Functional renormalisation group for Bose gases: From linear to hydrodynamic fluctuations

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Overview

1. Weakly-interacting Bose gases
2. Functional renormalisation group (FRG) and interpolating representation
3. Results
4. Conclusions

Weakly-interacting Bose gases

- Long standing theoretical interest in describing Bose-Einstein condensation (BEC) and superfluidity (Bogoliubov 1947)
- Interest greatly increased since the experimental realisation of BEC in cold atom gases

  Anderson *et al.* Science **269**, 198 (1995), Davis *et al.* Phys. Rev. Lett **75**, 3969 (1995)

- Behaviour of the system depends sensitively on the dimensionality

Weakly-interacting Bose gases

- Bare action of the system:

$$\mathcal{S}[\mathbf{\Phi}] = \int_x \left[\phi^\dagger \left(-\partial_\tau + \frac{\nabla^2}{2m} + \mu\right)\phi - \frac{g}{2}(\phi^\dagger\phi)^2\right]$$

g: repulsive contact interaction

- Interaction related to the two-body T-matrix:

$$T^{2B} = \begin{cases} \frac{4\pi a_{3D}}{m} & : d = 3 \\ \frac{4\pi/m}{\log(2/|\mu| a_{2D}^2) - 2\gamma_E} & : d = 2 \end{cases}$$

a: s-wave scattering length

Weakly-interacting Bose gases

Correlation function: $G_n(\mathbf{x}) = \langle \phi^\dagger(\mathbf{x})\phi(0) \rangle$ Long-distance limit: $|\mathbf{x}| \to \infty$

Three dimensions

$T < T_c$: $G_n(\mathbf{x}) \to \rho_c > 0$

- Long-range-order (LRO)
- $U(1)$ symmetry is broken, with ρ_c as order parameter

Weakly-interacting Bose gases

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Three dimensions

$$T < T_c: \quad G_n(\mathbf{x}) \to \quad \rho_c > 0$$

- Long-range-order (LRO)
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Two dimensions

$$T = 0: \quad G_n(\mathbf{x}) \to \quad \rho_c > 0$$

$$0 < T \leq T_c: \quad G_n(\mathbf{x}) \propto \quad |X|^{-\eta}$$

- Condensation only possible at $T = 0$ (Mermin-Wagner theorem)
- Quasi-long-range-order (QLRO) for $0 < T \leq T_c$ ($\rho_c = 0$, $\rho_s > 0$)

  ρ_s: superfluid density
- Phase transition driven by the unbinding of vortex pairs (BKT)

Weakly-interacting Bose gases

- Mean-field theory gives a reasonable qualitative description of Bose gases at low temperatures
- Perturbation theory plagued by IR divergences due to ungapped propagator of Goldstone mode

$$G_{||} = \frac{1}{\mathbf{q}^2 + q_c^2}, \qquad G_{\perp} = \frac{1}{\mathbf{q}^2}$$

- Divergences cancel, but cancellations are lost if expansions are truncated

Field representations

- IR divergences are present if we use the Cartesian representation

$$\phi(\mathbf{x}) = \sqrt{\rho_0} + \sigma(\mathbf{x}) + i\pi(\mathbf{x})$$

ρ_0: minimum of the action

- These can be avoided by using a convenient field representation
- Similar divergences arise when using a linear sigma model to describe broken chiral symmetry
- These can be solved by using a non-linear sigma model as in chiral-perturbation theory

Field representations

- Bose gases in the IR are described by the hydrodynamic effective theory introduced by Popov

Popov, *Functional Integrals and Collective Excitations* (1987)

- Popov introduced an Amplitude-Phase (AP) representation:

$$\phi(\mathbf{x}) = (\sqrt{\rho_0} + \delta\rho(\mathbf{x}))e^{i\theta(\mathbf{x})}$$

- Cartesian representation should be used in the UV, whereas AP representation in the IR
- Hydrodynamic theory widely used in modern calculations (QMC, Beliaev technique, etc)

