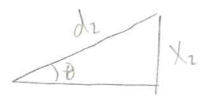
e) lever mass may or may not be neglected f, as per problem statement



Useful relations:

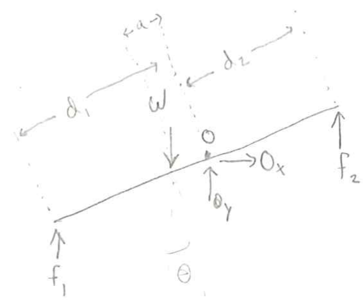


$\sin\theta = \theta = \frac{x_1}{d_1}$



$\sin\theta = \theta = \frac{x_2}{d_2}$

$$\theta = \frac{x_1}{d_1} = \frac{x_2}{d_2} \Rightarrow x_2 = \left(\frac{d_2}{d_1}\right)x_1 \quad (1) \qquad V_2 = \left(\frac{d_2}{d_1}\right)V_1 \quad (2)$$



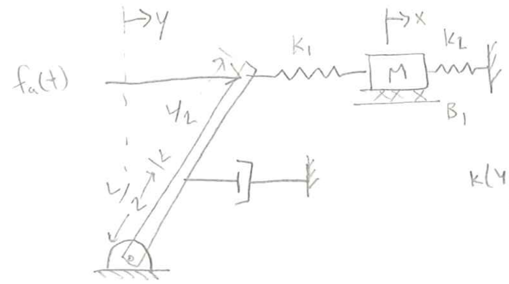
Also, sum of moments about pivot O $\circlearrowleft + (\Sigma\tau)_o = J_o\alpha$

$$-f_1(d_1\cos\theta) + W\cos\theta\left(\frac{d_1 - d_2}{2}\right) + f_2(d_2\cos\theta) = J_o\ddot{\theta}$$

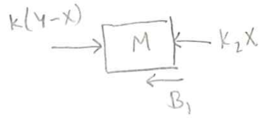
$$\ldots \quad \boxed{f_1 d_1 = f_2 d_2}$$

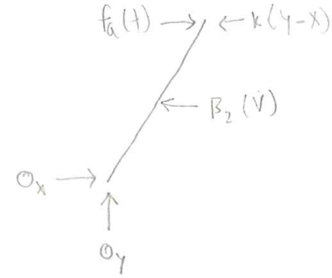
$$a = d_1 - \left(\frac{d_1 - d_2}{2}\right) = \left(\frac{d_1 - d_2}{2}\right)$$

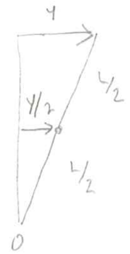
Notes 9-15-10



Ignore mass of the lever. assumption $y > x$







For mass $m \rightarrow \Sigma F_x = M\ddot{x}$ $\quad k_1(y-x) - k_2 x \ldots$

$M\ddot{x} + B_1\dot{x} + (k_1+k_2)x - k_1 y = 0$

$\Sigma \tau_0 = J_0 \alpha = 0 \qquad -\left(B_2\frac{\dot{y}}{2}\right)\left(\sqrt{\frac{L^2}{4}-\frac{y^2}{4}}\right) - k(y-x)\left(\sqrt{L^2-y^2}\right) + f_a(t)\sqrt{L^2-y^2} = 0$

Since θ is small, y is small, $y^2 \to 0$

$-B_2\left(\frac{\dot{y}}{2}\right)\left(\frac{L}{2}\right) - k_1(y-x)L + \underbrace{f_a(t)\ L}_{} = 0$

$B_2 \frac{L}{4}\dot{y} + [k_1 y - k_1 x]\cancel{L} = f_a(t)\cancel{L}$

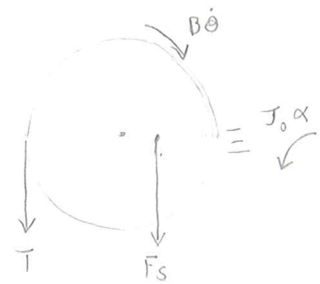
$\frac{B_2 \dot{y}}{4} + k_1 y - k_1 x = f_a(t)$

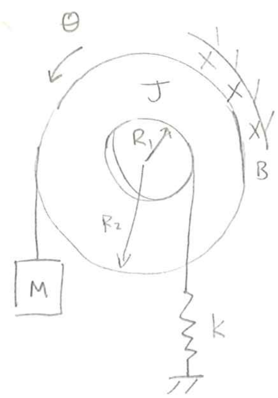
3 state vars

x	$\dot{x} = v$
$v \Rightarrow \dot{x}$	$\dot{v} = \frac{1}{m}\left[-B_1 v - (k_1+k_2)x + k_2 y\right]$
y	$\dot{y} = \frac{4}{B_2}\left[f_a(t) + k_1 x - k_1 y\right]$

Text p. 5.25   assume small θ

Find. D.E. describing the system





For drum $\circlearrowright + T = J_0\ddot{\theta}$

$T(R_2) - B\dot{\theta} - F_s(R_1) = J_0\ddot{\theta}$

& $F_s = k(R_1\theta)$

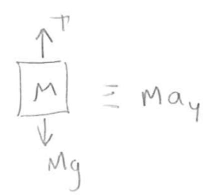
$T(R_2) - B\dot{\theta} - F_s(R_1) = J_0\ddot{\theta}$

& $a_t = R_2\alpha$

$\downarrow \Sigma F_y = Ma_y$

$-T + Mg = Ma_y$

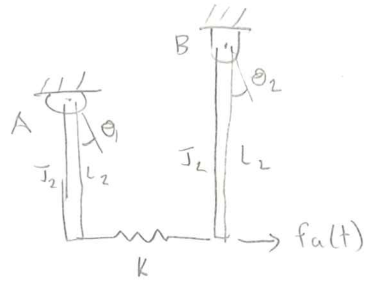
$T = (Mg - Ma_y) = Mg - M(R_2\ddot{\theta})$



$(Mg - MR_2\ddot{\theta})R_2 - B\dot{\theta} - k\theta R^2 = J_0\ddot{\theta}$

Only 1 eq. $\quad (J + MR_2^2)\ddot{\theta} + B\dot{\theta} + kR_1^2\theta = R_2 Mg \qquad$ State vars: $\theta, \dot{\theta}$

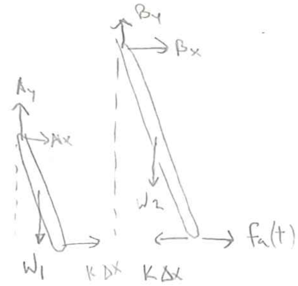
Notes 9-15-10



assume) $\theta_2 > \theta_1$

angular displacements are small

FBD



$\Delta x = \theta_2 L_2 - \theta_1 L_1$

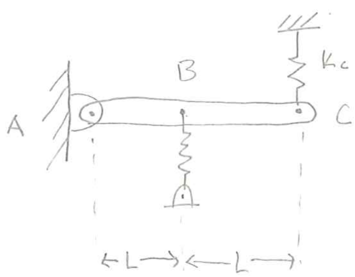
$+\circlearrowright \Sigma \tau_A = J_A \ddot{\theta}$

Answer: $J_A \ddot{\theta}_1 + (KL_1^2 + M_1 g \frac{L_1}{2})\theta_1 - K_1 L_1 L_2 \theta_2 = 0$

rest of this answer ON Blackboard

For B, $\circlearrowright + \Sigma \tau_B = J_B \ddot{\theta}_2$

Notes 9-17-10



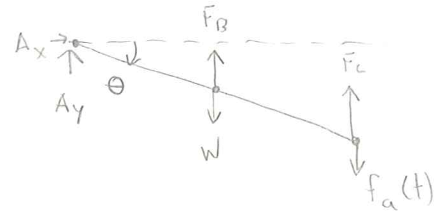
Given: mass 5kg pivot A is fixed

$k_B = 500\ N/m$

$k_c = 620\ N/m$

θ is small

$L = .7m$



$+\circlearrowleft \Sigma T_A = J\ddot{\theta}$

$W(L\cos\theta) + f_a(t)[2L\cos\theta]$ A_y $- F_b(L\cos\theta) - F_c[2L\cos\theta] = J\ddot{\theta}$

Small θ, $\cos\theta \simeq 1$ $F_B = k_B(L\sin\theta + Y_{eq})$ $F_c = k_c(2L\sin\theta + Y_{eq})$

$= k_B(L\theta + Y_{eq})$ $= k_c[2L\theta + Y_{eq})$

$WL + f_a(t)2L - k_B(L\theta + Y_{eq})(L) - k_c(2L\theta + Y_{eq})2L$

$\theta[-k_B L^2 - 4L^2 k_c] + WL + 2f_a L - k_B Y_{eq} L - 2k_c Y_{eq} L = J_A\ddot{\theta}$ At $\theta = 0$, $f_a(t) = 0$ } 2

then $\boxed{WL - k_B Y_{eq} L - 2k_c Y_{eq} L = 0}$, $\theta[-k_B L^2 - 4L^2 k_c] + 2f_a L = J_a\ddot{\theta}$

$J_a\ddot{\theta} + [k_B + 4k_c^2]L^2\theta = 2f_a L$ or $\boxed{\frac{J_a\ddot{\theta}}{2L} + \left[\frac{k_B + 4k_c}{2}\right]L\theta = f_a}$

$$\boxed{\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= \frac{2L}{J_a}\left[f_a - \left(\frac{k_B + 4k_c}{2}\right)L\theta\right]\end{aligned}}$$

Exam covers all material up to here

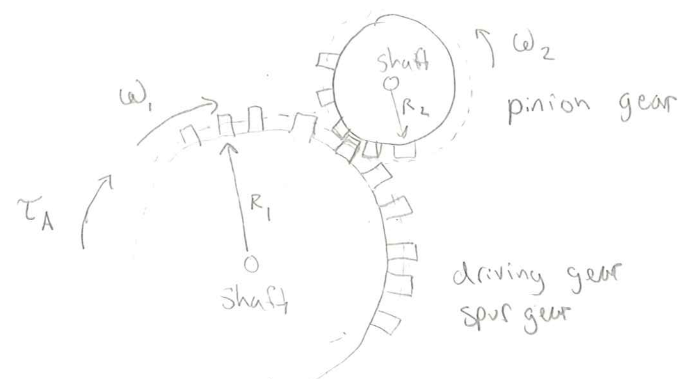
Notes 9-17-10

211 Exam 1   9/24   10-10:50 (Friday)

closed book/notes   Ch 1,2,3,5 (gears not included)
HW 1-4
3 problems
1 cheat sheet (staple it to test)

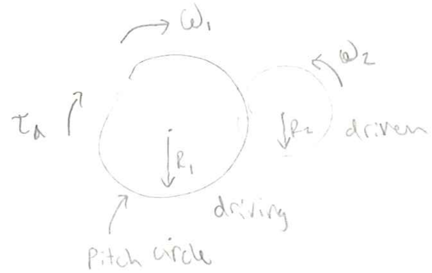
Office hrs
Tues 4-5
Wed → 12-4
Th 3:30 - 4:30
F → 9-10am

Gears:

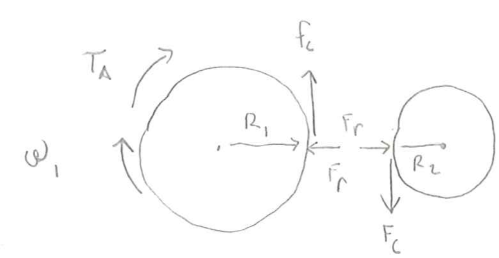


Gears
used to transmit power or torque
For this coursework we will assume: a) no moment of inertia
b) no friction
c) perfect meshing of teeth

For analysis, visualize 2 ideal gears as 2 pitch circles



At pitch point, P gear force → tangential component used to transmit power
→ radial component to bend gear shaft

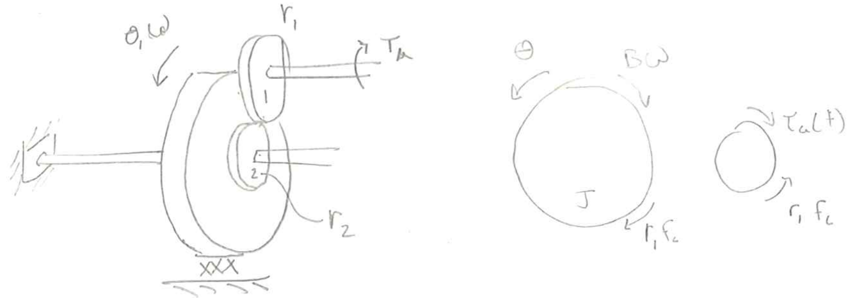


Important results for a gear
1) Radius proportional to # of teeth $\boxed{\frac{R_2}{R_1} = \frac{n_2}{n_1} = N(\text{Gear ratio}) = \frac{\omega_1}{\omega_2} = \frac{\theta_1}{\theta_2}}$

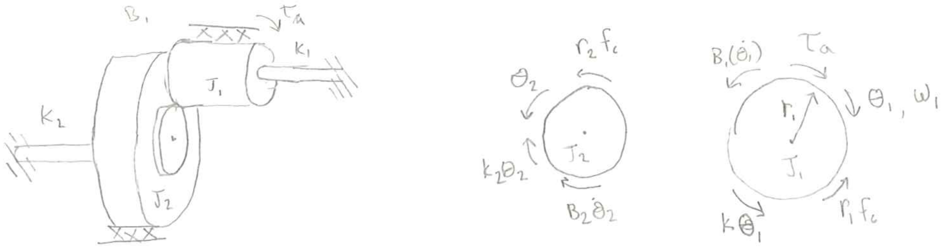
2) since the velocity at mesh point is same, $\frac{R_2}{R_1} = \frac{\omega_1}{\omega_2}$ $\boxed{V = R_1\omega_1 = R_2\omega_2}$

Example

answer on BB



$+\circlearrowleft \Sigma \tau = J\ddot{\theta}$



Eq for J_1 $\curvearrowright \Sigma \tau = J_1 \ddot{\theta}_1$

$$\tau_a - B\dot{\theta}_1 - k\theta_1 - r_1 f_c = J_1 \ddot{\theta}_1$$

J_2: $\downarrow + \quad \Sigma \tau = J_2 \ddot{\theta}_2$

$$-k_2\theta_2 - B_2\dot{\theta}_2 + r_2 f_c = J_2\ddot{\theta}_2 \Rightarrow \frac{J_2\ddot{\theta} + B_2\dot{\theta}_2 + k_2\theta_2}{r_2} = \cancel{r_2} f_c$$

$$J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + k_1\theta_1 + r_1\left(\frac{J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + k_2\theta_2}{r_2}\right) = \tau_a$$

Now, $\frac{\theta_1}{\theta_2} = N \Rightarrow \dot{\theta}_1 = N\dot{\theta}_2$

$\ddot{\theta}_1 = N\ddot{\theta}_2$

$$J_1 N\ddot{\theta}_2 + B_1(N\dot{\theta}_2) + k_1 N\theta_2 + \frac{1}{N}\left(J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + k_2\theta_2\right) = \tau_a$$

$$\ddot{\theta}_2\left(J_1 N + \frac{J_2}{N}\right) + \dot{\theta}_2\left(B_1 N + \frac{B_2}{N}\right) + \theta_2\left(k_1 N + k_2/N\right) = \tau_a$$

$$\underbrace{(J_1 N^2 + J_2)}_{J_{eq}}\ddot{\theta}_2 + \underbrace{(B_1 N^2 + B_2)}_{B_{eq}}\dot{\theta}_2 + \underbrace{(k_1 N^2 + k_2)}_{k_{eq}}\theta_2 = \tau_a N$$

$$J_{eq}\ddot{\theta}_2 + B_{eq}\dot{\theta}_2 + k_{eq}\theta_2 = \tau_a N$$

<u>State vars</u> θ_2
$\omega_2 \to \dot{\theta}_2$

these can be converted to eqn for 1st mass by using $\frac{\dot{\theta}_1}{N} = \frac{\omega_1}{N_1}$

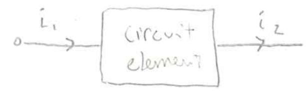
$$\begin{cases} \dot{\theta}_2 = \omega_2 \\ \dot{\omega}_2 = \frac{1}{J_{eq}}\left[\tau_a N - B_{eq}\omega_2 - K_{eq}\theta_2\right] \end{cases}$$

Ch. 6 Electrical Systems

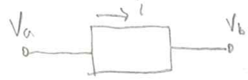
objectives: To model e. circuits by using ODE's (application of kirchoff's law)

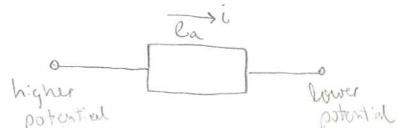
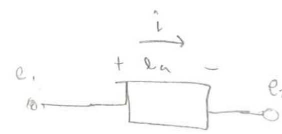
System variables:

1) charge, q: characteristic of matter (Two kinds, + & −) units: coulombs (C)
   proton, $e = 1.6 \times 10^{-19}$ C, electron $-e$
2) current, i: rate at which charge flows through given cross section
   measured in amperes $i = \frac{dq}{dt} = \dot{q}$ Ampere, A = $\frac{1 \text{ coulomb}}{\text{second}}$

   convention for denoting current  $i_1 = i_2$

3) Voltage, V: measured in volts. Voltage between 2 ends of a path is the work done to move an electric charge q along a path


$e = V_a - V_b$ where $V_a > V_B$

convention for denoting voltages:  or 

$e_1 - e_2 = e_a$

power supplied to a circuit element ~~$=$~~ $= e \cdot i = P$

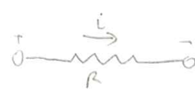
$\boxed{i = \frac{dq}{dt} = \dot{q}}$ $\boxed{e = V_a - V_b}$ $\boxed{P = ei}$ $\boxed{P_R = i^2 R \quad P_R = \frac{e^2}{R}}$ $\boxed{q = Ce \quad i = C\frac{de}{dt}}$ $\boxed{W_c = \frac{1}{2}Ce^2}$

Elements of a circuit

1) resistors
2) capacitors
3) inductors

} passive elements: cannot introduce new energy, but can store/dissipate energy

4) sources → Active element (like input)

resistor: dissipates energy 

for a linear resistor, voltage is proportional to current $e = Ri$ (Ohm's law)

R → resistance

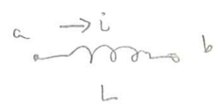
Power $p = ei = (Ri)(i) = Ri^2$ $\quad p = ei = e\left(\frac{e}{R}\right) = \frac{e^2}{R}$

capacitor: energy conserving device which stores electrical energy

for a linear capacitor $q = Ce$ where C → capacitance $\frac{dq}{dt} = C\frac{de}{dt} = i$

C, measured in Farads $= \frac{1 \text{ coulomb}}{\text{volt}}$ energy stored in capacitor $W = \frac{1}{2}Ce^2$

inductor: stores energy for linear inductor voltage drop $e = L\frac{di}{dt}$ L → inductance



Review for exam 1 Notes 9-22-10

Linear & Nonlinear Systems

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \dots a_1 \dots \quad g(t)$$

Important a) the coefficients a_i can be constants or a function of time

b) the presence of time dependent term has no effect on linear/nonlinear linearity is governed by the way states/dependent vars appear

$\ddot{x} + x\sin(t) = 0 \rightarrow$ linear

$\ddot{x} + (\sin x) = 0 \rightarrow$ Non linear $(\sin x)$

$\ddot{x} + x + x^3 = 0 \rightarrow$ non linear

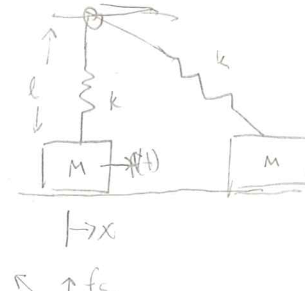
c) there is only one independent variable for an ODE however there can be multiple dependent variables & their derivatives (w.r.t time) present in an ODE

$$\left.\begin{aligned} \dot{x} + \dot{y} &= -3x + xz + te^{-t} \\ \dot{y} &= x - 2y \\ \dot{z} &= .4z + 2xy \end{aligned}\right\}$$ nonlinear system, dependent vars have a product term

$$\left.\begin{aligned} \dot{x} &= \sigma x + \beta y \\ \dot{y} &= \gamma y + \alpha zx \\ \dot{z} &= \delta z + \varepsilon xy \end{aligned}\right\}$$ Lorenz equation. Nonlinear

State variable form $\dot{q} = Aq + Bu$ $\quad \dot{q} = f(q, u)$

- for output eq'ns $y = Cq + Du$ $\quad y = f(q,u)$
- review how to put into matrix form



80% points $\left\{\begin{array}{l} \text{1) FBD} \\ \text{2) Equation} \end{array}\right.$

ODE

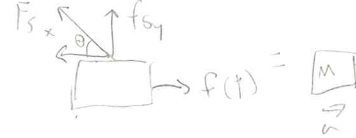
$+\circlearrowleft \Sigma T = J\ddot{\theta}$

planar

$\Sigma F_x = M\vec{a}_x$

$$f(t) - \left[k\left(\sqrt{l^2 + x^2} - l\right)\right]\cos\theta = ma_x$$

$$\cos\theta = \frac{x}{\sqrt{x^2 + l^2}}$$



$$m\ddot{x} + k\left(\sqrt{l^2+x^2} - l\right)\left(\frac{x}{\sqrt{l^2+x^2}}\right) = f_a$$

$$m\ddot{x} + kx - \frac{kxl}{\sqrt{l^2+x^2}} = f_a \Rightarrow \text{nonlinear}$$

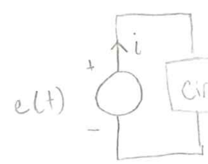
Last class: $i = \frac{dq}{dt} = \dot{q}$

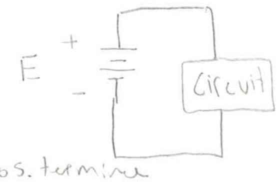
Resistance $e = Ri$

Capacitor $q = Ce$ — or

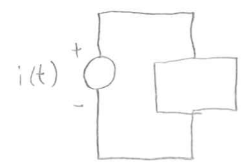
Inductor $e = L\frac{di}{dt}$

Sources: input to electrical circuit

a) 

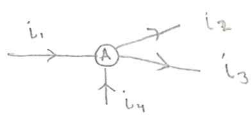
b) 

+ pos. terminal
− neg terminal
const. voltage source

c) 

Basic current laws: Two basic current laws (Kirchoff's law)

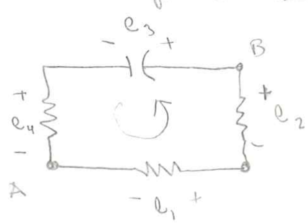
1st Law Kirchoff's Current law (KCL): the sum of current entering a node must equal sum of currents exiting the node (cons. of charge)



$i_1 + i_4 = i_2 + i_3$

$i_1 + i_2 + i_3 = 0$

2nd law for any closed circuit loop, the sum of the voltages around the loop must be equal to zero (does not matter which direction in the loop)

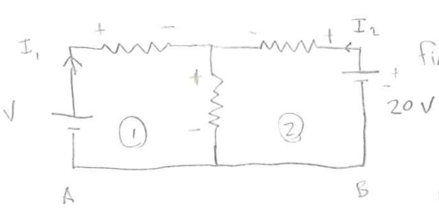


Start A, CCW

$e_1 + e_2 - e_3 - e_4 = 0$

Note: symbol reprents an arbitrarily established reference point having zero potential.

Will not be on HW or exams



Find current flowing through 40 Ω resistor

$I_1 + I_2$ flows through → 40 Ω

Use KVL for loop 1: CCW (at A)

$e_3 + e_1 - 10 = 0$

$40(I_1 + I_2) + 10(I_1) - 10 = 0$

$50 I_1 - 40 I_2 = 10$

$5I_1 + 4I_2 = 1 \quad (1)$

CCW @ B

$20 - e_2 - e_3 = 0$

$20 - 20(I_2) - 40(I_1 + I_2) = 0$

$20 - 60I_2 - 40I_1 = 0$

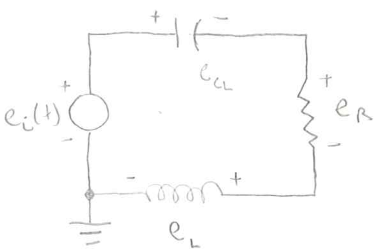
$2I_1 + 3I_2 = 1 \quad (2)$ ⟶

$I_1 = -.143$ amps $I_2 = .429$ amps

To model/generate ODE for a circuit:

1) label all unknown voltages
2) " positive directions for all currents
3) Apply KVL & KCL (where appropriate)
4) apply constitutive laws to each element
5) combine eqn's

Derive the model for the circuit



- same current in each element → series
- " voltage " " " → parallel

KVL A (CCW) $e_L + e_R + e_c - e_i(t) = 0$

$$L\frac{di}{dt} + C(R) + \frac{q}{C} = e_i(t)$$

$$L\frac{d}{dt}\left(\frac{di}{dt}\right) + R\frac{di}{dt} + \frac{1}{C}\frac{dq}{dt} = \frac{de_i}{dt}$$

$$\to L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{de_i}{dt}$$

$$\boxed{L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{e}_i}$$

$I = i$

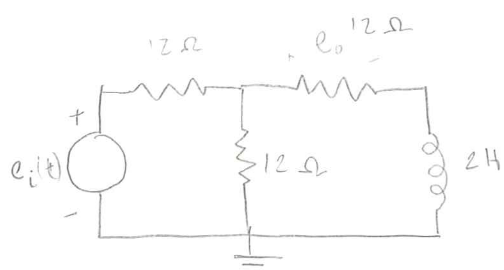
I O Model

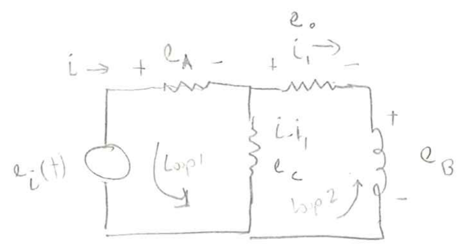
ODE Find ODE having only input, output terms

2 gen. procedures:
1) Loop-equation method
2) Node-equation method

Loop equation method

1) apply KCL to express current through each element
2) write KVL eqn for each loop
3) apply constitutive relations
4) combine results to get IO eqns





$i = \frac{V}{R}$ $V = Ri$

For loop 1: $e_c + e_A - e_i(t) = 0$

constitutive relations

1) $e_c = (i - i_1)12$
2) $e_A = (i)12$
3) $e_B = L\frac{di_1}{dt} = \frac{2di_1}{dt}$
4) $e_o = i_1(12)$

$(i - i_1)12 + 12i = e_i(t)$

$12(2i - i_1) = e_i(t)$ (1)

Loop 2: $e_B + e_o - e_c = 0$ (1)

$2\frac{di}{dt} + e_o - (i - i_1)(12) = 0$ (2)

$\frac{e_o}{12} = i_1$ (3)

From (1) & (3):

$2i - i_1 = \frac{e_i(t)}{12}$

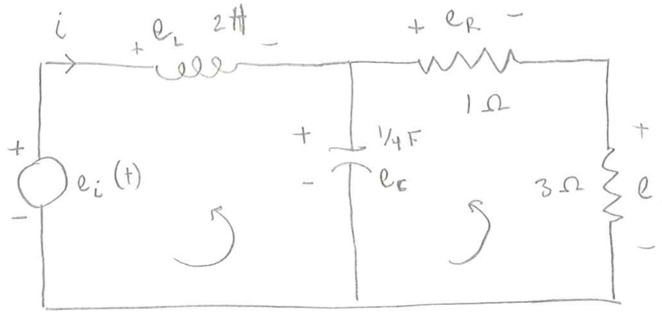
$2i = \frac{e_i(t)}{12} + \frac{e_o}{12}$

$i = \frac{1}{24}(e_i(t) + e_o)$

$$2\frac{d}{dt}\left(\frac{e_o}{12}\right) + e_o - \left[\frac{1}{24}e_i(t) + \frac{e_o}{12}\right](12) = 0$$

Simplify: $\boxed{\dot{e}_o + 9e_o = 3e_i(t)}$ input output eqn

Ex #2



Const. relations

1) $i_1 = C\frac{de_c}{dt}$

2) $e_L = L\frac{di}{dt} = 2\frac{di}{dt}$

3) $e_R = (i - i_1)$ (1)

4) $e_o = (i - i_1)$ (3)

- loop 1 KVL $e_c + e_L - e_i = 0$ take $\frac{d}{dt}$, $\Rightarrow \frac{de_c}{dt} + \frac{de_L}{dt} - \frac{de_i}{dt} = 0$

$4i + \frac{d}{dt}\left(2\frac{di}{dt}\right) - \dot{e}_i = 0 \Rightarrow 4i_1 + 2\frac{d^2 i}{dt} = \dot{e}_i$ (1)

- loop 2 KVL $e_o + e_R - e_c = 0$ $\frac{de_o}{dt} + \frac{de_r}{dt} - \frac{de_c}{dt} = 0$

$\dot{e}_o + \frac{d}{dt}(i - i_1) - 4i = 0$ (2)

$e_o + \frac{d}{dt}\left(\frac{e_o}{3}\right) = 4i \Rightarrow \dot{e}_o + \frac{\dot{e}_o}{3} = 4i$ $\boxed{i_1 = \frac{\dot{e}_o}{3}}$

$i - i_1 = \frac{e_o}{3}$ $i = \frac{e_o}{3} + i_1$ $\boxed{i = \frac{e_o}{3} + \frac{\dot{e}_o}{3}}$

