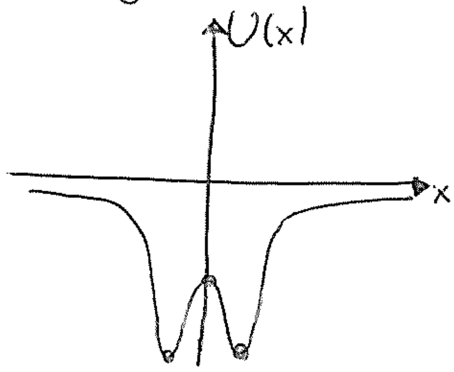
Auxiliar 14

P1 $U(x) = \dfrac{-W d^2(x^2+d^2)}{x^4+8d^4}$

a) Encontrar ptos. de equilibrios y su estabilidad

b) Ptos. de retorno para $E = -\frac{W}{8}$

Sol: Si graficamos U:



a) Queremos sacar $\frac{dU}{dx}, \frac{d^2U}{dx^2}$

Para simplificar los cálculos:

$$V(x) = \frac{U(x)}{W} = \frac{-d^2(x^2+d^2)}{x^4+8d^4} = \frac{-\cancel{d^4}\left(\frac{x^2}{d^2}+1\right)}{\cancel{d^4}\left(\frac{x^4}{d^4}+8\right)} =$$

$$= \frac{-(y^2+1)}{y^4+8}$$

donde $y = \frac{x}{d}$

dado que W y d son ctes, los ptos. de equilibrio de V serán los mismos de U.

Buscamos los ptos. de equilibrio;

$$\frac{dV}{dy} = \frac{-2y}{\cancel{y^4+8}} + \frac{\cancel{4}^{\,2}y^3(y^2+1)}{(y^4+8)^{\cancel{2}}} = 0 \rightarrow -y^5 - 8y + 2y^3(y^2+1) = 0$$

$$y(\underbrace{-y^4 - 8 + 2y^4 + 2y^2}_{y^4+2y^2-8}) = 0$$

$$y\,(\underbrace{y^2+4}_{\text{sols. imaginarias}})(y^2-2) = 0$$

$$\Rightarrow \quad \begin{array}{l} y_1 = 0 \\ y_2 = \sqrt{2} \\ y_3 = -\sqrt{2} \end{array} \Bigg\} \quad \boxed{\begin{array}{l} x_1 = 0 \\ x_2 = d\sqrt{2} \\ x_3 = -d\sqrt{2} \end{array}}$$

La estabilidad;

$$\frac{d^2V}{dy^2} = \frac{-2}{y^4+8} + \frac{8y^4}{(y^4+8)^2} + \frac{12y^2(y^2+1)}{(y^4+8)^2} + \frac{8y^4}{(y^4+8)^2} - \frac{32y^6(y^2+1)}{(y^4+8)^3}$$

evaluando:

$$\left.\frac{d^2V}{dy^2}\right|_0 = \frac{-2}{8} = -\frac{1}{4} < 0 \quad \Rightarrow \text{ inestable}$$

$$\left.\frac{d^2V}{dy^2}\right|_{\pm 2} = \frac{1}{12}\left[-2 + \frac{32}{12} + \frac{72}{12} + \frac{32}{12} - \frac{768}{12^2}\right] = \frac{4}{12} = \frac{1}{3} > 0 \quad \Rightarrow \text{ estables}$$

b) Los ptos. de retorno:

$$E = -\frac{\omega}{8} = \frac{-\omega(y^2+1)}{y^4+8} = U(y)$$

despejando: $\frac{1}{8} = \frac{y^2+1}{y^4+8} \rightarrow y^4+\cancel{8} = 8y^2+\cancel{8}$

$$y^4 = 8y^2$$

$$y^2(y^2-8)=0$$

$$\Rightarrow \left.\begin{matrix} y_1 = 0 \\ y_2 = 2\sqrt{2} \\ y_3 = -2\sqrt{2} \end{matrix}\right\} \boxed{\begin{matrix} x_1 = 0 \\ x_2 = 2d\sqrt{2} \\ x_3 = -2d\sqrt{2} \end{matrix}}$$

P2) $F = -kx + k\frac{x^3}{\alpha^2}$, con k, α positivas

Determinar $U(x)$, ¿Qué pasa cuando $E = \frac{k\alpha^2}{4}$?