Field representations

Long-distance behaviour of correlation function:

$$\lim_{|\mathbf{x}|\to\infty} G_n(\mathbf{x}) = \begin{cases} \rho_0 & : \textbf{(Cart.)} \qquad \rho_c = \rho_0, \\ \rho_0 e^{\langle(\vartheta(\mathbf{x})-\vartheta(0))^2\rangle} & : \textbf{(AP)} \qquad \rho_q = \rho_0, \end{cases}$$

- With AP representation long-distance behaviour driven by phase correlations
- ρ_q is the quasi-condensate density

  Al Khawaja *et al*, PRA **66**, 013615 (2002)
- In systems with QLRO $\rho_c = 0$ but $\rho_q > 0$

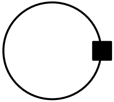
Functional Renormalisation Group (FRG)

- Bose gases have been widely studied within the FRG

Floerchinger *et al.* PRA (2009), Dupuis PRA (2009), Sinner *et al.* PRA (2010), Rançon *et al.* PRA (2012)

Flow equation

$$\partial_k \Gamma = \frac{1}{2} \,\mathrm{tr} \left[\partial_k \mathbf{R} (\mathbf{\Gamma}_k^{(2)} - \mathbf{R})^{-1} \right]$$



Wetterich, Phys. Lett. B (1993)

- As a non-perturbative method the divergences of perturbation theory are not present
- FRG has been successful describing three-dimensional Bose gases
- Less successful in two dimensions. Regulator needs to be fine-tuned to recover finite stiffness

Jakubczyk *et al*, PRE (2014); PRB (2017)

Functional Renormalisation Group (FRG)

- Can the AP representation be implemented within the FRG?
- It has been shown that for a $O(2)$-model in two dimensions it leads to stable solutions at the lowest order of the derivative expansion

  Defenu *et al* PRB **96**, 174505 (2017)

- AP representation not applicable in the UV

Interpolating representation

- Following Popov's ideas we use k-dependent fields (Lamprecht 2007):

$$\phi = (\sigma + b_k)e^{i\vartheta/b_k} - (b_k - \sqrt{\rho_0}), \quad b_k \in [\sqrt{\rho_0}, \infty)$$

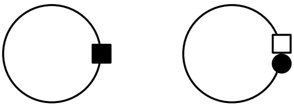
Lamprecht, Diploma thesis, Ruprecht-Karls-Universität Heidelberg (2007)

- In the limits ϕ take the forms:

$$\phi = \begin{cases} (\sqrt{\rho_0} + \sigma) + i\vartheta & : b_k \to \infty \quad \textbf{(Cartesian)}, \\ (\sqrt{\rho_0} + \sigma)e^{i\vartheta/\sqrt{\rho_0}} & : b_k = \sqrt{\rho_0} \quad \textbf{(AP)}. \end{cases}$$

Flow equation for k-dependent fields

$$\partial_k \Gamma + \dot{\mathbf{\Phi}} \cdot \frac{\delta \Gamma}{\delta \mathbf{\Phi}} = \frac{1}{2} \operatorname{tr} \left[\partial_k \mathbf{R} (\mathbf{\Gamma}_k^{(2)} - \mathbf{R})^{-1} \right] + \operatorname{tr} \left[\dot{\mathbf{\Phi}}^{(1)} \mathbf{R} (\mathbf{\Gamma}_k^{(2)} - \mathbf{R})^{-1} \right]$$



Pawlowski, Annals of Physics **322**, 2831 (2007)

$\mathbf{\Phi} = (\phi, \phi^\dagger)$, $\dot{\mathbf{\Phi}} = \partial_k \mathbf{\Phi}$, $\mathbf{\Gamma}^{(2)} = \delta^2_{\mathbf{\Phi},\mathbf{\Phi}} \Gamma$, $\dot{\mathbf{\Phi}}^{(1)} = \delta_{\mathbf{\Phi}} \dot{\mathbf{\Phi}}$

Interpolating representation

Ansatz: $$\Gamma[\boldsymbol{\Phi}] = \int_x \left[\phi^\dagger\left(-Z_\phi\partial_\tau + \frac{Z_m}{2m}\nabla^2\right)\phi + \frac{Y_m}{8m}\rho\nabla^2\rho - U(\rho,\mu)\right]$$