From (1) $4\left(\frac{\dot{e}_o}{3}\right) + 2\frac{d^2}{dt^2}\left(\frac{e_o}{3} + \frac{\dot{e}_o}{3}\right) = \dot{e}_i$ $\frac{4}{3}\dot{e}_o + \frac{2}{3}\ddot{e}_o + \frac{2}{3}\dddot{e}_o = \dot{e}_i$

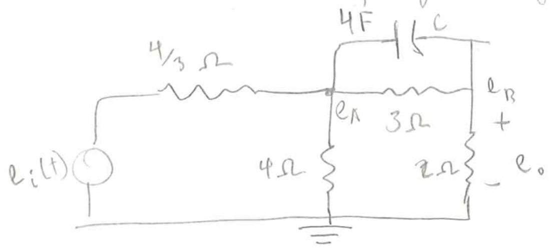
$\frac{d}{dt}\left[4e_o + 2\dot{e}_o + 2\ddot{e}\right] = \frac{d}{dt}[e_i] \times 3$

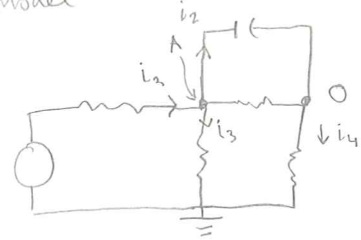
$\boxed{2\ddot{e} + 2\dot{e} + 4e_o = 3e_i}$

Notes 9-29-10

Node eqn method

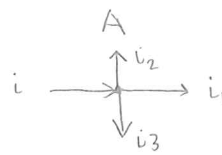
1) label all the voltage of each node wrt ground or reference node
2) Write KCL for each node
3) use constitutive relations
4) combine resulting eqns to find i/o model





10-1-10

Current flow at node



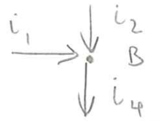
$i = i_1 + i_2 + i_3$

constitutive relations: (note $e_B - 0 = e_o$)

1) $e_i(t) - e_A = i(4/3) \Rightarrow i = 3/4(e_i(t) - e_A)$
2) $e_A - e_o = i_1(3) \Rightarrow i_1 = (e_A - e_o)/3$
3) $C\frac{d}{dt}(e_A - e_o) = i_2 \Rightarrow i_2 = 4(\dot{e}_A - \dot{e}_o)$
4) $e_A = i_3(4) \Rightarrow i_3 = e_A/4$
5) $e_o = i_4(2) \Rightarrow i_4 = e_o/2$

from 1 & A

$$\frac{3}{4}(e_i(t) - e_A) = \frac{e_A - e_o}{3} + 4(\dot{e}_A - \dot{e}_o) + \left(\frac{e_A}{4}\right) \quad (2)$$

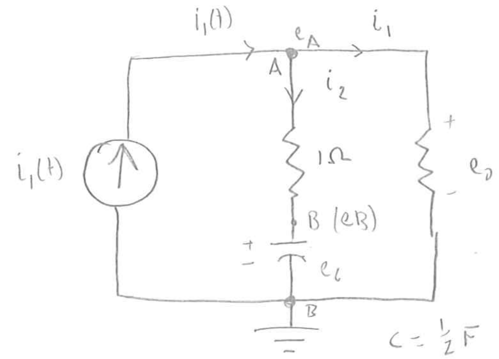


$i_1 + i_2 = i_4$ $\qquad \frac{e_A - e_o}{3} + 4(\dot{e}_A - \dot{e}_o) = \frac{e_o}{2}$ $\qquad (3)$

FROM (1): $\frac{3}{4}(e_i(t) - e_A) = \frac{e_o}{2} + \frac{e_A}{4} \Rightarrow e_A = \left[\frac{3}{4}e_i(t) - \frac{e_o}{2}\right]$ $\qquad (4)$

Plug (4) into (3) gives: $\frac{\left(\frac{3}{4}e_i(t) - \frac{e_o}{2}\right) - e_o}{3} + 4\left(\quad\right) - \dot{e}_o = \frac{e_o}{2}$

Simplified: $6\dot{e}_o + e_o = 3\dot{e}_i + \frac{1}{4}(e_i(t))$



Node A

$i_1(t) = i_1 + i_2$

Cons rel: $i_1 = (e_A - 0)/2$, $e_A - 0 = e_o$ $\Rightarrow i_1 = \frac{e_o}{2}$ (2)

$e_A - e_B = i_2$

$c\frac{de_c}{dt} = i_2$ also $e_B - 0 = e_c$ (3)

$i_1(t) = \frac{e_o}{2} + (e_A - e_B) = \frac{e_o}{2} + (e_o - e_B) = \frac{e_o}{2} + (e_o - e_c)$

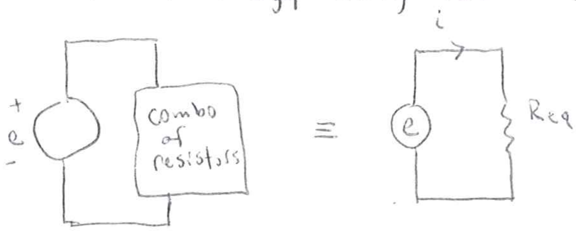
$e_c = \frac{3e_o}{2} - i_1(t)$ (4)

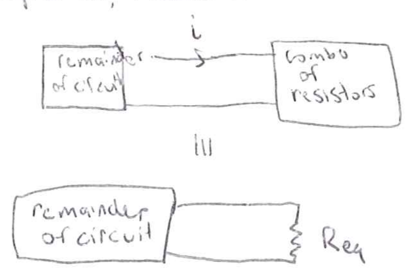
Now, $i_2 = e_A - e_B = \frac{1}{2}\dot{e}_c$ or $e_o - e_c = \frac{1}{2}\dot{e}_c$ $e_o - \left(\frac{3e_o}{2} - i_1(t)\right) = \frac{1}{2}\left(\frac{3\dot{e}_o}{2} - \frac{di_1}{dt}\right)$

$i_1(t) + \frac{1}{2}\frac{di}{dt} = \frac{3}{4}\dot{e}_o + \frac{1}{2}e_o$

6.5 Resistive Circuits

– contain only resistors & sources, ~~which contain~~ with no energy-storing element (eg. capacitor, inductor)



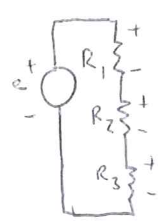


2 important cases of combination of resistors are series & parallel connections

Resistors in series

- 2 connected to a common node w/no other element connected to the com node
- the same current flows through the resistors

apply KVL to the loop



$e = i R_{eq}$

$R_{eq} = R_1 + R_2 + R_3$

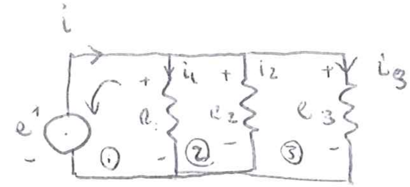
individual voltages across each resistor: $e_1 = \cancel{e_1 R_1}\ iR_1 = \frac{e}{R_{eq}}R$

$e_2 = iR_2 = \frac{e}{R_{eq}}(R_2)$ A

$e_N = \ldots$

set of eqn's A illustrate the principle of voltage division in series output also called "voltage divider" rule

Parallel resistors:



1) each node of resistor is connected to a separate node of another resistor

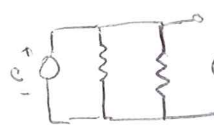
2) Apply KVL to loop 1 $\quad e_1 - e = 0 \Rightarrow e_1 = e$ $\quad$ apply to loop 2 $\quad e_2 - e_1 = 0 \Rightarrow e_2 = e_1$ $\quad$ Apply to L3

3) Voltages across each resistor is the same

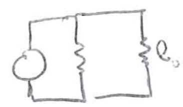
$i = i_1 + i_2 + i_3 \qquad i = e\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$

individual current: $i_1 = \frac{e_1}{R_1} = \frac{e}{R_1} = \frac{iR_{eq}}{R_1}$ $\quad$ For parallel resistors: Current divider Rule

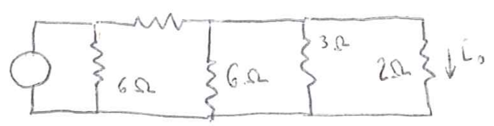
Note:



e_0 will be same as

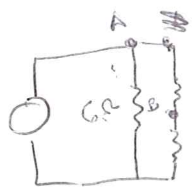


Ex.



find $R_{eq} = \frac{1}{6} + \frac{1}{2+6} + \frac{1}{}$

=



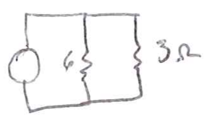
$\left(\frac{1}{6} + \frac{1}{3} + \frac{1}{2}\right) = \frac{1}{R_{eq}} \Rightarrow R_{eq} = 1$

$e_A - e_B = i(2) \qquad e_A - e_B = i(2)$

$e_B - 0 = i(1)$

$\boxed{i_0 = \frac{e_B}{2} = \frac{i_i(t)}{3}}$

=



$\frac{1}{R_{eq}} = \frac{1}{6} + \frac{1}{3}$ $\quad \boxed{R_{eq} = 2\Omega}$

State var form

$\dot{q} = Aq + Bu$

1) define appropriate set of state variables, usually, state vars are related to energy storing elements in a circuit
2) identify input & output of the circuit
3) derive an equation for the derivative of each state variables in only terms of state vars & inputs

$q = \begin{Bmatrix} e_c \\ i \end{Bmatrix} \qquad \dot{q} = \begin{Bmatrix} \dot{e}_c \\ \dot{i} \end{Bmatrix} \qquad \boxed{\dot{e}_c = \frac{i_c}{C}}$

$\boxed{\dot{I}_L = \frac{e_L}{L}}$

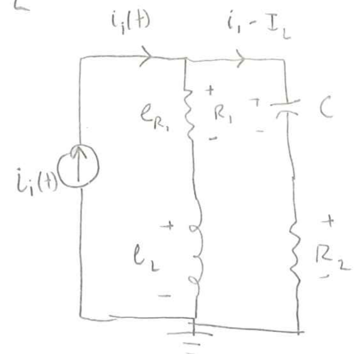
4 arrange eq's in matrix form to find AB matrix

Notes 10-6-10

State variable

1) define appropriate set of state variables, usually e_c, I_L
2) Identify input & output
3) eqn's for the derivative of each state (use loop or node eqn)

remember, $q_c = Ce_c \Rightarrow$ $\frac{I_c}{C} = \dot{e}_c$ $\frac{e_L}{L} = \dot{I}_L$



use loop eq. method let st. var be $q = \begin{bmatrix} e_c \\ I_L \end{bmatrix}$

$\dot{e}_c = \frac{I_c}{C} = \frac{i_1 - I_L}{C}$ (1)

$\dot{I}_L = \frac{e_L}{L}$ (2)

$e_L = f$

Apply KVL in loop 1

$e_{R_2} + e_c - e_{R_1} - e_L = 0$

$e_L = e_{R_2} + e_c - e_{R_1}$

$= (i_1 - I_L)(R_2) + e_c - I_L R_1$

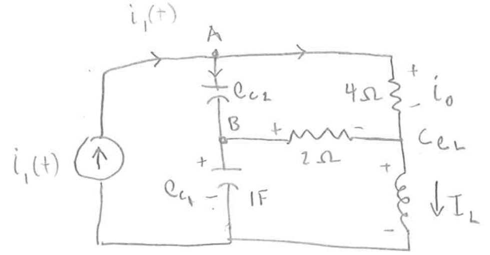
$e_L = i_1(t)R_2 - I_L(R_1 + R_2) + e_c$

$\dot{e}_c = (i_1(t) - I_L)/C$

$\dot{I}_L = \frac{e_L}{L} = \frac{1}{L}\left[i_1(t)R_2 - I_L(R_1+R_2) + e_c\right]$

$\dot{q} = Aq + Bu$ $\begin{bmatrix} \dot{e} \\ \dot{I}_L \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1/C \\ \frac{1}{L} & \frac{-(R_1+R_2)}{L} \end{bmatrix}}_{A} \begin{bmatrix} e_c \\ I_L \end{bmatrix} + \underbrace{\begin{bmatrix} 1/C \\ R_2/L \end{bmatrix}}_{B} i_1(t)$

Part b) y = current through R_2 = $i_1(t) - I_L$ $y = Cq + Du$ $y = \underbrace{[0\ -1]}_{C} \begin{bmatrix} e_c \\ I_L \end{bmatrix} + \underbrace{[1]}_{D} e_i(t)$



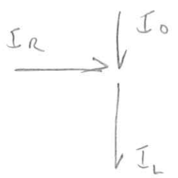
State var eqns

$\dot{e}_{c1} = I_{c1}$

$\dot{e}_{c2} = 0.5\, I_{c2}$

$\dot{I}_L = e_L/3$

summation about C



$I_L = I_R + i_o$

$I_L = \left(\frac{e_{c1} - e_L}{2}\right) + \frac{1}{4}(e_{c1} + e_{c2} - e_L)$ Find state var eqn

$e_L = \frac{e_{c1} + e_{c2}}{3} - \frac{4}{3} I_L$ $i_o = \frac{e_{c1} + e_{c2} - e_L}{4} = \frac{e_{c1} + e_{c2} - \left(\frac{e_{c1} + e_{c2}}{3} - \frac{4}{3} I_L\right)}{4}$

$i_o = \frac{e_{c2}}{6} + \frac{I_L}{3}$ $\dot{I}_L = \frac{e_L}{3} = \frac{1}{3}\left(\frac{e_{c1} + e_{c2}}{3} - \frac{4}{3} I_L\right)$

$\dot{I}_L = \frac{1}{9}(3e_{c1} + e_{c2} - 4I_L)$

Node A

$\xrightarrow{i_1(t)}$ $\xrightarrow{i_o}$ $\downarrow i_{c2}$

$i_1 = i_o + i_{c2}$ $i_{c2} = i(t) - i_o = i(t) - \frac{e_{c2}}{6} - \frac{I_L}{3}$

$\boxed{\dot{e}_{c2} = 0.5\left(i_1(t) - \frac{e_{c2}}{6} - \frac{I_L}{3}\right)}$ $\boxed{\dot{I}_L = \frac{1}{9}(3e_{c1} + e_{c2} - 4I_L)}$

ground node! $\downarrow I_{c1}$ $\leftarrow I_L$ $\cancel{i_1(t)}$ $I_{c1} = i_1(t) - I_L$
$\overleftarrow{I_i(t)}$

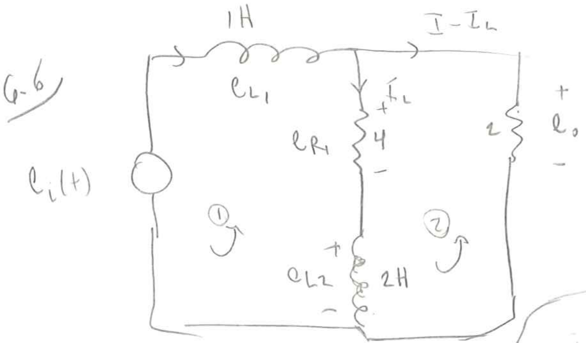
$\dot{e}_{c1} = i_i(t) - I_L$ $\dot{e}_{c1} = i_1(t) - I_L$

$\dot{e}_{c2} = \frac{1}{12}(6 i_i(t) - e_{c2} - 2I_L)$

$\dot{I}_L = \frac{1}{9}(3e_{c1} + e_{c2} - 4I_L)$

$$\begin{bmatrix} \dot{e}_{c1} \\ \dot{e}_{c2} \\ \dot{I}_L \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & -\frac{1}{12} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{9} & -\frac{4}{9} \end{bmatrix}}_{A} \begin{bmatrix} e_{c1} \\ e_{c2} \\ I_L \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}}_{B} i_i(t)$$

6.6



Loop eqn ①

$$e_{L2} + e_{R_1} + e_{L_1} - e_i(t) = 0 \quad (1)$$

$$2\frac{dI_L}{dt} + 4I_L + \frac{dI}{dt} = e_i(t) \quad (3)$$

② $e_o - e_{R_1} - e_{L2} = 0$

$e_o - 4I_L - 2\frac{dI_L}{dt} = 2$ → add eq's together

$$\frac{dI}{dt} = i(t) - e_o \qquad \dot{e}_o = (\dot{I} - \dot{I}_L)(2)$$

$$\frac{dI_L}{dt} = e_i(t) - e_o - \frac{\dot{e}_o}{2}$$

take $\frac{d}{dt}$ of eq (3) to get i/o

So far, electrical & mechanical systems → generating ODE & state variable. consider the DE $M\ddot{x} + B\dot{x} + Kx = F_a(t)$

Computational method (MATLAB)

analytical method (laplace transform)

To determine a system solution/response a) modeling equations b) inputs c) set of initial conditions ($x(0)$, $\dot{x}(0)$)

Laplace transform solution: the application of laplace to any modeling eqn transforms the entire eqn from time domain to a new domain "s". The original modeling eqn's are changed to purely algebraic eqn's

For $M\ddot{x} + B\dot{x} + Kx = 0$ Let $M=1$, $B=2$, $K=5$. $X(0)=1$ $\dot{X}(0)=0$

applying laplace, laplace of $(x(t)) = X(s) = \dfrac{s+2}{s^2+2s+5}$

now an algebraic function of s

taking the inverse laplace, $x(t) = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t$

Notes 10-8-10

Representation of laplace ($\mathcal{L}$ or L)
for any function $f(t)$ $\mathcal{L}(f(t)) = F(s)$ where $t \to$ time domain
$s \to$ complex frequency domain

Mathematical definition of laplace transform: consider any func. $f(t)$ defined for $t \geq 0$. Then laplace trans. of $f(t)$

$$\boxed{\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt}$$

Note 1) $f(t)$ is func of time or constant
2) s is a complex qty, $s = \sigma + i\omega$ (σ real, $i\omega$ imaginary)
3) while integrating, s is treated as a constant
4) laplace transforms can only be applied to linear systems

Brief review of integral calc.

1) $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

2) $\int f(x) g(x)\, dx \neq \int f(x)\, dx \int g(x)\, dx$

3) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ $\quad n \neq -1$ when $n = -1$, $\int \frac{dx}{x} = \ln(x) + c$

4) integration by parts $\int_a^b u\,dv = uv \Big|_a^b - \int_a^b v\,du$

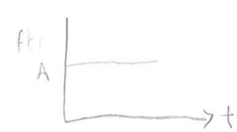
ex: $\int x \sin x\, dx$ $\quad x = u \quad dx = du \quad -\cos x = v \Rightarrow \sin x\,dx = dv$
now $\int = \int u\,dv = x(-\cos x) - \int(-\cos x)\,dx = [-x\cos x + \sin x] + c$

5) $\int e^{ax} dx = \frac{e^x}{a} + c$ $\quad \int \sin x\,dx = -\cos x + c \quad \int \cos x\,dx = \sin x + c$

6) integ. by substitution $\int \frac{dx}{\sqrt{1-x^2}}$ $\quad x = \sin\theta \quad dx = \cos\theta\, d\theta$
$\Rightarrow \int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta\, d\theta}{\cos\theta} = \theta = \sin^{-1}(x)$

Laplace transform of common functions (Refer to appendix E, p 548)

1) step function suppose $f(t) = A \;\forall\, t > 0$

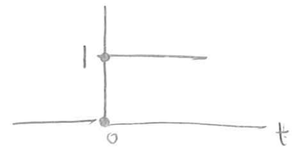


$$\boxed{\mathcal{L}(A) = \frac{A}{s}}$$

$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t) e^{-st} dt$
$= A \int_0^\infty e^{-st} dt = A \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{A}{-s}\left[e^{-s\infty} - e^0\right] = \boxed{\frac{A}{s}}$

Unit step function $(u(t))$

$u(t) = \begin{Bmatrix} 0 & t<0 \\ 1 & t>0 \end{Bmatrix}$



$\boxed{\mathcal{L}(u(t)) = \frac{1}{s}}$ $\qquad \mathcal{L}(A) = \frac{A}{s}$

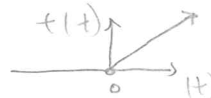
exponential function $\quad f(t) = e^{at}$

$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t) e^{-st}\,dt = \int_0^\infty (e^{at} \cdot e^{-st})\,dt = \int_0^\infty e^{-(s-a)t}\,dt$

$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{-1}{s-a}\left[\cancel{\ }^{\,0} - e^{-(s-a)(0)}\right] = -\frac{1}{s-a}[-1] = \frac{1}{s-a}$

$\mathcal{L}\, e^{(at)} = \frac{1}{s-a} \qquad \mathcal{L}\, e^{-at} = \frac{1}{s+a}$

Ramp func. $\quad f(t) = \begin{Bmatrix} t & \text{for } t>0 \\ 0 & \text{for } t<0 \end{Bmatrix}$



$\mathcal{L}\, f(t) = \int_0^\infty f(t) e^{-st}\,dt = \int_0^\infty t\, e^{-st}\,dt$ $\qquad u = t \quad du = dt$

$v = e^{-st} \Rightarrow \cancel{dt}\ dv = -se^{-st}\,dt \Rightarrow -\frac{dv}{s} = e^{-st}\,dt$

$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du = -\frac{1}{s}\int_a^b u\,dv = -\frac{1}{s}\left[uv\Big|_a^b - \int_a^b v\,du\right]$

$= -\frac{1}{s}\left\{\underbrace{\left(te^{-st}\right)_0^\infty}_{0} - \int_0^\infty e^{-st}\,dt\right\} \qquad -\frac{1}{s}\left[\left(-\frac{e^{-st}}{-s}\right)_0^\infty\right] \qquad \frac{1}{s^2}\left(e^{-s\infty} - e^{0}\right) = \frac{1}{s}\int_a^b$

$$\boxed{\begin{aligned} &\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt \\ &\mathcal{L}(A) = \frac{A}{s} \quad A \to \text{const} \\ &\mathcal{L}\, e^{at} = \frac{1}{s-a} \quad a \to \text{real } \# \\ &\mathcal{L}(t) = \frac{1}{s^2} \end{aligned}}$$

Trig. func's $\quad f(t)$ is replaced by $\sin \omega t$ or $\cos \omega t$

$\mathcal{L}(\sin \omega t) = \int_0^\infty (\sin \omega t)^{-st}\,dt$ $\qquad$ Euler's eqn: $e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{i\omega t} = \cos\omega t + i\sin\omega t$

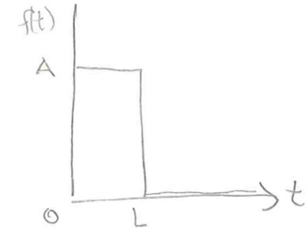
$\ldots \sin\omega t = e^{i\omega} - e \qquad\qquad e^{-i\omega} = \cos\omega t - i\sin\omega t$

$\boxed{\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}} \qquad \boxed{\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}}$

Notes 10-11-10

Rectangular Pulse: a step function for duration $t = L$ then back to zero

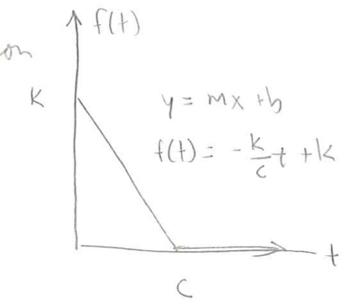
$$f(t) = \begin{cases} A & 0 < t \leq L \\ 0 & t > L \end{cases} \qquad \mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}\,dt$$



$$\mathcal{L}(f(t)) = \int_0^L Ae^{-st}\,dt = A\int_0^L e^{-st}\,dt$$

$$\ldots \quad \boxed{\mathcal{L}(f(t)) = \frac{A}{s}\left[1 - e^{-st}\right]}$$

Find a laplace transform of following function



$$f(t) = \int_0^\infty f(t)e^{-st}\,dt$$

$$f(t) = \begin{cases} -\frac{k}{c}t + k & 0 < t \leq c \\ 0 & t > c \end{cases}$$

$$f(t) = \int_0^c \left(-\frac{kt}{c} + k\right)e^{st}\,dt = \int_0^c -\frac{kt}{c}e^{-st}\,dt + \int_0^c ke^{-st}\,dt$$

taken from notes last

$$= \int_0^c te^{-st}\,dt + k\int_0^c e^{-st}\,dt \qquad = \int_0^c te^{-st}\,dt = \frac{1}{e}\left\{\left[te^{-st}\right]_0^c - \int_0^c e^{-st}\,dt\right\}$$

$$= \frac{1}{s}\left\{\left[ce^{-sc} - 0\right] - \left[\frac{e^{-st}}{-s}\right]_0^c\right. \qquad = -\frac{1}{s}\left\{ce^{-st} + \frac{1}{s}\left[e^{-st} - e^0\right]\right\}$$

$$\ldots \Rightarrow \int_0^c te^{-st}\,dt = \frac{1}{s^2}\left[1 - (1+cs)e^{-sc}\right]$$

$$= -\frac{k}{c}\left(\frac{1}{s^2}\right)\left[1 - (1+cs)e^{-sc}\right] + k\left(\frac{1}{s} - \frac{e^{-sc}}{s}\right) \qquad \boxed{\mathcal{L}(f(t)) = \frac{k}{s} - \frac{k}{cs^2}\left(1 - e^{-sc}\right)}$$

Laplace transform properties let $f(t)$ & $g(t)$ be arbitrary two functions such that

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

$$\mathcal{L}(g(t)) = G(s) = \int_0^\infty g(t)e^{-st}\,dt$$

1) multiplication by a const. $\mathcal{L}(a f(t)) = \int_0^\infty [a f(t)]e^{-st}\,dt = a\underbrace{\int_0^\infty f(t)e^{-st}\,dt}_{F(s)}$

$$\Rightarrow \mathcal{L}(a f(t)) = a F(s)$$

Notes 10-11-10

2) Superposition

the transform of the sum of the two functions $f(t)$ & $g(t)$ is

$\mathcal{L}(f(t) + g(t)) = F(S) + G(S)$

the general superposition property : $\mathcal{L}[a f(t) + b g(t)] = aF(s) + bG(s)$

3) Multiplication of functions:

$$\mathcal{L}[f(t) \cdot g(t)] \neq \mathcal{L}(f(t)) \cdot \mathcal{L}[g(t)] \neq F(s) \cdot G(s)$$

3a) mult. of exponential func w/ $f(t)$ ∴ replace $f(t)$ w/ $f(t)e^{-at}$

$$\mathcal{L}(f(t)e^{-at}) = \int_0^{\infty}[f(t)e^{-at}]e^{-st}\,dt \underset{\substack{\uparrow \\ F(s)}}{=} \int_0^{\infty} f(t)e^{(-st-at)}\,dt$$

$$= \int_0^{\infty} f(t)e^{-(s+a)t}\,dt = F(s+a)$$

$\mathcal{L}\, f(t)e^{-at} = F(s+a)$ where $\mathcal{L}(f(t)) = F(s)$ Note $a = -a$

$\Rightarrow a = 0, \quad \mathcal{L}(f(t)) = F(s)$

3b) $f(t)$ multiplied by t, $\mathcal{L}(tf(t)) = -\frac{d}{ds}F(s)$ where $F(s) = \mathcal{L}(f(t))$ Notes 10-13-10

$$-\frac{d}{ds}F(s) = -\frac{d}{ds}\int_0^{\infty} f(t)e^{-st}\,dt = -\int_0^{\infty}\frac{d}{ds}(f(t)e^{-st})\,dt$$

$$= +\int_0^{\infty} f(t)(+t)e^{-st}\,dt \qquad \int_0^{\infty}[tf(t)]e^{-st}\,dt = \mathcal{L}\, t(f(t)) = -\frac{d}{ds}F(s)$$

$\mathcal{L}(t\sin 6t) = \int_0^{\infty} t\sin 6t\, e^{-st}\,dt$ let $f(t) = \sin 6t$

$\mathcal{L}\, f(t) = F(s) = \frac{6}{s^2+6^2}$ $\quad \mathcal{L}(t\sin 6t) = -\frac{d}{ds}\left(\frac{6}{s^2+6^2}\right) = -\left(\frac{-12s}{(s^2+6^2)^2}\right) = \left(\quad\right)$

- Since $\mathcal{L}\, t f(t) = -\frac{d}{ds}(F(s))$ let $f(t) = 1$, $F(s) = \mathcal{L}(1) = \frac{1}{s}$

$\mathcal{L}(t(1)) = -\frac{d}{ds}\left(\frac{1}{s}\right) = \left(\frac{1}{s^2}\right)$ Let $f(t) = t$ $F(s) = \frac{1}{s^2}$

$\mathcal{L}(t(t)) = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$ $\quad \mathcal{L}(t(t^2)) = \frac{2\cdot 3}{s^4}$ $\quad \mathcal{L}(t^n) = \frac{1\cdot 2\cdot 3 \cdots n}{s^{n+1}}$

$$\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$$

Notes 10-13-10

example: Evaluate $\mathcal{L}(e^{5t}t^3) \Rightarrow \mathcal{L}[f(t)e^{-at}]$

where $a = -5$ $\quad f(t) = t^3 \quad \mathcal{L}(f(t)) = F(s) \quad F(s) = \mathcal{L}(t^3)$

$F(s) = \frac{3!}{s^4} = \frac{3\times 2\times 1}{s^4} = \frac{6}{s^4} \Rightarrow \mathcal{L}(e^{5t}t^3) = F(s-5) = \frac{6}{(s-5)^4}$

