Cold quantum gases from functional Renormalisation

Felipe Isaule

University of Manchester

felipe.isaule@postgrad.manchester.ac.uk

Collaborators: Michael C. Birse, Niels R. Walet

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1 Weakly-interacting Bose gases

2 Functional renormalisation group

3 Fermi gases and the BCS-BEC crossover

4 Conclusions

Weakly-interacting Bose gases

- Dilute gas of bosons interacting through weak repulsive short-range interactions
- Long standing theoretical interest for describing Bose-Einstein condensation (BEC) and superfluidity (Bogoliubov 1947)
- Interest greatly increased since the experimental realisation of BEC in cold atom gases

  M.H. Anderson *et al.* Science **269**, 198 (1995), K. B. Davis *et al.* PRL **75**, 3969 (1995)

- Behaviour of the system depends sensitively on the dimensionality

- Bare action of the system:

$$\mathcal{S} = \int_x \left[\phi^\dagger \left(-\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) \phi - \frac{g}{2} (\phi^\dagger \phi)^2 \right]$$

g: weak repulsive contact interaction

- Interaction related to the two-body T-matrix:

$$T^{2B} = \begin{cases} \frac{4\pi a_{3D}}{m} & : d = 3 \\ \frac{4\pi/m}{\log(2/|\mu| a_{2D}^2) - 2\gamma_E} & : d = 2 \end{cases}$$

a: s-wave scattering length

- Weak interaction:
  - $d = 3$: $n_0 a_{3D}^3 \ll 1$
  - $d = 2$: $1/\log(1/n_0 a_{2D}^2) \ll 1$

n_0: atom density

- Mean-field theory gives a qualitative description of Bose gases at low temperatures
- Fluctuations need to be considered
- Weakly-interacting Bose gases are now well described:
  MC simulations, Beliaev technique, RG approaches

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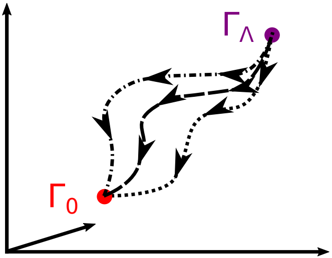
Functional Renormalisation Group (FRG)

- Properties of the system extracted from partition function $Z = \int \mathcal{D}\mathbf{\Phi} e^{S[\mathbf{\Phi}]}$
- Within the FRG the effective action Γ is obtained by solving a RG equation
- A regulator function R_k is added to the theory which suppresses all fluctuations for momenta $q < k$: k-dependent action Γ_k

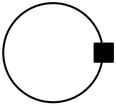


Flow equation (Wetterich equation)

$$\partial_k \Gamma = \frac{1}{2}\,\mathrm{tr}\left[\partial_k \mathbf{R}(\mathbf{\Gamma}_k^{(2)} - \mathbf{R})^{-1}\right]$$

C. Wetterich, Phys. Lett. B **301**, 90 (1993)

$$\partial_k \Gamma = \frac{1}{2}$$



- This equation is exact. FRG is a non-perturbative framework
- In general the flow equation cannot be solved and approximations need to be made: ansatz for Γ
- Encouraging results for cold quantum gases

  I. Boettcher *et al*, Nucl. Phys. B **228**, 63 (2015)
- Some application to nuclear matter,

  M. Drews and W. Weise, Prog. Part. Nucl. Phys. **93**, 69 (2016)

  and few-nucleon systems

  M. Birse *et al.*, PRC **87**, 054001 (2013)

We use the following ansatz based on a gradient expansion:

$$\Gamma = \int_x \left[\phi^\dagger \left(-Z_\phi \partial_\tau + \frac{Z_m}{2m}\nabla^2\right)\phi + \frac{Y_m}{8m}\rho\nabla^2\rho - U(\rho,\mu)\right]$$

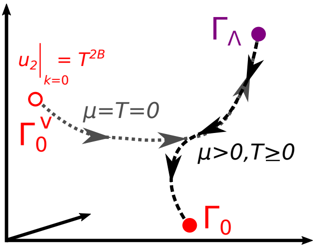
U is the effective potential dependent on $\rho = \phi^\dagger\phi$. It is expanded as