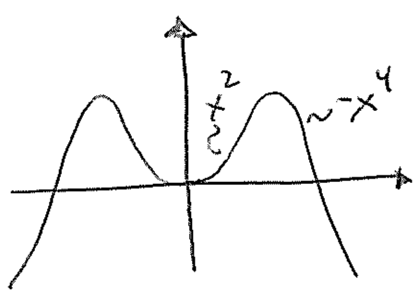
Sol. Recordamos que:

$$F = -\frac{dU}{dx}$$

para sacar U integramos F:

$$\Rightarrow \boxed{U = \frac{k}{2}x^2 - \frac{k}{4}\frac{x^4}{\alpha^2}}$$

Si graficamos:



Sacaremos los ptos. de equilibrio:

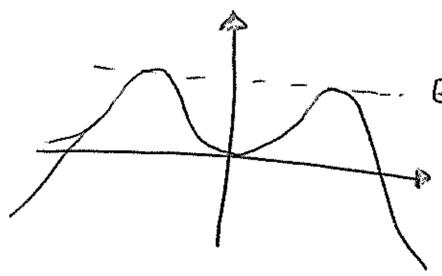
$$\frac{dU}{dx} = -F = kx - k\frac{x^3}{\alpha^2} = 0$$

$$kx\left(1 - \frac{x^2}{\alpha^2}\right) = 0 \quad \Rightarrow \boxed{x_1 = 0 \quad x_2 = \alpha \quad x_3 = -\alpha}$$

Para $E = \frac{k\alpha^2}{4}$;

$$\frac{k\alpha^2}{4} = U(x) \Rightarrow x = \alpha$$

Corresponde a un movimiento encerrado en el pozo de la energía potencial: (si es que $|x| < \alpha$):



Sacaremos el período de oscilación con esa energía E:

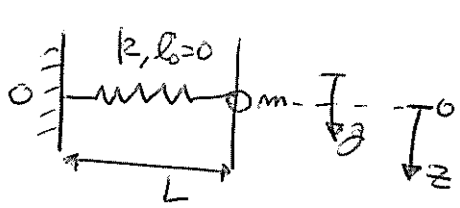
$$T = 2\sqrt{\frac{m}{2}} \int_{-\alpha}^{\alpha} \frac{dx}{\sqrt{E-U}} = 4\sqrt{\frac{m}{2}} \int_0^{\alpha} \frac{dx}{\sqrt{\frac{k\alpha^2}{4} - \frac{k}{2}x^2 + \frac{k}{4}\frac{x^4}{\alpha^2}}} =$$

$$= 4\sqrt{\frac{m}{\cancel{2}}} \int_0^{\alpha} \frac{dx}{\sqrt{\frac{k}{\cancel{4}_2\alpha^2}}\sqrt{\underbrace{\alpha^4 - 2\alpha^2x^2 + x^4}_{(\alpha^2-x^2)^2}}} = 4\alpha\sqrt{\frac{m}{k}} \int_0^{\alpha} \frac{dx}{\alpha^2 - x^2} =$$

$$= 4\cancel{\alpha}\sqrt{\frac{m}{k}} \left.\frac{\operatorname{arctgh}(x/\alpha)}{\cancel{\alpha}}\right|_0^{\alpha} = 4\sqrt{\frac{m}{k}} \operatorname{arctgh}(1) \longrightarrow \infty$$

$$\Rightarrow \boxed{T \to \infty}$$

P3)

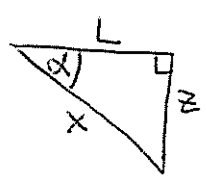


a) Rapidez máxima si parte del reposo en $z = 0$

b) Ptos. eq.

c) T

Sol: Necesitamos el estiramiento del resorte cuando el anillo está a una altura z:



$$x = \sqrt{z^2 + L^2}$$

La energía potencial: $U(z) = \frac{k}{2}\overbrace{(z^2 + L^2)}^{x^2} - mgz$

La energía total:

$$E = \frac{m}{2} v^2 + U$$

la rapidez máxima ocurre cuando $U = U_{min}$ ← ptos. eq. estables

$$\frac{dU}{dz} = kz - mg = 0 \Rightarrow z = \frac{mg}{k}$$

Entonces la rapidez máxima: $v_{max}^2 = \frac{2}{m}\left(E - U\left(\frac{mg}{k}\right)\right)$