Transform of derivatives $\quad f'(t) \quad \mathcal{L}(f'(t)) = \mathcal{L}\,\frac{df(t)}{dt} = \int f'(t)e^{-st}\,dt$

use integ. by-parts $\int u\,dv = uv - \int v\,du \quad$ let $v = f(t) \quad dv = f'(t)$

let $u = e^{-st} \quad du = -se^{-st}\,dt \quad \int \underbrace{e^{-st}}_{u}\,\underbrace{f'(t)\,dt}_{dv} = e^{-st}f(t)\big|_0^\infty - \int_0^\infty f(t)(-se^{-st}\,dt)$

$= -f(0) + s\int_0^\infty f(t)e^{-st}\,dt \quad \boxed{\mathcal{L}(f'(t)) = -f(0) + sF(s)}$

evaluate $\mathcal{L}\,f'(t)$ where $f(t) = \cos\omega t$ & verify answer

$\mathcal{L}(f'(t)) = -f(0) + sF(s) = -1 + s\,\mathcal{L}(\cos\omega t) = -1 + s\left(\frac{s}{s^2+\omega^2}\right)$

$= \ldots \frac{\omega^2}{s^2+\omega^2}$

Transform of 2nd & higher order derivatives

let $f(t)$ be a func. of time s.t. $\mathcal{L}\,f(t) = F(s)$ $\quad$ let $\frac{d\,f(t)}{dt} = \dot{f}(t) = g(t)$

$\Rightarrow \ddot{f}(t) = \dot{g}(t)$ then

$\mathcal{L}(\dot{g}(t)) = s\,\mathcal{L}(g(t)) - g(0) = s\,\mathcal{L}(\dot{f}(t)) - g(0) = s[s\,\mathcal{L}(f(t)) - f(0)] - \dot{f}(0)$

$= s^2\,\mathcal{L}(f(t)) - sf(0) - \dot{f}(0) \quad \boxed{\mathcal{L}(\ddot{f}(t)) = s^2F(s) - sf(0) - \dot{f}(0)}$

generalized formula for transform of n^{th} derivative

$\mathcal{L}\left(\frac{d^n f(t)}{dt^n}\right) = s^nF(s) - s^{n-1}f(0) + \ldots f^{n-1}(0)$

consider the spring-mass damper system $M\ddot{x} + B\dot{x} + kx = 0 \quad x(0) = 1$

$M = 1 \quad B = 2 \quad k = 5 \quad \dot{x}(0) = 0 \quad$ Find $X(s)$ $\quad \mathcal{L}(\ddot{x} + 2\dot{x} + 5x) = 0$

$= \underbrace{[s^2X(s) - sX(0) - \dot{x}(0)]}_{\ddot{x}} + 2\underbrace{[sX(s) - X(0)]}_{\dot{x}} + 5X(s) = 0$

$X(s)[x^2 + 2s + 5] = s + 2 \Rightarrow \boxed{X(s) = \frac{s+2}{s^2+2s+5}}$

Laplace transform of integrals

consider function $\int_0^t f(\lambda)\,d\lambda$ $\quad \mathcal{L}\left(\int_0^t f\lambda d\lambda\right) = \int_0^\infty \left[f(\lambda)\,d\lambda\right] e^{-st}\,dt$

$\mathcal{L} = \frac{F(s)}{s}$ where $\mathcal{L}\, f(t) = F(s)$

Laplace inversion * final step to evaluate $f(t)$ *

$\mathcal{L}[f(t)] = F(s) \quad f(t) = \mathcal{L}^{-1}(F(s)) \quad \mathcal{L}(1) = \frac{1}{s} \Rightarrow 1 = \mathcal{L}^{-1}\left(\frac{1}{s}\right)$

$\mathcal{L}(t) = \frac{1}{s^2} \Rightarrow t = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$

Find $\mathcal{L}^{-1}\left(\frac{1}{s^5}\right) \quad \mathcal{L}(t^n),\ n=4 = \frac{\lfloor 4}{s^5} = \frac{24}{s^4} \quad \boxed{\mathcal{L}^{-1} = \frac{t^4}{24}}$

Find $\mathcal{L}^{-1}\left(\frac{-2s+6}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + 6\,\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$

$\mathcal{L}\cos\omega t = \frac{s}{s^2+\omega^2} \quad \mathcal{L}(\sin\omega t) = \frac{\omega}{\omega^2+s^2} \quad \boxed{\mathcal{L}^{-1} = -2\cos 2t + 3\sin 2t}$

eg: $F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} \ldots + a_0}$ (1)

Note: coeff of highest power of $s^n = 1$. func's that can be written in form (1) are called rational functions a) if $m \le n \Rightarrow$ proper rational function $m < n$ strictly proper r.f. for any strictly p.r.f., we define the method of "partial fraction expansion" to find $\mathcal{L}^{-1}$

Now $D(s) = s^n + a_{n-1}s^{n-1} \ldots + a_0$
$= (s-s_1)(s-s_2)\ldots(s-s_n)$

where s_1, s_2, s_3 are the n roots of $D(s)=0$ qty's $(s_1, s_2, \ldots)$ called "poles" of $F(s)$

3 possible cases!

a) all poles are distinct & real

b) 2 or more poles equal

c) poles are complex #'s

case a) distinct real poles:

$\mathcal{L}(f(t)) = F(s) = \frac{N(s)}{D(s)} = \frac{b^m s^m + \ldots b_0}{s^n + a_{n-1}s^{n-1} + \ldots a_0} = \frac{b^m s^m + \ldots b_0}{(s-s_1)(s-s_2)(s-s_n)} = \frac{A_1}{(s-s_1)} + \frac{A_2}{(s-s_2)}$

A_1, A_2, A_n are constants $\quad s_1, s_2, s_n$ distinct real poles of $D(s)=0$

$f(t) = \mathcal{L}^{-1}\left(\frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \ldots \frac{A_n}{s-s_n}\right) \quad f(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \ldots A_n e^{s_n t}$

$F(s) = \frac{s+1}{s^2-4s}$ $\quad D(s)=0 \quad s(s-4)=0 \quad s=0, s=4$

$= \frac{s+1}{s(s-4)} = \frac{A_1}{s} + \frac{A_2}{s-4}$ Method 1: Multiply both sides by $D(s)$

$s+1 = \frac{A_1(s^2-4s)}{s} + \frac{A_2(s^2-4s)}{(s-4)}$ Compare coefficient of s & constants

$s+1 = A_1(s-4) + A_2(s)$ $\quad s: 1 = A_1 + A_2 \rightarrow A_2 = 1 - A_1 = 1-(-\frac{1}{4}) = 5/4$

const: $1 = -4A_1 \Rightarrow A_1 = -\frac{1}{4}$

$F(s) = -\frac{1}{4s} + \frac{5}{4(s-4)}$ $\quad f(t) = \mathcal{L}^{-1}(F(s)) = -\frac{1}{4}\mathcal{L}^{-1}(\frac{1}{s}) + \frac{5}{4}\mathcal{L}^{-1}(\frac{1}{s-4})$

$= -\frac{1}{4}(1) + \frac{5}{4}e^{4t}$ $\quad f(t) = \frac{1}{4}[5e^{4t} - 1]$

Method 2: $\frac{s+1}{s(s-4)} = \frac{A_1}{s} + \frac{A_2}{s-4}$ $\quad A_1$: put $s=0$ in the LHS expression except the s factor

$A_1 = \frac{0+1}{0-4} = -\frac{1}{4}$ $\quad A_2$: put $s-4=0$ in LHS & ignore $s-4$ factor

$A_2 = \frac{4+1}{4} = 5/4$

$F(s) = \frac{s+1}{s^3+s^2-6s}$ $\quad D(s) = s^3+s^2-6s = s(s^2+s-6) = s(s-2)(s+3)$

$D(s) = s(s-2)(s+3)$ $\quad F(s) = \frac{s+1}{D(s)} = \frac{A_1}{s} + \frac{A_2}{(s-2)} + \frac{A_3}{(s+3)}$

A_1: put $s=0$ $\quad A_1 = \frac{1}{(0+3)(0-2)} = -\frac{1}{6}$ $\quad A_2$: $s+3=0$ $\quad A_2 \; \frac{-3+1}{(-3)(-3-2)} = \frac{-2}{15}$

A_3: $(s-2)=0 \Rightarrow s=2$ $\quad A_3 = \frac{2+1}{2(2+3)} = \frac{3}{10}$

$F(s) = -\frac{1}{6s} - \frac{2}{15(s-3)} + \frac{3}{10(s-3)}$ $\quad f(t) = -\frac{1}{6}\mathcal{L}^{-1}(\frac{1}{s}) - \frac{2}{15}\mathcal{L}^{-1}(\frac{1}{s+3}) + \frac{3}{10}\mathcal{L}^{-1}(\frac{1}{s-2})$

Case b: repeated poles (repeating roots)

$F(s) = \frac{N(s)}{D(s)} = \ldots$ $\quad F(s) = \frac{A_1}{(s-s_1)} + \frac{A_2}{(s-s_1)^2} + \cdots \frac{A_k}{(s-s_1)^k} + \frac{A_{k+1}}{(s-s_{k+1})} + \cdots \frac{A_n}{(s-s_n)}$

where $n-k$ roots are distinct

$F_s = \frac{s^4+1}{(s-1)^3} = \frac{A_1}{(s-1)} + \frac{A_2}{(s-1)^2} + \frac{A_3}{(s-1)^3}$ or $F_s = \frac{s^4+1}{(s-1)^2(s-2)^2} = \frac{A_1}{s-1} + \frac{A_2}{(s-1)^2} + \frac{A_2}{(s-2)^1} + \frac{A_3}{(\;)^2} \ldots$

10-15-10

find $f(t)$ $\quad F(S) = \frac{(S^2+S-2)}{(S-1)^3} = \frac{A_1}{(S-1)} + \frac{A_2}{(S-1)^2} + \frac{A_3}{(S-1)^3}$

To find $A_{1,2,3}$

Method 1, Method 2

2 pages back: $\frac{N(S)}{D(S)}$ where $m<n$

for distinct roots $\quad F(S) = \frac{A_1}{(S-S_1)} + \frac{A_2}{(S-S_2)}$

repeated roots $\quad F(S) = \frac{N(S)}{D(S)} = \frac{A_1}{(S-S_1)} + \frac{A_2}{(S-S_2)} \cdots$

10-18-10

repeated r. $\quad F(S) = \frac{S^2+S-2}{(S+1)^3}$ Method 1 $= \frac{A_1}{(S+1)} + \frac{A_2}{(S+1)^2} + \frac{A_3}{(S+1)^3}$

Mult. both sides by $D(S)$ $\quad (S^2+S-2) = A_1(S+1)^2 + A_2(S+1) + A_3(1) = S^2(A_1) + S(2A_1+A_2) + (A_1+A_2+A_3)$

compare the coef: S^2; $1 = A_1$ $\quad S$: $1 = 2A_1 + A_2 \Rightarrow A_2 = -1$ $\quad$ Const: $-2 = A_1 + A_2 + A_3$

$-2 = 1 - 1 + A_3 \Rightarrow A_3 = -2$

$f(t) = \mathcal{L}^{-1}\left(\frac{1}{S+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{(S+1)^2}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{(S+1)^3}\right)$

$= e^{-t} - te^{-t} - t^2e^{-t}$

Method (2) Recommended

$F(S) = \frac{(S^2+S-2)}{(S+1)^3} = \frac{A_1}{(S+1)} + \frac{A_2}{(S+1)^2} + \frac{A_3}{(S+1)^3}$ $\quad$ mult. by highest power of $(S+1)$

(1) $\quad S^2+S-2 = A_1(S+1)^2 + A_2(S+1) + A_3$ $\quad$ plug $S+1=0$ into eqn or $\underline{S=-1}$

$(-1)^2 + (-1) - 2 = A_1(0) + A_2(0) + A_3 \Rightarrow A_3 = -2$ $\quad$ take derivative of eqn (1) wrt $\underline{\underline{S}}$

(2) $\quad 2S+1 = 2A_1(S+1) + A_2$ $\quad$ plug $(S=-1) \Rightarrow A_2 = -1$ $\quad$ take der. of (2) wrt S

$2 = 2A_1 \Rightarrow A_1 = 1$

Find $f(t)$ for given $F(S)$: $\quad F(S) = \frac{S^3+2S+4}{S(S+1)^2(S+2)}$ $\quad$ method 1 shown in class first. not recommended

Sol'n using differentiation $\quad A_1$: $S=0$ in LHS

$\frac{0^3+2(0)+4}{(0+1)^2(0+2)} = A_1 = 2$ $\quad$ A_4: Plug $S+2=0$ $(S=-2)$ $\quad \frac{-2^3 + \cancel{2(-2)} \cancel{+4}}{-2(-2+1)^2} = A_4 = 4$

(1) $\quad \frac{S^3+2S+4}{S(S+2)} = \frac{A_1(S+1)^2}{S} + A_2(S+1) + A_3 + \frac{A_4(S+1)^2}{S+2}$ $\quad$ plug $S+1=0$ $\quad \frac{(-1)^3+2(-1)+4}{(-1)(-1+2)} = 0+0+A_3+0$

$\Rightarrow A_3 = -1$ $\quad \frac{-1-2+4}{-1(1)}$ $\quad$ For A_2, take d. of eqn (1) wrt S

$\frac{S(S+2)[3S^2+2] - [S^3+2S+4][2S+2]}{[S(S+2)]^2} = A_2 + 0 + \cancel{(S+1)}\, g(S)$ $\quad$ plug $S=-1$

plug $S=-1$ $\quad \frac{(-1)(-1+2)[3(-1)^2+2] - [(-1)^3+2(-1)+4][-2+2]}{[-1(-1+2)]^2} = A_2 \Rightarrow A_2 = -5$

Notes 10-18

$$F(s) = \frac{2}{s} - \frac{5}{s+1} - \frac{1}{(s+1)^2} + \frac{4}{(s+2)} \Rightarrow f(t) = \mathcal{L}^{-1}(F(s)) = 2\mathcal{L}^{-1}\left(\frac{1}{s}\right) - 5\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$- \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) + 4\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

Case (c): Complex Poles: when poles have the form:

$s_1 = -a + i\omega$
$s_2 = -a - i\omega$ } Note, comp. poles always occur in complex conjugate

method of analysis → Partial-fraction method
→ Completing the square

$F(s) = \frac{4s+8}{s^2+2s+5}$

$s^2 + 2s + 5 \Rightarrow s = -2 \pm \sqrt{}$ ... $(-1 \pm 2i)$

Notes 10-20-10

Ex $= \frac{4s+8}{(s^2+2s+5)}$ express in terms of $(s+1)$ $= \frac{4(s+1)+4}{(s+1)^2+4}$

$\mathcal{L}^{-1}\, e^{-at}$

Ex: $F(s) = \frac{5s^2+8s-5}{s^2(s^2+2s+5)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3 s + A_4}{(s^2+2s+5)}$

Find A_1 & A_2 1st; mult by s^2 $\Rightarrow$ $\frac{5s^2+8s-5}{(s^2+2s+5)} = A_1 s + A_2 + \frac{s^2(A_3 s + A_4)}{(s^2+2s+5)}$ (1)

plug $s=0$ $\frac{-5}{5} = \boxed{A_2 = -1}$

Take derivative wrt(s) of (1) $\frac{(s^2+2s+5)[10s+8] - (5s^2+8s-5)(2s+2)}{(s^2+2s+5)^2} = A_1 + 0 + s[G(s)]$ (2)

Plug $s=0$ $\frac{5 \cdot 8 - (-5)(2)}{5^2} = A_1 = \frac{40+10}{25} = \boxed{A_1 = 2}$

$\frac{5s^2+8s-5}{s^2(s^2+2s+5)} = \frac{-1}{s} + \frac{2}{s^2}$ mult. both sides by D(s)

$5s^2 + 8s - 5 = \frac{2}{s} s^2(s^2+2s+5) - \frac{1}{s^2}(s^2)(s^2+2s+5) + (A_3 s + A_4)(s^2)$ now $s^3 \Rightarrow$

$0 = 2 + A_3 \Rightarrow \boxed{A_3 = -2}$ $s^2 \Rightarrow$ $5 = 4 - 1 + A_4$ $\Rightarrow \boxed{A_4 = 2}$

s: $8 = 10 - 2 \Rightarrow 8 = 8$

$F(s) = \frac{2}{s} - \frac{1}{s^2} + \frac{-2s+2}{s^2+2s+5}$

$$\frac{-2s+2}{s^2+2s+5} = \frac{-2s+2}{(s^2+2s+1)+4} = \frac{-2s+2}{(s+1)^2+4} = \frac{-2(s+1)+4}{(s+1)^2+4} = \frac{-2(s+1)}{(s+1)^4+4} + \frac{4}{(s+1)^2+4}$$

$$f(t) = \mathcal{L}^{-1}(F(s)) = 2\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - 2\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) + 2\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+2^2}\right)$$

$$f(t) = 2 - t - 2te^{-t}\cos 2t + 2e^{-t}\sin 2t$$

Case $m=n$ $\quad F(s) = \frac{N(s)}{D(s)}$ where $N(s)$ & $D(s)$ are polynomials of same degree

<u>Example:</u> $F(s) = \frac{2s^2+7s+8}{s^2+3s+2} = \frac{2(s^2+3s+2)+s+4}{(s^2+3s+2)} = 2 + \frac{s+4}{(s^2+3s+2)}$

$$F(s) = 2 + \frac{s+4}{(s^2+3s+2)}$$

$$f(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(2) + \mathcal{L}^{-1}\left(\frac{s+4}{s^2+3s+2}\right) = 2\delta(t) + \mathcal{L}^{-1}\left(\frac{s+4}{(s+1)(s+2)}\right)$$

impulse func.

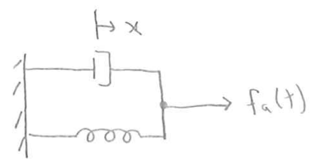
$$\frac{s+4}{(s+1)(s+2)} = \frac{A_1}{(s+1)} + \frac{A_2}{(s+2)} \qquad A_1: \frac{-1+4}{-1+2} = 3 \qquad A_2: \frac{-2+4}{-2+1} = -2$$

$$\boxed{f(t) = 2\delta(t) - 2e^{-2t} + 3e^{-t}} = 2\delta(t) + 3\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

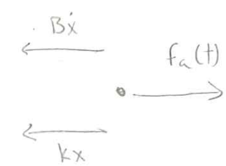
<u>Modeling of dynamic system</u> general procedure

1) derive/write ODE/integral - differential Eqn's
2) evaluate transform of ODE
3) solve the algebraic eqn's for the transform of the output
4) evaluate inverse transform to get output as a func. of time

<u>ex:</u>



<u>find expression for $x(t)$</u>



$\xrightarrow{+} \Sigma F_x = \cancel{m}\ddot{x} \qquad -B\dot{x} - kx + f_a(t) = 0 \qquad B\dot{x} + kx = f_a(t) = A$

Notes 10-27-10

$\mathcal{L}(B\dot{x}) + \mathcal{L}(kx) = \mathcal{L}(A) \Rightarrow B[sX(s) - x(0)] + k[X(s)] = \frac{A}{s}$ where $\mathcal{L}(x(t)) = X(s)$

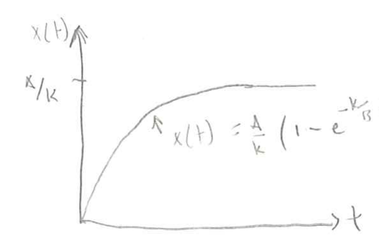
$X(s)\,[Bs+k] = A/s \qquad X(s) = \frac{A}{(Bs+k)s} = \frac{A/B}{(s+k/B)s}$

$= \frac{A_1}{s} + \frac{A_2}{s+k/B} \qquad A_1: s=0 \quad \frac{A/B}{k/B} = A_1 = \frac{A}{k} \qquad A_2: s=-k/B \quad \frac{A/B}{-k/B} = A_2 = -A/k$

$X(s) = \frac{A}{k}\left(\frac{1}{s}\right) + \left(-\frac{A}{k}\right)\left(\frac{1}{s+\frac{k}{B}}\right); \quad \frac{A/B}{k/B} = A_1 = \frac{A}{k}$

$x(t) = \mathcal{L}^{-1}(X(s)) = \frac{A}{k}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{A}{k}\mathcal{L}^{-1}\left(\frac{1}{s+k/B}\right) = \frac{A}{k} - \frac{A}{k}e^{-\frac{k}{B}t} \qquad x(t) = \frac{A}{k}\left[1 - e^{-\frac{k}{B}t}\right]$

$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1 \qquad \mathcal{L}(e^{-at}) = \frac{1}{s+a}$



General 1^{st}-order System

Consider a linear, single input, single output system desc. by a 1^{st} order D.E.

$$a_2 \dot{y} + a_1 y = a_0 g(t) \quad \text{divid by } a_2 \Rightarrow \dot{y} + \frac{a_1}{a_2} y = \frac{a_0}{a_2} g(t)$$

$$\dot{y} + \frac{y}{a_2/a_1} = \frac{a_0 g(t)}{a_2} \quad \text{or} \quad \underbrace{\boxed{\dot{y} + \frac{y}{\tau} = f(t)}}_{\text{Gen. Form}} \quad \text{where } \tau = \left(\frac{a_2}{a_1}\right)$$

To solve 1^{st} order sys. let $y(0) = 0$ $\quad f(t) = A$

Transform the eqn $\mathcal{L}(\dot{y}) + \mathcal{L}\left(\frac{y}{\tau}\right) = \mathcal{L}(A)$

$$[sY(s) - Y(0)] + \frac{1}{\tau} Y(s) = A/s \qquad Y(s)\left[s + \frac{1}{\tau}\right] = \frac{A}{s} + Y_0$$

$$\boxed{Y(s) = \frac{A}{s(s+\frac{1}{\tau})} + \frac{Y_0}{(s+1/\tau)}}$$

General 1^{st} O. DE system

$$\frac{A}{s(s+\frac{1}{\tau})} = \frac{A_1}{s} + \frac{A_2}{s+\frac{1}{\tau}} \qquad A_1 = \frac{A}{1/\tau} = \tau A \qquad A_2 ? \; s = -\frac{1}{\tau}, \; A_2 = \frac{A}{-1/\tau} = -A\tau$$

$$Y(s) = \frac{\tau(A)}{s} - \frac{\tau(A)}{s+\frac{1}{\tau}} + \frac{Y_0}{(s+\frac{1}{\tau})} \qquad y(t) = \tau A \, \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \tau A \, \mathcal{L}^{-1}\left(\frac{1}{s+\frac{1}{\tau}}\right) + Y_0 \, \mathcal{L}^{-1}\left(\frac{1}{s+1/\tau}\right)$$

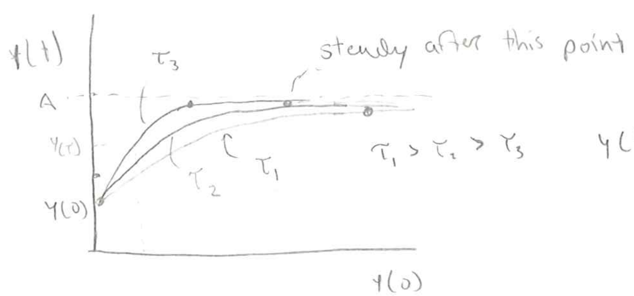
$$y(t) = \tau A - \tau A e^{-t/\tau} + Y_0 e^{-t/\tau} \qquad \boxed{y(t) = \tau A + (Y_0 - \tau A) e^{-t/\tau}}$$

Analysis of sol'n to 1^{st} od. system

The complete response $(y(t))$ can be viewed as $y(t) = Y_{tr}(t) + Y_{ss}(t)$

where $Y_{tr}(t) \rightarrow$ transient response

$Y_{ss}(t) \rightarrow$ steady state response



$\tau_1 > \tau_2 > \tau_3 \qquad y(t) = \tau A + [Y(0) - \tau A] e^{-t/\tau}$

The value of y varies with time for ~~these~~ 3 plots

$Y_{tr} \rightarrow$ Transient Response: the behavior of system immediately after it is turned on.

$Y_{ss} \rightarrow$ Steady st. response, part of the sol'n after trans. terms have disappeared as $t \rightarrow \infty$

— for the given sol'n $\quad y(t) = \tau A + [Y(0) - \tau A] e^{-t/\tau}$

$$Y_{ss} = \lim_{t \to \infty} Y(t) \Rightarrow Y_{ss} = \tau A$$

$y(t) = y_{tr} + y_{ss}$ Notes 10-22-10

Significance of "τ" $y(t) = \tau A + [y(0) - \tau A]e^{-t/\tau}$

$= y_{ss} + [y(0) - y_{ss}]e^{-t/\tau}$

evaluate $y(t)$ @ $t = \tau \Rightarrow y(\tau) = y_{ss} + [y(0) - y_{ss}](0.3679)$

$y(\tau) = y_{ss}(1 - .3679) + .3679(y_0)$

$= .6321(y_{ss}) + (1 - 0.6321)y_0$

$= y_0 + .6321(y_{ss} - y_0)$ for $t = \tau$

Similarly, for $t = 2\tau$ $y(2\tau) = y_0 + .865(y_{ss} - y_0)$

$y(3\tau) = y_0 + .95(y_{ss} - y_0)$

$" \; 4\tau = " \; .98(\quad)$

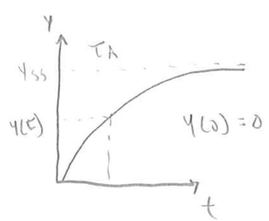
From $y(\tau)$, $y(2\tau)$ $y(3\tau)$ the response of a const. input is approximately 63%, 86%, 95%, & 98% respectively from initial value $y(0)$ to the steady state value

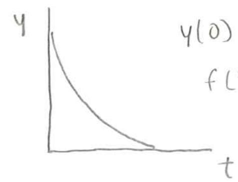
$\tau \Rightarrow$ "Time constant" having units of seconds

it is a measure of how quickly the system responds & reaches steady state

a common rule of thumb is to say that transients have died out when it is reduced to about 2% of initial value ($t = 4\tau$)

Special cases





$y(0) \neq 0$

$f(t) = 0$ (zero input case

Notes 10-25-10

General 1st order system

$\dot{y} + \frac{y}{\tau} = f(t)$ $\tau \to$ time constant $y \to$ output $f(t) \to$ input

for a constant input, $f(t) = A$ $y(t) = \tau A + [y(0) - \tau A]e^{-t/\tau} = y_{ss} + y_{tr}$

$y_{ss} = y(t)$ $\lim_{t\to\infty}$

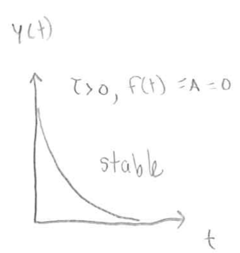
Stability of 1st order system

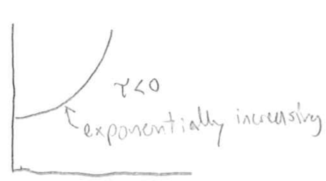
the general sol'n of 1st order system $y(t) = \tau A + [y(0) - \tau A]e^{-t/\tau}$

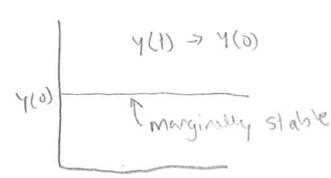
a 1st order system is: · stable if $\tau > 0$

unstable if $\tau < 0$

marginally stable if $\tau \to \infty$







$\dot{y} + .5y = F(t)$ where $F(t) = e^{-t/2}$; $y(0) = 1$

a) find $y(t)$

b) τ and y_{ss}

c) y_{tr}

d) sketch the complex response

e) comment on stability

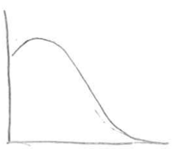
Lap. transf. $\mathcal{L}(\dot{y}) + 0.5\,\mathcal{L}(y) = \mathcal{L}(F(t))$ $\quad [sY(s) - y(0)] + .5Y(s) = \mathcal{L}(e^{-t/2})$

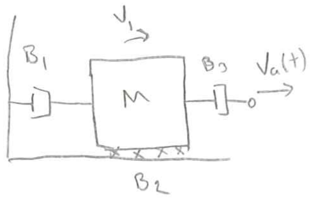
$Y(s)[s + 0.5] = \frac{1}{s+.5} + y(0) = \frac{1}{s+0.5}$

$Y(s) = \frac{1}{(s+.5)^2} + \frac{1}{s+.5}$ $\quad y(t) = \mathcal{L}^{-1}(Y(s)) = te^{-t/2} + e^{-t/2}$ $\Rightarrow$ a) $\boxed{y(t) = e^{-t/2}\,[t+1]}$

$\dot{y} + \frac{y}{\tau} = F(t)$ $\quad \tau = 2$ $\quad y(t) = \underbrace{y_{ss}}_{0} + y_{tr}$

so system is stable

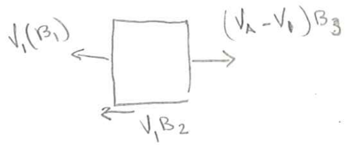


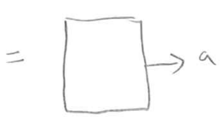


a) D.E. for M in terms of V_1

b) time const

c) sketch response when $V_a(t) = 0$ for $t \geq 0$ & $V_1(0) = 10$





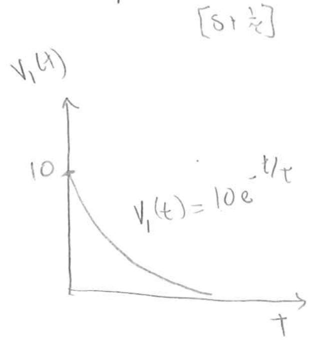
---- $M\dot{V}_1 = (B_1 + B_2 + B_3)V_1 = B_3 V_a(t)$

