Application of the functional renormalisation group to Bose gases

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Overview

Weakly-interacting Bose gases

- Dilute Bose gases: $r_0 \ll d$

  r_0:range of the interaction
  d:inter-particle distance

  "Almost" an ideal Bose gas
- Long theoretical interest for studying Bose-Einstein condensation (BEC) and superfluidity (Bogoliubov 1947)
- Behaviour of the system depends sensitively on the dimensionality
- Interest greatly increased with the experimental realisation of BEC in cold atom gases

  Anderson *et al.* Science **269**, 198 (1995), Davis *et al.* Phys. Rev. Lett **75**, 3969 (1995)
- Systems with different shapes (dimensions) and temperatures have been produced
- Interaction between atoms can be tuned using Feshbash resonances

Weakly-interacting Bose gases

- Bare action of the system:

$$\mathcal{S}[\mathbf{\Phi}] = \int_x \left[\phi^\dagger \left(-\partial_\tau + \frac{\nabla^2}{2m} + \mu\right)\phi - \frac{g}{2}(\phi^\dagger\phi)^2\right]$$

g: repulsive contact interaction, $\tau = it$

- Interaction related to the two-body T-matrix:

$$T^{2B} = \begin{cases} \frac{4\pi a_{3D}}{m} & : d = 3 \\ \frac{4\pi/m}{\log(2/|\mu|a_{2D}^2) - 2\gamma_E} & : d = 2 \end{cases}$$

a_d: s-wave scattering length

- Weakly-interacting regime:
  - $d = 3$: $n_0 a_{3D}^3 \ll 1$
  - $d = 2$: $1/\log(1/n_0 a_{2D}^2) \ll 1$
  - $d = 1$: $(-n_0 a_{1D})^{-1} \ll 1$

n_0: atom density

Weakly-interacting Bose gases

Correlation function: $G_n(\mathbf{x}) = \langle \phi^\dagger(\mathbf{x})\phi(0)\rangle$

- **Three dimensions**: Long-distance limit $|\mathbf{x}| \to \infty$

$$G_n(\mathbf{x}) \to \quad \rho_c > 0 \qquad : T < T_c$$

$$G_n(\mathbf{x}) \propto \begin{cases} |X|^{-(1+\eta^*)} & : T = T_c,\ \eta^* > 0 \\ e^{-|X|/\xi} & : T > T_c \end{cases}$$

ρ_c: Condensate density

  - System shows long-range-order (LRO)
  - $U(1)$ symmetry is broken, with ρ_c as order parameter

Weakly-interacting Bose gases

- **Two dimensions**: Long-distance limit $|\mathbf{x}| \to \infty$

$$G_n(\mathbf{x}) \to \quad \rho_c > 0 \qquad\qquad : T = 0$$

$$G_n(\mathbf{x}) \propto \begin{cases} |X|^{-\eta} & : T \leq T_c,\ \eta > 0 \\ e^{-|X|/\xi} & : T > T_c \end{cases}$$

  - Condensation only possible at $T = 0$ (Mermin-Wagner theorem)
  - System shows quasi-long-range-order (QLRO)
  - $\rho_c = 0$ for $0 < T \leq T_c$, but superfluid density $\rho_s > 0$
  - Phase transition driven by the unbinding of vortex pairs: Berenzinskii-Kosterlitz-Thouless (BKT) transition

Weakly-interacting Bose gases

- Mean-field theory gives a reasonable qualitative description of the three-dimensional gas at low temperatures
- Perturbation theory plagued by IR divergences due to ungapped propagator of Goldstone mode

$$G_{||} = \frac{1}{\mathbf{q}^2 + q_c^2}, \qquad G_{\perp} = \frac{1}{\mathbf{q}^2}$$

- Divergences cancel, but cancellations are lost if expansions are truncated
- Need of non-perturbative approaches

Field representations

- IR divergences are present if we use the straightforward Cartesian representation

$$\phi(\mathbf{x}) = \sqrt{\rho_0} + \sigma(\mathbf{x}) + i\pi(\mathbf{x})$$

ρ_0: minimum of the action

- Divergences can be avoided by using a convenient field representation
- Similar issues arise when using a linear sigma model to describe broken chiral symmetry.
- These can be solved by using a non-linear sigma model as in chiral-perturbation theory