Necesitamos E, lo sacamos de la condición inicial:

$$E = 0 + U(0) = \frac{k}{2} L^2$$

además:

$$U\left(\frac{mg}{k}\right) = \frac{k}{2}\left(\frac{m^2 g^2}{k^2} + L^2\right) - mg \cdot \frac{mg}{k} = \underbrace{\frac{m^2 g^2}{2k} - \frac{m^2 g^2}{k}}_{-\frac{m^2 g^2}{2k}} + \frac{kL^2}{2} = \frac{kL^2}{2} - \frac{m^2 g^2}{2k}$$

entonces:

$$v_{max}^2 = \frac{2}{m}\left(\frac{k}{2}L^2 - \frac{k}{2}L^2 + \frac{m^2 g^2}{2k}\right)$$

$$\Rightarrow \boxed{v_{max} = g\sqrt{\frac{m}{k}}}$$

b) Ya sacamos el pto. de equilibrio:

$$\boxed{z_{eq} = \frac{mg}{k}}$$

c) Queremos expandir, sacamos la sgda. derivada de U:

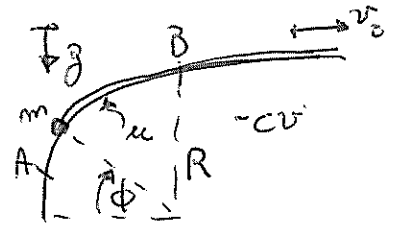
$$\frac{d^2U}{dz^2} = k \rightarrow U \approx U(z_{eq}) + \frac{1}{2} \cdot U''(z_{eq}) \cdot (z - z_{eq})^2 =$$

$$= U(z_{eq}) + k(z - z_{eq})^2$$

$$\rightarrow E = \frac{m}{2}\dot{z}^2 + U(z_{eq}) + k(z - z_{eq})^2 \quad / \frac{d}{dt}()$$

$$0 = m\ddot{z} + k(z - z_{eq}) \Rightarrow \omega_0^2 = \frac{k}{m} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

P4)



$A: \phi = \frac{\pi}{6}$

a) Determinar mayor v_0 para que no se separe m entre A-B

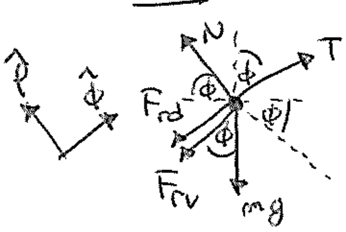
b) W de c/u de las fuerzas entre A - B

c) W del motor, ¿puede ser nulo?

Sol: a) En cilíndricas: $\vec{v} = R\dot{\phi}\hat{\phi} \Rightarrow \dot{\phi} = v_0/R = cte$

$\vec{a} = -R\dot{\phi}^2\hat{\rho} = -\frac{v_0^2}{R}\hat{\rho}$

DCL:



Newton:

$\hat{\rho}: \; -m\frac{v_0^2}{R} = N - mg\,\text{sen}\phi \rightarrow N = mg\,\text{sen}\phi - m\frac{v_0^2}{R}$

$\hat{\phi}: \quad 0 = T - mg\cos\phi - F_{rd} - F_{rv}$

Para que no se despegue; $N > 0 \Rightarrow g\,\text{sen}\phi > \frac{v_0^2}{R}$

Como queremos que lo anterior se cumpla para todo ϕ entre $\frac{\pi}{6}$ y $\frac{\pi}{2}$:

$$g\,\text{sen}\phi > \underbrace{(g\,\text{sen}\phi)\big|_{min}}_{\text{sen}\frac{\pi}{6} = \frac{1}{2}} > \frac{v_0^2}{R}$$