$\Rightarrow V_1 + \left[\frac{(B_{1,2,3})V_1}{M}\right] = \frac{B_3 V_a(t)}{M}$

$\left(\frac{M}{B_1 + B_2 + B_3}\right) = \tau$ $\quad \dot{V}_1 + \frac{V_1}{\tau} = \cancel{\frac{B_3}{M} V_a(t)}^{\,0}$ $\quad \mathcal{L}(\dot{V}_1) + \mathcal{L}\left(\frac{V_1}{\tau}\right) = 0$

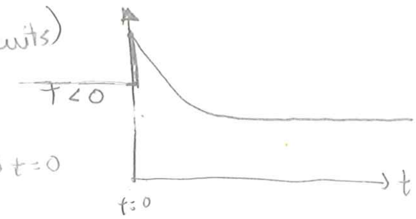
$\Rightarrow [sV_1(s) - V_1(0)] + \frac{1}{\tau}V_1(s) = 0$ $\quad \Rightarrow V_1(s)[s + \frac{1}{\tau}] = V_1(0) = 10$

$V_1(s) = \frac{10}{[s + \frac{1}{\tau}]}$ $\quad v_1(t) = \mathcal{L}^{-1}(V_1(s)) = 10\,\mathcal{L}^{-1}\left(\frac{1}{s + \frac{1}{\tau}}\right)$ $\quad V_1(t) = 10e^{-t/\tau}$



Notes 10-25

Initial Conditions: (usually 1^{st} order electrical circuits)



$$y(t) = y_{ss} + [y(0) - y_{ss}]e^{-t/\tau}$$

general sol'n

discontinuity @ $t=0$

change this to $y(0+)$, so you are evaluating on the (+) side of discontinuity

physical qty's that do not change at $t=0$, even if input is suddenly applied

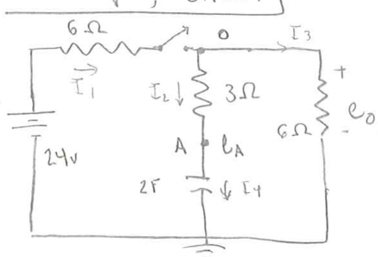
- Mechanical systems
  - a) elastic potential energy $= \frac{1}{2}kx_k^2$
  - b) potential energy $= \frac{1}{2}mv_m^2$
  - c) rotational potential $= \frac{1}{2}k(\Delta\theta)^2$
  - d) rotational kinetic $= \frac{1}{2}J\omega^2$

$x_k, v_M, \Delta\theta$, and ω do not change instantly

- electrical systems
  - a) capacitor $E_c = \frac{1}{2}Ce_c^2$
  - b) inductor $E_L = \frac{1}{2}LI_L^2$

e_c and I_L do not change instantly on application of input

example, Circuit



$t<0$, switch is open & no energy is stored in the capacitor

switch closes @ $t=0$

Find : $e_o(t)$

$e_c = 0$ @ $t<0$ but cannot say that $e_o(0) = 0$ because it is a dissipative element

Node eqn method to solve $e_o(t)$ @ o: $I_1 = I_2 + I_3$ @ A: $I_2 = I_4$

10-27-10

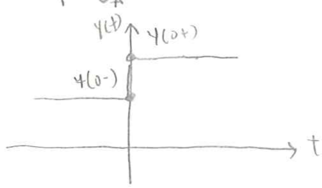
$24 - e_o = I_1(6)$

$e_o - e_A = I_2(3)$

$e_o = I_3(6)$

plug. $\frac{24-e_o}{6} = \frac{e_o - e_A}{3} + \frac{e_o}{6}$ (1), $\frac{e_o - e_A}{3} = 2\dot{e}_A$ (2)

- $I_4 = c\dot{e}_A$



use (1), get $e_A = 2e_o - 12$ (3)

$\boxed{\frac{e_A - (2e_o - 12)}{3} = 2(2\dot{e}_o) \text{ then plug into 2 to get}}$

* dont do this step. avoid $\dot{e}$'s

apply laplace: to (3) $E_A(s) = 2E_o(s) - \frac{12}{s}$ (4) and to (2) $\frac{E_o(s) - E_A(s)}{3} = 6[sE_A(s) - \cancel{e_A(0)}]$

$E_o(s)[1-2] + \frac{12}{s} = 6s[2E_o(s) - \frac{12}{s}]$

Missed some notes here

...

$E_o(s) = \frac{12(1+6s)}{12s(\frac{1}{12}+s)}$

$e_o(t) = 12 - 6e^{-t/12}$

last page

init. conditions:

$t(0-) = t(0+)$

e_c

I_L

however, $e_r(0-) \neq e_r(0+)$

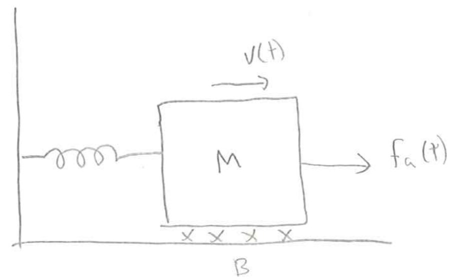
Example

$M = 1\,kg$

$B = 3$

$k = 2$

$X(0) = V(0) = 0$



a) find D.E. in V.

b) V(t) assuming $f_a(t) = 1 + \sin t$

c) Trans. & SS response

$M\ddot{x} + B\dot{x} + kx = f_a(t)$ $\quad M(V(t)) + BV(t)$ $\quad V(t) = \frac{dx}{dt}$

$\int_0^t V(t)\,dt = \int_{x=0}^{x} dx$ $\quad \int_0^t v(t)\,dt = x(t) - x(0)$

$x(t) = x(0) + \int_0^t v(t)\,dt$

Part b)

$M\dot{v} + BV + k[x(0) + \int_0^t v(t)\,dt] = f_a(t)$

$M\mathcal{L}(\dot{v}) + B\mathcal{L}(V) + k[\mathcal{L}(x(0)) + \mathcal{L}\int_0^t v(t)\,dt] = \mathcal{L}(f_a(t))$

$M[sV(s) - \cancel{V(0)}^{0}] + BV(s) + k\,\cancel{\frac{x(0)}{s}}^{0} + \frac{V(s)}{s} = \mathcal{L}(1 + \sin t)$

$\mathcal{L}\int_0^t v(t)\,dt = \frac{V(s)}{s}$ $\quad V(s)\left[s + 3 + \frac{2}{s}\right] = \frac{1}{s} + \frac{1}{s^2+1}$

$V(s)\left[\frac{s^2+3s+2}{\cancel{s}}\right] = \frac{s^2+1+s}{\cancel{s}(s^2+1)} \Rightarrow V(s) = \frac{s^2+1+s}{(s^2+3s+2)(s^2+1)}$

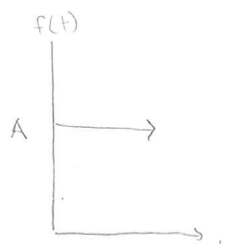
$V(s) = \frac{s^2+1+s}{(s+1)(s+2)(s^2+1)} = \frac{A_1}{(s+1)} + \frac{A_2}{(s+2)} + \frac{A_3 s + A_4}{(s^2+1)}$

Solving the transform, $\boxed{V(t) = \underbrace{0.5e^{-t} - 0.6e^{-2t}}_{Y_{tr}} + \underbrace{.1\cos t + .3\sin t}_{Y_{ss}}}$

Inputs: → unit step function
→ impulse function

multiply the unit step func. by A:

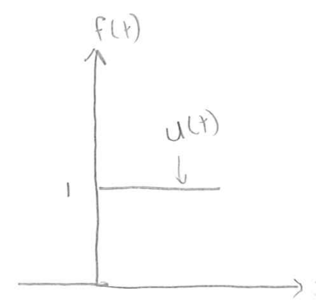
$f(t)u(t) = 0 \quad t \le 0$
$\qquad\qquad A \quad t > 0$



unit step func. (U(t))

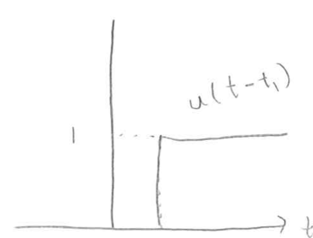
$u(t) = 0 \quad t \le 0$
$\qquad\quad 1 \quad t > 0$

discontinuity @ t=0



discontinuity @ $t = t_1$

$u(t-t_1) = 0 \quad t \le t_1$
$\qquad\qquad\quad 1 \quad t > t_1$



Exam Wed

Exam 2 11-3-2010

closed book/notes

$\mathcal{L}$ transform sheet provided

ch 5, 6, 7.1-7.4, 7.6

HW 4-8

3 problems

crib sheet A-4 front & back

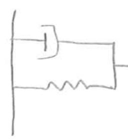
review 11-1 (monday)

BRING PAPER

Notes 10-29-10

if input to system is $u(t)$, we define sol^n (or output) as unit step response $[y(t)]$

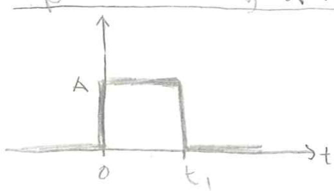
$$x(t) = \frac{A}{K}\left[1 - e^{-\left(\frac{K}{B}\right)t}\right] \qquad y_{u(t)} = x(t) = \frac{1}{K}\left[1 - e^{-\left(\frac{K}{B}\right)t}\right] \quad \text{where } A = 1$$



$f(t) = A$

Inputs consisting of horizontal & vertical lines.

$$f(t) = \begin{cases} 0 & t \le 0 \\ A & 0 < t \le t_1 \\ 0 & t > t_1 \end{cases}$$

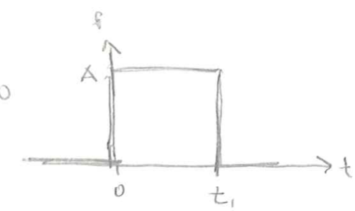


to show the input as sum of unit step input functions define two unit step functions @ pt. of discontinuity

$$u(t) = \begin{cases} 0 & t \le 1 \\ 1 & t > 0 \end{cases} \qquad u(t-t_1) = \begin{cases} 0 & t \le t_1 \\ 1 & t > t_1 \end{cases}$$

$$f(t) = Au(t) - Au(t-t_1)$$

ex: linear 1st order system $\dot{y} + \frac{y}{\tau} = u(t)$ $y(0) = 0$



general sol^n of 1st order sys. for const. input

$$y(t) = \tau A - [y(0) - \tau A]e^{-t/\tau} = \tau A[1 - e^{-t/\tau}] \quad \text{for } 0 < t < t_1$$

$$y(t) = \tau A[1 - e^{-t/\tau}] - Au(t-1)$$

for $t > t_1$ $f(t) = Au(t) - AU(t-t_1)$ Hence, we may use superposition & sum the responses to the components $Au(t)$ & $-AU(t-t_1)$

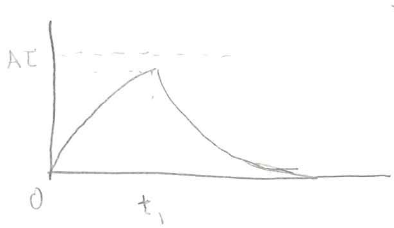
hence for $t > t_1$: $y(t) = [\tau A(1 - e^{-t/\tau}) - \tau A(1 - e^{-(t-t_1)/\tau})]$

$$\tau A[1 - e^{-t/\tau} - 1 + e^{-(t-t_1)/\tau}] \quad \tau A[-e^{-t/\tau} + e^{-t/\tau} \cdot e^{t_1/\tau}]$$

$$y(t) = \tau A[e^{t_1/\tau} - 1]e^{-t/\tau}$$

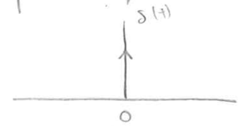
complete sol^n $y(t) = \tau A(1 - e^{-t/\tau})$ for $0 \le t \le t_1$

$\tau A(e^{t_1/\tau} - 1)e^{-t/\tau}$ for $t > t_1$



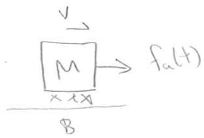
<u>Impulse input</u>: ($\delta(t)$) Usually used in signal processing

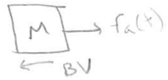
<u>Dirac delta function</u> $\delta(t) = \begin{cases} \infty & t=0 \\ 0 & t \neq 0 \end{cases}$



mathematical property of impulse input

1) $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$ 2) $\mathcal{L}\,\delta(t) = 1$ 3) Impulse response $h(t) = \frac{d}{dt}(y_{u}(t))$





<u>find</u> a) $y_u(t)$ & $h(t)$

b) plot responses

$\Sigma F_x = M\dot{v}_x$ $f_a - BV = M\dot{V}$ $M\dot{V} - BV = f_a(t)$ $\dot{V} = \frac{V}{\left(\frac{M}{B}\right)} = \frac{f_a(t)}{M}$ $\tau = \left(\frac{M}{B}\right)$

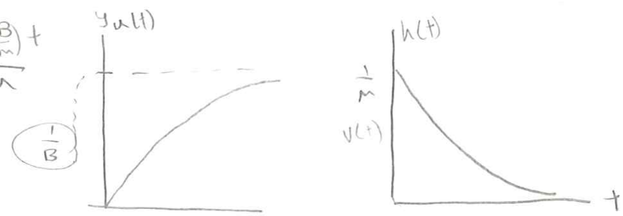
general sol^n $\dot{y} + \frac{y}{\tau} = A$ $y(t) = \tau A \leftarrow$ A becomes $\frac{1}{M}$

$y(t) = \tau A + [y(0) - \tau A]e^{-t/\tau}$ $A = \frac{1}{M}$ $y_u(t) = V(t) = \frac{M}{B}\left(\frac{1}{M}\right) + \left[V(0) - \frac{M}{B}\left(\frac{1}{M}\right)\right]e^{-t\left(\frac{B}{M}\right)}$

$\tau = \frac{M}{B}$ $y_u(t) = \frac{1}{B} - \frac{1}{B}e^{-\frac{B}{M}t}$ $h(t) = \frac{d}{dt}(y_u(t)) = 0 - \frac{1}{B}\left(-\frac{B}{M}\right)e^{-\left(\frac{B}{M}\right)t}$

$h(t) = \frac{e^{-\left(\frac{B}{M}\right)t}}{M}$

<u>Responses</u>:



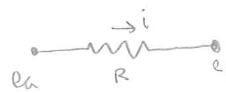
$y_u(t) = \frac{1}{B} - \frac{1}{B}e^{-\frac{B}{M}t}$

$h(t) = \frac{1}{M}e^{-\left(\frac{B}{M}\right)t}$

Review

<u>chapter 6</u> variables q, i, e chg. current, voltage

<u>elements</u>: resistor



$e_R = iR$ or $e_a - e_b = iR$

2) capacitor $q = Ce_c \Rightarrow \frac{dq}{dt} = i_c = C\dot{e}_c$ —||— or —|(—

3) inductor $e_L = L\frac{di_L}{dt}$

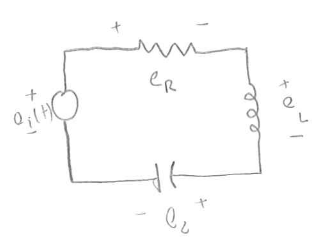
$e_L = e_a - e_b$

4) sources (voltage or current) $e_i(t)$ or $i_i(t)$

Basic Current laws: KCL

$\Rightarrow i_1 = i_2 + i_3$

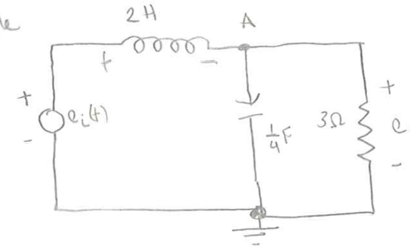
KVL $e_i(t)$



Exam will have a 2-loop problem

- Can use EITHER method, loops or nodes

Example 6.71



find I/O eqn

KCL $I_1 = I_2 + I_3$

constitutive relations

1) $e_i - e_o = L\frac{dI_1}{dt} = 2\frac{dI_1}{dt}$

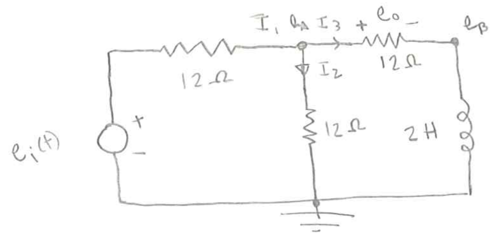
2) $I_2 = C\dot{e}_c = \frac{1}{4}\dot{e}_o$

3) $e_o = I_3(R) = I_3(3)$

Now) $\dot{I}_1 = \dot{I}_2 + \dot{I}_3$

$\frac{e_i - e_o}{2} = \frac{1}{4}\ddot{e}_o + \frac{\dot{e}_o}{3}$ $\quad \frac{e_i}{2} = \frac{\ddot{e}_o}{4} + \frac{\dot{e}_o}{3} + \frac{e_o}{2}$

$$\boxed{3\ddot{e}_o + 4\dot{e}_o + 6e_o = 6e_i}$$



KCL ⇒ $I_1 = I_2 + I_3$

Const. rel. 1) $e_i(t) - e_A = I_1(12)$

$e_A - 0 = I_2(12)$

$e_A - e_B = e_o = I_3(12)$

$e_B - 0 = 2\frac{dI_3}{dt}$

(2)

$\frac{e_i(t) - e_A}{12} = \frac{e_A}{12} + \frac{e_B}{12}$ (3) $\quad \dot{I}_1 = \dot{I}_2 + \dot{I}_3 \quad \frac{\dot{e}_i(t) - \dot{e}_A}{12} = \frac{\dot{e}_A}{12} + \frac{e_B}{12} \Rightarrow$ (4)

$e_i(t) - e_A = e_A + e_o \qquad \frac{e_i(t) - e_o}{2} = e_A \qquad e_A - e_B = e_o \qquad e_A - e_o = e_B$

now it's just math

$\dot{e}_i = 2\dot{e}_A + 6e_B \qquad \dot{e}_i = \cancel{2}\left(\frac{\dot{e}_i(t) - \dot{e}_o}{\cancel{2}}\right) + \overset{3}{\cancel{6}}\left(\frac{e_i(t) - 3e_o}{\cancel{2}}\right) \qquad \cancel{\dot{e}_i} = \cancel{\dot{e}_i(t)} - \dot{e}_o + 3e_i(t) - 9e_o$

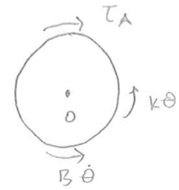
$$\boxed{\dot{e}_o + 9e_o = 3e_i(t)}$$

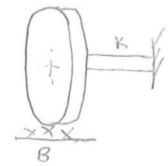
state variable form:

$\dot{e}_c = \frac{I_c}{C} \qquad \dot{I}_L = \frac{e_L}{L} \qquad I_c$ and $e_L = f($input, e_c, and $I_L)$

$\dot{q} = Aq + Bu$

Ch 5



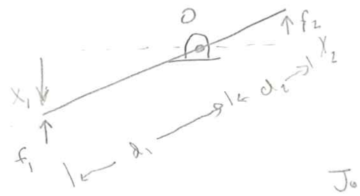


$\Sigma M_o = J\ddot{\theta}$

$\tau_A - k\theta - B\dot{\theta} = J_o\ddot{\theta}$

or $J_o\ddot{\theta} + B\dot{\theta} + k\theta = \tau_a(t)$

levers

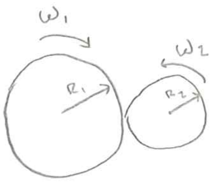


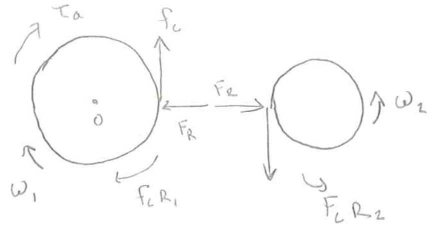
$\Sigma M_o = J\alpha = J\ddot{\theta}$

$-f_1(d_1\cos\theta) + f_2(d_2\cos\theta) + w\cos\theta(a) = J\ddot{\theta}$ where $a = \left(\frac{d_1+d_2}{2}\right)$

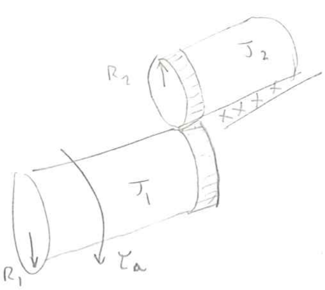
$J_o = J_a + ma^2$

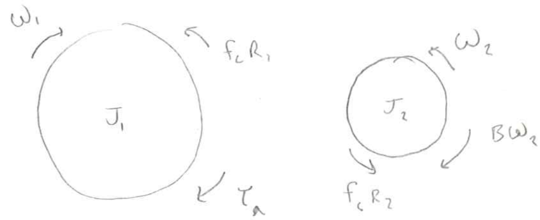
Gears





ex)





$\Sigma M_o = J_1\dot{\omega}_1$

$-f_cR_1 + \tau_a = J_1\dot{\omega}_1$ —(1)

$\Sigma M_o = J_2\dot{\omega}_2$

$f_cR_2 - B\omega_2 = J_2\dot{\omega}_2$ (2)

R_2[eqn 1] + R_1[eqn 2] $\tau_a R_2 - BR_1\omega_2 = J_1\dot{\omega}_1 + J_2 R_1 \dot{\omega}_2$

Then use $\frac{R_2}{R_1} = \frac{\omega_1}{\omega_2}$

Ch 8 Transfer function Analysis

Consider an n^{th} order, linear DE.

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots a_1\dot{y} + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \dots b_0 u(t)$$

Take laplace transform of above eqn

let $\mathcal{L}(y(t)) = Y(s)$
$\mathcal{L}(u(t)) = U(s)$

$$a_n[s^n Y(s) - s^{n-1}y(0) + \dots - y^{n-1}(0)] + a_{n-1}[s^{n-1}Y(s) - s^{n-2}y(0)]$$
$$+ \dots + a_1[sY(s) - y(0)] + a_0 Y(s) = b_m[s^m U(s) - s^{m-1}u(0) \dots - u^{m-1}(0)]$$
$$+ b_{m-1}[s^{m-1}U(s) - s^{m-2}u(0) \dots u^{m-2}(0)] + \dots b_0 U(s)$$

set initial conditions equal to zero

$$[a_n s^n + a_{n-1}s^{n-1} + \dots a_1 s + a_0]Y(s) = [b_m s^m + b_{m-1}s^{m-1} + \dots + b_0]U(s)$$

or $$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots b_0}{a_n s^n + a_{n-1}s^{n-1} + \dots a_1 s + a_0} = \frac{\mathcal{L}\text{ of output}}{\mathcal{L}\text{ of input}}$$

Note: all the states (ie $y, \dot{y}, \ddot{y}, \dots \frac{d^{n-1}y(0)}{dt^{n-1}}$) are zero

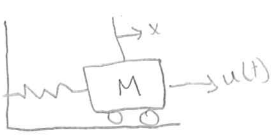
Transfer function for any dynamic system:

ratio of the laplace transform of output to laplace trans of input evaluated at zero initial conditions

ie, $\text{Trans. Function} = H(s) = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$

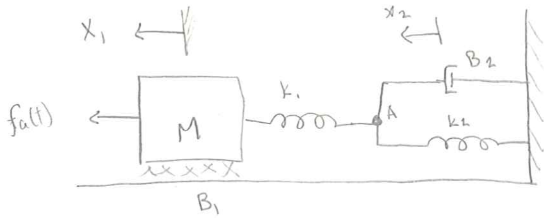
or $$H(s) = \frac{b_m s^m + \dots b_0}{a_n s^n + a_{n-1}s^{n-1} \dots a_1 s + a_0}$$

consider spring mass system



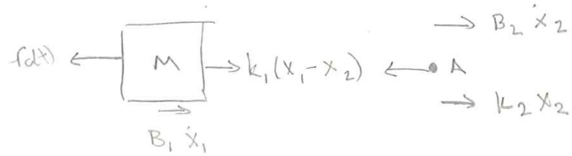
$H(s) = \frac{\mathcal{L}(x(t))}{\mathcal{L}(u(t))} = \frac{X(s)}{U(s)}$ the trans. function is identical w/ zero state response

$\dddot{y} + 3\ddot{y} - \dot{y} = u(t)$ $\quad y(0)\ \dot{y}(0)\ \ddot{y}(0) = 0$ $\quad q = \begin{bmatrix} y \\ \dot{y} \to v \\ \ddot{y} \to a \end{bmatrix}$



Find
a) EOM
b) H(s) from (a)
c) output $x_1(t)$
d) H(s) from state-space form

$M_1 = 1\,kg$
$B_1 = B_2 = 1\,N\text{-}s/m$ $\qquad X_1 = X_2 = 0$ (springs undeflected
$K_1 = K_2 = 1\,N/m$



For mass M: $\overset{+}{\leftarrow} \Sigma F_x = M\ddot{x}$

$$f_a(t) - k_1(x_1 - x_2) - B_1\dot{x}_1 = M\ddot{x}_1 \Rightarrow M\ddot{x}_1 + B_1\dot{x} + k_1(x_1 - x_2) = f_a(t)$$

$$(1) \quad \ddot{x}_1 + \dot{x}_1 + x_1 - x_2 = f_a(t)$$

For junc. A $\quad \Sigma F_x = 0$

$$K(x_1 - x_2) - B_2\dot{x}_2 - k_2 x_2 = 0 \qquad x_1 - \dot{x}_2 - 2x_2 = 0 \qquad (2)$$

$$\boxed{\ddot{x}_1 + \dot{x}_1 + x_1 - x_2 = f_a(t) \qquad \dot{x}_2 + 2x_2 - x_1 = 0}$$

$$H(s) = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})} = \frac{\mathcal{L}(x_1(t))}{\mathcal{L}[f_a(t)]} = \frac{X_1(s)}{F_a(s)}$$

$$(1) \quad [s^2 X_1(s) - s\cancel{x_1(0)}^{\,0} - \cancel{\dot{x}_1(0)}^{\,0}] + [sX_1(s) - \cancel{x_1(0)}^{\,0}] + X_1(s) - X_2(s) = F_a(s)$$

$$X_1(s)[s^2 + s + 1] - X_2(s) = F_a(s)$$

$$(2) \quad [sX_2(s) - X_2(0)] + 2X_2(s) - X_1(s) = 0 \qquad X_2(s)[s+2] = X_1(s) \Rightarrow X_2(s) = \frac{X_1(s)}{s+2}$$

$$X_1(s)\left[s^2 + s + \frac{1}{s+2}\right] = F_a(s)$$

$$X_1(s)\left[\frac{(s^2+s+1)(s+2)}{(s+2)}\right] = F_a(s) \qquad \frac{X_1(s)}{F_a(s)} = \frac{s+2}{(s^2+s+1)(s+2) - 1} = \frac{s+2}{s^3 + 3s^2 + 3s + 1}$$

$$= \boxed{\frac{s+2}{(s+1)^3} = H(s)}$$

c) $\quad \frac{X_1(s)}{F_a(s)} = H(s) \qquad X_1(s) = \frac{s+2}{(s+1)^3} F_a(s) \qquad X_1(t) = \frac{s+2}{(s+1)^3}\left(\frac{1}{s}\right) \qquad \underbrace{\frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{(s+1)^2} + \frac{A_4}{(s+1)^3}}_{\text{I do this}}$

(3)

$A_1 = 2 \qquad A_2 = -2 \qquad A_3 = -2 \qquad A_4 = -1$

$$\mathcal{L}^{-1}(3) = \boxed{x_1 t = 2 - \left(\tfrac{1}{2}t^2 + 2t + 2\right)e^{-t}}$$

x_1 ✓	$\dot{x}_1 = v_1$
$\dot{x}_1 \to v_1$ ✓	$\dot{v}_1 = f_a(t) - v_1 - x_1 + x_2$
x_2	$\dot{x}_2 = x_1 - 2x_2$

Apply transform to all 3 eqns

$$\begin{cases} [sX_1(s) - x_1(0)] = V_1(s) \\ [sV_1(s) - v_1(0)] = F_a(s) - V_1(s) - X_1(s) + X_2(s) \\ [sX_2(s) - x_2(0)] = X_1(s) - 2X_2(s) \end{cases}$$

$$s[sX_1(s)] = sV_1(s) = F_a(s) - sX_1(s) - X_1(s) + X_2(s)$$

$$X_2(s)[s+2] = X_1(s) \qquad X_2(s) = \frac{X_1(s)}{s+2}$$

$$\frac{X_1}{F_a(s)} = \frac{s+2}{(s^2+s+1)(s+2) - 1}$$

11-8-10

<u>Complete soln of higher order system</u>

for 1st order system $y(t) = y_{ss}(t) + y_{tr}(t)$

the same concept can be extended to higher order system

consider 2nd order D.E (linear) $a_2\ddot{y} + a_1\dot{y} + a_0 y = f_a(t)$ where $y(0) = y_0$ & $\dot{y}(0) = \dot{y}_0$

1) take laplace transform $a_2[s^2Y(s) - sy(0) - \dot{y}(0)] + a_1[sY(s) - y(0)] + a_0 Y(s) = F_a(s)$

$$Y(s)[a_2s^2 + a_1s + a_0] - a_2sy_0 - a_2\dot{y}_0 - a_1y_0 = F_a(s)$$

$$Y(s)[a_2s^2 + a_1s + a_0] = F_a(s) + a_2sy_0 + a_2\dot{y}_0 - a_1y_0$$

$$Y(s) = \frac{F_a(s) + [a_2sy_0 + a_2\dot{y}_0 - a_1y_0]}{a_2s^2 + a_1s + a_0}$$

input term

contains initial values of state vars

a) if initial values of st. vars $= 0$ $[y_0 = \dot{y}_0 = 0]$ Then $Y(s) = \underbrace{\left(\frac{1}{a_2s^2 + a_1s + a_0}\right) F_a(s)}_{Y_{zs}}$

zs = zero state response

b) if input values $= 0$, $\Rightarrow f_a(t) = 0 \Rightarrow F_a(s) = 0$

$$Y(s) = \frac{a_2(sy_0 + \dot{y}_0) + a_1y_0}{a_2s^2 + a_1s + a_0} = Y_{zi} \qquad zi = \text{zero input}$$

$$Y(s) = Y_{zs}(s) + Y_{zi}(s) \qquad y(t) = \mathcal{L}^{-1}(Y_{zs}(s)) + \mathcal{L}^{-1}(Y_{zi}(s))$$

$$\boxed{y(t) = y_{zs}(t) + y_{zi}(t)}$$

note: 1) Transfer function $H(s)$ is identified by zero state response

ie $Y_{zs}(s) = \dfrac{F_a}{a_2s^2 + a_1s + a_0} \Rightarrow \dfrac{Y_{zs}}{F_a(s)} = H(s) = \dfrac{1}{a_2s^2 + a_1s + a_0}$

2) $Y_{zi}(t)$ means zero input $\Rightarrow f_a(t) = 0 \Rightarrow$ homogeneous D.E. (free response)

Note cont. 3) Sum of $y_{zs}(t) + y_{zi}(t) = y(t) = y_{ss} + y_{tr}$

• stability of higher order system (using transfer function)

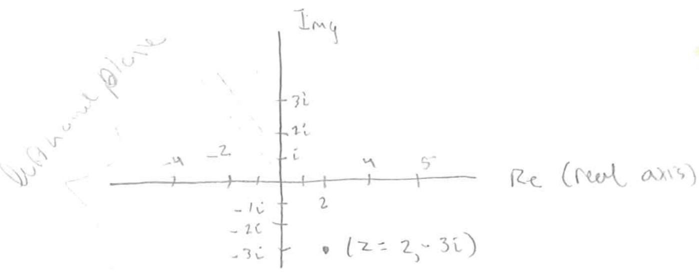
recall, $H(s) = \underline{\text{big expression}}$ for n^{th} order DE $\underbrace{\text{big expressions}}$

Let $P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots a_0$ the nth order polynomial $P(s) \rightarrow$

$\rightarrow$ characteristic polynomial, AND $P(s) = 0$ is called char. poly. equation

The roots of $P(s)$ are called eigen values or Poles of the trans. function

Poles are plotted on a complex (s or z) plane



Criteria for stability

1) all poles have neg. real points ⇒ STABLE
2) " " " " " " and one or more distinct poles lie on imaginary axis
⇒ MARGINALLY STABLE or critically stable system
3) UNSTABLE system - at least one pole lies in right hand plane OR have double roots on imaginary axis

11-10-2010

For n^{th} order DE $\longrightarrow$ $a_n \frac{d^n y}{dt^n} + \dots a_1 \frac{dy}{dt} + a_0 = b_m \frac{d^m u}{dt^m} + b_0 u(t)$

$Y(s) = Y_{zs}(s) + Y_{zi}(s)$

$y(t) = y_{zs}(t) + y_{zi}(t) = y_{ss}(t) + y_{tr}(t)$

☆ Transfer function $= \frac{Y_{zs}}{F_a(s)} = H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots a_1 s + a_0}$

where $P(s) = a_n s^n + \dots a_0$

Roots of $P(s) = 0$ are poles of transf. func.