Field representations

- Bose gases in the IR are described by the hydrodynamics effective theory introduced by Popov

Popov, *Functional Integrals and Collective Excitations* (1987)

- Popov introduced an Amplitude-phase (AP) representation

$$\phi(\mathbf{x}) = (\sqrt{\rho_0} + \delta\rho(\mathbf{x}))e^{i\theta(\mathbf{x})}$$

ρ_0: minimum of the action

- IR divergences not present in correlator
- Cartesian representation should be used in the UV

$$\phi(\mathbf{x}) = \phi_<(\mathbf{x}) + \phi_>(\mathbf{x})$$

- Hydrodynamic theory widely used in modern calculations (QMC, Beliaev technique, etc)

Field representations

Long-distance behaviour of correlation function:

$$\lim_{|\mathbf{x}|\to\infty} G_n(\mathbf{x}) = \begin{cases} \rho_0 & : \textbf{(Cart.)} \qquad \rho_c = \rho_0, \\ \rho_0 e^{\langle(\vartheta(\mathbf{x})-\vartheta(0))^2\rangle} & : \textbf{(AP)} \qquad \rho_q = \rho_0, \end{cases}$$

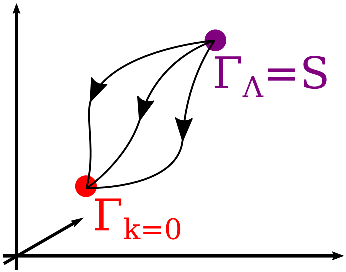
- With AP representation long-distance behaviour driven by phase correlations
- ρ_q is the quasi-condensate density
- In systems with QLRO $\rho_c = 0$ but $\rho_q > 0$

Functional Renormalisation Group (FRG)

- Properties of the system extracted from partition function $Z = \int \mathcal{D}\boldsymbol{\Phi} e^{S[\boldsymbol{\Phi}]}$
- Within the FRG the effective action Γ is obtained by solving a RG equation
- The FRG has been applied with reasonable success to high-energy physics, condensed matter and statistical physics
- Some application on nuclear matter

  Drews *et al* Phys. Rev. C **91**, 035802 (2015), Pósfay *et al* Phys. Rev. C **97**, 025803 (2018)

- A regulator function R_k is added to the theory which suppresses all fluctuations for momenta $q < k$: k-dependent action Γ_k

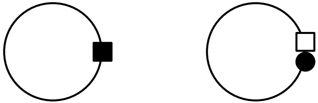


Functional Renormalisation Group (FRG)

Flow equation (Wetterich equation):

$$\partial_k \Gamma + \dot{\mathbf{\Phi}} \cdot \frac{\delta \Gamma}{\delta \mathbf{\Phi}} = \frac{1}{2} \operatorname{tr} \left[\partial_k \mathbf{R} (\mathbf{\Gamma}_k^{(2)} - \mathbf{R})^{-1} \right] + \operatorname{tr} \left[\dot{\mathbf{\Phi}}^{(1)} \mathbf{R} (\mathbf{\Gamma}_k^{(2)} - \mathbf{R})^{-1} \right]$$

$\mathbf{\Phi} = (\phi, \phi^\dagger)$, $\dot{\mathbf{\Phi}} = \partial_k \mathbf{\Phi}$, $\mathbf{\Gamma}^{(2)} = \delta^2_{\mathbf{\Phi},\mathbf{\Phi}} \Gamma$, $\dot{\mathbf{\Phi}}^{(1)} = \delta_{\mathbf{\Phi}} \dot{\mathbf{\Phi}}$



- This equation is exact. FRG is a non-perturbative framework
  (FRG also known as Exact RG and Non-perturbative RG)
- In general it cannot be solved and approximations need to be made
- A truncated ansatz for Γ is proposed

Functional Renormalisation Group (FRG)

- Bose gases have been widely studied within the FRG using the Cartesian representation
- The extreme IR regime cannot be accessed due to truncation of the action
- FRG has been successful describing three-dimensional Bose gases
- Not as successful in two dimensions. Superfluid phase is not recovered
- The truncation of the action result in incorrect β-functions