Regimes can be characterised by: $w_k = \dfrac{Z_\sigma k^2/2m}{2u_2\rho_0}$ $\qquad Z_\sigma = Z_m + Y_m\rho_0$

- UV regime ($w_k \gg 1$): fluctuations are Gaussian
- IR regime ($w_k \ll 1$): Goldstone (phase) fluctuations dominate

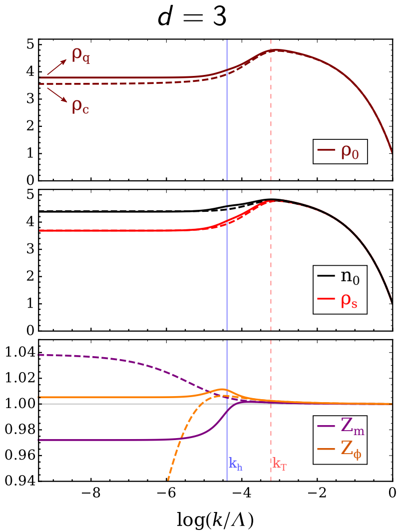
$\phi = (\sigma + b_k)e^{i\vartheta/b_k} - (b_k - \sqrt{\rho_0})$:

$$b_k = \sqrt{\rho_0}\,[1 + (\alpha w_k)^\nu]$$

Transition should be made around the healing scale k_h ($w_{k_h} = 1$)

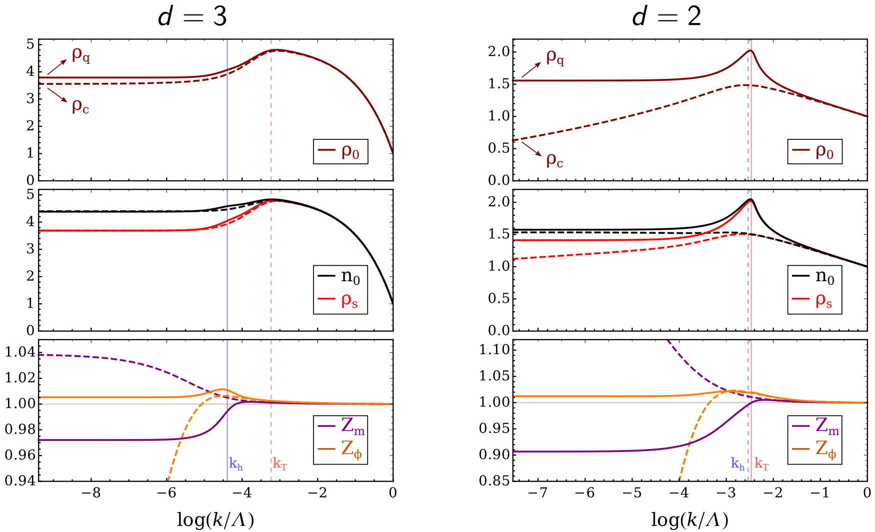
Capogrosso-Sansone *et al*, New J. Phys. **12**, 043010 (2010)

Results: Flows $0 < T < T_c$ ($\nu = 3$)



Dashed: Cartesian representation, **Solid:** Interpolating representation

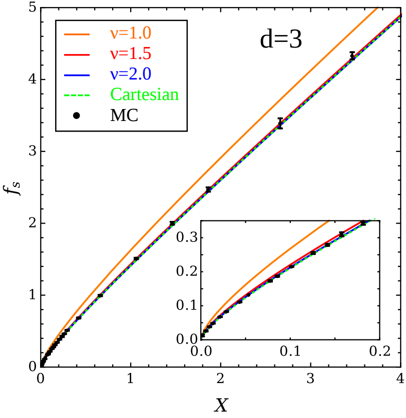
Results: Flows $0 < T < T_c$ $(\nu = 3)$



Dashed: Cartesian representation, **Solid:** Interpolating representation

Results phase transition region

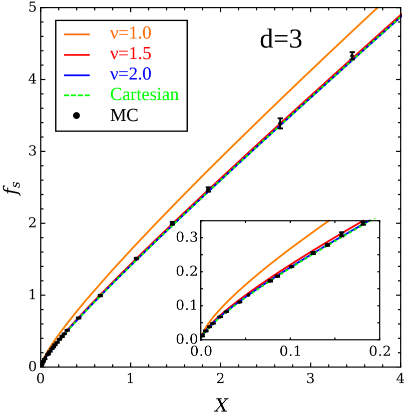
$$f_s = \rho_s/(m^3 T^2 u_{2,\Lambda})$$