Example: $\frac{d^2y}{dy^2} + \frac{5dy}{dt} + 6y = 2\frac{du}{dt} + 1$ find system poles & comment on stability

$\underline{\text{Sol}^n}$ $H(s) = \frac{b_n s^n + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} \dots + a_1 s + a_0} = \frac{b_1 s + b_0}{a_2 s + a_1 s + a_0} = \frac{2s + 1}{s^2 + 5s + 6}$

$P(s) = s^2 + 5s + 6$ find poles $\rightarrow P = 0$ $s^2 + 3s + 2s + 6$ $s(s+3) + 2(s+3)$ $(s+2)(s+3)$

Poles: $\underline{s = -2,\ s = -3}$ $\rightarrow$ stable example cont'd

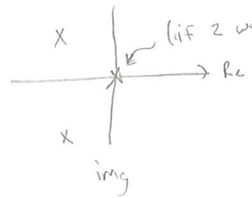
11-10-2010

$H(s) = \dfrac{30(s-6)}{s(s^2+4s+13)}$ $\quad s = \dfrac{-4 \pm \sqrt{16-4(1)(13)}}{2}$ $\quad s = -4 \pm \sqrt{-36}$

$s = -2 \pm 3i$

$s_2 = -2+3i$

$s_3 = -2-3i$



(if 2 were here, would be

system is marginally/critically stable

---

ex) $\ddot{y}(t) = 4u(t)$ where $u(t) = 7\delta(t)$ $\quad y(0-) = y_0$ & $\dot{y}(0-) = \dot{y}_0$

Find a) y_{zs}, y_{zi} b) $y(t)$ c) y_{tr}, y_{ss} d) $P(s)$ & Poles, e) comment on stability

Input $\Rightarrow f_a = 4u(t) = 4(7\delta(t)) = 28\delta(t)$ $\quad F_a(s) = 28$

$\ddot{y}(t) = f_a(t)$

laplace transform both sides: $s^2Y(s) - sy(0) - \dot{y}(0) = F_a(s)$

$Y(s)[s^2] = F_a(s) + sy(0) + \dot{y}(0)$

$Y(s) = \underbrace{\dfrac{F_a(s)}{s^2}}_{Y_{zs}(s)} + \underbrace{\dfrac{sy(0)+\dot{y}(0)}{s^2}}_{Y_{zi}(s)}$ $\qquad Y_{zs}(s) = \dfrac{F_a(s)}{s^2} \Rightarrow \dfrac{Y_{zs}(s)}{F_a(s)} = \boxed{\dfrac{1}{s^2} = H(s)}$

$Y_{zs}(s) = \dfrac{F_a(s)}{s^2} = \dfrac{28}{s^2} \Rightarrow y_{zs}(t) = 28t$

$Y_{zi}(s) = \dfrac{sy_0 + \dot{y}_0}{s^2} = y_0\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \dot{y}_0\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = y_0 + \dot{y}_0 t$

$y(t) = y_{zs}(t) + y_{zi}(t)$ $\qquad \boxed{y(t) = 28t + y_0 + \dot{y}_0(t)}$

- no transient terms
- all is steady (y_{ss})

$P(s) = s^2$ $\quad s_1 = s_2 = 0 \Rightarrow$ unstable

$\dot{y}(t) = x(t)$

$\dot{x}(t) = -4y(t) - 3u(t)$

$y(0) = y_0$ & $\dot{y}(0) = \dot{y}_0$ & $u(t) = -5$ $\qquad y(0) = y_0$ & $\dot{y}(0) = \dot{y}_0$ & $u(t) = -5$

$\ddot{y}(t) = \dot{x}(t) = -4y(t) - 3u(t)$ $\qquad \ddot{y}(t) + 4y(t) = -3u(t)$

Let $f_a(t) = -3u(t) = -3(-5) = 15$ $\quad F_a(s) = \dfrac{15}{s}$

$\ddot{y}(t) + 4y(t) = f_a(t)$ laplace both sides $s^2Y(s) - sy(0) - \dot{y}(0) + 4Y(s) = F_a(s)$

$Y(s) = \underbrace{\left(\dfrac{1}{s^2+4}\right)F_a(s)}_{Y_{zs}} + \underbrace{\left(\dfrac{sy_0+\dot{y}_0}{s^2+4}\right)}_{Y_{zi}}$ $\qquad \dfrac{Y_{zs}}{F_a(s)} = \dfrac{1}{s^2+4} = H(s)$

y_{tr} is not present. $y_{ss} = y(t)$

11-10-2010

$$Y_{zs}(s) = \frac{1}{s^2+4} F_a(s) = \frac{1}{s^2+4}\left(\frac{15}{s}\right)$$

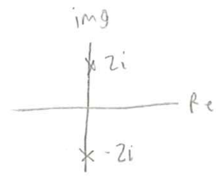
$$y_{zs}(t) = \mathcal{L}^{-1} \frac{15}{s(s^2+4)} \cdots \frac{A}{s} + \frac{A_2 s + A_3}{s^2+4} = \cancel{\frac{15}{4}\mathcal{L}^{-1}\left(\frac{1}{s}\right)} \cdots$$

$$y_{zs}(t) = \frac{15}{4} - \frac{15}{4}\cos 2t$$

$$y_{zi} = \frac{s y_0}{s^2+4} + \frac{\dot{y}_0}{s^2+4}$$

$$Y_{zi}(t) = y_0 \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \frac{\dot{y}_0}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) \qquad y_{zi}(t) = y_0 \cos 2t + \frac{\dot{y}_0}{2}\sin 2t$$

$$y(t) = \cos 2t\left[y_0 - \frac{15}{4}\right] + \frac{\dot{y}_0}{2}\sin 2t + \frac{15}{4} = y_{ss}$$



$P(s) = s^2+4$

$P(s) = 0 \Rightarrow s = \pm 2i$

analysis of 2^{nd} order system

general $a_2\ddot{y} + a_1\dot{y} + a_0 y = f_a(t) \longrightarrow (1)$

characteristic polynomial $P(s) = a_2 s^2 + a_1 s + a_0 \quad (n=2)$

$\Rightarrow$ poles $s_1, s_2 = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$ complete sol^n $y(t) = y_H + y_p$

- 3 cases are possible
  - a) when $a_1^2 - 4a_2a_0 > 0$, s_1 & s_2 are real & distinct, $y_H = k_1 e^{s_1 t} + k_2 e^{s_2 t}$
    where k_1, k_2 are constants and are evaluated using initial conditions
  - b) when " $= 0$ $s_1 = s_2$, k_1 & k_2 from IC $y_H = k_1 e^{s_1 t} + k_2 t e^{s_2 t}$
  - c) when " < 0 s_1 & s_2 are complex conjugates $y_H = e^{\alpha t}[k_1\cos\beta t + k_2 \sin\beta t]$
    where $\alpha \rightarrow$ real part $= \frac{a_1}{2a_2}$ $\beta \rightarrow$ img part $= \dfrac{\pm\sqrt{a_1^2 - 4a_2a_0}}{2a_2}$

standard form of a 2^{nd} order d.E. Note 11-12

take general form, above. divide by a_2: $\ddot{y} + \frac{a_1}{a_2}\dot{y} + \frac{a_0}{a_2}y = \frac{f_a(t)}{a_2}$

define 2 important parameters a) $\omega_n = \sqrt{\frac{a_0}{a_2}}$ where ω_n is called the natural freq. of the system

b) ζ = zeta $= \dfrac{a_1}{2\sqrt{a_0 a_2}}$ where ζ is called the damping ratio

$$\ddot{x} + \frac{B\dot{x}}{M} + \frac{kx}{M} = \frac{f_a(t)}{M} \qquad \frac{K}{M} = \left(\frac{1}{s^2}\right)\text{units} \qquad \zeta \text{ dimensionless}$$

$$\frac{a_1}{a_2} = 2\omega\zeta$$

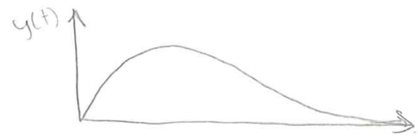
Characteristic eqn:

$$P(s) = s^2 + 2\omega_n \zeta s + \omega_n^2$$

roots: $\dfrac{-2\omega_n \zeta \pm \sqrt{4\omega_n^2 \zeta^2 - 4\omega_n^2}}{2}$ $\quad s_1, s_2 = \omega_n \zeta \pm \left(\sqrt{\zeta^2 - 1}\right) \omega_n$

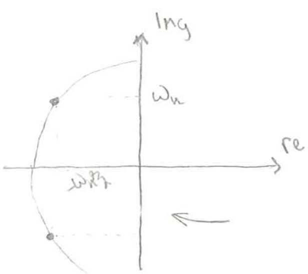
analysis of system (damping)

1) for $\zeta < 0$, (roots have (+) real part) unstable
2) for $\zeta > 1$ (distinct (−) roots) stable, zero input response for $\zeta > 1$,
   $y(t) = y_H + \cancel{y_P}^{0} = y_{ZI}$



for $\zeta > 1$, system is overdamped
3) for $0 \le \zeta < 1$, roots are complex conjugate system is stable
   $\zeta = 0 \Rightarrow$ marginally stable

system is called underdamped

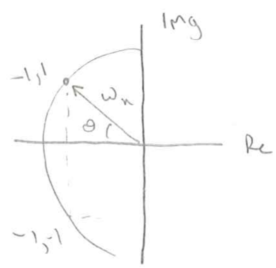


roots of the underdamped system lie on a circle w/ radius ω_n centered @ origin

4) for $\zeta = 1$, two equal roots at $-\omega_n$ system is stable, called critically damped

soln) compare $\ddot{y} + 2\dot{y} + 2y = u(t)$ & $\ddot{y} + 2\omega_n \zeta \dot{y} + \omega_n^2 y = f(t)$
$2\omega_n \zeta = 2$
$2 = \omega_n^2 \Rightarrow \omega_n = \sqrt{2}$ rad/s
$\zeta = \ldots .707$

for part(B) $s^2 + 2s + 2 = 0$ $\quad s = \dfrac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm i$ $\quad s_1 = -1 + i \quad s_2 = -1 - i$



$\theta = \cos^{-1}(\zeta) = 45°$

Ex

2nd order system

$J\ddot{\theta} + B\dot{\theta} + K\theta = T_a(t)$ assume no torque, $t>0$

$J = 1 kg\text{-}m^2$ $B = 5 N\text{-}\frac{m}{s}$ $k = 6\frac{N}{m}$

a) ξ & ω_n

$\ddot{\theta} + 5\dot{\theta} + 6\theta = T_a(t)$

$\ddot{y} + 5\dot{y} + 6y = u(t)$

$\ddot{y} + 2\omega_n\xi\dot{y} + \omega_n^2 y = f(t)$

Notes 11-15-10

Method (1) $\mathcal{L}(\ddot{\theta} + 5\dot{\theta} + 6\theta) = 0$ $\Theta(s) = \dfrac{s\theta_0 + \dot{\theta}_0 + 5\theta_0}{s^2+5s+6}$

$\theta(t) = (3\theta_0 + \dot{\theta}_0)e^{-2t} - (2\theta_0 + \dot{\theta})e^{-3t}$

Method: $y(t) = y_H + y_P^{\,0}$

$y_{zi}(t) = y_H$ $P(s) = s^2+5s+6=0 = (s+2)(s+3) \Rightarrow s = -2, -3$

$y(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} = k_1 e^{-2t} + k_2 e^{-3t}$

$\theta(0) = \theta_0 = k_1 e^0 + k_2 e^0 = k_1 + k_2$ $\dot{\theta}(t) = -2k_1 e^{-2t} - 3k_2 e^{-3t}$

$\dot{\theta}_0 = -(2k_1 + 3k_2) = -(2k_1 + 3(\theta_0 - k_1)) = +k_1 - 3\theta_0$

$\dot{\theta}_0 + 3\theta_0 = k_1$

$k_2 = \theta_0 - k_1 = -(\dot{\theta}_0 + 2\theta_0)$

$\boxed{\theta(t) = (\dot{\theta}_0 + 3\theta_0)e^{-2t} - (\dot{\theta}_0 + 2\theta_0)e^{-3t}}$

Pole-zero plot of transfer function

for n^{th} order D.E, (see 11-10-2010 notes) $\rightarrow = \dfrac{b_m\left[s^m + \frac{b_{m-1}}{b_m}s^{m-1} \dots \frac{b_0}{b_m} u(t)\right]}{a_n\left[s^n + \frac{a_{n-1}}{a_n}s^{n-1} \dots + \frac{a_1 s}{a_n} + \frac{a_0}{a_n}\right]}$

another form of $H(s)$:

$$H(s) = \frac{K[(s-z_1)(s-z_2)\dots(s-z_m)]}{[(s-p_1)(s-p_2)\dots(s-p_n)]}$$

where z_1, z_2, z_m are zeros of $H(s)$

p_1, p_2, p_n are "Poles" of $H(s)$

z_m & p_n can be distinct, repeating or complex

$K = \frac{b_m}{a_n}$ = transfer function Gain

in compact form $H(s) = \dfrac{k\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$

plot of z_m & p_n on a s-plane is pole-zero plot

zeros - O

poles - X

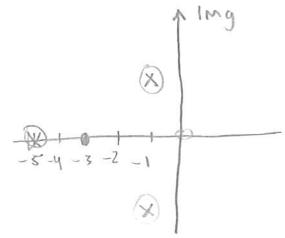
zeros of system have no effect on syst. stability

11-15-10

Example: Draw pole-zero plot in s-plane

$$H(s) = \frac{2s^2+6s}{(s+5)(s^2+2s+5)} = \frac{2(s)(s+3)}{(s+5)(s-[-1+2i])(s-[-1-2i])}$$

$K = 2$ zeros $z_1 = 0,\ z_2 = -3$ $\quad p = -5,\ p_2, p_3 = -1 \pm 2i$



Effect of zeros on a system

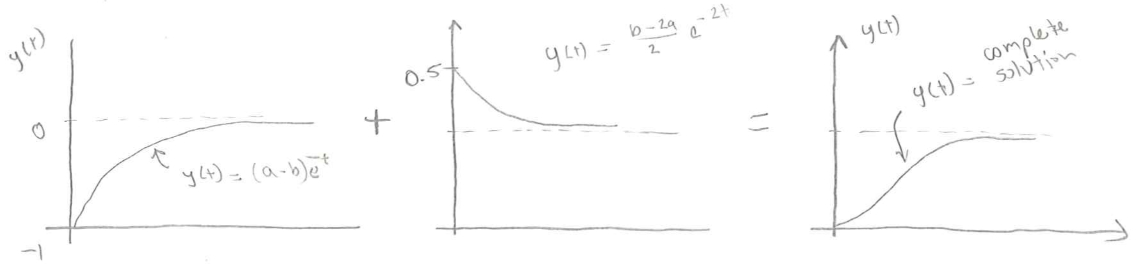
Consider any trans func. $H(s) = \dfrac{as+b}{(s^2+3s+2)} = \dfrac{as+b}{(s+1)(s+2)}$

let $u(t) = 1 \Rightarrow u(s) = \frac{1}{s}$ $\quad H(s) = \dfrac{Y(s)}{U(s)} \Rightarrow Y(s) = U(s)H(s)$

$$Y(s) = \frac{1}{s}\,\frac{as+b}{(s+1)(s+2)} = \frac{A_1}{s} + \frac{A_2}{(s+1)} + \frac{A_3}{(s+2)}$$

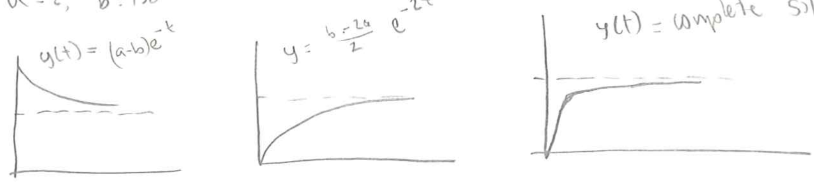
$$\boxed{y(t) = \frac{b}{2} + (a-b)e^{-t} + \frac{(b-2a)}{2}e^{-2t}}$$

Case a $\quad a = 0,\ b = 2$ (no zeros, poles @ $s = -1, -2$)



in absence of any zero, the pole @ $s = -1$ is the dominant pole

$a = 2,\ b = 1.8$ $\qquad 2s + 1.8 =$



1) Poles closest to origin are dominant poles (in absence of zeros

2) zeros cancel the effect due to poles closer to them, making other poles dominant

11-15-10

Properties of transfer function

for any input $u(t) \Rightarrow \mathcal{L}\, u(t) = U(s)$

$\frac{Y(s)}{U(s)} = H(s) \Rightarrow Y(s) = H(s)U(s)$

special cases:

a) for a unit step response $u(s) = \frac{1}{s}$ $\Rightarrow y_u(t) = \mathcal{L}^{-1}\left(\frac{H(s)}{s}\right)$

b) for unit impulse response $u(t) = \delta(t) \leftarrow$ "shortcut method" to $\frac{d y_u(t)}{dt} = h(t)$

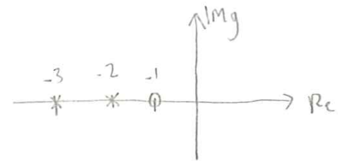
$\Rightarrow h(t) = \mathcal{L}^{-1}(H(s))$

c) for constant input $u(t) = A$ $y_{ss}(t) = AH(s=0)$

$y_u(t) = [\mathcal{L}(te^{-t})]s$

8.27 $H(s) = \frac{s+1}{s^2+5s+6}$

a) Plot pole,-zero pattern
b) general form of y_{zi}
c) gen form of I/O D.E.



zeros: $s+1=0 \rightarrow s=-1$

poles: $(s+2)(s+3)=0$

b) y_{zi} for 2^{nd} order DE. $y_{zi} = y_H = k_1 e^{s_1 t} + k_2 e^{s_2 t}$ $\quad y_{zi} = k_1 e^{-2t} + k_2 e^{-3t}$

c) general form of I/O D.E. $a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \dots a_1 \dot{y} + a_0 = b_m \frac{d^m u}{dt^m} + \dots b_0 u(t)$

$H(s) = \frac{b_m s^m \dots}{\dots} \Rightarrow \ddot{y} + 5\dot{y} + 6y = \dot{u} + u$ $\quad (a_2 = 1 \; a_1 = 5 \; a_0 = 6)$

d) $y_u(t) = \mathcal{L}^{-1}\left(\frac{H(s)}{s}\right) = \mathcal{L}^{-1}\left[\left(\frac{s+1}{s^2+5s+6}\right)\left(\frac{1}{s}\right)\right]$ $\quad y_u(t) = \frac{1}{6} + \frac{1}{2}e^{-2t} - \frac{2}{3}e^{-3t}$

$h(t) = \frac{d}{dt} y_u(t) = \dots -e^{-2t} + 2e^{-3t}$

8.36 the unit step response of a certain system $y_u(t) = 2 + 2e^{-t}\cos(2t)$

find a) $H(s)$ b) ζ & ω_n c) $h(t)$

a)
$H(s) = s\mathcal{L}[y_u(t)]$
$s\mathcal{L}[2 + 2e^{-t}\cos 2t]$
$s\left[\frac{2}{s} + 2\mathcal{L}e^{-t}\cos 2t\right]$ $\quad \mathcal{L}[e^{-at} f(t)] = F(s+a)$

$H(s) = 2 + 2s\frac{(s+1)}{(s+1)^2+4} = \frac{2((s+1)^2+4) + 2s(s+1)}{(s+1)^2+4}$

$\boxed{H(s) = \frac{4s^2 + 6s + 10}{s^2 + 2s + 5}}$ Part (a)

b) standard form (2^{nd} order)

$\ddot{y} + 2\omega_n \zeta \dot{y} + \omega_n^2 y = f(t)$ $\quad P(s) = s^2 + 2\omega_n \zeta s + \omega_n$ $\quad P(s) = s^2 + 2s + 5$

$\omega_n = \sqrt{5}$ $\quad \zeta = \frac{1}{\sqrt{5}}$

c) $h(t) = \frac{d}{dt}[y_u(t)] = \frac{d}{dt}[2 + 2e^{-t}\cos 2t] = \dots \boxed{h(t) = 2e^{-t}[-\cos 2t - 2\sin 2t]}$

11-15-2010

Forced response using Frequency Response function (FRF)

for n^{th} order d.e. $a_n \frac{d^n y}{dt^n} + a_{n-1} \cdots + a_0 y = f(t)$

the complete solution $y(t) = y_H(t) + y_p(t)$ H → homogeneous P → particular

$f(t) = 0$ Free response or zero input

when $f(t) \neq 0 \Rightarrow$ forced response, then $y(t) = y_H + y_p$ ← steady state

FRF - used to find a steady state response for harmonic inputs

$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = 1 + \sin t$

The steady state response (y_{ss}) to an input of the form $u(t) = B \sin(\omega t + \phi)$

is given by $y_{ss}(t) = y_p = BM \sin(\omega t + \phi + \theta)$

where M & θ are evaluated using FRF $M \rightarrow$ magnitude of frequency response function

$FRF = H(s = j\omega)$

ex) $H(s) = \frac{s+1}{s^2+1}$ $H(s = j\omega) = \frac{j\omega + 1}{(j\omega)^2 + 1} = \boxed{\frac{j\omega + 1}{-\omega^2 + 1} = FRF}$

ME 211 Exam III Dec 1 (Wed) ch 7 & 8

HW 7, 8, 9, 10, 11

HW 11, to be given on Friday, due 11/29

3 problems Review 11/29 (Monday) Bring: Crib sheet, paper,

provided. transforms

a = (1 2 3; 4 5 6; 7 8 9)

det(a)
b = [1 2 3 4]' ⇒ column vector

roots of polynomial → coefficient matrix,
then take cd = roots( ) gives roots
create new script file (an m-file)

put notes in your ~~do~~ M file type. % comment
secondordersys.m on bb.

Matlab notes from Project meeting

Notes 11-19-10

Forced Response using frequency response

for n^{th} order D.E. $\ldots a_0 y = f(t)$ where $f(t)$ = sin or cos func. of time
$y(t) = y_{tr} + y_p$

the steady state response for an input of form
$u(t) = B\sin(\omega t + \phi)$ $\quad y_{ss}(t) = y_p = BM\sin(\omega t + \phi + \theta)$
where M & θ are evaluated using freq. response function FRF = $H(s = j\omega)$

$M = \dfrac{\text{Mag of numerator of } H(j\omega)}{\text{Mag of denominator of } H(j\omega)}$ $\quad \theta = \arg(\text{Num}) - \arg(\text{den})$

ex) The transfer function for an elec. circuit $H(s) = \dfrac{6s^2 - 1}{6s^2 + 6s + 1}$

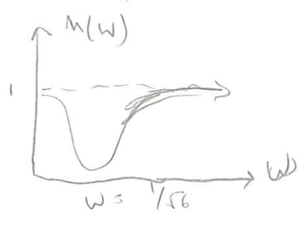
find 1) FRF 2) mag of FRF (or M) 3) angle θ 4) plot ω vs M

1) $FRF = H(j\omega) = \dfrac{6(j\omega)^2 + 1}{6(j\omega)^2 + 6(j\omega) + 1} = \dfrac{6(-\omega^2) + 1}{6(-\omega^2) + 6j\omega + 1}$

$$H(j\omega) = \frac{1 - 6\omega^2}{(1 - 6\omega^2) + (6\omega)j}$$

$M|H(j)| = \dfrac{\text{Mag(num.)}}{\text{Mag(denom)}}$ $\quad M(\omega) = \dfrac{1 - 6\omega^2}{\sqrt{(1-6\omega)^2 + (6\omega)^2}}$

$\theta = \arg(\text{numerator}) - \arg(\text{denom}) = \tan^{-1}\left(\cancel{\frac{0}{1-6\omega^2}}\right)^{0} - \tan^{-1}\left(\frac{6\omega}{1-6\omega^2}\right) = -\tan^{-1}\left(\frac{6\omega}{1-6\omega^2}\right)$



11-19-2010

8.49) y_{ss} of a stable syst.

$u(t) = \sin 5t + \sin 10t + \sin 15t$ has the form

$y_{ss} = A\sin(5t+\theta_1) + B\sin(10t+\theta_2) + C\sin(15t+\theta_3)$

For a system, $H(s) = \frac{5}{s^2+.2s+100}$ find A, B, & C

$u_1(t) = \sin 5t \quad u(t) = B\sin(\omega t+\phi) \quad B=1 \quad \omega=5 \quad \phi=0$

$y_{ss} = BM\sin(\omega t+\phi+\theta) = (1)M\sin(5t+0+\theta)$

$H(j\omega) = H(5j) = \frac{5j}{(5j)^2+0.2(5j)+100} \qquad H(j\omega) = \frac{5j}{75+j}$

$M = \frac{|Num|}{|Denom|} = \frac{5}{\sqrt{(75)^2+1^2}} = 0.067$

$u_2(t) = \sin(10t) \qquad u = B\sin(\omega t+\phi) \quad B=1 \quad \omega=10 \quad \phi=0$

. . . . .

Ex) . . .

