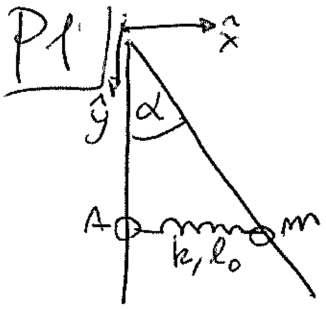
P1

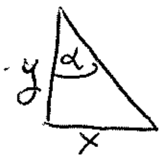


a) Si se sueltan los anillos del vértice de las barras, encuentre la distancia a que llegan usando ptos. de retorno.

b) ptos. de equilibrio

c) Ec. de mov. y T

Sol: a) Por geometría:



$\Rightarrow \operatorname{tg}\alpha = \frac{x}{y} \Rightarrow y = x\operatorname{ctg}\alpha$

$\dot{y} = \dot{x}\operatorname{ctg}\alpha$

La energía cinética del anillo:

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}\underbrace{(1+\operatorname{ctg}^2\alpha)}_{\operatorname{csec}^2\alpha}\dot{x}^2 = \frac{m}{2}\operatorname{csec}^2\alpha\,\dot{x}^2$$

La potencial:

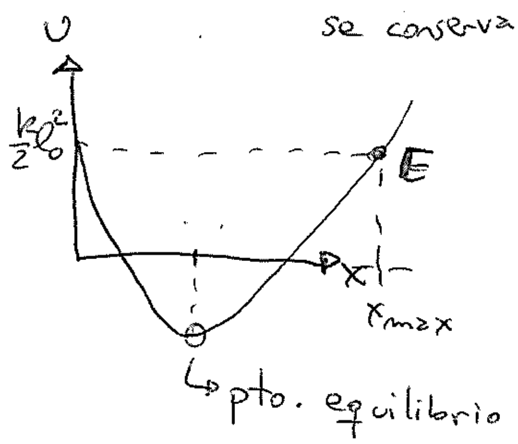
Grav: $U_g = -mg\,y = -mg\,x\operatorname{ctg}\alpha$

Resorte: $U_r = \frac{k}{2}(x-l_0)^2 \quad \Rightarrow U = -mg\,x\operatorname{ctg}\alpha + \frac{k}{2}(x-l_0)^2$

Si es soltado de $x=0$, la energía:

$$E(x=0) \underset{\uparrow}{=} E = \frac{k}{2}l_0^2$$

se conserva



Igualo el potencial a la energía:

$$E = U(x_{retorno})$$

$$\cancel{\frac{k}{2}l_0^2} = -mg\,x\operatorname{ctg}\alpha + \frac{k}{2}x^{\cancel{2}} - kx\,l_0 + \cancel{\frac{k}{2}l_0^2}$$

$$0 = -mg\operatorname{ctg}\alpha + \frac{k}{2}x_{max} - kl_0$$

$$\Rightarrow \boxed{x_{max} = \frac{2}{k}mg\operatorname{ctg}\alpha + 2l_0}$$

b) El pto. de equilibrio corresponde al mínimo de potencial:

$$\frac{dU}{dx} = -mg\operatorname{ctg}\alpha + k(x_{eq} - l_0) = 0 \quad \Rightarrow \boxed{x_{eq} = l_0 + \frac{mg}{k}\operatorname{ctg}\alpha}$$

Notar que $x_{max} = 2x_{eq}$

c) ¿Es estable?

$$\left.\frac{d^2U}{dx^2}\right|_{x_{eq}} = k > 0 \quad \Rightarrow \text{estable}$$

Pequeñas oscilaciones: $U \approx U(x_{eq}) + \underbrace{\left.\frac{dU}{dx}\right|_{x_{eq}}}_{0}(x-x_{eq}) + \frac{1}{2}\underbrace{\left.\frac{d^2U}{dx^2}\right|_{x_{eq}}}_{k}(x-x_{eq})^2$

$$= U(x_{eq}) + \frac{k}{2}(x-x_{eq})^2$$

Entonces la energía:

$$E = \frac{m}{2}\csc^2\alpha\,\dot{x}^2 + U(x_{eq}) + \frac{k}{2}(x-x_{eq})^2 \qquad / \frac{d}{dt}()$$

$$0 = m\csc^2\alpha\,\cancel{\dot{x}}\,\ddot{x} + k(x-x_{eq})\cancel{\dot{x}}$$

$$\ddot{x} + \underbrace{\frac{k}{m\csc^2\alpha}}_{\omega_0^2}(x-x_{eq}) = 0 \qquad \Rightarrow \omega_0 = \frac{1}{\csc\alpha}\sqrt{\frac{k}{m}} = \operatorname{sen}\alpha\sqrt{\frac{k}{m}}$$