Effective action

We use the following ansatz:

$$\Gamma[\mathbf{\Phi}] = \int_x \left[\phi^\dagger \left(-Z_\phi \partial_\tau + \frac{Z_m}{2m}\nabla^2\right)\phi + \frac{Y_m}{8m}\rho\nabla^2\rho - U(\rho,\mu)\right]$$

U is the effective potential dependent on $\rho = \phi^\dagger\phi$ and μ. It is expanded as

$$U = u_0 - n_0(\mu-\mu_0) + (u_1 - n_1(\mu-\mu_0))(\rho-\rho_0) + \frac{1}{2}(u_2 - n_2(\mu-\mu_0))(\rho-\rho_0)^2$$

- Z_ϕ, Z_m, Y_m, u_i and n_i depend on k.
- $\rho_0 = \langle\rho\rangle$ is the k-dependent minimum of U, and μ_0 is the k-independent physical chemical potential
- Initial conditions of the flow completely defined by scattering length a, μ_0 and T

Effective action

$$\Gamma[\mathbf{\Phi}] = \int_x \left[\phi^\dagger \left(-Z_\phi \partial_\tau + \frac{Z_m}{2m} \nabla^2 \right) \phi + \frac{Y_m}{8m} \rho \nabla^2 \rho - U(\rho, \mu) \right]$$

$$U = u_0 - n_0(\mu - \mu_0) + (u_1 - n_1(\mu - \mu_0))(\rho - \rho_0) + \frac{1}{2}(u_2 - n_2(\mu - \mu_0))(\rho - \rho_0)^2$$

- Broken phase: $\rho_0 > 0, u_1 = 0$
  Symmetric phase: $\rho_0 = 0, u_1 > 0$
- The flow starts in the broken phase
- If $\rho_0 > 0$ at $k = 0$ the system is in its superfluid phase, otherwise is its normal phase
- $\rho_s = Z_m \rho_0$ is the k-dependent superfluid density.

Thermodynamics

$U(\rho_0, \mu_0)$ corresponds to the density of the grand canonical potential Ω_G

$$d\Omega_G = -PdV - SdT - Nd\mu$$

Thus

$$n_0 = -\frac{\partial U}{\partial \mu}\bigg|_{\rho_0,\mu_0}, \qquad s = -\frac{\partial U}{\partial T}\bigg|_{\rho_0,\mu_0},$$

are the k-dependent boson density and entropy density, respectively.
We can easily extract the thermodynamics properties of the system.

$$E/N = -P/n_0 + \mu_0 + s\,T/n_0$$

Interpolating representation

- Following Popov's ideas we use k-dependent fields (Lamprecht 2007):

$$\phi = (\sigma + b_k)e^{i\vartheta/b_k} - (b_k - \sqrt{\rho_0}), \quad b_k \in [\sqrt{\rho_0}, \infty)$$

- The field representation changes smoothly with k
- In the limits ϕ take the forms:

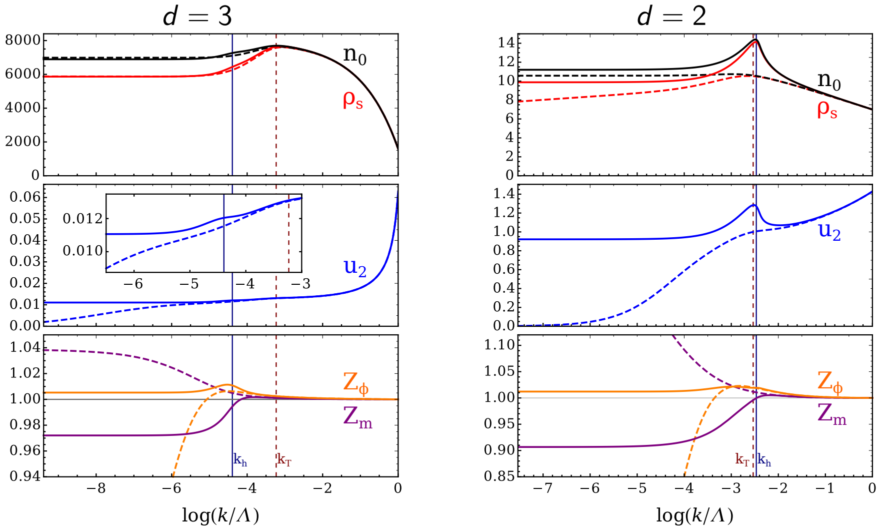
$$\phi = \begin{cases} (\sqrt{\rho_0} + \sigma) + i\vartheta & : b_k \to \infty \quad \textbf{(Cartesian)}, \\ (\sqrt{\rho_0} + \sigma)e^{i\vartheta/\sqrt{\rho_0}} & : b_k = \sqrt{\rho_0} \quad \textbf{(AP)}. \end{cases}$$