$u_2(t) = 3\sin(t+\frac{\pi}{3}) \qquad u = B\sin(\omega t+\phi) \quad B=3 \quad \omega=1 \quad \phi=\frac{\pi}{3}$

$H(j\omega) = H(j) = \frac{5(j)^2+20}{j^2+4j+5} = \frac{15}{4+4j} \qquad M = \frac{15}{\sqrt{4^2+(4)^2}} = 2.652$

$y_{ss} = (3)(2.652)(\sin(t+\frac{\pi}{3}-\frac{\pi}{4}) = 7.956\sin(t+\frac{\pi}{12})$

$u_2(t) = 3\sin(t+\frac{\pi}{3}) \qquad H(2j) = \frac{5(2j)^2+20}{(2j)^2+4(2j)+5} = 0 \qquad M=0$

$y_{ss} = BM\cos(\omega t+\phi+\theta)$

complete $\quad y_{ss} = 16 + 7.956\sin(t+\frac{\pi}{12})$

consider n^{th} order general $\quad a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \dots a_1 \frac{dy}{dt} + a_0 = b_m \frac{d^m u}{dt^m} + \dots b_1 \frac{du}{dt} + b_0 u$

then $H(s) = \dfrac{b_m s^m + \dots b_1 s + b_0}{\underbrace{a_n s^n + a_{n-1} s^{n-1} + \dots a_1 s + a_0}} = \dfrac{Y(s)}{U(s)}$

$P(s)$ = characteristic eqn $\quad$ where $\mathcal{L}(u(t)) = U(s)$

$\mathcal{L}(y(t)) = Y(s)$

ex) given: $2\ddot{y} + 12\dot{y} + 50y = u(t)$

method 1) $H(s) = \mathcal{L}(2\ddot{y} + 12\dot{y} + 50y) = \mathcal{L}(u(t))$

$= 2[s^2 Y(s) - sY(0) - \dot{y}(0)] + 12[sY(s) - Y(0)] + 50Y(s) = U(s)$

$Y(s)[2s^2 + 12s + 50] = U(s)$

$\dfrac{Y(s)}{U(s)} = \dfrac{1}{2s^2 + 12s + 50}$

dont need initial cond. to find $H(s)$. it is always taken at zero state

method 2)

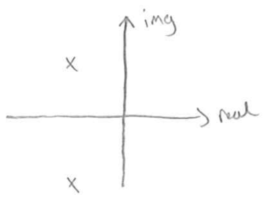
compare differential eqn with general forms above

$a_2 = 2 \quad a_1 = 12, \quad a_0 = 50 \quad b_0 = 1$

$H(s) = \dfrac{1}{2s^2 + 12s + 50}$

note: be able to work this backwards to find differential equations

2) stability: $P(s)$ = denominator $H(s)$ = $2s^2 + 12s + 50 = 0$



$s^2 + 6s + 25 = 0$

$s = \dfrac{-6 \pm \sqrt{6^2 - 4(25)}}{2}$ $\quad s_{1,2} = -3 \pm 4i$

know the rules for stability based on pole location

Damping ratio &

damping ratio & natural frequency

$$\ddot{y} + 6\dot{y} + 25y = \frac{u(t)}{2}$$

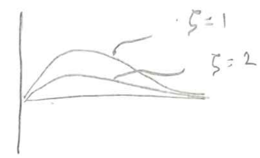
compare with standard form of 2nd order $\quad \ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = f(t)$

$$2\zeta\omega_n = 6 \qquad \omega_n^2 = 25 \quad \Rightarrow \quad \boxed{\zeta = .6 \qquad \omega_n = 5}$$

classify system $\quad 0 < \zeta < 1$ underdamped

$\zeta > 1$ overdamped

$\zeta = 1$ critically damped



general form of zero input $\quad 2\ddot{y} + 12\dot{y} + 50y = \cancel{u(t)}^{\,0}$

$$\mathcal{L}[\qquad\qquad] = 0$$

cannot take $y(0)$ and $\dot{y}(0) = 0$ for Y_{zi}, only for transfer func.

$$Y(s)\left[2s^2 + 12s + 50\right] = 2sy(0) + 12y(0) + 2\dot{y}(0)$$

$$\Rightarrow Y(s) = \frac{2s\,y(0) + 12y(0) + 2\dot{y}(0)}{2s^2 + 12s + 50}$$

Y_{zi}, zero input

2) $Y_{zi} = k_1 e^{s_1 t} + k_2 e^{s_2 t} \qquad s_1 \,\&\, s_2$ are real & distinct

$y_{zi} = k_1 e^{s_1 t} + k_2 t e^{s_2 t} \qquad s_1 = s_2$

$= k e^{\alpha t}\cos(\beta t + \phi) \qquad s_{1,2} = \alpha \pm i\beta$

for this case, $\quad Y_{zi} = k e^{-3t}\cos(4t + \phi) \qquad s_{1,2} = -3 \pm 4i$

initial cond. must be given to find k, ϕ.
need $y(0)$ & $\dot{y}(0)$

~~Unit step response~~

unit step response $u(t)=1$ $(y_u(t))$

<u>method 2</u> from properties of Laplace transform $y_u(t) = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right]$

$$y_u(t) = \mathcal{L}^{-1}\left(\frac{1}{s(2s^2+12s+50)}\right) = \frac{A_1}{s} + \frac{A_2 s + A_3}{2s^2+12s+50}$$

$$A_1 : A_1 = \frac{1}{0+0+50} = \frac{1}{50}$$

$$1 = \frac{1}{50}(2s^2+12s+50) + A_2 s^2 + A_3 s$$

$$s^2 : 0 = \frac{2}{50} + A_2 \qquad A_2 = \frac{-1}{25}$$

$$\mathcal{L}^{-1} : \quad \frac{1}{50}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{-\frac{2}{50}s - \frac{12}{50}}{2s^2+12s+50}\right)$$

$$\frac{1}{50} - \frac{2}{2(50)}\mathcal{L}^{-1}\left(\frac{s+3-3}{(s+3)^2+4^2}\right) - \frac{12}{2(50)}\mathcal{L}^{-1}\left(\frac{1}{(s+3)^2+4^2}\right)$$

$$\mathcal{L}^{-1} = \boxed{y_u(t) = \frac{1}{50} - \frac{1}{50}e^{-3t}\cos 4t - \frac{3}{200}e^{-3t}\sin 4t}$$

<u>Impulse input</u> $h(t) = \dfrac{d\,y_u(t)}{dt}$

$H(s)$: $h(t) = \mathcal{L}^{-1}(H(s))$

<u>Steady state response:</u> $u(t) = 2 + 3\sin\left(t+\frac{\pi}{3}\right)$

$u(t)=2$ $\qquad y_{ss} = AH(0) = 2\left(\frac{1}{0+0+50}\right) = \frac{1}{25}$

$u(t) = 3\sin\left(t+\frac{\pi}{3}\right)$ $\qquad B:3,\ \omega=1,\ \phi=\frac{\pi}{3}$

$$y_{ss} = BM\sin(\omega t + \phi + \theta)$$

find M; $M(\omega) =$ mag num / mag denom

$\theta =$ arg num ...

notes : topic - inverse laplace
order of numerator $\geq$ order of denom (divide fractions)
order num < denom $\Rightarrow$ partial fractions

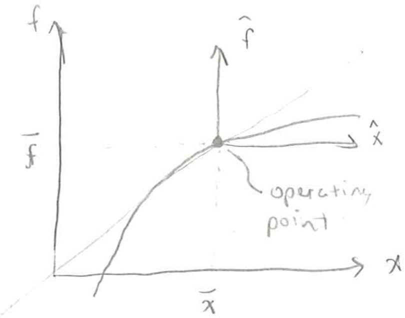
12-3-10

Chapter 9. developing a linear Model linearization is the process of finding a linear model that approximates a nonlinear one

terminology:

1) operating point: a specific point that lies on the non-linear curve
2) $\bar{X}$: the x-coordinate of operating point (a const. nominal value)
$\dot{\bar{X}} = \ddot{\bar{X}} = \overset{n}{\bar{X}} = 0$
3) $\bar{f}$. y-coord. of operating point

4) incremental values: a coordinate system whose axis are $\hat{x}(t)$ & $\hat{f}(t)$ having origin at operating point. The nonlinear function f(x) is then expressed as $\hat{f}(\hat{x})$ where $\hat{f}(\hat{x})$ is a linear function



For any st. line having slope m & passing through points (x, y)

$y - y_1 = m(x - x_1)$ $\qquad m = \left.\frac{df}{dx}\right|_{x=\bar{x}}$ $\qquad x - \bar{x} = \hat{x}$

$f - \bar{f} = \hat{f}$

$f - \bar{f} = k(x - \bar{x})$

Calculate operating point: Operating point of a system will be a condition of equilibrium

a) output terms (say y(t) or x(t)) are replaced by $\bar{y}$ or $\bar{x}$
b) input " (say u(t)) " " $\bar{u}$
c) all derivatives of $\bar{y}$ & $\bar{u}$ = 0 since $\bar{y}$ & $\bar{u}$ are const.
d) for input terms, u(t), only constant terms are retained and any time dependent terms are ignored

ex) consider the nonlinear input-output eqn -

$\ddot{x} + 2\dot{x} + 2x^3 = u(t)$ where $u(t) = 2 + A\sin t$

find op. point,

$x \to \bar{x}$ $u \to \bar{u}$ $\cancel{\ddot{\bar{x}}}^0 + \cancel{2\dot{\bar{x}}}^0 + 2\bar{x}^3 = \bar{u}$ $2\bar{x}^3 = 2 \Rightarrow \bar{x} = (1)^{1/3} = 1$ $\boxed{\bar{x} = 1}$

consider nonlinear output eqn: $\ddot{x} + 2\dot{x} + f(x) = A + B\sin 3t$

where $f(x) = \begin{cases} -4\sqrt{|x|} & \text{for } x < 0 \\ 4\sqrt{x} & \text{for } x \geq 0 \end{cases}$

modulus function or (absolute value function) $|X| = X$ for $x \geq 0$

ie the modulus function returns the value of a variable

replace $x \to \bar{x}$ $\cancel{\ddot{\bar{x}}}^0 + \cancel{2\dot{\bar{x}}}^0 + f(\bar{x}) = u \Rightarrow f(\bar{x}) = 8$

$-4\sqrt{|\bar{x}|} = 8$ for $\bar{x} < 0$ $4\sqrt{\bar{x}} = 8$ $\boxed{\bar{x} = 4 \text{ for } \bar{x} \geq 0}$

this sol^n is Not possible; it can be ignored

9.12 System described by non-linear eqn

$\ddot{y} + 2(\dot{y} + \dot{y}^3) + 2y + |y|y = \underbrace{A + B\cos t}_{u(t)}$ for $A = -3$ find op. point

$y \to \bar{y} \Rightarrow 2\bar{y} + |\bar{y}|\bar{y} = -3$

case a) $\bar{y} \geq 0$, $\bar{y}^2 + 2\bar{y} + 3 = 0$ $\bar{y} = \frac{-2 \pm \sqrt{4 - 4(3)}}{2}$

case b) $\bar{y} < 0$ $-\bar{y}^2 + 2\bar{y} + 3 = 0$ $\bar{y} = \frac{-2 \pm \sqrt{4 - 4(-1)(3)}}{-2}$ $\bar{y} = \frac{-2 \pm 4}{-2} = 3, -1$ $\boxed{\bar{y} = -1}$

Procedures for linearization

step 1) determine operating points (may have multiple o.p's)

step 2) identify the nonlinear term & linearize it using taylor series expansion for any function $f(x)$,

the Taylor Series about pt $\bar{x}$ is $f(x) = f(\bar{x}) + \frac{df}{dx}\Big|_{\bar{x}}(x - \bar{x}) + \frac{1}{2!}\frac{d^2f}{dx^2}\Big|_{\bar{x}}(x - \bar{x})^2 + \ldots$

since $x - \bar{x} = \hat{x}$ (increment variable) $\Rightarrow \hat{x}^2 \cong 0$

$f(x) = f(\bar{x}) + \frac{df}{dx}\Big|_{\bar{x}}\hat{x} + \frac{1}{2!}\frac{d^2f}{dx^2}\Big|_{\bar{x}}\hat{x}^2 + \ldots$ $f(x) = f(\bar{x}) \frac{df}{dx}\Big|_{\bar{x}}\hat{x} + \text{H.O.T.}$

step 3) rewrite all linear terms as $x = \hat{x} + \bar{x}$

step 4) solve the linear eqn

Exam 3 average 23.9 / 30 Notes 12-6

my score 25 / 30

Example - use taylor series expansion to solve the linearized model

1) $f(x) = 0.5x^3$ at $\bar{x} = \pm 2$

2) $f(y) = \frac{1}{y}$ at $\bar{y} = 0.5$

3) $f(\dot{y}) = \dot{y}^2$ at $\bar{y} = 1$

1) $f(x) = f(\bar{x}) + \frac{df}{dx}\Big|_{\bar{x}} \hat{x} = 0.5\bar{x}^3 + \frac{d}{dx}(.5x^3)\Big|_{\bar{x}} \hat{x} = 0.5(\pm 2)^3 + 0.5(3x^2)\Big|_{\bar{x}} \hat{x}$

$\Rightarrow .5(2)^3 + 1.5(\pm 2)^2 \hat{x}$ for $\bar{x} = +2$ $0.5(8) + 6\hat{x}$ $\bar{x} = 2: 4 + 6\hat{x}$ | $\bar{x} = -2$ $-4 + 6\hat{x}$

$0.5x^3 = \begin{matrix} 4 + 6\hat{x} \text{ at } \bar{x} = 2 \\ -4 + 6\hat{x} \text{ at } \bar{x} = -2 \end{matrix}$

2) $f(y) = f(\bar{y}) + \frac{df}{dy}\Big|_{\bar{y}} \hat{y}$ $f(y) = \frac{1}{\bar{y}} + \frac{d}{dy}\left(\frac{1}{y}\right)\Big|_{\bar{y}} \hat{y} = 2 + \left(-\frac{1}{y^2}\right)\Big|_{\bar{y}} \hat{y} = 2 + \left(\frac{-1}{(.5)^2}\right)\hat{y}$

$= 2 - 4\hat{y}$

3) $f(\dot{y}) = \dot{y}^2$ @ $\bar{y} = 1$ $f(\dot{y}) = f(\dot{\bar{y}}) + \frac{df}{d\dot{y}}\Big|_{\dot{\bar{y}}} \hat{\dot{y}}$ $\boxed{f(y) \approx (0)}$

$= \underset{0}{\cancel{\dot{\bar{y}}^2}} + \underset{0}{\frac{d}{dy}\cancel{(\dot{y}^2)}}\Big|_{\dot{\bar{y}}} \hat{\dot{y}}$

---

o-output
i-input

$\frac{1}{2}\dot{e}_o + 2e_o^3 + e_o = e_i(t)$ where $e_i(t) = 18 + A\cos\omega t$

Find a) operating point b) linearized model c) comment on stability of linearized model

$e_o \to \bar{e}_o$ $\frac{1}{2}\dot{\bar{e}}_o + 2\bar{e}_o^3 + \bar{e}_o = \bar{e}_i(t) = 18$

$e_i \to \bar{e}_i$ $\bar{e}_o + 2\bar{e}_o^3 = 18$ $\Rightarrow \bar{e}_o = 2$ (guess & check method)

$f(x) = f(\bar{x}) + \frac{df}{dx}\Big|_{\bar{x}} \hat{x}$ $f(e_o) = f(\bar{e}_o) + \frac{df}{de_o}\Big|_{\bar{e}} \hat{e}_o$

where does this come from?

$e_o - \bar{e}_o = \hat{e}_o$

$e_o = \hat{e}_o + \bar{e}_o$

$2\bar{e}_o^3 := 2\bar{e}_o^3 + \frac{d}{de_o}(2e_o^3)\Big|_{\bar{e}_o} \hat{e}_o$

$= 16 + 6e_o^2\Big|_{\bar{e}_o} \hat{e}_o$

$2e_o^3 = 16 + 24\hat{e}_o$

$\frac{1}{2}\left(\dot{\hat{e}}_o + \dot{\bar{e}}_o\right) + \left(16 + 24\hat{e}_o\right) + \left(\hat{e}_o + \bar{e}_o\right) = 18 + A\cos\omega t$

$\boxed{\frac{\dot{\hat{e}}_o}{2} + 25\hat{e}_o = A\cos\omega t}$

$\tau > 0$ = stable

$\tau < 0$ = unstable

Ex $\ddot{y} + 2(\dot{y} + \dot{y}^3) + 2y + |y|y = A + B\cos t$ derive linear model for $A=15$

Step 1 $\ddot{\bar{y}} + 2(\dot{\bar{y}} + \dot{\bar{y}}^3) + 2\bar{y} + |\bar{y}|\bar{y} = A$

$2\bar{y} + |\bar{y}|\bar{y} = 15$

$\bar{y}^2 + 2\bar{y} - 15 = 0$ $\quad \bar{y} = \dfrac{-2 \pm \sqrt{4 - 4(1)(-15)}}{2} = 3, -5 \quad [\text{if } \bar{y} > 0]$

$\bar{y} = \dfrac{-2 \pm \sqrt{4 - 4(-1)(-15)}}{-2}$ not possible

note $\bar{y}$ is always const

Step 2 identify nonlinear terms

1) $2\dot{y}^3 = 0$

2) $|y|y = y^2$ $\quad f(x) = f(\bar{x}) + \left.\dfrac{df}{dx}\right|_{\bar{x}} \hat{x}$

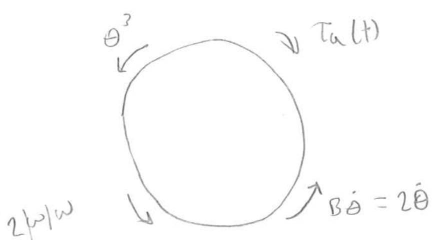
$y^2 = \bar{y}^2 + 2y|_{\bar{y}}\,\hat{y}$

$y^2 = 9 + 6\hat{y}$

Step 3 $y = \bar{y} + \hat{y}$ $\quad (\ddot{\hat{y}} + \ddot{\bar{y}}) + 2(\dot{\hat{y}} + \dot{\bar{y}}) + 0 + 2(\bar{y} + \hat{y}) + 9 + 6\hat{y} = A + B\cos t$

$$\boxed{\ddot{\hat{y}} + 2\dot{\hat{y}} + 8\hat{y} = B\cos t}$$

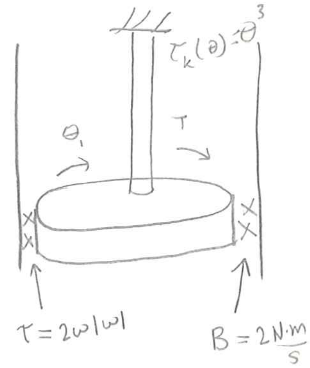
Ex



Given: $\tau_a = 8 + \hat{\tau}_a t$

find: a) O.P.

b)



$\curvearrowleft + \Sigma M = J\ddot{\theta}$

$$\tau_a(t) - \theta^3 - 2|\dot{\theta}|\dot{\theta} - 2\dot{\theta} = J\ddot{\theta}$$

$$\boxed{\ddot{\theta} + 2|\dot{\theta}|\dot{\theta} + 2\dot{\theta} + \theta^3 = \tau_a(t)}$$

$$\cancel{\ddot{\bar{\theta}}}_0 + 2|\cancel{\dot{\bar{\theta}}}|\cancel{\dot{\bar{\theta}}}_0 + 2\cancel{\dot{\bar{\theta}}}_0 + \bar{\theta}^3 = \bar{\tau} = 8$$

$$\bar{\theta}^3 = 8 \rightarrow \boxed{\bar{\theta} = 2}$$

for θ^3

$$f(\theta) = f(\bar{\theta}) + \frac{d}{d\theta} f(\theta)\Big|_{\bar{\theta}} \hat{\theta} = \bar{\theta}^3 + 3\bar{\theta}^2\hat{\theta}$$

$$= 8 + 12\hat{\theta}$$

now $(\ddot{\hat{\theta}} + \cancel{\ddot{\bar{\theta}}}^0) + 0 + 2(\dot{\hat{\theta}} + \cancel{\dot{\bar{\theta}}}^0) + (8 + 12\hat{\theta}) = \tau_a(t) = 8 + \hat{\tau}_a(t)$

$$\boxed{\ddot{\hat{\theta}} + 2\dot{\hat{\theta}} + 12\hat{\theta} = \hat{\tau}_a(t)}$$

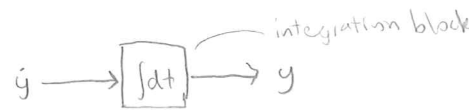
Block diagrams

* this mat'l will help with controls

<u>Block diagrams:</u> pictorial representation of a dynamic system using rectangular blocks & circles

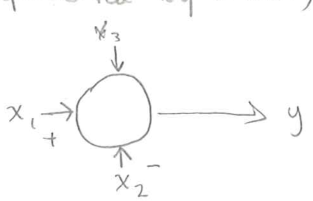
Consider the eqn $y = Ax$

$y = \int_0^y \dot{y}\,dt$



summer or summing junction - the addition or subtraction of variables is done using a summer (usually represented by a circle)

$y = x_1 - x_2 + x_3$

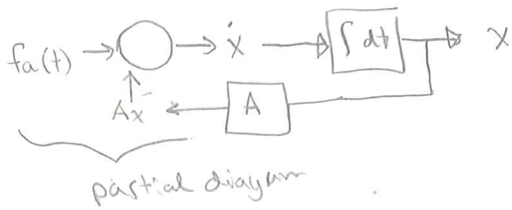


block diagram for nth order ODE

1) Solve the eqn for highest derivative of output variable
2) Connect integrator blocks in series to produce output variable

ex 1)

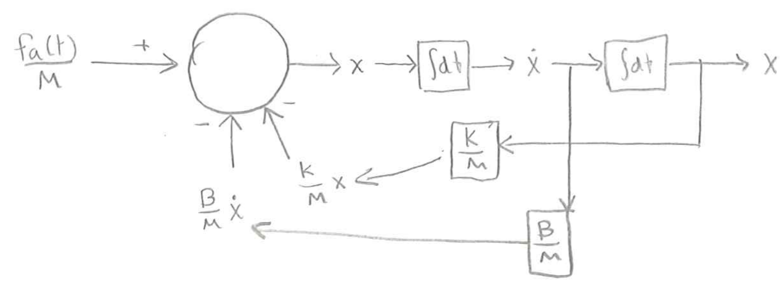
$$\dot{x} + Ax = f_a(t)$$
$$\dot{x} = f_a(t) - Ax$$



ex 2)

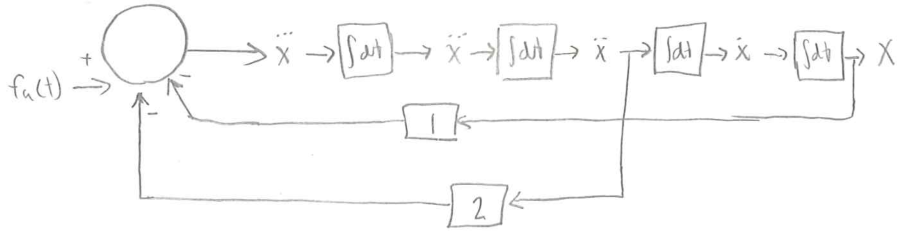
$$M\ddot{x} + B\dot{x} + kx = f_a(t)$$

$$\ddot{x} = \frac{f_a(t)}{M} - \frac{B\dot{x}}{M} - \frac{kx}{M}$$



ex 3)

$$\ddddot{x} + 2\ddot{x} + x = f_a(t)$$
$$\ddddot{x} = f_a(t) - x - 2\ddot{x}$$



12-8-10

Review - Final Exam

ch 1

linearity: $a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots$

→ t independent variable
→ x dependent variable
→ n order of D.E.

homogeneous or zero input $g(t)=0$
non-homogeneous $g(t)\neq 0$

$y=f(x)$ ← independ
↑ depend.

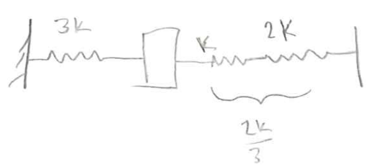
$\frac{d^2y}{dt^2} + 2\frac{dy}{dx} + 6y = x^2$ ODE

$z=f(x,y)$ $\quad \frac{\partial z}{\partial x} = z + 2x\frac{\partial z}{\partial y}$ PDE

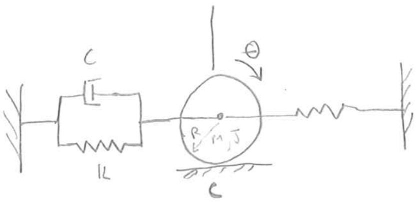
$\ddot{x} + t^3 x = \sin t$

$y\left(\frac{dy}{dx}\right) - x\left(\frac{dy}{dx}\right)^2 = x$ nonlinear, $n=1$

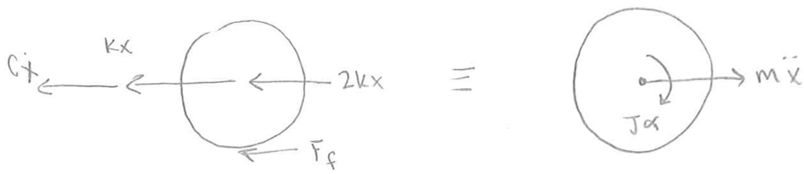
all springs $=k$



use Newton's law $\quad Fa(t) - 3kx - \frac{2}{3}kx = m\ddot{x}$ $\quad \boxed{m\ddot{x} + \frac{11}{3}kx = Fa(t)}$ $\quad \boxed{k_{eq} = \frac{11}{3}}$



Find Eq of motion of wheel in terms of coord. x.
assume wheel rolls w/o slipping
J = mass moment of inertia



EOM $\quad \Sigma F_x = m\ddot{x}$

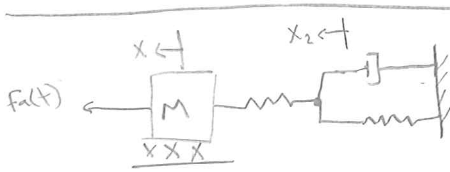
$$-c\dot{x} - kx - 2kx - F_f = m\ddot{x}$$
$$m\ddot{x} + c\dot{x} + 3kx + F_f = 0$$

$$\circlearrowright + \Sigma M = J\ddot{\theta}$$
$$F_f(R) = J\ddot{\theta}$$
$$F_f = \frac{J\ddot{\theta}}{R}$$

$$m\ddot{x} + c\dot{x} + 3kx + \frac{J\ddot{\theta}}{R} = 0$$

$$\frac{J\ddot{\theta}}{R} = \frac{J}{R}\left(\frac{\ddot{x}}{R}\right)$$

$$\boxed{\ddot{x}\left[m + \frac{J}{R^2}\right] + c\dot{x} + 3kx = 0}$$



$$M\ddot{x} + B_1\dot{x}_1 + k_1(x_1 - x_2) = f_a(t)$$
$$B_2\dot{x}_2 + k_2x_2 + k_1(x_2 - x_1) = 0$$

State var form

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = \frac{1}{M_1}\left[-k_1v_1 - B_1v_1 + k_1x_2 + f_a(t)\right]$$
$$\dot{x}_2 = \frac{1}{B_2}\left[k_1x_1 - (k_1+k_2)x_2\right]$$
$$\dot{q} = Aq + Bu$$

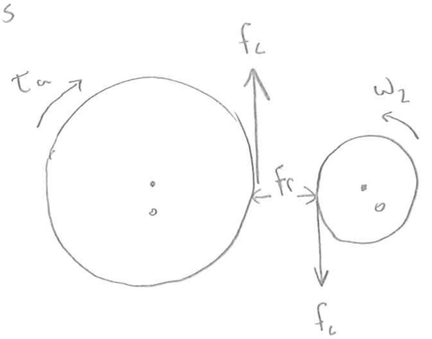
state var

$$q = \begin{bmatrix} x_1 \\ v_1 \\ x_2 \end{bmatrix} \rightarrow \dot{x}$$

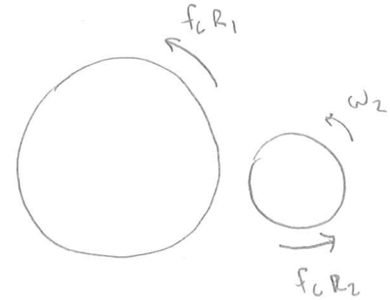
$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k_1}{m} & -\frac{B_1}{m} & \frac{k_1}{m} \\ \frac{k_1}{B_2} & 0 & -\frac{k_1+k_2}{B_2} \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \end{bmatrix}$$

output $\quad y = Cq + Du \qquad y = [k_1 \;\; 0 \;\; -k_1] \begin{bmatrix} x_1 \\ v_1 \\ \dot{x}_2 \end{bmatrix} + [0] \; f_a(t)$

Gears



OR



$$\boxed{\frac{R_2}{R_1} = \frac{\omega_1}{\omega_2} = \frac{\theta_1}{\theta_2} = N}$$

review sessions

bring paper ~~6~~ 5 of 6 problems

Questions

11-15 Unit impulse response, why is there a $y(t)$ in notes?
wtf is $u(t) = \delta(t)$?

Exams	.4366		
Homework	.1914		.272

EXAMS	30/30	x 20%	0.2
	21/30	x 10%	.07
	25/30	x 20%	.1667
HW avg:	95.7/100	x 20%	.1914
Project:	9.5/10	x 10%	.095
TOTAL		8	.72307 / 1

average before final = 90.39%

.890 − .72307 = .1669

need on final $\boxed{83.5\%}$ (to get .89%)

Exam Practice

notes

11-19- finding y_{ss} of a stable system -ex

Frequency Response Function, elec. circuit -ex

Forced Response using FRF, $M =$ __ $\theta =$ __

11-15 FRF - finds steady-state response to harmonic input

$y_p = BM \sin(\omega t + \phi + \theta)$

$FRF = H(j\omega)$

free response = zero input $\Rightarrow f(t) = 0$

FRF complete sol'n = $y(t) = y_H(t) + y_p(t)$

unit step response -ex 8.36

- find $H(s)$
- find $h(t)$

find gen. form, plot zero-pole pattern, -ex 8.27

Properties of a trans. function

- special cases (unit step resp., impulse resp., const input)

ex notes here

effects of zeros on system

- graphs

pole-zero plot, + ex.