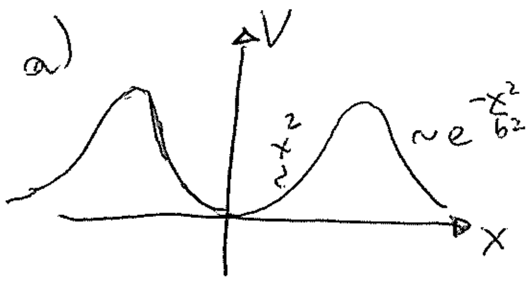
$$\Rightarrow T = \frac{2\pi}{\omega_0} \qquad \Rightarrow \boxed{T = \frac{2\pi}{\operatorname{sen}\alpha}\sqrt{\frac{m}{k}}}$$

P2) $V(x) = Ax^2 e^{-\left(\frac{x}{b}\right)^2}$

a) Ptos. eq. y frecuencias de pequeñas oscilaciones

b) Se agrega un resorte que transforma los ptos. inestables en estables. Determine k y l_0.

Sol: a)



Los ptos. de equilibrio:

$$\frac{dV}{dx} = 0 = 2Axe^{-\left(\frac{x}{b}\right)^2} - 2\frac{x}{b}xAe^{-\left(\frac{x}{b}\right)^2} =$$

$$= 2Ax\left[1 - \frac{x^2}{b^2}\right]e^{-\left(\frac{x}{b}\right)^2} = 0$$

$$\Rightarrow \boxed{\begin{aligned} x_1 &= 0 \\ x_2 &= b \\ x_3 &= -b \end{aligned}}$$

$x_1 = 0 \rightarrow$ estable

$x_2 = b$, $x_3 = -b$ } inestable

Sacamos ω para x_1:

$$\frac{d^2V}{dx^2} = 2A\left[1 - \frac{x^2}{b^2}\right]e^{-\left(\frac{x}{b}\right)^2} - 4A\frac{x^2}{b^2}e^{-\left(\frac{x}{b}\right)^2} - 4A\frac{x^2}{b^2}\left[1 - \frac{x^2}{b^2}\right]e^{-\left(\frac{x}{b}\right)^2}$$

evaluando en $x_1 = 0$:

$$\left.\frac{d^2V}{dx^2}\right|_0 = 2A$$

Expandiendo: $V(x) \approx V(0) + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_0 (x-0)^2 = Ax^2$

En vez de usar energía usamos fuerza:

$$F = -\frac{dV}{dx} = -2Ax = m\ddot{x} \Rightarrow \ddot{x} + \underbrace{\frac{2A}{m}}_{\omega_0^2}x = 0$$

$$\Rightarrow \boxed{\omega_0 = \sqrt{\frac{2A}{m}}}$$

b) Ahora: $V(x) = Ax^2 e^{-\left(\frac{x}{b}\right)^2} + \frac{k}{2}(|x| - l_0)^2$

$$\rightarrow \frac{dV}{dx} = 2Axe^{-\left(\frac{x}{b}\right)^2}\left[1 - \frac{x^2}{b^2}\right] + k(|x| - l_0)$$

Impongo que $\pm b$ sean ptos. de eq:

$$\left.\frac{dV}{dx}\right|_{\pm b} = 0 = 0 + k(b - l_0) \Rightarrow \boxed{l_0 = b}$$

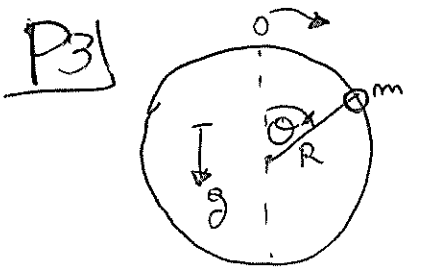
Ahora impongo que sean estables:

$$\left.\frac{d^2V}{dx^2}\right|_{\pm b} = 2A\left(1 - \frac{x^2}{b^2}\right)e^{-\left(\frac{x}{b}\right)^2} - 4A\frac{x^2}{b^2}e^{-\left(\frac{x}{b}\right)^2} - 4A\frac{x^2}{b^2}\left(1 - \frac{x^2}{b^2}\right)e^{-\left(\frac{x}{b}\right)^2} + k > 0$$