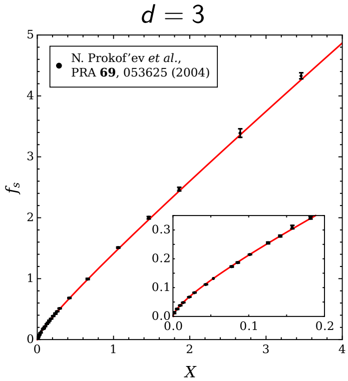
$$U = u_0 + u_1(\rho-\rho_0) + \tfrac{u_2}{2}(\rho-\rho_0)^2 \qquad \rho_0 = \langle\rho\rangle$$

- Z_ϕ, Z_m, Y_m, u_i and ρ_0 depend on k
- $\rho_s = Z_m\rho_0$ is the superfluid density
- $U(\rho_0)$ is the density of the grand canonical potential Ω_G

  $d\Omega_G = -PdV - SdT - Nd\mu$
- In the IR: $\phi(x) = (\sqrt{\rho_0} + \sigma(x))e^{i\theta(x)}$

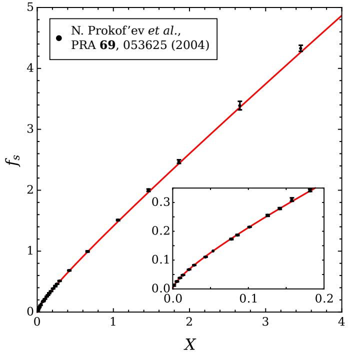
Popov, Functional Integrals and Collective Excitations (1987)





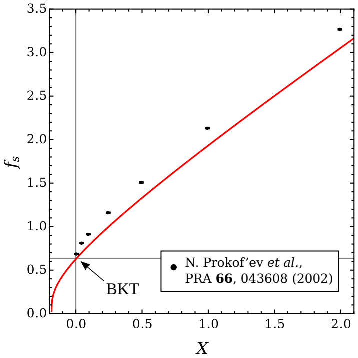
$$f_s = \frac{\rho_s}{m^3 T^2 u_{2,\Lambda}}, \qquad X = \frac{\mu - \mu_c}{m^3 T^2 u_{2,\Lambda}^2}$$

$d = 3$



$$f_s = \frac{\rho_s}{m^3 T^2 u_{2,\Lambda}}, \qquad X = \frac{\mu - \mu_c}{m^3 T^2 u_{2,\Lambda}^2}$$

$d = 2$



$$f_s = \frac{\rho_s}{m T}, \qquad X = \frac{\mu - \mu_c}{m\, T\, u_{2,\Lambda}}$$

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Fermi gases and the BCS-BEC crossover

- Dilute two-component Fermi gases interacting through attractive short-range interactions

$$\mathcal{S} = \int_x \left[\psi_s^\dagger \left(-\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) \psi_s - u_d \phi^\dagger \phi - g \left(\phi^\dagger \psi_1 \psi_2 + \phi \psi_2^\dagger \psi_1^\dagger \right) \right]$$

ψ_s: fermionic *atom* fields

ϕ: bosonic *dimer* fields $\sim \langle \psi_1 \psi_2 \rangle$

- They can be studied with cold-atom experiments

C. A. Regal *et al.*, PRL **92**, 040403, (2004). C. Chin, Science **305**, 1128 (2004)

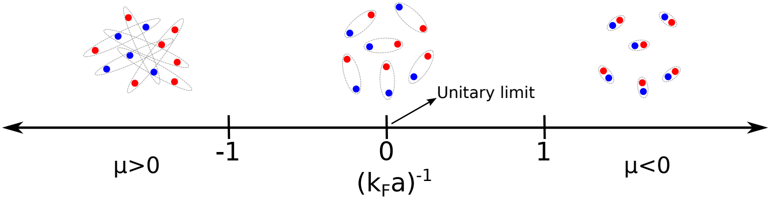
BCS

Condensation of Cooper pairs
Large pair size
Weak coupling

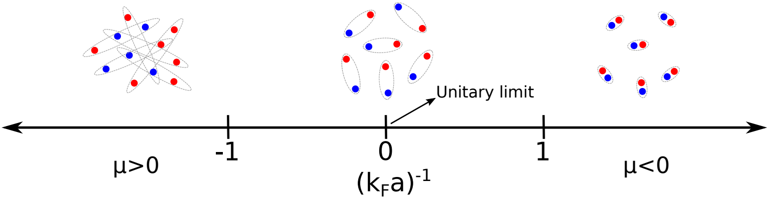
Crossover

BEC

Condensation of bound bosons
Small pair size
Strong coupling



BCS	Crossover	BEC
Condensation of Cooper pairs Large pair size Weak coupling		Condensation of bound bosons Small pair size Strong coupling