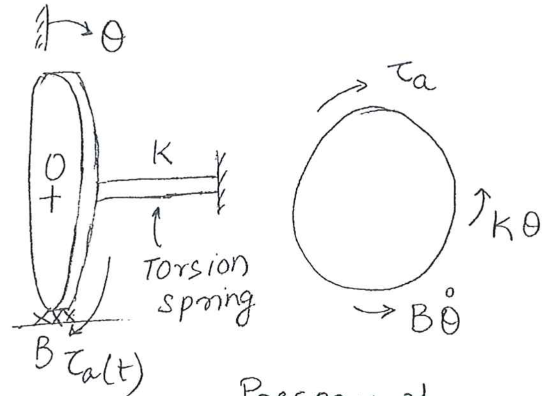
?

analysis of system (stability)

11-12-10 Std. form analys. 2nd order sys. - finding S_1 & S_2 Poles given general form/char. polynomial

Review / Exam - 2

chapter 5



$+\curvearrowleft \Sigma M_o = J\ddot{\theta}$

$J \rightarrow$ Mass moment of inertia (resistance to rotation)

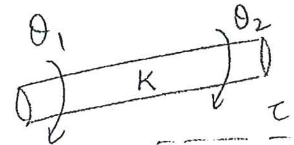
$\tau_a - K\theta - B\dot{\theta} = J_o\ddot{\theta}$

applied torque, Torsional torque, Friction or Viscous Torque

or $J\ddot{\theta} + B\dot{\theta} + K\theta = \tau_a(t)$

Presence of weight & normal reaction

For Torsional bar



$\tau = K(\theta_1 - \theta_2)$

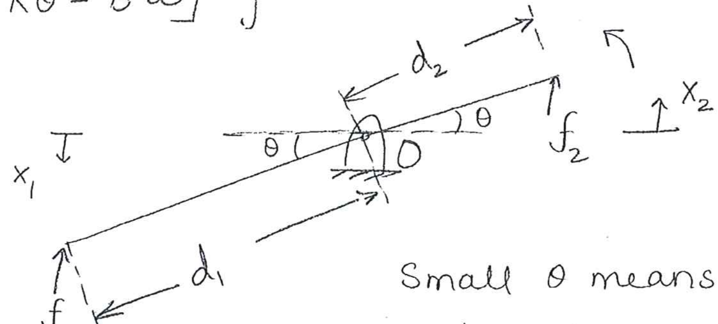
In state variable form: $q = \begin{bmatrix} \theta \\ \omega \end{bmatrix} \rightarrow \dot{\theta}$

$\Rightarrow \quad \dot{\theta} = \omega$

$\dot{\omega} = \frac{1}{J}\left[\tau_a(t) - K\theta - B\omega\right]$

can put in $\dot{q} = Aq + Bu$

Levers: Rigid bars pivoted at a point.

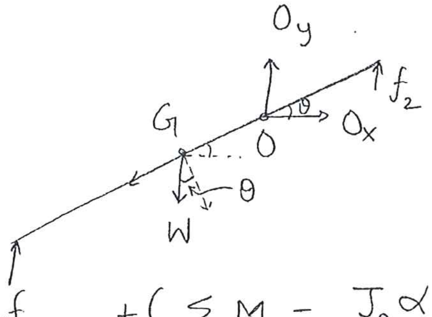


Small θ means:

$\sin\theta \approx \theta$

$\cos\theta \approx 1$

Lever mass may or may-not be neglected



$+\curvearrowleft \Sigma M_o = J_o\alpha$

$-f_1(d_1\cos\theta) + f_2(d_2\cos\theta) + W\cos\theta\,(a) = J_o\alpha = J_o\ddot{\theta}$

where $a = d_1 - \left(\frac{d_1+d_2}{2}\right)$

For small θ, $\cos\theta \approx 1$, $J_o = J_{G_1} + ma^2$

where, $J_{G_1} = \frac{mL^2}{12}$

$a = \frac{d_1 - d_2}{2}$

$$\boxed{-f_1 d_1 + f_2 d_2 + W\left(\frac{d_1-d_2}{2}\right) = J_o\ddot{\theta}}$$

CRIB SHEETS

David Malawey

General 1^{st} order system

$$\dot{y} + \frac{y}{\tau} = f(t) \Rightarrow y(t) = \tau A + (y_0 - \tau A)e^{-t/\tau}$$

- if $f(t)$ = const, $f(t)$ is your A
- find response means find $\mathcal{L}(f(t))$

Stable $\tau > 0$ Marginally stable $\tau \to \infty$

unstable $\tau < 0$

τ = time const., measures how quickly sys. reaches steady state

Definition of Laplace:

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

Transform derivatives:

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) + \dots f^{n-1}(0)$$

Int. by Parts

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

- make e^{xt} the V term then $dv = xe^{xt}dt$

matrix, st. var form

$Y = Cq + Du$ outputs

$\dot{q} = Aq + Bu$

st. vars input

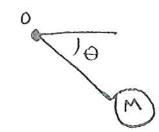
$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ & & \\ & & \end{bmatrix}\begin{bmatrix} q_1\\ q_2\\ q_3\end{bmatrix} + \begin{bmatrix} b\\ b\\ b\end{bmatrix}\begin{bmatrix} f_a(t)\end{bmatrix}$$

H_0 = angular momentum of particle about O

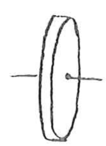
H_0 for mass at a distance

$H_0 = \vec{r} \times \vec{p} = \vec{r} \times mv = ml^2\dot{\theta}$

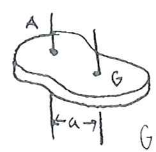
$M_0 = \dot{H}_0 = ml^2\ddot{\theta} = F d_{perp.} = I\alpha$



Parallel Axis thm



$J = \frac{MR^2}{2}$



$J_{AA'} = J_0 + Ma^2$

G = centroid

Pendulum, moving support

- create r vector, get $\vec{r}$ in $\hat{i}$ & $\hat{j}$

static equilibrium deflection

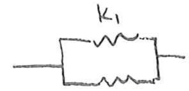
- $\dot{y} \to 0$ $\ddot{y} \to 0$ $y \to y_0$

Gears

$N = \frac{R_2}{R_1} = \frac{\theta_1}{\theta_2}$ $v = r\omega$ $\Sigma T = J\ddot{\theta}$

K_{eq} or B_{eq}

parallel



$= k_1 + k_2$

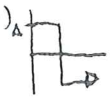
series

$= \frac{k_1 k_2}{k_1 + k_2}$

State Var form: terms w/ x = terms w/out

Y_{tr}, transient response $t \to \infty$, $Y_{tr} \to 0$

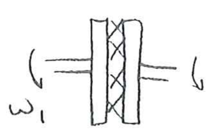
Y_{ss}, steady state, $= \lim_{t\to\infty} f(t)$



$u(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 1 \end{cases}$ $u(t-t_1) = \begin{cases} 0 & t \leq t_1 \\ 1 & t > t_1 \end{cases}$

$f(t) = Au(t) - 2Au(t-t_1)$

torque



$\tau = B\Delta\omega$

levers: $\circlearrowleft + \Sigma M_0 = J_0\,\alpha$

small θ: $\sin\theta = \theta$, $\cos\theta = 1$

$J = \frac{ML^2}{12}$

$J = \frac{ML^2}{3}$

EXAM III 211

David Malawey

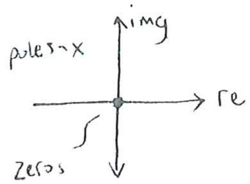
n^{th} order differential eqn $a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^n u}{dt^n} + \ldots b_1 \frac{du}{dt} + b_0 u$

then $H(s) = \dfrac{b_m s^n + \ldots b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots a_1 s + a_0} = \dfrac{Y(S)}{U(S)} = \dfrac{Y_{zs}}{Fa(S)} = \dfrac{\text{output}}{\text{input}}$

Dont need initial condition to find H(s). It is always taken at zero st.

$P(s)$ = characteristic eqn where $\mathcal{L}[u(t)] = U(s)$, $\mathcal{L}[y(t)] = Y(s)$

stability: P(s), denominator of H(s)



1) all poles have neg. real points ⇒ stable
2) all " " " " " & 1 or more distinct poles lies on img. axis → → Marginally (critically) stable
3) at least one pole lies in R.H. plane OR has double roots on imaginary axis

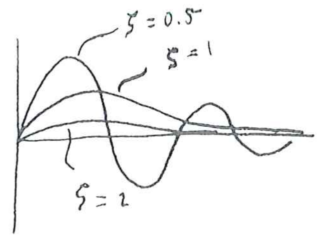
Given: $P(s) = as^2 + bs + c$ zeros of $p(s)$ ⇒ $s = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Damping & natural frequency

standard form, 2nd order:

$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = f(t)$

• stable if $0 < \zeta < 1$



CLASSIFICATION

$0 < \zeta < 1$ underdamped

$\zeta > 1$ overdamped

$\zeta = 1$ critically damped

Cannot take $y(0)$ & $\dot{y}(0)$ $= 0$ for Y_{zi}, only for transfer function

Zero input response, Y_{zi} general form

$a_2\ddot{y} + a_1\dot{y} + a_0 y = \cancel{u(t)}\ 0$ $\quad \mathcal{L}[\quad] = 0$

$Y_{zi} = k_1 e^{s_1 t} + k_2 e^{s_2 t}$ $\quad s_1$ & s_2 are real & distinct

$Y_{zi} = k_1 e^{s_1 t} + k_2 t e^{s_2 t}$ $\quad s_1 = s_2$

$Y_{zi} = k e^{\alpha t} \cos(\beta t + \phi)$ $\quad s_{1,2} = \alpha \pm i\beta$

initial conditions must be given to find k, ϕ. need $y(0)$ & $\dot{y}(0)$. Set $t=0$ then $y(0) = y_{zi}$

Unit step response $u(t) = 1$, for $\dfrac{1}{s(as^2+bs+c)}$ use $\dfrac{A}{s} + \dfrac{Bs+C}{(as^2+bs+c)}$ Also $\dfrac{1}{(s+c)^2} = \dfrac{A}{(s+c)} + \dfrac{B}{(s+c)^2}$

$y_u(t) = \mathcal{L}^{-1}\left[\dfrac{H(s)}{s}\right]$

impulse input response $h(t) = \dfrac{d\, y_u(t)}{dt}$ OR $H(s)$ $h(t) = \mathcal{L}^{-1}(H(s))$

Given $U(s)$ find $y(t)$
$Y(s) = H(s)\,U(s)$

steady state response $u(t)$ = const. A. $\quad y_{ss} = A\,H(0)$

$u(t) = B\sin(\omega t + \phi) \Rightarrow y_{ss}(t) = y_p = BM\sin(\omega t + \phi + \theta)$

FRF, Frequency Response function: $= H(s = j\omega)$

M = magnitude of FRF groups $\sqrt{(\text{real part})^2 + (\text{img part})^2}$ ← ignore j's within this pt

$\arg = \tan^{-1}\left(\dfrac{\text{img}}{\text{real}}\right)$ $\quad \theta(\omega) = \arg(\text{num}) - \arg(\text{denom})$

David Malawey

2nd order D.E

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = b_2\ddot{u} + b_1\dot{u} + b_0 u$$

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

Operating Point

$x \to \bar{x}$ $\quad u \to \bar{u}$ $\quad$ where $\dot{\bar{x}} = \ddot{\bar{x}} = 0$ always

$\bar{x}$ = x-coord of O.P.
$\bar{f}$ = y-coord of O.P.

a) replace output terms $y(t)$ or $x(t)$ by $\bar{y}$ or $\bar{x}$
b) same for input
c) for input terms, $u(t)$, all time-dependent terms are ignored

General form of 3rd order DE

$$a_3\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_3\dddot{u} + b_2\ddot{u} + b_1\dot{u} + b_0 u$$

$$H(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

Procedure for linearization:

1) determine O.P.'s
2) identify nonlinear terms, linearize using Taylor S.

Taylor Series: about pt. $\bar{x}$

$$f(x) = f(\bar{x}) + \frac{df}{dx}\Big|_{\bar{x}}(x-\bar{x}) + \frac{1}{2!}\frac{d^2f}{dx^2}\Big|_{\bar{x}}(x-\bar{x})^2 + \dots$$

since $x - \bar{x} = \hat{x}$ (increment. var) $\quad \hat{x}^2 \approx 0$

$$f(x) = f(\bar{x}) + \frac{df}{dx}\Big|_{\bar{x}}\hat{x} + \cancel{\frac{1}{2!}\frac{d^2f}{dx^2}\Big|_{\bar{x}}(\hat{x})^2}^{\,0} + \text{H.O.T}$$

3) rewrite all linear terms as $x = \hat{x} + \bar{x}$
4) Solve the linear eqn

- if you get a $|x|x$, choose $x > 0$
  ignore $\bar{x}$ values that are (−)

EXAM III 211

n^{th} order differential eqn $a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots a_1 \frac{dy}{dt} + a_0 = b_m \frac{d^m u}{dt^m} + \ldots b_1 \frac{du}{dt} + b_0 u$

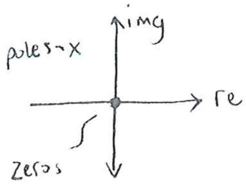
then $H(s) = \frac{b_m s^m + \ldots b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots a_1 s + a_0} = \frac{Y(s)}{U(s)} = \frac{Y_{zs}}{F_a(s)} = \frac{\text{output}}{\text{input}}$

Dont need initial condition to find $H(s)$. It is always taken at zero st.

$P(s) =$ characteristic eqn where $\mathcal{L}[u(t)] = U(s)$, $\mathcal{L}[y(t)] = Y(s)$

Given: $H(s)$ Find general form: Take coefficients and plug in backwards

stability: $P(s)$, denominator of $H(s)$



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3) at least one pole lies in R.H. plane OR has double roots on imaginary axis

Given: $P(s) = as^2 + bs + c$ zeros of $p(s) \Rightarrow s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Damping & natural frequency

standard form, 2nd order: $\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = f(t)$ (input)

Given $\ddot{y} + a\dot{y} + by = f(t)$ find ω_n & $\zeta \Rightarrow$ compare to std. fm.

• stable if $0 < \zeta < 1$

CLASSIFICATION
$0 < \zeta < 1$ underdamped
$\zeta > 1$ overdamped
$\zeta = 1$ critically damped

general form of zero input $a\ddot{y} + b\dot{y} + cy = \cancel{u(t)}\ 0$

$\mathcal{L}[\qquad\qquad] = 0$

Cannot take $y(0)$ & $\dot{y}(0)$ $= 0$ for Y_{zi}, only for transfer function

Y_{zi}, zero input

2) $Y_{zi} = k_1 s e^{s_1 t} + k_2 e^{s_2 t}$ s_1 & s_2 are real & distinct

$Y_{zi} = k_1 e^{s_1 t} + k_2 t e^{s_2 t}$ $s_1 = s_2$

$Y_{zi} = k e^{\alpha t} \cos(\beta t + \phi)$ $s_{1,2} = \alpha \pm i\beta$

initial conditions must be given to find k, ϕ. need $y(0)$ & $\dot{y}(0)$. Set $t=0$ then $y(0) = y_{zi}$

Unit step response $u(t) = 1$

$y_u(t) = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right]$

for $\frac{1}{s(as^2 + bs + c)}$ use $\frac{A}{s} + \frac{Bs + C}{(as^2 + bs + c)}$

Also $\frac{\sim}{(s+c)^2} = \frac{A}{(s+c)} + \frac{B}{(s+c)}$

~~Y_{tr} = goes to zero as $t \to \infty$~~
~~$x_{ss} = \lim_{t \to \infty} x(t)$~~

impulse input response $h(t) = \frac{d\, y_u(t)}{dt}$ OR $H(s)$ $h(t) = \mathcal{L}^{-1}(H(s))$

Given $U(s)$ find $Y(t)$
$Y(s) = H(s)\, U(s)$

steady state response $u(t) =$ const. A. $y_{ss} = A H(0)$

$u(t) = B\sin(\omega t + \phi) \Rightarrow y_{ss}(t) = y_p = BM\sin(\omega t + \phi + \theta)$

FRF, Frequency Response function: $= H(s = j\omega)$

$M =$ magnitude of FRF groups $\sqrt{(\text{real part})^2 + (\text{img part})^2}$ [ignore j's within this p

$\arg = \tan^{-1}\left(\frac{\text{img}}{\text{real}}\right)$ $\theta(\omega) = \arg(\text{num}) - \arg(\text{denom})$

EXAM III 211

n^{th} order differential eqn $a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots a_1 \frac{dy}{dt} + a_0 = b_m \frac{d^m u}{dt^m} + \ldots b_1 \frac{du}{dt} + b_0 u$

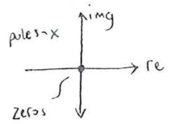
then $H(s) = \frac{b_m s^m + \ldots b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots a_1 s + a_0} = \frac{Y(s)}{U(s)} = \frac{Y_{zs}}{F_a(s)} = \frac{\text{output}}{\text{input}}$ ✓

$P(s)$ = characteristic eqn where $\mathcal{L}[u(t)] = U(s)$, $\mathcal{L}[y(t)] = Y(s)$

Dont need initial conditions to find H(s). It is always taken at zero st.

Given: H(s) Find general form: Take coefficients and plug in backwards

stability: P(s), denominator of H(s)



1) all poles have neg. real points → stable
2) all " " " " " & 1 or more distinct poles lies on img. axis → → Marginally (critically) stable
3) at least one pole lies in RH. plane OR has double roots on imaginary axis

Given: $P(s) = as^2 + bs + c$ zeros of $p(s) \Rightarrow s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Damping & natural frequency

standard form, 2nd order: $\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = f(t)$ (input)

Given $\ddot{y} + a\dot{y} + by = f(t)$ find ω_n & $\zeta \Rightarrow$ compare to std. fm.

• stable if $0 < \zeta < 1$

CLASSIFICATION
$0 < \zeta < 1$ underdamped
$\zeta > 1$ overdamped
$\zeta = 1$ critically damped

general form of zero input $a\ddot{y} + b\dot{y} + cy = \cancel{u(t)}\ 0$

$\mathcal{L}[\qquad\qquad] = 0$

Cannot take $y(0)$ & $\dot{y}(0)$ $= 0$ for Y_{zi}, only for transfer function

Y_{zi}, zero input response

2) $Y_{zi} = k_1 e^{s_1 t} + k_2 e^{s_2 t}$ s_1 & s_2 are real & distinct
$Y_{zi} = k_1 e^{s_1 t} + k_2 t e^{s_2 t}$ $s_1 = s_2$
$Y_{zi} = k e^{\alpha t} \cos(\beta t + \phi)$ $s_{1,2} = \alpha \pm i\beta$

initial conditions must be given to find k, ϕ. need $y(0)$ & $\dot{y}(0)$. Set $t=0$ then $y(0) = y_{zi}$

Unit step response $u(t) = 1$
$y_u(t) = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right]$

for $\frac{1}{s(as^2 + bs + c)}$ use $\frac{A}{s} + \frac{Bs + C}{(as^2 + bs + c)}$

Also $\frac{\sim}{(s+c)^2} = \frac{A}{(s+\,)} + \frac{B}{(s+c)^2}$

Y_{tr} = goes to zero as $t \to \infty$
$X_{ss} = \lim_{t\to\infty} x(t)$

impulse input response $h(t) = \frac{d\, y_u(t)}{dt}$ OR $H(s)$ $h(t) = \mathcal{L}^{-1}(H(s))$

Given $U(s)$ find $Y(t)$
$Y(s) = H(s)\,U(s)$

steady state response $u(t) = $ const A. $y_{ss} = AH(0)$

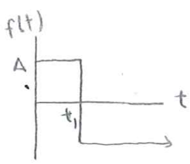
$u(t) = B\sin(\omega t + \phi) \Rightarrow y_{ss}(t) = y_p = BM\sin(\omega t + \phi + \theta)$

FRF, Frequency Response Function: $= H(s = j\omega)$

M = magnitude of FRF groups $\sqrt{(\text{real part})^2 + (\text{img part})^2}$ ← ignore j's within this prt

$\theta(\omega) = \arg(\text{num}) - \arg(\text{denom})$

$\arg = \tan^{-1}\left(\frac{\text{img}}{\text{real}}\right)$



$u(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 1 \end{cases}$ $u(t - t_1) = \begin{cases} 0 & t \le t_1 \\ 1 & t > t_1 \end{cases}$

$f(t) = Au(t) - 2A\,u(t - t_1)$

$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

$y = y_{ss} + y_{tr}$ y_{tr} = part that $\to 0$ as $t \to \infty$

System stabil if $y(t) = y_{ss}$ as $t \to \infty$

State Var form

$Y = Cq + Du$ outputs

$\dot{q} = Aq + Bu$ → inputs
↓
state vars

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} b \\ b \\ b \end{bmatrix} (f_a(t))$$

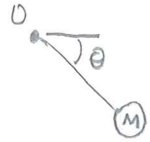
David Malawey Exam I

H_o = angular momentum of a particle about o

H_o for mass at a distance

$H_o = \vec{r} \times \vec{p} = (\vec{r} \times m\vec{v}) = ml^2\theta$

$M_o = \dot{H}_o = ml^2\dot{\theta} = F d_{perp.} = I\alpha$



State Var form

$y = Cq + Du$ outputs

$\dot{q} = Aq + Bu$

Aq → state vars; Bu → input

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} a_1\ a_2\ a_3 \\ \\ \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \end{bmatrix} + \begin{bmatrix} b \\ b \\ b \end{bmatrix} [fa(t)]$$

state vars; inputs

Pendulum w/moving support

- create r vector, get $\ddot{r}$ in $\hat{i}$ & $\hat{j}$

Static equilibrium deflection

$\dot{y} \to 0 \quad \ddot{y} \to 0 \quad y \to y_o$

state var form

terms with x = terms w/o x or, F(t)

Keq or Beq

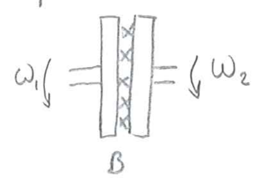
parallel

$= k_1 + k_2$

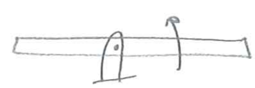
series

$= \dfrac{k_1 k_2}{k_1 + k_2}$

torque

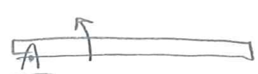


$\tau = B\Delta\omega$

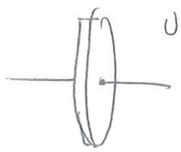


$J = \dfrac{ML^3}{12}$

ok



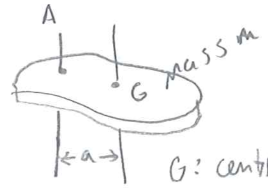
$J = \dfrac{ML^2}{3}$



Uniform disk

$J = \frac{1}{2}MR^2$

Parallel axis thm.



$J_{AA'} = J_o + Ma^2$

G: centroid

$\perp \Rightarrow V = 0$

KCL equ: Current in each node sums to zero

KVL equ: Voltage around loop sums to zero - try to get only 1 inductor in loop

$e = L\frac{di}{dt} = Ri \quad \dot{e}_c = \frac{I_c}{c}$

$i = C\frac{di}{dt} \quad i_L = i_L(+0) + \frac{1}{L}\int_0^t e_L(\lambda)\,d\lambda$

R resistance, Ω

c capacitance,

e voltage

L inductance, Henries

State var eqns

$\dot{I}_L = \frac{e_L}{L} \quad \dot{e}_c = \frac{I_c}{c}$

- one state var eqn for each ~~reer~~ & $\dashv\vdash$
- nodes for capacitors

Gears $\frac{R_2}{R_1} = n = \frac{\theta_1}{\theta_2} \quad V = r\omega \quad \Sigma T = J\ddot{\theta}$

$y_{tr} \rightarrow$ transient response : immediately after turned on

$y_{ss} \rightarrow$ steady state response = after y_{tr} has disappeared $t \rightarrow \infty$

$\tau \rightarrow$ time constant (seconds) measures how quickly system reaches steady state

1st order system : Stable $\tau > 0$
unstable $\tau < 0$
marginally stable $\tau \rightarrow \infty$

definition of laplace

$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$

common transforms

$\mathcal{L}(A) = \frac{A}{s} \quad A \rightarrow$ const

$\mathcal{L}e^{at} = \frac{1}{s-a} \quad a \rightarrow$ real #

$\mathcal{L}(t) = \frac{1}{s^2}$

$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$

$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$

$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

transform of derivatives

$\mathcal{L}(f'(t)) = -f(0) + sF(s)$

$\mathcal{L}[\ddot{f}(t)] = s^2F(s) - sf(0) - \dot{f}(0)$

$\mathcal{L}\left(\frac{d^nf(t)}{dt^n}\right) = s^nF(s) - s^{n-1}f(0) + \ldots f^{n-1}(\ )$

General 1st order sys. where $\tau = \frac{a_2}{a_1}$

$a_2\dot{y} + a_1y = a_0g(t) \qquad \frac{a_0}{a_2}g(t) = f(t)$

$\dot{y} + \frac{a_1}{a_2}y = \frac{a_0}{a_2}g(t) \Rightarrow \dot{y} + \frac{y}{\tau} = f(t)$

$\Rightarrow y(t) = \tau A + (y_0 - \tau A)e^{-t/\tau}$

if $f(t)$ is constant, $f(t)$ is your A

find response means find $\mathcal{L}[f(t)]$

Levers

Small θ :
$\sin\theta = \theta$
$\cos\theta = 1$

$\circlearrowleft^+ \Sigma M_0 = J_0\alpha \qquad J_{\text{slender rod}} = \frac{1}{12}ML^2$ center axis
$\frac{1}{3}ML^2$ end

Integration by parts

$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du \quad$ sub $\frac{-dv}{s} = e^{-st}dt$

ME 211: MODELING AND ANALYSIS OF DYNAMIC SYSTEMS

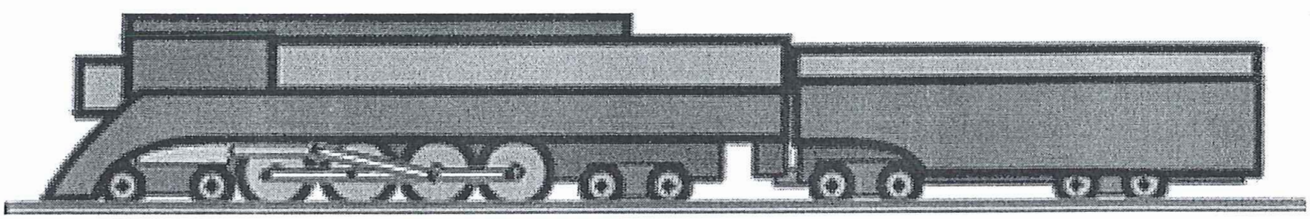
PROJECT

(Due 12-6-2010 in class or in my office TMH 129 by 2pm)

The course project for ME-211 is a group project. Each group comprises of 4-5 students. The objective behind assigning a group project is make you learn how to cooperate with others who have different skills than their own, how to work in multicultural teams and, finally, how to work in a way that is widely used in real-life today.

MODELING A TOY TRAIN SYSTEM

Consider a toy train consisting of an engine and a caboose. The engine-caboose system has a rectilinear motion. The mass of the engine and the caboose are M_1 and M_2, respectively. The two are held together by a spring, which has the stiffness coefficient of k. F_a represents the force applied by the engine. For modeling purpose, take step input and the rolling friction acting on the wheels as B(dx/dt), where x represent the displacement coordinate and B is the coefficient of rolling friction.



The following values of constants are given:

$M_1 = 1$ kg; $M_2 = 0.5$ kg; k = 1 N/m; $F_a = U(t)$; $B_{engine} = 0.02$ kg/s; $B_{caboose} = 0.01$ kg/s;

do on your own

(a): Find the differential equation of motion for engine and caboose. (5 Points)
(b): Let the output be the velocity of the engine. Put (a) into state-space form and identify the A, B, C and D matrices. (10 Points)

example available on BB

(c): Assuming initial conditions; x(0) = v(0) = 0; find the transfer function of the system. Using MATLAB command "roots" find the zeros and poles of the transfer function. Then, using MATLAB command "zplane", show the zero-pole plot on the complex plane. Comment on stability of the system, based on the plot. (30 Points)
(d). Model the above system in MATLAB using "ss" and "lsim" commands. Plot time (on x axis) versus output (y axis) graph. Show the step input velocity response for 60 seconds, using 0.01 seconds as time increment. (10 Points) state space
(e). Repeat (d) using "tf" and "lsim" MATLAB commands. (10 Points)
(f). Plot the unit impulse velocity response for the engine. (5 Points)
(g). Submit project report (30 Points)

INSTRUCTIONS

ıe objective of this course project is to model and analyze a toy train system consisting of an engine and a caboose. You are expected to simulate the model for step and impulse input and perform system analysis based on the plots generated.