$\pi/6$

$$\Rightarrow \frac{g}{2} > \frac{v_0^2}{R} \Rightarrow \boxed{v_{max}^2 = \frac{gR}{2}}$$

b) • Peso: $W^{mg}_{A\to B} = -\Delta U_{A\to B} = -mgR(\overbrace{\underbrace{\text{sen}\frac{\pi}{2}}_{1}}^{B} - \overbrace{\underbrace{\text{sen}\frac{\pi}{6}}_{\frac{1}{2}}}^{A}) = -\frac{mgR}{2} < 0$

$U = mgR\,\text{sen}\phi$

• Roce dinámico: $W^{d}_{A\to B} = \int_A^B (-\mu N\hat{\phi} \cdot R\,d\phi\,\hat{\phi}) = -\mu m \int_{\pi/6}^{\pi/2} (g\,\text{sen}\phi - \frac{v_0^2}{R}) R\,d\phi =$

$$= -\mu m R\left[\underbrace{g(-\cos\phi)\Big|_{\pi/6}^{\pi/2}}_{\sqrt{3}/2} - \frac{v_0^2}{R}\frac{\pi}{3}\right] = -\mu m\left(g\frac{R\sqrt{3}}{2} - v_0^2\frac{\pi}{3}\right)$$

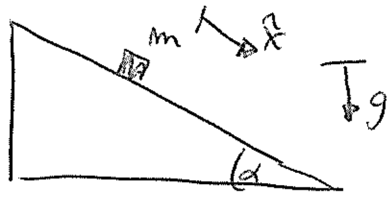
• Roce viscoso: $W^{v}_{A\to B} = -\int_{\pi/6}^{\pi/2} c \cdot v_0 R\,d\phi = -\frac{c v_0 R\pi}{3} < 0$

c) El trabajo realizado total:

$W = \Delta K = 0$ } porque v es cte

$W = W_{motor} + W^{mg} + W^{d} + W^{v} = 0 \Rightarrow \boxed{W_{motor} = \frac{mgR}{2} + \mu m\left(g\frac{R\sqrt{3}}{2} - \frac{v_0^2\pi}{3}\right) + c}$

P5)



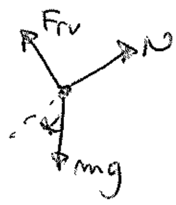
$F_{rv} = -kmv^2$

$\dot{x}(0) = 0$

Mostrar que el tiempo que toma recorrer d:

$$t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{kg\,\text{sen}\alpha}}$$

Sol: DCL:



Newton:

$$\cancel{m}\ddot{x} = \cancel{m}g\,\text{sen}\alpha - k\cancel{m}\dot{x}^2$$

$$\Rightarrow \quad \frac{d\dot{x}}{dt} = g\,\text{sen}\alpha - k\dot{x}^2$$

$$\underbrace{\frac{1}{k}\int_0^{\dot{x}} \frac{d\dot{x}}{\frac{g}{k}\text{sen}\alpha - \dot{x}^2}}_{\left.\frac{\tanh^{-1}\left(\frac{\dot{x}}{\sqrt{\frac{g}{k}\text{sen}\alpha}}\right)}{\sqrt{\frac{g\,\text{sen}\alpha}{k}}}\right|_0^{\dot{x}}} = \int_0^t dt$$

$$\Rightarrow \quad t = \frac{1}{k}\sqrt{\frac{k}{g\,\text{sen}\alpha}}\,\tanh^{-1}\left(\frac{\dot{x}}{\sqrt{\frac{g}{k}\text{sen}\alpha}}\right)$$

despejando: $\tanh^{-1}\left(\frac{\dot{x}}{\sqrt{\frac{g}{k}\text{sen}\alpha}}\right) = t\sqrt{gk\,\text{sen}\alpha} \quad \Rightarrow \quad \frac{dx}{dt} = \sqrt{\frac{g}{k}\text{sen}\alpha}\,\tanh\left(t\sqrt{gk\,\text{sen}\alpha}\right)$

$$\int_0^d dx = d = \sqrt{\frac{g}{k}\,\text{sen}\alpha}\;\underbrace{\int_0^t \tanh\left(t\sqrt{gk\,\text{sen}\alpha}\right)dt}_{\frac{\ln\left(\cosh\left(\sqrt{gk\,\text{sen}\alpha}\,t\right)\right)}{\sqrt{gk\,\text{sen}\alpha}}}$$

$$d = \frac{1}{k}\ln\left(\cosh\left(\sqrt{gk\,\text{sen}\alpha}\,t\right)\right)$$

$$\Rightarrow \quad \cosh\left(\sqrt{gk\,\text{sen}\alpha}\,t\right) = e^{kd}$$

$$\sqrt{gk\,\text{sen}\alpha}\,t = \cosh^{-1}(e^{kd})$$

$$\Rightarrow \quad \boxed{t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{gk\,\text{sen}\alpha}}}$$