- UV regime: fluctuations are Gaussian
  IR regime: system dominated by Goldstone (phase) fluctuations

$$b_k = \sqrt{\rho_0}\left[1 + \left(\frac{Z_\sigma k^2/2m}{2u_2\rho_0}\right)^\nu\right]$$

$$Z_\sigma = Z_m + Y_m\rho_0$$

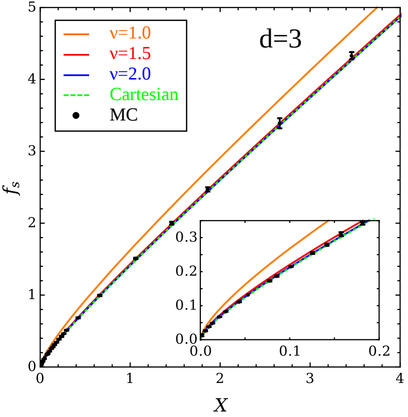
Results: Flows $0 < T < T_c$



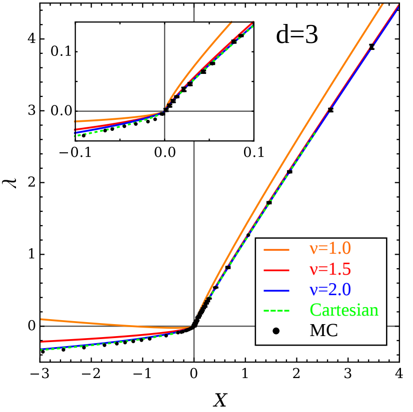
Dashed: Cartesian representation, **Solid:** Interpolating representation

Results phase transition region ($d = 3$)

$$f_s = \frac{\rho_s}{m^3 T^2 u_{2,\Lambda}}$$



$$\lambda = \frac{n_0 - n_c}{m^3 T^2 u_{2,\Lambda}^2}$$

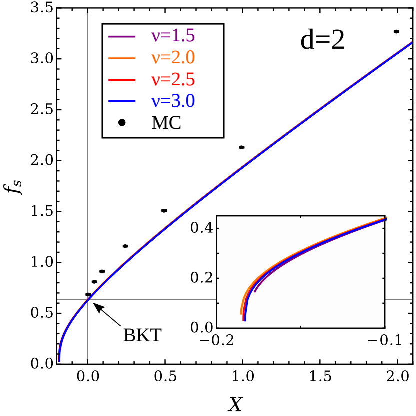


MC: Prokof'ev *et al.*, Phys. Rev. A **69**, 053625 (2004)

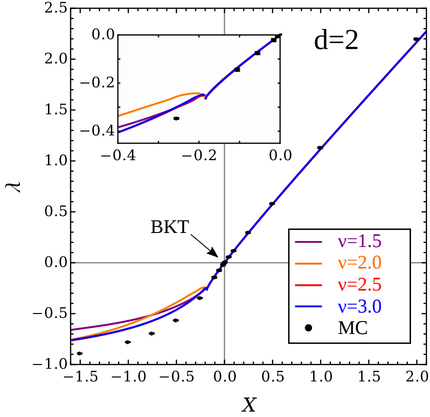
$$X = \frac{\mu_0 - \mu_c}{m^3 T^2 u_{2,\Lambda}^2}$$

Results phase transition region ($d = 2$)

$$f_s = \rho_s/(m^2 T^2)^{\frac{1}{2}}$$



$$\lambda = (n_0 - n_c)/(m^2 T^2 u_{2,\Lambda}^2)^{\frac{1}{2}}$$



MC: Prokof'ev *et al.*, Phys. Rev. A **66**, 043608 (2002)