The ME-211/ Blackboard website has a "Project" folder (in the contents section) that contains helpful examples and m files you may require to generate MATLAB plots:

Report Format
The report must be created in a word processing package, include page numbers, and be written clearly and concisely. Documents must be printed on a laser printer and major sections and subsections of the report must be clearly denoted. Each group must submit a hard copy of the report before/by due date/time. **Electronic submission (via email as scanned files) of the report is not acceptable**. Extension/ late submission of project work is not acceptable. The report must contain the following sections:

Title Page: The Title Page includes the following information:- project title, student's names, email addresses, course title, instructor's name, and submission date.

Abstract: (not to exceed 50 – 100 words): The Abstract provides a brief overview of the report that includes a clear, concise explanation of the problem, approach, and results.

Introduction: The Introduction is an explanation of problem. You may paraphrase what is given in the problem statement.

Results & Conclusion: The results and conclusion section is a detailed description of methods that are used to solve the problem and the resulting outcomes. In this section you will attach:

a) Calculation work (to derive equation of motion, state space form, transfer equation, etc.). Hand written calculation work is acceptable, but it should be clearly written, F.B.D's drawn should have clearly marked axis, forces labeled along with directions shown using arrows.
b) Figure(s)/ Plot(s) (properly labeled).
c) MATLAB m files.
d) Any other supporting document/ file.
e) Conclusion: Provide a brief discussion of the project outcomes. Do not simply state what you see in the graphs. Interpret the results.

Note: The project work will be evaluated along three primary axes:

Correctness: E.g., Math calculation(s), Matlab code free of bugs.
Quality of Report: E.g., formatting of word document, quality of plots, clarity of thought and writing skills, etc.
Originality: Overall originality of your work.

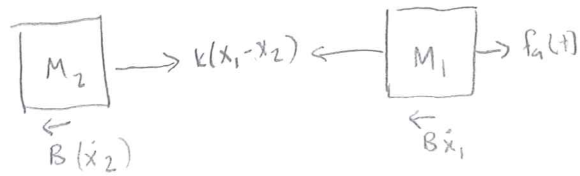
Based on the above three criteria, groups will be given an **extra credit** of up-to 2 points (on a 10 point basis) if your work is simply outstanding.

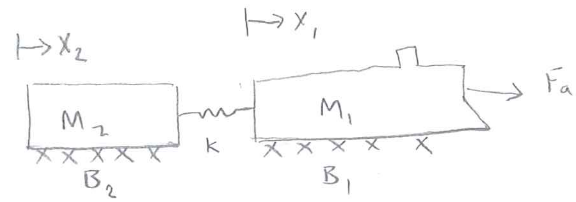
S.NO	NAME	GROUP #
1	Allen,Joshua	**GROUP - 1**
2	Barlow,Jerry	**Meeting Time**
3	Barnes,Brian	**Wednesday**
4	Binns,Crawford	**11/10/2010**
5	Bird,James	**11.30 - 1.30 pm**
1	Davis,Nicholas	**GROUP - 3**
2	Doerner,Trent	**Meeting Time**
3	Ekholm,Michelle	**Monday**
4	Fennell,Sean	**11/15/2010**
5	Garvey,Kyle	**11.30 - 1.30 pm**
1	Kekec,David	**GROUP - 5**
2	Lange,Matthew	**Meeting Time**
3	Liston,Paul	**Friday**
4	Livergood,Jacob	**11/19/2010**
5	Malawey,David	**11.30 - 1.30 pm**
1	Perry,Zachariah	**GROUP - 7**
2	Quade,Justin	**Meeting Time**
3	Richert,Jacob	**Wednesday**
4	Ridenhour,Krishawn	**11/17/2010**
5	Riechers,Stephan	**1.30 - 3.30 pm**
1	Smith Jr,Todd	**GROUP - 9**
2	Smith,Caleb	**Friday**
3	Stanley,Chad	**11/12/2010**
4	Stowe,Dalton	**1.30 - 3.30 pm**

S. NO	NAME	GROUP #
1	Bock,Benjamin	**GROUP - 2**
2	Boyert,Joshua	**Meeting Time**
3	Brasher,Grant	**Friday**
4	Braun,Andrew	**11/12/2010**
5	Cawvey,Zachary	**11.30 - 1.30 pm**
1	Hamilton,Turquiose	**GROUP - 4**
2	Heimsoth,William	**Meeting Time**
3	Jenke,Robert	**Wednesday**
4	Johnson II,Garrick	**11/17/2010**
5	Johnson,Jeremiah	**11.30 - 1.30 pm**
1	Michael,Joseph	**GROUP - 6**
2	Moesch,Nikolas	**Meeting Time**
3	Newman,Jacob	**Friday**
4	Ohlms,Gerald	**11/19/2010**
5	Otto,Emily	**1.30 - 3.30 pm**
1	Rose,Megan	**GROUP - 8**
2	Rusher,Charles	**Meeting Time**
3	Russell,William	**Monday**
4	Schaeperkoetter,Matthew	**11/15/2010**
5	Shaffer,Samuel	**1.30 - 3.30 pm**
1	Tigue,Adam	**GROUP - 10**
2	Welch,Jacob	**Wednesday**
3	Wommack,Daniel	**11/10/2010**
4	Wood,Adam	**1.30 - 3.30 pm**

Note: The group meeting should not take more than 45-60 minutes. Please email me with a suitable time when all/ max. number of group members can attend the meeting.

David Mahoney 211 modeling project





$$M_1\ddot{x}_1 = f_a(t) - B_1\dot{x}_1 - k(x_1 - x_2)$$

$$M_1\ddot{x}_1 + B_1\dot{x}_1 + k(x_1 - x_2) = f_a(t) \quad -(1)$$

$$(1) \,\&\, (2) \Rightarrow \quad M_1\ddot{x}_1 + B_1\dot{x}_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 = f_a(t)$$

$$\dot{x}_1 = \frac{1}{B_1}\left(f_a(t) - M_1\ddot{x}_1 - M_2\ddot{x}_2 - B_2\dot{x}_2\right)$$

$$M_2\ddot{x}_2 = k(x_1 - x_2) - B_2\dot{x}_2$$

$$\ddot{x}_2 = \frac{1}{M_2}\left[k(x_1 - x_2) - B_2\dot{x}_2\right]$$

<u>state-space eq's</u>

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = \frac{1}{M_1}\left[f_a(t) - B_1 v_1 - k(x_1 - x_2)\right]$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 = \frac{1}{M_2}\left(k(x_1 - x_2) - B_2 v_2\right)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{M_1} & -\frac{B_1}{M_1} & \frac{k}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{M_2} & 0 & -\frac{k}{M_2} & -\frac{B_2}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M_1} \\ 0 \\ 0 \end{bmatrix} \left[f_a(t)\right]$$

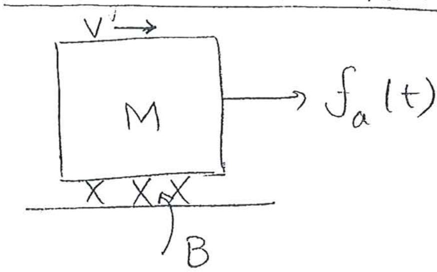
<u>output eq'n</u> $\quad Y = V_1$

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + [0]\left[f_a(t)\right]$$

Tra

Ex 7-6 Pg. 213 | Using ss command in MATLAB

FIRST ORDER SYSTEM



$$M\mathring{V} + BV = f_a(t)$$

$$\mathring{V} + \frac{B}{M}V = \frac{1}{M}f_a(t) \quad \text{— (1)}$$

In state space form:

Let $B = M = 1$

$$\mathring{V} = -\frac{B}{M}V + \frac{f_a(t)}{M}$$

(one state "V" & one input $f_a(t)$)

Put in $\mathring{q} = Aq + Bu$

$$\mathring{V} = \underbrace{\left[-\frac{B}{M}\right]}_{A} V + \underbrace{\left[\frac{1}{M}\right]}_{B} f_a(t)$$

since output is $V(t)$: ∴ output eqn, $V = V$

$$y = Cq + Du$$

$$V = \underbrace{[1]}_{C} V + \underbrace{[0]}_{D} f_a(t)$$

For simulation in MATLAB

$A = [-1]$

$B = [1]$

$C = [1]$

$D = [0]$

$$\left\{\begin{array}{l}\text{Initial condition} \quad V(0) = 0 \\ \text{Input} \rightarrow f_a(t) = 1 \text{ for } y_u(t)\end{array}\right\}$$

Using tf (transfer function) in MATLAB

Apply Laplace transform to eqn (1)

$$sV(s) - \cancel{V(0)}^{\,0} + \frac{B}{M}V(s) = \frac{1}{M}F(s)$$

$$\Rightarrow V(s)\left[s + \frac{B}{M}\right] = \frac{1}{M}F(s) \Rightarrow \frac{V(s)}{F(s)} = \frac{(1/M)}{(s + B/M)} = H(s)$$

For simulation

num = $[1/M]$; den = $[1 \quad B/M]$

$= [1]$ $\qquad = [1 \quad 1]$

SECOND ORDER SYSTEM

Ex 8.1: Plot of unit step response for system in example 8.1/ Page 242. Input $= f_a(t)$ & Output x_1

Initial conditions $= 0$

$x_1(0) = x_2(0) = v_1(0) = 0$

(a) Using state space method:

Equation is:

$$M\ddot{x}_1 + B_1\dot{x}_1 + K_1(x_1 - x_2) = f_a(t)$$

$$B_2\dot{x}_2 + K_2x_2 + K_1(x_2 - x_1) = 0$$

choose state variables,

$$q = \begin{Bmatrix} x_1 \\ v_1 \\ x_2 \end{Bmatrix} \rightarrow \dot{x}$$

$$\Rightarrow \quad \dot{x}_1 = v_1$$

$$\dot{v}_1 = \frac{1}{M}\left[f_a(t) - B_1v_1 - K_1x_1 + K_1x_2\right]$$

$$\dot{x}_2 = \frac{K_1}{B_2}x_1 - \frac{(K_1+K_2)}{B_2}x_2$$

Putting in $\dot{q} = Aq + Bu$ form

$$\underset{3\times1}{\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \end{bmatrix}} = \underbrace{\underset{3\times3}{\begin{bmatrix} 0 & 1 & 0 \\ -\frac{K_1}{M} & -\frac{B_1}{M} & \frac{K_1}{M} \\ \frac{K_1}{B_2} & 0 & -\frac{(K_1+K_2)}{B_2} \end{bmatrix}}}_{[A]} \underset{3\times1}{\begin{bmatrix} x_1 \\ v_1 \\ x_2 \end{bmatrix}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \end{bmatrix}}_{[B]} f_a(t)$$

Output eqn:

$x_1 = x_1$; put in $y = Cq + Du$ form,

$$x_1 = \underbrace{\underset{1\times3}{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}}_{[C]} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \end{bmatrix} + \underbrace{[0]}_{[D]} f_a(t)$$

(c) Using transfer function

First obtain the laplace transform of diff^n eqn & put in form of eqn(2) / Page 244

$$X_1(s) = \left[\frac{s+2}{s^3+3s^2+3s+1}\right] F_a(s)$$

since $\underbrace{\text{transfer function}}_{\text{T.F}} = \frac{\mathcal{L}(\text{Output})}{\mathcal{L}(\text{Input})} = \frac{\mathcal{L}(x_1(t))}{\mathcal{L}(f_a(t))}$

$$\Rightarrow H(s) = T.F = \frac{X_1(s)}{F_a(s)} = \frac{s+2}{s^3+3s^2+3s+1}$$

Once you have the transfer function, finding response of the system is easy.

Define numerator, num = [1 2] ← In Matlab
i.e. write the coefficients of s, starting with the highest power of 's' and ending with constant (if any)

Define denominator, $\text{den} = \underbrace{[1 \quad 3 \quad 3 \quad 1]}_{\text{same rule as above}}$

Please refer to the matlab file for rest of code

Modeling a Toy Train System

ME 211 Modeling and Analysis of Dynamic Systems

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Abstract:

A model engine and caboose are modeled for inputs and outputs. The system involves a unit step input, friction coefficients, and a spring. Output values of velocity and other parameters are to be produced.

Free body diagrams were created and equations are derived from these free body diagrams. The transfer function, damping ratio, and natural frequency were then found. Using MATLAB, the system was modeled and plots were produced.

The results show how the engine velocity results from the unit step input and unit impulse input.

Introduction:

The problem is to model an engine and caboose, with a force input and displacement as the output. A second order differential equation and a transfer function are to be used with MATLAB to create plots of the transfer function zero pole plot and the outputs given unit step and unit impulse inputs. The constant values for the system are listed at the beginning of the calculation work.

Results and Conclusion:

a) Calculation Work

$$M_1 = 1Kg$$
$$M_2 = 0.5Kg$$
$$K = 1\ {}^{N}/_{M}$$
$$F_a = U(t)$$
$$B_e = .02\ {}^{Kg}/_{S}$$
$$B_c = .01\ {}^{Kg}/_{S}$$

$$\sum F_x = M_1\ddot{x}$$

$$M_1\ddot{x}_1 + B_2\dot{x}_1 = k(x_1 - x_2) = U(t)$$

$$\ddot{x} + .02\dot{x} + (x_1 - x_2) = U(t)$$

$$\sum F = M_2\ddot{x}_2$$

$$M_2\ddot{x}_2 + B_e\dot{x}_2 - K(x_1 - x_2) = 0$$

$$.5\ddot{x}_2 + .01\ddot{x}_2 - (x_1 - x_2) = 0$$

$$\dot{X}_1 = V_1$$

$$\dot{V}_1 = \ddot{X}_1$$

$$\dot{V}_1 = \frac{1}{M_1}[U(t) - B_2V_1 - K(x_1 - x_2)]$$

$$\dot{V}_1 = U(t) - .02V_1 - x_1 + x_2$$

$$\dot{X}_2 = V_2$$

$$\dot{V}_2 = \ddot{X}_2$$

$$\dot{V}_2 = \frac{1}{M_2}[-B_cV_2 + K(x_1 - x_2)]$$

$$\dot{V}_2 = -.02V_2 + 2x_1 - 2x_2$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{V}_1\\ \dot{x}_2\\ \dot{V}_2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -1 & -.02 & 1 & 0\\ 0 & 0 & 0 & 1\\ 2 & 0 & -2 & -.02\end{bmatrix}\begin{bmatrix}x_1\\ V_1\\ x_2\\ V_2\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix}U(t)$$

$$[V_1] = [0 \quad 1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ V_1\\ x_2\\ V_2\end{bmatrix} + [0]U(t)$$

$$\dot{X}_1 = V_1$$

$$\mathrm{L}(\dot{X}_1) = \mathrm{L}(V_1) \longrightarrow sX_1 - x(0) = V_1(s)$$

$$\dot{V}_1 = u(t) - .02V_1 - X_1 + X_2$$

$$\mathrm{L}(V) = U(s) - .02(V_1) - \mathrm{L}(X_1) + \mathrm{L}(X_2)$$

$$sV_1(s) - v(0) = U(s) - .02V_1(s) - X_1(s) + X_2(s)$$

$$v(0) = 0$$

$$\dot{X}_2 = V_2$$

$$sV_2(s) = -(.02)V_2(s) + X_1(s) - X_2(s)$$

$$(s+.02)V_2(s) = -\frac{2}{s}V_2(s) + \frac{2}{s}V_1(s)$$

$$\left(s+.02+\frac{2}{s}\right)V_2(s) = \frac{2}{s}V_1(s)[s]$$

$$V_2(s) = \frac{2V_1(s)}{(s^2+.02s+2)}$$

$$X_2(s) = \frac{2V_1(s)}{s(s^2+.02s+2)}$$

$$sV_1(s) = U(s) - .02V_1(s) - \frac{V_1(s)}{s} + \frac{2V_1(s)}{s(s^2+.02s+2)}$$

$$U(s) = sV_1(s) + .02V_1(s) + \frac{V_1(s)}{s} - \frac{2V_1(s)}{s(s^2+.02s+2)}$$

$$U(s) = \left[s+0.2+\frac{1}{s} - \frac{2}{s(s^2+.02s+2)}\right]V_1(s)$$

$$s(s^2+.02s+2)U(s) = [(s^2+.02s+1)(s^2+.02s+2)-2]$$

$$V_1(s) = \frac{s(s^2+.02s+2)U(s)}{[(s^2+.02s+1)(s^2+.02s+2)-2]}$$

$$H(s) = \frac{V_1(s)}{U(s)}$$

$$H(s) = \frac{s(s^2+.02s+2)}{(s^2+.02s+1)(s^2+.02s+2)-2}$$

$$H(s) = \frac{s^2+.02s+2}{s^3+.04s^2+3.0004s+.06}$$

b) Figures

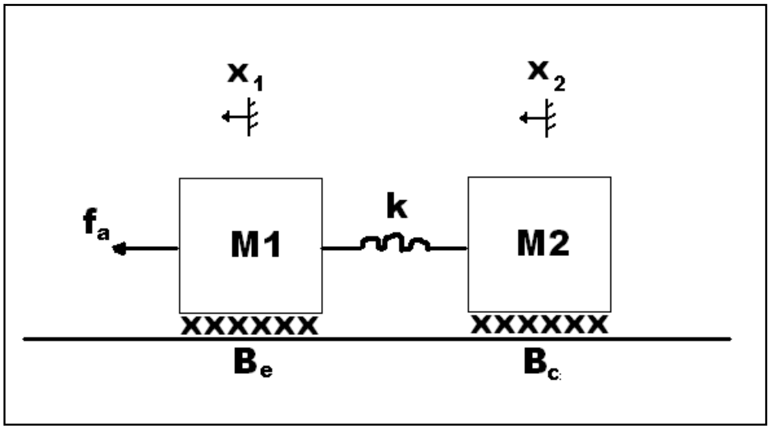

Figure 1- System Sketch

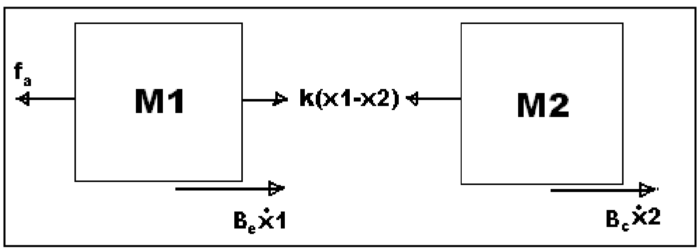

Figure 2 – Free Body Diagram

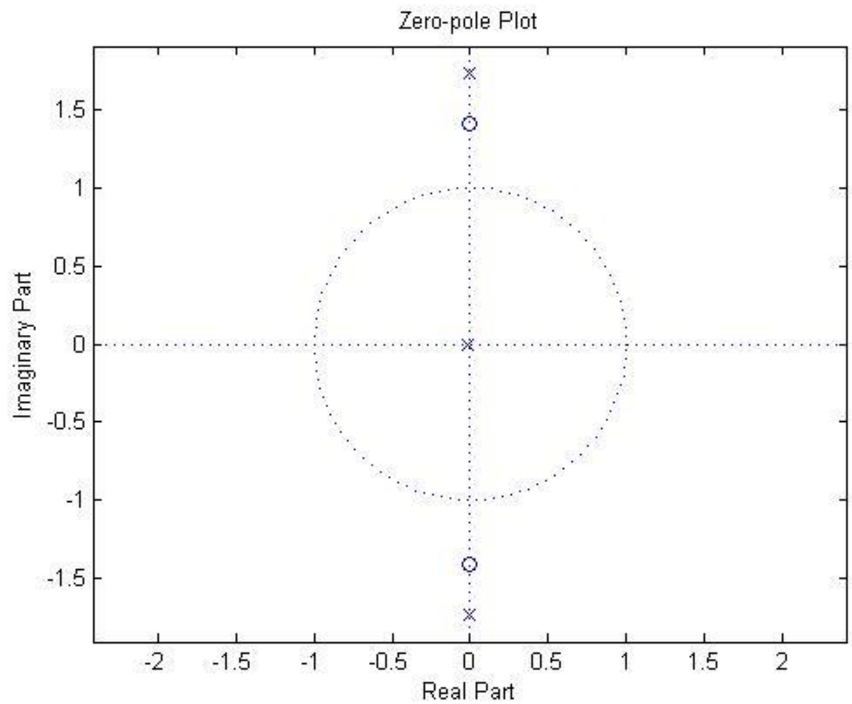

Figure 3 – Zero-pole plot of Transfer Function

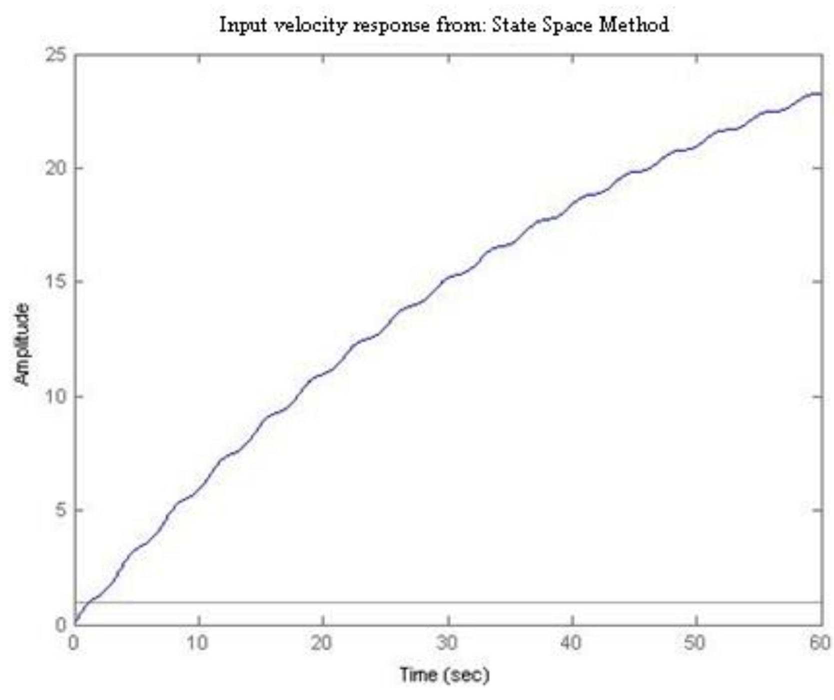

Figure 4 – Velocity vs. Time, State Space Method, Unit Step

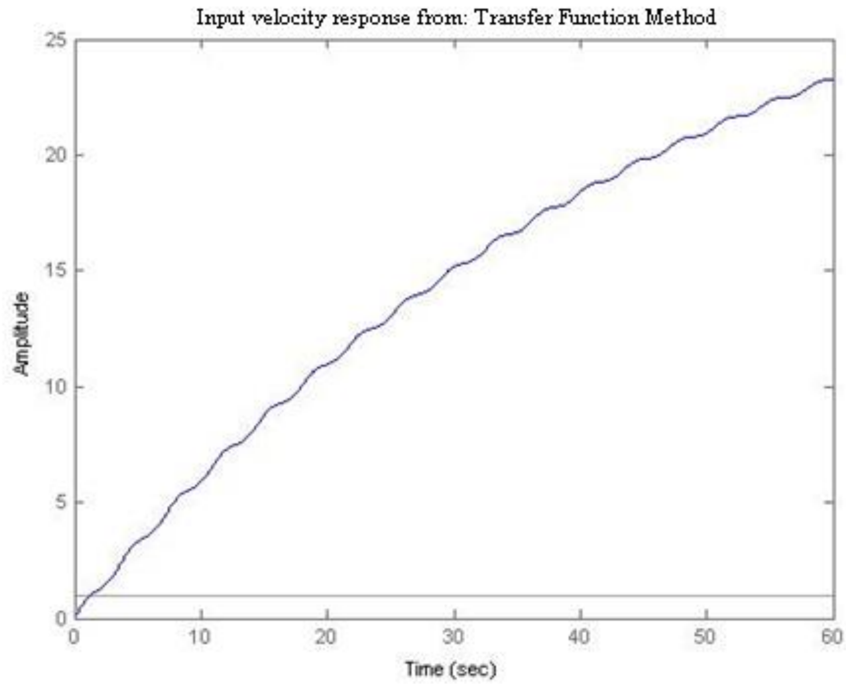

Figure 5 – Velocity vs. Time, Transfer Function Method, Unit Step

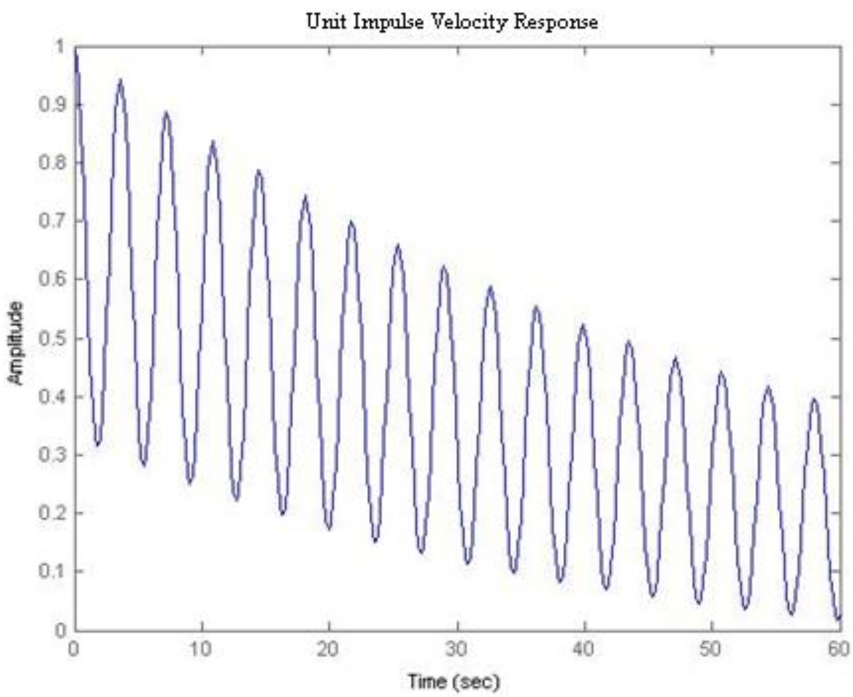

Figure 6 – Velocity vs. Time, Unit Impulse

c) MATLAB m files

```
%FINDING ZEROS AND POLES OF TRANSFER FUNCTION AND PLOTTNG ZERO-PLOT GRAPH
num=[1 .02 2];% DEFINES COEFICIENT'S OF s IN NUMERATOR
den=[1 .04 3.0004 .06];% DEFINES COEFICIENT'S OF s IN DENOMINATOR
rn=roots (num);% FINDING ZEROS OR ROOTS OF NUMERATOR
rd=roots (den);% FINDING POLES OR ROOTS OF DENOMINATOR
figure (1)
zplane (rn,rd)


% PLOT OF OUTPUT VS TIME GRAPH FOR STEP INPUT USING STATE SPACE METHOD (i.e.
ss command in MATLAB)
tfinal=60;
t=0:0.01:tfinal;
U=ones (size (t));% DEFINING STEP INPUT
A=[0 1 0 0;-1 -.02 1 0;0 0 0 1;2 0 -2 -.02];% DEFINING A MATRIX
B=[0 1 0 0]';% DEFINING B MATRIX
C=[0 1 0 0];% DEFINING C MATRIX
D=0;% DEFINING D MATRIX
X0=[0 0 0 0];% DEFINING INITIAL CONDITION (MUST HAVE ENTRIES EQUAL TO THE
NUMBER OF STATE VARIABLES)
sys=ss (A,B,C,D);
figure (2)
lsim (sys,U,t,X0)% SIMULATING LINEAR SYSTEM


% PLOT OF OUTPUT VS TIME GRAPH FOR STEP INPUT USING TRANSFER FUNCTION METHOD
(i.e. tr command in MATLAB)
systf=tf (num,den);
figure (3)
lsim (systf,U,t);

% PLOT OF OUTPUT VS TIME GRAPH FOR IMPULSE INPUT USING IMPULSE COMMAND
figure (4)
impulse (systf,tfinal)
```

d) Basic Equations used:

<u>Definiton of a transfer function</u>

H(s)= $\mathcal{L}$(output)/$\mathcal{L}$(input)= $\mathcal{L}(x_1(t))/\mathcal{L}(f_a(t))$

<u>Poles of a second order equation</u>

S_1, S_2 such that characteristic equation P(s) = $a_2 s^2 + a_1 s + a_0 = 0$

Zeros of a second order equation: zeros of numerator of H(t)

<u>Matrix form of input/output equations</u>

dq/dt=[A][q] + [B][u(t)]

For any input u(t), $y_u(t)=\mathcal{L}^{-1}\left(\frac{H(s)}{s}\right)$

For unit impulse response, $h(t)=\mathcal{L}^{-1}(H(s))$

e) Conclusion:

The zero-pole plot **(Figure 3)** displays two zeros and three poles all lying on the imaginary axis. This plot shows that the system is *marginally stable***.**

For a *unit step input*, the engine of the train accelerates forward, but not with a constant acceleration. The acceleration varies with the system's natural frequency. The cause for humps in the velocity vs. time graph **(Fig. 4 & 5)** can be described as followed:

As the engine moves forward, the spring stretches and reduces the engine's forward acceleration. Then the caboose accelerates due to the spring tension and "catches up" with the engine. Finally, the spring force is reduced and the engine accelerates at a greater rate. The cycle then continues. The velocity of the train will increase until the friction forces equal the input force.

Given a *unit impulse input*, the engine velocity follows a similar pattern with the same natural frequency **(Figure 6)**. However, the maximum velocity occurs at the first cycle, and the train continues to slow due to friction forces. For each cycle, the velocity amplitude reduces due to friction forces. The velocity and amplitude will both decrease until they equal zero.