Propuesto 1; $\vec{F} = -\vec{\nabla} U = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}$

a) $\vec{F} = \underbrace{(ayz+bx+c)}_{-\frac{\partial U}{\partial x}}\hat{x} + \underbrace{(axz+bz)}_{-\frac{\partial U}{\partial y}}\hat{y} + \underbrace{(axy+by)}_{-\frac{\partial U}{\partial z}}\hat{z}$

$\hat{x}$: $U = -ayzx - \frac{b}{2}x^2 - cx + f_1(y,z)$

$\hat{y}$: $U = -axzy - bzy + f_2(x,z)$

$\hat{z}$; $U = -axyz - byz + f_3(x,y)$

"empalmando" los tres U;

$$\boxed{U = -axyz - \frac{b}{2}x^2 - byz - cx}$$

b) $\vec{F} = \underbrace{-ze^{-x}}_{-\frac{\partial U}{\partial x}}\hat{x} + \underbrace{\ln z}_{-\frac{\partial U}{\partial y}}\hat{y} + \underbrace{\left(e^{-x} + \frac{y}{z}\right)}_{-\frac{\partial U}{\partial z}}\hat{z}$

$\hat{x}$; $U = -ze^{-x} + f_1(y,z)$

$\hat{y}$: $U = -y\ln z + f_2(x,z)$

$\hat{z}$: $U = -ze^{-x} - y\ln z + f_3(x,y)$

$$\Rightarrow \boxed{U = -ze^{-x} - y\ln z}$$

Propuesto 2: $y(x) = \frac{1}{4}\frac{(x^2-x_0^2)^2}{x_0^3} + \frac{2a}{3}\frac{x^3}{x_0^2}$

a) $U = mgy = mg\left[\frac{1}{4}\frac{(x^2-x_0^2)^2}{x_0^3} + \frac{2a}{3}\frac{x^3}{x_0^2}\right]$

b) Los ptos. de equilibrio:

$$\frac{dU}{dx} = mg\left[\frac{1}{\cancel{2}}\frac{(x^2-x_0^2)}{x_0^3}\cancel{2}x + 2a\frac{x^2}{x_0^2}\right] =$$

$$= \frac{mg}{x_0^2}\left[\frac{x(x^2-x_0^2)}{x_0} + 2ax^2\right] = 0$$

$$\Rightarrow \underbrace{x}_{x_1=0}\left[\frac{(x^2-x_0^2)}{x_0} + 2ax\right] = 0$$

$$\rightarrow \quad x^2 - x_0^2 + 2ax_0x = 0$$

$$\Rightarrow \quad x = \frac{-\cancel{2}ax_0 \pm \sqrt{\cancel{4}a^2x_0^2 + \cancel{4}x_0^2}}{\cancel{2}} =$$

$$= -ax_0 \pm x_0\sqrt{a^2+1}$$

$\Rightarrow$ Los ptos. de eq. son:

$$\boxed{x_1 = 0} \qquad \boxed{x_{2,3} = -ax_0 \pm x_0\sqrt{a^2+1}}$$

Lo estabilidad:

$$\frac{d^2U}{dx^2} = \frac{mg}{x_0^2}\left[\frac{(x^2-x_0^2)}{x_0} + 2ax + \frac{2ax^2}{x_0} + 2ax\right] =$$

$$= \frac{mg}{x_0^2}\left[\frac{x^2}{x_0}(1+2a) + 4ax - x_0\right]$$

evaluando en los puntos de eq:

$x_1=0$: $\left.\frac{d^2U}{dx^2}\right|_0 = -\frac{mg}{x_0} < 0 \quad \Rightarrow$ inestable