MC: Prokof'ev *et al.*, Phys. Rev. A **69**, 053625 (2004)

$$X = \frac{\mu_0 - \mu_c}{m^3 T^2 u_{2,\Lambda}^2}$$

$$f_s = \rho_s/(m^3 T^2 u_{2,\Lambda})$$



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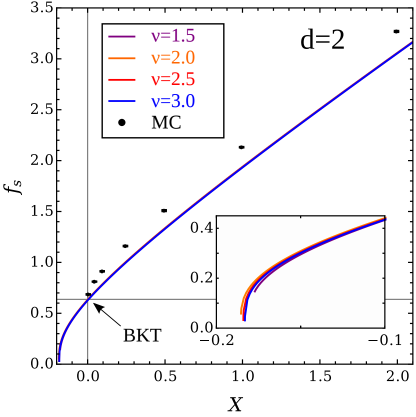
$$X = \frac{\mu_0 - \mu_c}{m^3 T^2 u_{2,\Lambda}^2}$$

$$f_s = \rho_s/(m^2 T^2)^{\frac{1}{2}}$$



MC: Prokof'ev *et al.*, Phys. Rev. A **66**, 043608 (2002)

$$X = (\mu_0 - \mu_c)/(m^2 T^2 u_{2,\Lambda}^2)^{\frac{1}{2}}$$

Conclusions

- The interpolating representation allow us to treat the Gaussian fluctuations in the UV with a Cartesian representation, and the Goldstone (phase) fluctuations in the IR with an AP representation
- We obtain a stable superfluid phase in low dimensions
- Vortex effects need to be explicitly included in two dimensions
- Interpolating representation can be implemented in the study of Fermi gases
- AP representation can be generalised to include periodic effects for strongly interacting one dimensional gases

Cazalilla *et al.*, Reviews of Modern Physics **83**, 1405 (2011)

More details in: F. Isaule, M. Birse and N. Walet, Phys. Rev. B. **98**, 144502 (2018)
F. Isaule, M. Birse and N. Walet, arXiv:1902.07135 (2019)

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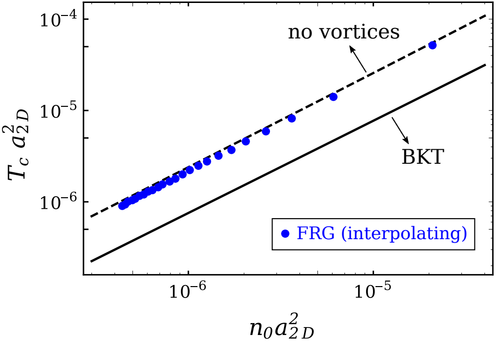
Critical temperature ($d = 2$)



No vortex effects

$$T_c = \frac{2\pi n_0}{m} \frac{1}{\log(\log(1/n_0 a_{2D}^2))}$$

Fisher *et al.*, PRB **37**, 4936 (1988)

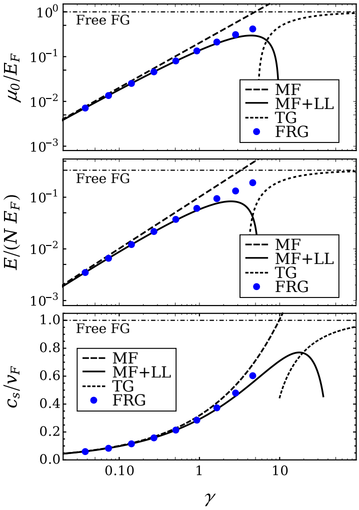
$$\longrightarrow$$

Including vortex effects

$$T_c = \frac{2\pi n_0}{m} \frac{1}{\log(\xi/4\pi) + \log(\log(1/n_0 a_{2D}^2))}$$

Prokof'ev *et al.* PRL **87**, 270402 (2001)