- BCS-BEC crossover regime is strongly-interacting
- Nuclear physics applications:
  low-density neutron matter, neutron-rich nuclei.
  Zinner et al., J. Phys. G: Nucl. Part. Phys. **40**, 053101 (2013)
- BCS-BEC crossover also present in one and two dimensions:
  High-temperature superconductors, nuclear pastas, etc

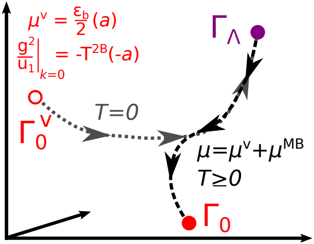
We use the following ansatz based on a gradient expansion:

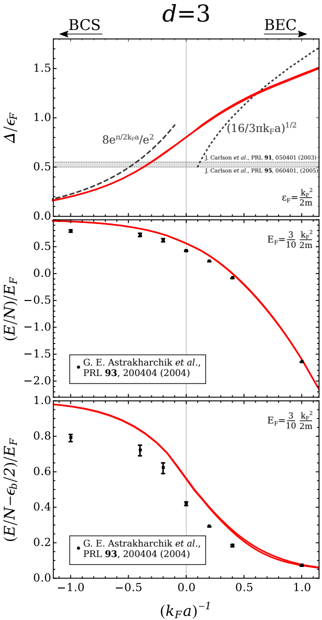
$$\Gamma = \int_x \left[\psi_s^\dagger \left(-Z_\psi \partial_\tau + \frac{Z_M}{2m}\nabla^2 + \Sigma_{\psi_s} \right) \psi_s + \phi^\dagger \left(-Z_\phi \partial_\tau + \frac{Z_m}{4m}\nabla^2 \right) \phi - g\left(\phi^\dagger \psi_1 \psi_2 + \phi \psi_2^\dagger \psi_1^\dagger \right) - U(\rho, \mu) \right]$$

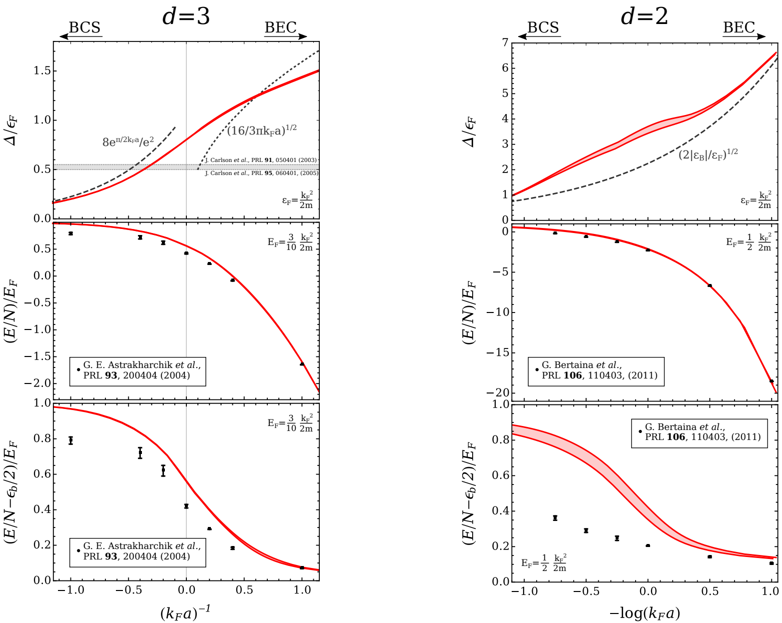
where $\rho = \phi^\dagger \phi$ and $U = u_0 + u_1(\rho - \rho_0) + \frac{u_2}{2}(\rho - \rho_0)^2$. $\rho_0 = \langle \rho \rangle$