$$0 - 4Ae^{-1} + 0 + k > 0$$

$$\Rightarrow \boxed{k > \frac{4A}{e}}$$

P3

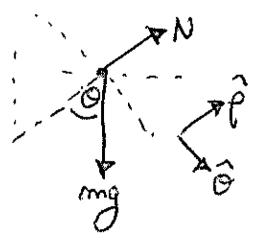


a) Fuerza entre aro y anillo en $\theta = \frac{\pi}{2}, \pi$

b) Periodo pequeñas oscilaciones

Sol: a) $\vec{a} = -R\dot{\theta}^2\hat{\rho} + R\ddot{\theta}\hat{\theta}$

DCL:



Newton:

$\hat{\rho}: -mR\dot{\theta}^2 = N - mg\cos\theta \qquad (1)$

$\hat{\theta}: mR\ddot{\theta} = mg\,\text{sen}\,\theta \qquad (2)$

Queremos evaluar en $\theta = \frac{\pi}{2}, \pi$, luego necesitamos $N(\theta)$

Usamos (2):

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{g}{R}\,\text{sen}\,\theta$$

$$\int_0^{\dot{\theta}} \dot{\theta}\, d\dot{\theta} = \frac{g}{R}\int_0^{\theta} \text{sen}\,\theta\, d\theta$$

$$\frac{\dot{\theta}^2}{2} = \frac{g}{R}(1-\cos\theta)$$

remplazando $\dot{\theta}^2$ en (1):

$$-m\cancel{R}\cdot\frac{2g}{\cancel{R}}(1-\cos\theta) = N - mg\cos\theta$$

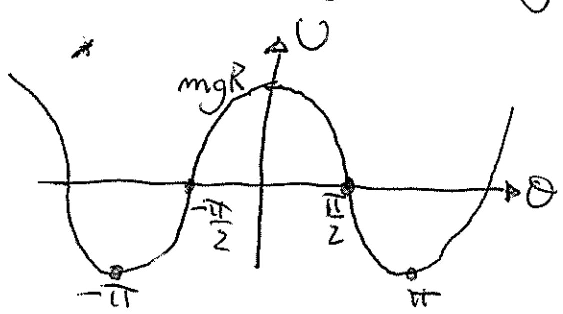
$$\Rightarrow N = mg(3\cos\theta - 2)$$

ahora evaluando en $\frac{\pi}{2}$ y π:

$$\boxed{N\left(\frac{\pi}{2}\right) = -2mg} \qquad \boxed{N(\pi) = -5mg}$$

b) Usaremos energía potencial:

$U = mgh = mgR\cos\theta$ } tomamos al centro del aro como $z=0$



Al "ojo" se ven que los equilibrios son $\theta = 0$ (inestable) y $\theta = \pi$ (estable)

Pero los podemos calcular:

$$\frac{dU}{d\theta} = 0 = -mgR\,\text{sen}\,\theta^* \Rightarrow \theta^* = 0, \pi$$

y la estabilidad:

$$\frac{d^2U}{d\theta^2} = -mgR\cos\theta \begin{cases} > 0 & \text{para } \theta = \pi \rightarrow \text{estable} \\ < 0 & \text{para } \theta = 0 \rightarrow \text{inestable} \end{cases}$$

Ahora expandemos U en torno al punto de equilibrio:

$$U(\theta) \approx U(\theta^*) + \cancel{\left.\frac{dU}{d\theta}\right|_{\theta^*}}^{\,0}(\theta - \theta^*) + \frac{1}{2}\left.\frac{d^2U}{d\theta^2}\right|_{\theta^*}(\theta - \theta^*)^2$$

Nos interesa $\theta = \pi$, pues es el eq. estable, es decir, el único que corresponde a una oscilación.