Results zero temperature ($d = 1$)

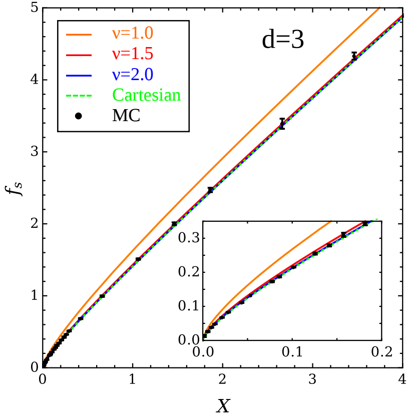


$$\gamma = -\frac{2}{n_0 a_{1D}}$$

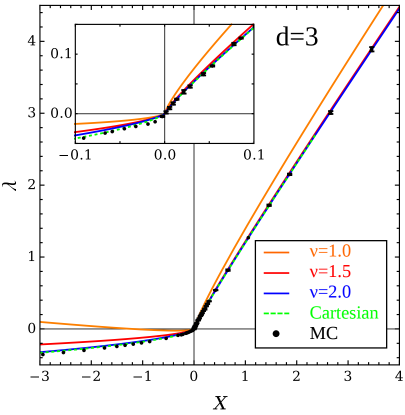
For more on thermodynamics of 1D Bose Gase see De Rosi *et al.*, PRA **96**, 013613 (2017)

Results phase transition region ($d = 3$)

$$f_s = \frac{\rho_s}{m^3 T^2 u_{2,\Lambda}}$$



$$\lambda = \frac{n_0 - n_c}{m^3 T^2 u_{2,\Lambda}^2}$$

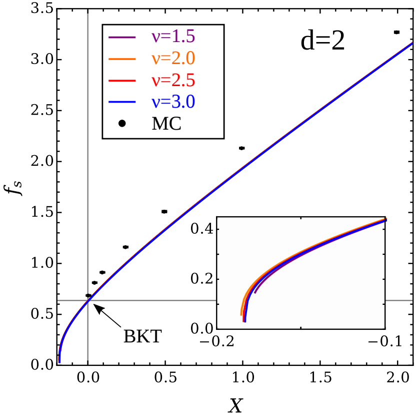


MC: Prokof'ev *et al.*, Phys. Rev. A **69**, 053625 (2004)

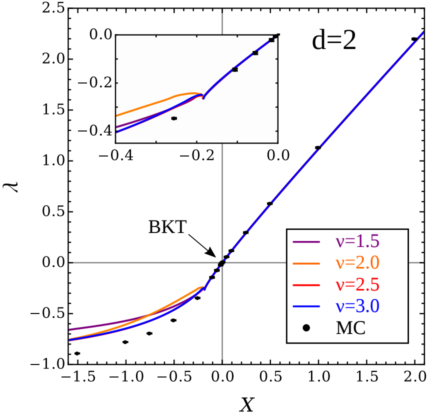
$$X = \frac{\mu_0 - \mu_c}{m^3 T^2 u_{2,\Lambda}^2}$$

Results phase transition region ($d = 2$)

$f_s = \rho_s/(m^2 T^2)^{\frac{1}{2}}$



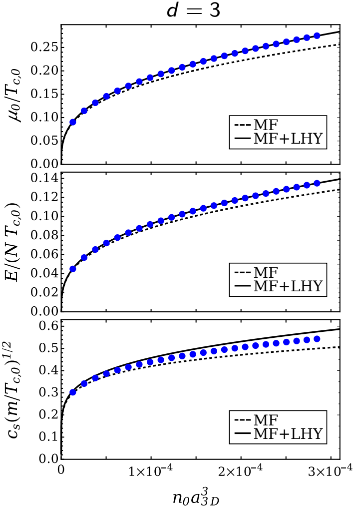
$\lambda = (n_0 - n_c)/(m^2 T^2 u_{2,\Lambda}^2)^{\frac{1}{2}}$