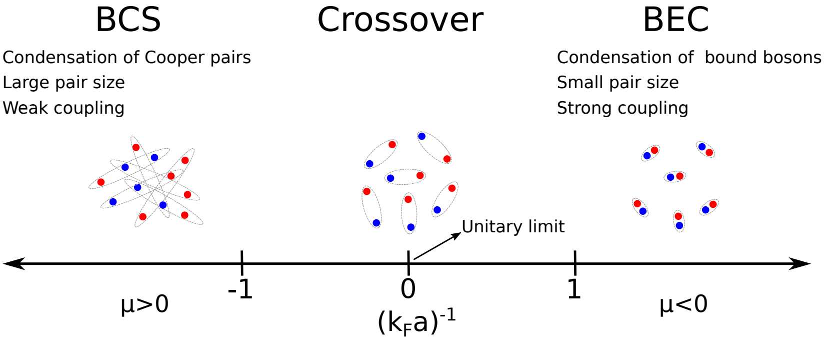
$$X = (\mu_0 - \mu_c)/(m^2 T^2 u_{2,\Lambda}^2)^{\frac{1}{2}}$$

Conclusions

- The FRG is a powerful yet simple non-perturbative formalism to study dilute Bose gases
- However, truncations of the effective action in systems with a broken symmetry can result in incorrect flows in the IR
- The interpolating representation allow us to correctly treat the Gaussian fluctuations in the UV, and the Goldstone (phase) fluctuations in the IR, recovering Popov's hydrodynamic effective action
- We obtain a stable superfluid phase in low dimensions
- Vortex effects need to be explicitly included by considering the periodicity of the phase fields in order to describe the BKT transition

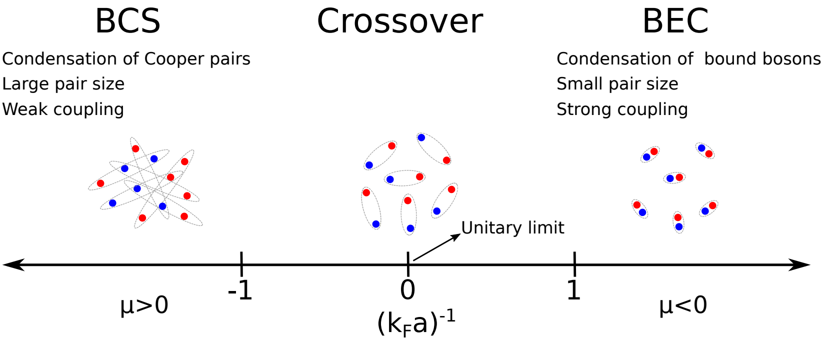
More details on Isaule *et al.*, Phys. Rev. B. **98**, 144502 (2018)

Future work: Fermi gases and the BCS-BEC crossover



- It can be studied through cold atom experiments
- Of interest in nuclear physics:
  low-density neutron matter, neutron-rich nuclei, etc
  Zinner *et al.*, J. Phys. G: Nucl. Part. Phys. **40**, 053101 (2013)
- BCS-BEC crossover also present in one and two dimensions:
  High-temperature superconductors, nuclear pastas, etc
  Turlapov *et al.*, J. Phys.: Condens. Matter **29**, 383004 (2017)

Future work: Fermi gases and the BCS-BEC crossover



- The FRG can be used to study the BCS-BEC crossover
- Our approach used with the Bose gas will be extended to the study of the Fermi gas
- Systems with spin-imbalance and more than one species of fermions can be studied

Application of the functional renormalisation group to Bose gases

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Weakly-interacting Bose gases

- **One dimension**: Long-distance limit $|\mathbf{x}| \to \infty$

$$G_n(\mathbf{x}) \propto \begin{cases} |X|^{-\eta} & : T = 0, \\ e^{-|X|/\xi} & : T > 0, \end{cases}$$

  - System shows quasi-long-range-order (QLRO) at $T = 0$

Initial conditions

At a UV scale Λ, $\Gamma_\Lambda = S$ and thus:

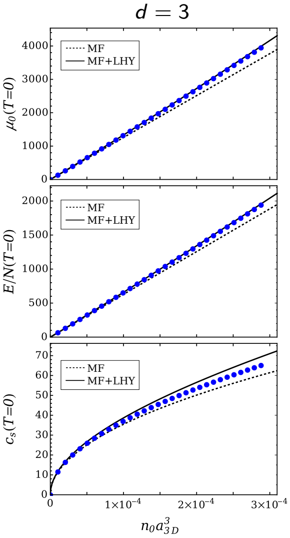
$$\rho_{0,\Lambda} = n_{0,\Lambda} = \frac{\mu_0}{u_{2,\Lambda}}\Theta(\mu_0), \quad u_{1,\Lambda} = -\mu_0\Theta(-\mu_0),$$
$$Z_{m,\Lambda} = Z_{\phi,\Lambda} = 1, \quad Y_{m,\Lambda} = 0, \quad n_{1,\Lambda} = 1, \quad n_{2,\Lambda} = 0, \quad s_\Lambda = 0.$$

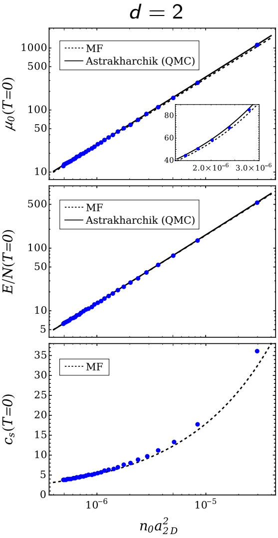
Interaction term u_2 needs to be renormalised. In vacuum ($T = 0$, $\mu_0 \leq 0$):

$$u_2(k=0) = \begin{cases} \frac{4\pi a_{3D}}{m}, & : d = 3, \\ \frac{4\pi/m}{\log(2/|\mu| m a_{2D}^2) - 2\gamma_E} & : d = 2, \end{cases}$$

The only inputs are a_d, μ_0 and T.

Results zero temperature

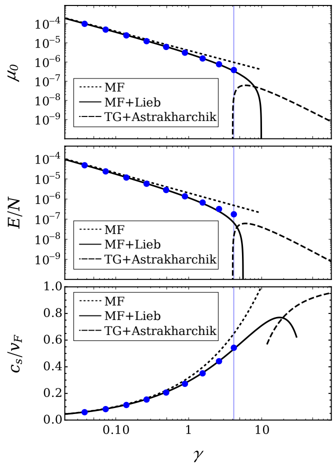




LHY: Lee *et al.*, Phys. Rev. **106**, 1135 (1957)

QMC: Astrakharchik *et al.*, Phys. Rev. A **79**, 051602 (2009)

Results zero temperature



$d = 1$

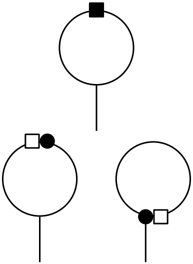
$\gamma = -\frac{2}{n_0 a_{1D}}$

Lieb *et al.*,Phys. Rev. **130**, 1605 (1963)
Astrakharchik *et al.*,Phys. Rev. Lett. **95**, 190407 (2005)