$$U(\theta) = -mgR + \frac{mg}{2}R(\theta - \pi)^2$$

entonces la energía: $E = \frac{m}{2}v^2 + U$, con $v = R\dot{\theta}$

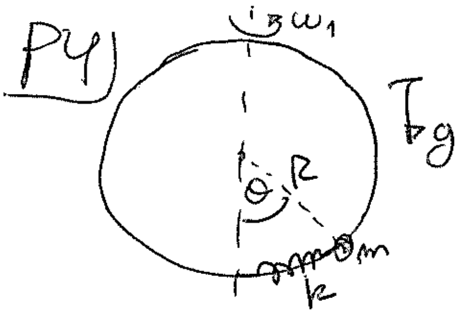
$$\Rightarrow E = \frac{m}{2}R^2\dot{\theta}^2 - mgR + \frac{mg}{2}R(\theta - \pi)^2 \quad / \frac{d}{dt}()$$

$$0 = mR^2\cancel{\dot{\theta}}\ddot{\theta} + mgR(\theta - \pi)\cancel{\dot{\theta}}$$

$$\Rightarrow \ddot{\theta} + \underbrace{\frac{g}{R}}_{\omega_0^2}(\theta - \pi) = 0 \Rightarrow \omega_0 = \sqrt{\frac{g}{R}}$$

y el período: $\boxed{T = 2\pi\sqrt{\frac{R}{g}}}$

P4)



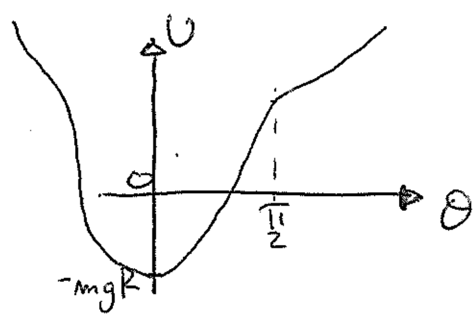
a) Ptos. de equilibrio y T

b) ¿Qué pasa si el aro rota con $\omega_1 = \dot{\phi}$? Encuentre T

Sol: a) El potencial:

$$U = -mgR\cos\theta + \frac{k}{2}R^2\theta^2$$

graficando:



$$\Rightarrow \boxed{\theta^* = 0}$$

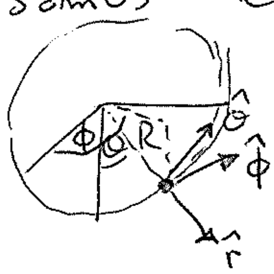
La energía: $E = \frac{m}{2}R^2\dot{\theta}^2 - mgR\cos\theta + \frac{k}{2}R^2\theta^2 \quad / \frac{d}{dt}()$

$$0 = mR^2\cancel{\dot{\theta}}\ddot{\theta} + mgR\underbrace{\sin\theta}_{\sim\theta}\cancel{\dot{\theta}} + kR^2\theta\cancel{\dot{\theta}}$$

$$\ddot{\theta} + \frac{g}{R}\theta + \frac{k}{m}\theta = 0$$

$$\ddot{\theta} + \underbrace{\omega_0^2}_{\frac{g}{R}+\frac{k}{m}}\theta = 0 \Rightarrow \omega_0 = \sqrt{\frac{g}{R}+\frac{k}{m}} \Rightarrow \boxed{T = \frac{2\pi}{\sqrt{\frac{g}{R}+\frac{k}{m}}}}$$

b) Usamos esféricas: $\vec{v} = R\omega_1\sin\theta\,\hat{\phi} + R\dot{\theta}\,\hat{\theta}$



$$\Rightarrow v^2 = R^2\omega_1^2\sin^2\theta + R^2\dot{\theta}^2$$

El potencial queda igual:

$$E = \frac{m}{2}(R^2\dot{\theta}^2 + R^2\omega_1^2\sin^2\theta) - mgR\cos\theta + \frac{k}{2}R^2\theta^2 \quad / \frac{d}{dt}()$$

$$mR^2\cancel{\dot{\theta}}\ddot{\theta} + mR^2\omega_1^2\underbrace{\sin\theta}_{\sim\theta}\underbrace{\cos\theta}_{\sim 1-\cancel{\frac{\theta^2}{2}}}\cancel{\dot{\theta}} + mgR\underbrace{\sin\theta}_{\sim\theta}\cancel{\dot{\theta}} + kR^2\theta\cancel{\dot{\theta}} = 0$$

$$\ddot{\theta} + \omega_1^2\theta + \underbrace{\frac{g}{l}\theta + \frac{k}{m}\theta}_{\omega_0^2\theta} = 0$$

$$\ddot{\theta} + (\underbrace{\omega_1^2 + \omega_0^2}_{\text{nuevo frecuencia}})\theta = 0 \quad \Rightarrow \quad \boxed{T = \frac{2\pi}{\sqrt{\omega_1^2 + \omega_0^2}}}$$