A piecewise linear approximation of $C(f)$ leads also to an approximation of the success probability $P(f)$ of [3]

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Abstract

Solving reliable payment flows on the Lightning network with the methods proposed in [3] involves maximizing the success probability $P(f)$ at best optimal. In order to solve the optimization problem it is reformulated to a minimization problem with convex but non-linear cost function $C(f)$. When for example optimal piecewise linear approximation (PWL) with N linear segments [2] is applied to $C(f)$, it can be solved more efficiently with established methods that require linearity[1]. However, the introduced approximation error ϵ can lead to sub-optimality in $C(f)$ and $P(f)$. We show, how such an approximation error for $C(f)$ reflects on the original objective $P(f)$ (in the worst case) and that (in theory) the approximation of $C(f)$ leads also to an approximation of $P(f)$ that can be improved by improving the PWL of $C(f)$.

1 Notation

Lets use the following notation:

- $C(f) := \sum_{e \in E} -\log\left(P_e(X_e \geq f_e)\right) \ldots$ cost function of the min-cost flow problem of [3]
- $c_e(f_e) := -\log\left(P_e(X_e \geq f_e)\right) \ldots$ (convex) summand of the cost function for a given channel $e \in E$ of [3]
- $g_e(f_e) \ldots$ minimax solution that minimizes the error $\epsilon := log\,(\delta)\,, \delta \geq 1$ of the piecewise linear approximation of $c_e(f_e)$ with N elements as proposed in [2], i.e. $log\,(\delta) = \max_{0 \leq P_e \leq 1} |g_e(f_e) - c_e(f_e)|$
- $\tilde{C}(f) := \sum_{e \in E} g_e(f_e) \ldots$ approximation of $C(f)$ with lower and upper bounds on approximation error
- $P(f) := \prod_{e \in E} P_e\left(X_e \geq f_e\right) \ldots$ success probability of entire flow of [3].
- $\tilde{P}(f) := \exp(-\tilde{C}(f)) \ldots$ approximation of $P(f)$ with lower and upper bounds on approximation error

2 Application of PWL to $c_e(f_e), \forall e \in E$

2.1 Deduce bounds for $\tilde{C}(f)$

By the definition of the minimax solution, i.e. $log\,(\delta) = \max_{0 \leq P_e \leq 1} |g_e(f_e) - c_e(f_e)|$, the following condition holds true:

$$c_e(f_e) - \log(\delta) \leq g_e(f_e) \leq c_e(f_e) + \log(\delta)$$

Applying these inequalities to $C(f)$ we can establish the following approximation bounds for $\tilde{C}(f)$:

$$\sum_{e \in E} \left(-\log\left(P_e(X_e \geq f_e)\right) - \log(\delta)\right) \leq \tilde{C}(f) \leq \sum_{e \in E} \left(-\log\left(P_e(X_e \geq f_e)\right) + \log(\delta)\right)$$

[1] See for example the implementations developed for [3]: `https://github.com/renepickhardt/mpp-splitter/blob/master/MinimalLinearizedmincostflowexampleforMPP.ipynb`

2.2 Upper bound for $\tilde{P}(f)$

We start with the lower bound of $\tilde{C}(f)$:

$$\sum_{e\in E} \left(-\log\left(P_e(X_e \geq f_e)\right) - \log(\delta)\right) \leq \tilde{C}(f)$$

This can be reformulated to:

$$\sum_{e\in E} \left(-\log\left(\delta P_e(X_e \geq f_e)\right)\right) \leq \tilde{C}(f)$$

Multiplying the inequality with (-1) leads to:

$$\sum_{e\in E} \left(\log\left(\delta P_e(X_e \geq f_e)\right)\right) \geq -\tilde{C}(f)$$

Multiplying the inequality with exp() leads to:

$$\delta^{|E|} P(f) = \delta^{|E|} \prod_{e\in E} P_e\left(X_e \geq f_e\right) \geq \tilde{P}(f)$$

2.3 Lower bound for $\tilde{P}(f)$

We proceed with the upper bound of $\tilde{C}(f)$:

$$\tilde{C}(f) \leq \sum_{e\in E} \left(-\log\left(P_e(X_e \geq f_e)\right) + \log(\delta)\right)$$

Multiplying the inequality with (-1) leads to:

$$-\tilde{C}(f) \geq \sum_{e\in E} \left(\log\left(P_e(X_e \geq f_e)\right) - \log(\delta)\right)$$

Multiplying the inequality with exp() leads to:

$$\tilde{P}(f) \geq \frac{\prod_{e\in E} P_e\left(X_e \geq f_e\right)}{\delta^{|E|}} = \frac{P(f)}{\delta^{|E|}}$$

2.4 Deduce bounds for $P(f)$

The previous bounds can be reformulated to:

$$\frac{\tilde{P}(f)}{\delta^{|E|}} \leq P(f) \leq \delta^{|E|} \tilde{P}(f)$$

3 Application of PWL to $C(f)$ directly

In [3] it is shown, that $C(f)$ is convex on the entire domain. Hence, the same arguments of previous sections can be applied to $C(f)$ directly. For an PWL $\hat{C}(f)$ of $C(f)$ the following condition holds true:

$$C(f) - \log(\delta) \leq \hat{C}(f) \leq C(f) + \log(\delta)$$

Multiplying the inequality with $-\exp()$ leads to:

$$\delta P(f) \geq \hat{P}(f) \geq \frac{P(f)}{\delta}$$

This can be reformulated to:

$$\frac{\hat{P}(f)}{\delta} \leq P(f) \leq \delta \hat{P}(f)$$

4 Conclusions

- For $\delta \to 1^+$, $\tilde{P}(f)$ approximates $P(f)$ (and vice versa).
- For $\delta \to 1^+$, $\tilde{C}(f)$ approximates $C(f)$ (and vice versa).

5 Future work

- Explore [1] further to find **unknown** optimal or near-optimal N for a desired approximation error ϵ.
- Explore further literature besides [2] on piecewise linear approximation.
- Explore optimization literature on min-cost flow problems with uncertain capacities in more detail.
- An optimal solution f_C^* to $\tilde{C}(f)$ should also lead to a near-optimal solution of $P(f)$ with some deducible error. Our intuition says, this should be the case given the problem properties. However, it should be argued properly.
- Is the approach with **optimal** piecewise linear approximation for a desired approximation quality for $P(f)$ even practical and allows decent run-times?
- Finding a decent N and a corresponding piecewise linear approximation with desired approximation error can be calculated offline / cached.

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References

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