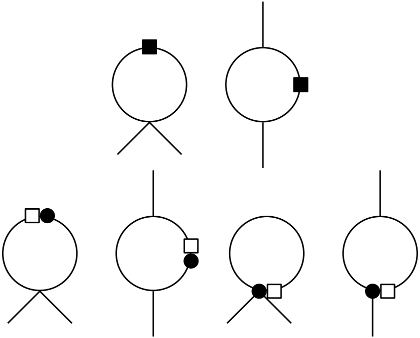
Flow equations Bose gas

$$
\begin{aligned}
2u_2\sqrt{\rho_0}\dot{\rho}_0 &= \dot{\Gamma}^{(1)}_{\sigma}\Big|_{\rho_0,\mu_0}, \\
-4\rho_0\dot{u}_2 + 2u_2\dot{\rho}_0 &= \dot{\Gamma}^{(2)}_{\sigma\sigma}\Big|_{\rho_0,\mu_0}, \\
\dot{n}_0 - n_1\dot{\rho}_0 &= \partial_\mu\dot{\Gamma}\Big|_{\rho_0,\mu_0}, \\
2\sqrt{\rho_0}\dot{n}_1 - 2n_2\sqrt{\rho_0}\dot{\rho}_0 &= \partial_\mu\left(\dot{\Gamma}^{(1)}_{\sigma}\right)\Big|_{\rho_0,\mu_0}, \\
4\rho_0\dot{n}_2 + 2\dot{n}_1 - 2n_2\dot{\rho}_0 &= \partial_\mu\left(\dot{\Gamma}^{(2)}_{\sigma\sigma}\right)\Big|_{\rho_0,\mu_0}, \\
2\dot{Z}_\phi &= \partial_{p_0}\left(\partial_k\Gamma^{(2)}_{\sigma\vartheta}\right)\Big|_{\phi_0,\mu_0,p=0}, \\
-\frac{\dot{Z}_\vartheta}{m} &= \partial_{\mathbf{p}^2}\left(\dot{\Gamma}^{(2)}_{\vartheta\vartheta}\right)\Big|_{\rho_0,\mu_0,p=0}, \\
-\frac{\rho_0\dot{Y}_m}{m} - \frac{\dot{Z}_\vartheta}{m} &= \partial_{\mathbf{p}^2}\left(\dot{\Gamma}^{(2)}_{\sigma\sigma}\right)\Big|_{\rho_0,\mu_0,p=0},
\end{aligned}
$$

$\dot{\Gamma}^{(1)}$



$\dot{\Gamma}^{(2)}$



The BCS-BEC crossover within the FRG

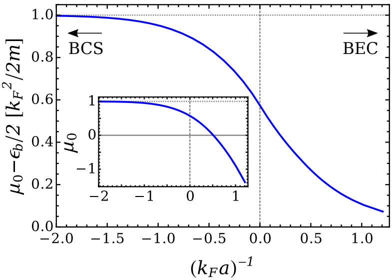
We consider a fermionic system with one species of fermions:

$$\Gamma[\boldsymbol{\Phi}] = \int_x \Bigg[\sum_{\sigma=1,2} \psi_\sigma^\dagger \left(-Z_\psi \partial_\tau + \frac{Z_M}{2m} \nabla^2 + \Sigma_\psi \right) \psi_\sigma + \phi^\dagger \left(-Z_\phi \partial_\tau + \frac{Z_m}{4m} \nabla^2 \right) \phi + \frac{Y_m}{16m} \rho \nabla^2 \rho - g \left(\phi^\dagger \psi_1 \psi_2 + \phi \psi_2^\dagger \psi_1^\dagger \right) - U(\rho, \mu) \Bigg].$$

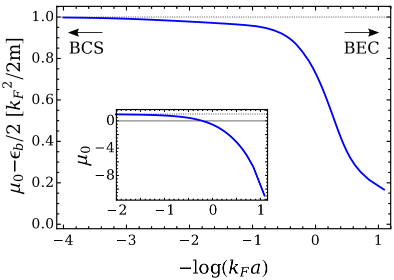
- ψ_σ are fermion fields and ϕ bosons fields that represent pairs of fermions
- The k-dependent pairing gap is given by $\Delta = g\sqrt{\rho_0}$
- The only inputs are the s-wave scattering length a, μ_0 and T

Preliminary results

d=3

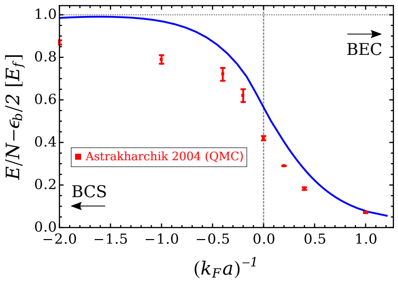


d=2



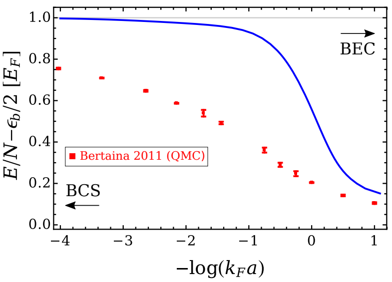
Preliminary results

d=3



Astrakharchik *et al.*, Phys. Rev. Lett. **93**, 200404 (2004)

d=2



Bertaina *et al.*, Phys. Rev. Lett. **106**, 110403 (2011)