- $\Delta = g\rho_0^{1/2}$ is the pairing gap
- $d = 3$: $\epsilon_b = \frac{-1}{ma_{3D}^2}\Theta(a_{3D})$
  $T^{2B} = \frac{4\pi a_{3D}}{m}\Theta(-a_{3D})$
- $d = 2$: $\epsilon_b = \frac{-4}{me^{2\gamma_E} a_{2D}^2}$
- In the IR: $\phi(x) = (\phi_0 + \sigma(x))e^{i\theta(x)}$

L. Salasnich *et al.*, PRA **88**, 053612 (2013)




d=3
BCS
BEC
Δ/ε_F
8e^{π/2k_F a}/e^2
(16/3πk_F a)^{1/2}
J. Carlson et al., PRL 91, 050401 (2003)
J. Carlson et al., PRL 95, 060401, (2005)
ε_F = k_F^2/2m
(E/N)/E_F
E_F = 3/10 k_F^2/2m
G. E. Astrakharchik et al., PRL 93, 200404 (2004)
(E/N−ε_b/2)/E_F
E_F = 3/10 k_F^2/2m
G. E. Astrakharchik et al., PRL 93, 200404 (2004)
(k_F a)^{-1}


d=3
BCS
BEC
$8e^{\pi/2k_Fa}/e^2$
$(16/3\pi k_F a)^{1/2}$
J. Carlson et al., PRL 91, 050401 (2003)
J. Carlson et al., PRL 95, 060401, (2005)
$\varepsilon_F = \frac{k_F^2}{2m}$
Δ/ϵ_F
$E_F = \frac{3}{10}\frac{k_F^2}{2m}$
$(E/N)/E_F$
G. E. Astrakharchik et al., PRL 93, 200404 (2004)
$(E/N-\epsilon_b/2)/E_F$
$(k_F a)^{-1}$
d=2
$(2|\varepsilon_B|/\varepsilon_F)^{1/2}$
$E_F = \frac{1}{2}\frac{k_F^2}{2m}$
G. Bertaina et al., PRL 106, 110403, (2011)
$-\log(k_F a)$

Conclusions

- The FRG is a powerful yet simple non-perturbative formalism to study quantum gases
- It allows us to give a consistent description of few- and many-body systems in different dimensions
- Quantitative accuracy in Fermi gases can be achieved by including missing physics
- Future work: gases with a mixture of different particles
  low-density nuclear matter

More details: F. Isaule, M. Birse and N. Walet, PRB **98**, 144502 (2018)
F. Isaule, M. Birse and N. Walet, arXiv:1902.07135 (2019)

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CONICYT Ministerio de Educación Gobierno de Chile

Correlation function: $G_n(\mathbf{x}) = \langle \phi^\dagger(\mathbf{x}) \phi(0) \rangle$

Three dimensions

$$T < T_c: \quad G_n(\mathbf{x}) \to \quad \rho_c > 0$$

- Long-range-order (LRO)
- $U(1)$ symmetry is broken, with ρ_c as order parameter

Two dimensions

$$T = 0: \quad G_n(\mathbf{x}) \to \quad \rho_c > 0$$

$$0 < T \leq T_c: \quad G_n(\mathbf{x}) \propto \quad |X|^{-\eta}$$

- Condensation only possible at $T = 0$ (Mermin-Wagner theorem)
- Quasi-long-range-order (QLRO) for $0 < T \leq T_c$ ($\rho_c = 0$, $\rho_s > 0$)
  ρ_s: superfluid density
- Phase transition driven by the unbinding of vortex pairs (BKT)

Initial conditions

Bose gas:

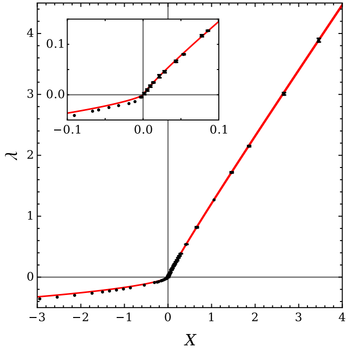
$$\rho_{0,\Lambda} = \frac{\mu_0}{u_{2,\Lambda}}\Theta(\mu_0), \quad u_{1,\Lambda} = -\mu_0\Theta(-\mu_0), \quad Z_{m,\Lambda} = Z_{\phi,\Lambda} = 1, \quad Y_{m,\Lambda} = 0.$$