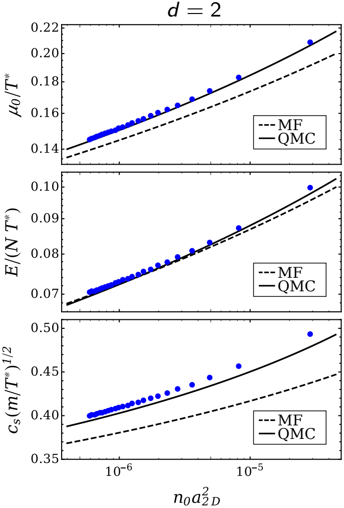


MC: Prokof'ev *et al.*, Phys. Rev. A **66**, 043608 (2002)

$$X = (\mu_0 - \mu_c)/(m^2 T^2 u_{2,\Lambda}^2)^{\frac{1}{2}}$$

Results zero temperature





LHY: Lee *et al.*, Phys. Rev. **106**, 1135 (1957)

QMC: Astrakharchik *et al.*, Phys. Rev. A **79**, 051602 (2009)

Initial conditions

At a UV scale Λ, $\Gamma_\Lambda = S$ and thus:

$$\rho_{0,\Lambda} = n_{0,\Lambda} = \frac{\mu_0}{u_{2,\Lambda}}\Theta(\mu_0), \quad u_{1,\Lambda} = -\mu_0\Theta(-\mu_0),$$

$$Z_{m,\Lambda} = Z_{\phi,\Lambda} = 1, \quad Y_{m,\Lambda} = 0, \quad n_{1,\Lambda} = 1, \quad n_{2,\Lambda} = 0, \quad s_\Lambda = 0.$$

Interaction term u_2 needs to be renormalised. In vacuum ($T = 0$, $\mu_0 \leq 0$):

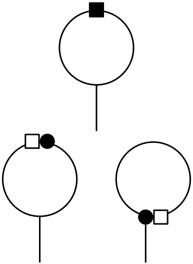
$$u_2(k=0) = \begin{cases} \frac{4\pi a_{3D}}{m}, & : d = 3, \\ \frac{4\pi/m}{\log(2/|\mu| m a_{2D}^2) - 2\gamma_E} & : d = 2, \end{cases}$$

The only inputs are a_d, μ_0 and T.

Flow equations Bose gas

$$
\begin{aligned}
2u_2\sqrt{\rho_0}\dot{\rho}_0 &= \dot{\Gamma}^{(1)}_{\sigma}\Big|_{\rho_0,\mu_0}, \\
-4\rho_0\dot{u}_2 + 2u_2\dot{\rho}_0 &= \dot{\Gamma}^{(2)}_{\sigma\sigma}\Big|_{\rho_0,\mu_0}, \\
\dot{n}_0 - n_1\dot{\rho}_0 &= \partial_\mu\dot{\Gamma}\Big|_{\rho_0,\mu_0}, \\
2\sqrt{\rho_0}\dot{n}_1 - 2n_2\sqrt{\rho_0}\dot{\rho}_0 &= \partial_\mu\left(\dot{\Gamma}^{(1)}_{\sigma}\right)\Big|_{\rho_0,\mu_0}, \\
4\rho_0\dot{n}_2 + 2\dot{n}_1 - 2n_2\dot{\rho}_0 &= \partial_\mu\left(\dot{\Gamma}^{(2)}_{\sigma\sigma}\right)\Big|_{\rho_0,\mu_0}, \\
2\dot{Z}_\phi &= \partial_{p_0}\left(\partial_k\Gamma^{(2)}_{\sigma\vartheta}\right)\Big|_{\phi_0,\mu_0,p=0}, \\
-\frac{\dot{Z}_\vartheta}{m} &= \partial_{\mathbf{p}^2}\left(\dot{\Gamma}^{(2)}_{\vartheta\vartheta}\right)\Big|_{\rho_0,\mu_0,p=0}, \\
-\frac{\rho_0\dot{Y}_m}{m} - \frac{\dot{Z}_\vartheta}{m} &= \partial_{\mathbf{p}^2}\left(\dot{\Gamma}^{(2)}_{\sigma\sigma}\right)\Big|_{\rho_0,\mu_0,p=0},
\end{aligned}
$$

$\dot{\Gamma}^{(1)}$



$\dot{\Gamma}^{(2)}$