Fermi gas:

$$\rho_{0,\Lambda} = Z_{m,\Lambda} = Z_{\phi,\Lambda} = u_{2,\Lambda} = 0, \quad Z_{M,\Lambda} = Z_{\psi,\Lambda} = 1, \quad \Sigma_{\psi,\Lambda} = \mu$$

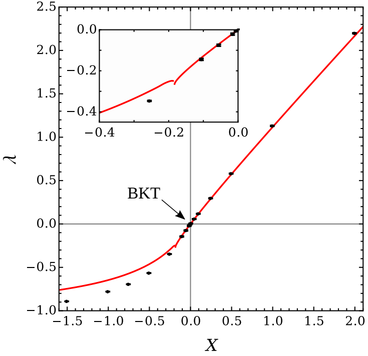
Results Bose gas finite temperature

$d = 3$



$\lambda = \frac{n - n_c}{m^3 T^2 u_{2,\Lambda}^2}, \qquad X = \frac{\mu - \mu_c}{m^3 T^2 u_{2,\Lambda}^2}$

$d = 2$



$\lambda = \frac{n - n_c}{m T u_{2,\Lambda}}, \qquad X = \frac{\mu - \mu_c}{m\, T\, u_{2,\Lambda}}$

Field representations

- Perturbation theory plagued by IR divergences due to ungapped propagator of Goldstone mode

$$\phi(\mathbf{x}) = \sqrt{\rho_0} + \sigma(\mathbf{x}) + i\pi(\mathbf{x}) : \qquad G_{||} = \frac{1}{\mathbf{q}^2 + q_c^2}, \qquad G_\perp = \frac{1}{\mathbf{q}^2}$$

- Divergences cancel, but cancellations are lost if expansions are truncated
- These can be avoided by using a convenient field representation

$$\phi(\mathbf{x}) = (\sqrt{\rho_0} + \delta\rho(\mathbf{x}))e^{i\theta(\mathbf{x})}$$

- Similar divergences arise when using a linear sigma model to describe broken chiral symmetry
- These can be solved by using a non-linear sigma model as in chiral-perturbation theory

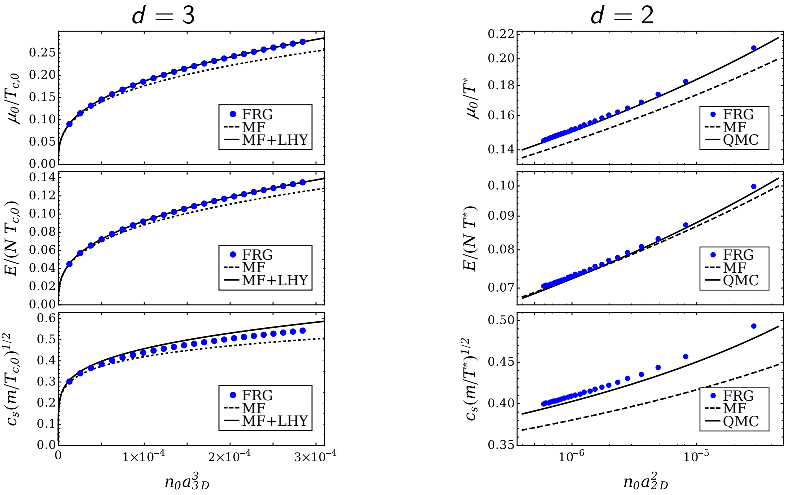
Field representations

Long-distance behaviour of correlation function:

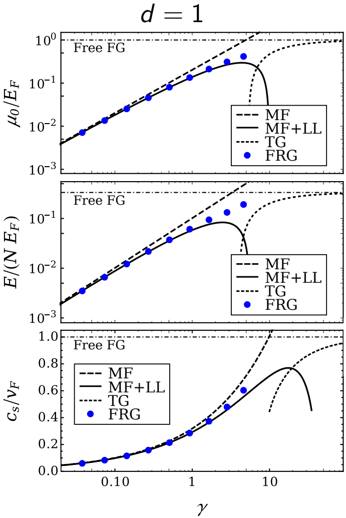
$$\lim_{|\mathbf{x}|\to\infty} G_n(\mathbf{x}) = \begin{cases} \rho_0 & : \textbf{(Cart.)} \qquad \rho_c = \rho_0, \\ \rho_0 e^{\langle(\vartheta(\mathbf{x})-\vartheta(0))^2\rangle} & : \textbf{(AP)} \qquad \rho_q = \rho_0, \end{cases}$$

- With AP representation long-distance behaviour driven by phase correlations
- ρ_q is the quasi-condensate density
  Al Khawaja *et al*, PRA **66**, 013615 (2002)
- In systems with QLRO $\rho_c = 0$ but $\rho_q > 0$

Results Bose gas zero temperature

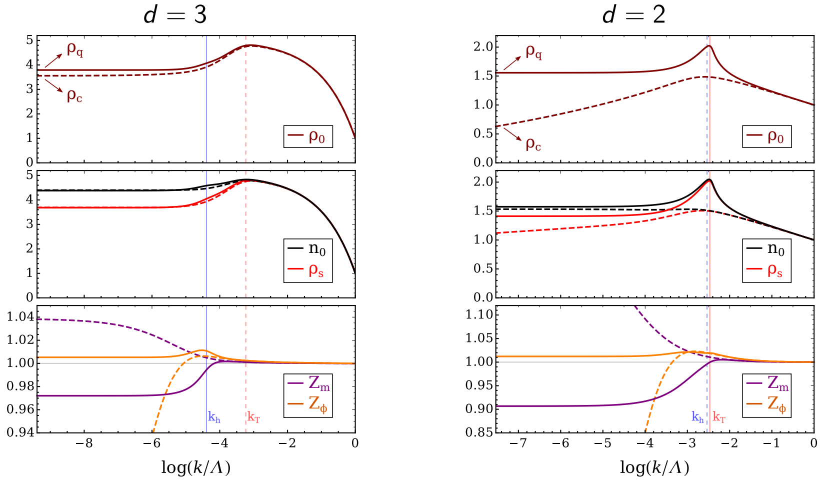


Results Bose gas zero temperature



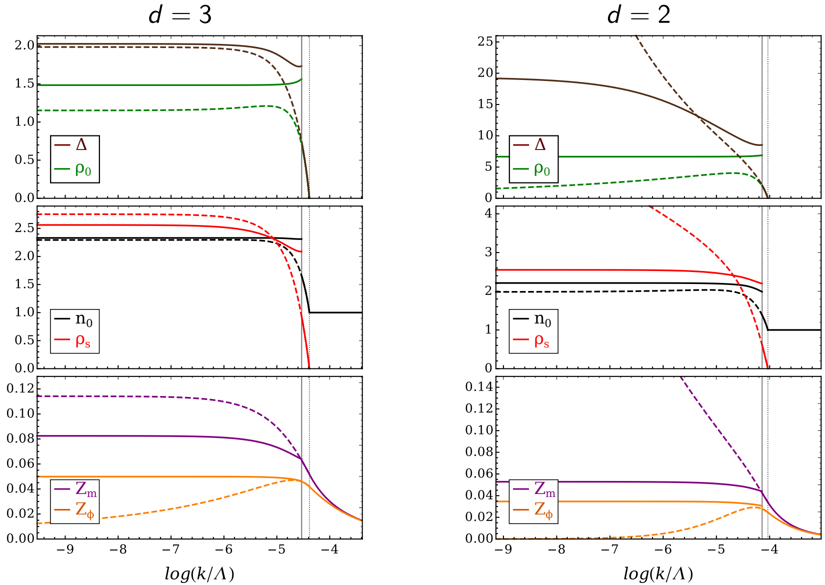
$$\gamma = -2/m\,a_{1D}$$

Flows Bose gas $0 < T < T_c$



Dashed: Cartesian representation, **Solid:** Interpolating representation

Flows Fermi gas $T = 0$



Dashed: Cartesian representation, **Solid:** Interpolating representation