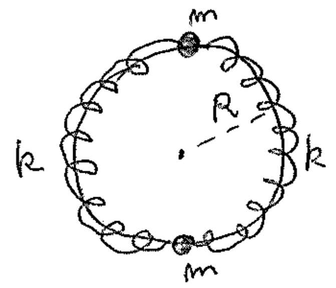
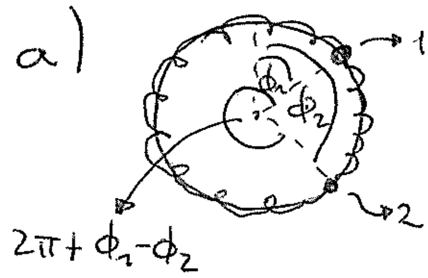
P1]



a) $\frac{d^2}{dt^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -M\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

b) Encontrar valores y autovalores de M

Sol: a)



$\vec{a}_1 = -R\dot{\phi}_1\hat{\rho}_1 + R\ddot{\phi}_1\hat{\phi}_1$

$\vec{a}_2 = -R\dot{\phi}_2\hat{\rho}_2 + R\ddot{\phi}_2\hat{\phi}_2$

Sólo nos interesan las ecuaciones de Newton en $\hat{\phi}_i$:

$\hat{\phi}_1$: $m\cancel{R}\ddot{\phi}_1 = k\cancel{R}(\phi_2-\phi_1) - k\cancel{R}(2\pi-\phi_2+\phi_1)$

$\Rightarrow \ddot{\phi}_1 = 2\frac{k}{m}(\phi_2-\phi_1-\pi)$ (1)

$\hat{\phi}_2$: $m\cancel{R}\ddot{\phi}_2 = k\cancel{R}(2\pi-\phi_2+\phi_1) - k\cancel{R}(\phi_2-\phi_1)$

$\Rightarrow \ddot{\phi}_2 = 2\frac{k}{m}(\phi_1-\phi_2+\pi)$ (2)

llamando $\theta_2 = \phi_2 - \pi \rightarrow \ddot{\theta}_2 = \ddot{\phi}_2$:

(1) $\rightarrow$ $\ddot{\phi}_1 = 2\frac{k}{m}(-\phi_1+\theta_2)$ (3)

(2) $\rightarrow$ $\ddot{\theta}_2 = 2\frac{k}{m}(\phi_1-\theta_2)$ (4)

Las ecs. (3) y (4) las puedo escribir como:

$$\boxed{\underbrace{\frac{d^2}{dt^2}\begin{pmatrix} \phi_1 \\ \theta_2 \end{pmatrix}}_{\vec{y}} = -\underbrace{\begin{pmatrix} 2\frac{k}{m} & -2\frac{k}{m} \\ -2\frac{k}{m} & 2\frac{k}{m} \end{pmatrix}}_{M}\underbrace{\begin{pmatrix} \phi_1 \\ \theta_2 \end{pmatrix}}_{\vec{y}}}$$

b) Llamando $\frac{k}{m} = \omega_0^2$:

$$M = \begin{pmatrix} 2\omega_0^2 & -2\omega_0^2 \\ -2\omega_0^2 & 2\omega_0^2 \end{pmatrix}$$

El problema de autovalores: $(M - \omega^2 \mathbb{1})\vec{v} = 0$

$$\det\left(M-\omega^2 \mathbb{1}\right)=\begin{vmatrix} 2\omega_0^2-\omega^2 & -2\omega_0^2 \\ -2\omega_0^2 & 2\omega_0^2-\omega^2 \end{vmatrix}=0$$

$$(2\omega_0^2-\omega^2)^2-4\omega_0^4=0$$

$$\cancel{4\omega_0^4}-4\omega_0^2\omega^2+\omega^4-\cancel{4\omega_0^4}=0$$

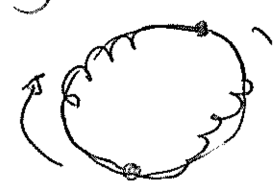
$$\omega^2(\omega^2-4\omega_0^2)=0$$

$$\Rightarrow \boxed{\begin{array}{l}\omega_1^2=0 \\ \omega_2^2=4\omega_0^2\end{array}}$$

Ahora los vectores propios asociados a cada valor propio:

$\underline{\omega_1=0}$: $M\vec{e}_1=\omega_1^2\vec{e}_1 \rightarrow \begin{pmatrix} 2\omega_0^2 & -2\omega_0^2 \\ -2\omega_0^2 & 2\omega_0^2\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}=\vec{0}$

$$\left.\begin{array}{l}\cancel{2\omega_0^2}x-\cancel{2\omega_0^2}y=0 \\ \Rightarrow x=y \quad \Rightarrow \boxed{\vec{e}_1=\begin{pmatrix}1\\1\end{pmatrix}}\end{array}\right\}$$ modo traslaci



$\underline{\omega_2=2\omega_0}$: $\begin{pmatrix} 2\omega_0^2 & -2\omega_0^2 \\ -2\omega_0^2 & 2\omega_0^2\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}=4\omega_0^2\begin{pmatrix} x \\ y\end{pmatrix}$

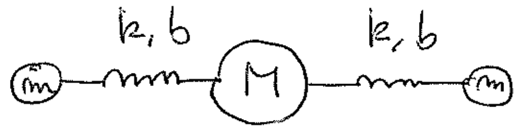
$$\left.\begin{array}{l}\cancel{2\omega_0^2}x-\cancel{2\omega_0^2}y=\overset{2}{\cancel{4\omega_0^2}}x \\ -x=y \quad \Rightarrow \boxed{\vec{e}_2=\begin{pmatrix}1\\-1\end{pmatrix}}\end{array}\right\}$$

Modo antisimé

La solución general:

$$\boxed{\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=(At+B)\begin{pmatrix}1\\1\end{pmatrix}+\left(C\cos(2\omega_0 t)+D\sin(2\omega_0 t)\right)\begin{pmatrix}1\\-1\end{pmatrix}}$$

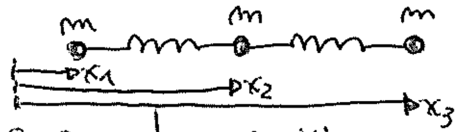
P2



$M = m$

a) $\frac{d^2}{dt}\vec{y} = -M\vec{y}$ b) Valores y vectores propios

Sol: a)



Las ecs. de mov:

$\hat{x}_1$: $m\ddot{x}_1 = k(x_2 - x_1 - b)$

$\hat{x}_2$: $m\ddot{x}_2 = k(x_3 - x_2 - b) - k(x_2 - x_1 - b)$

$= k(x_1 + x_3 - 2x_2)$

$\hat{x}_3$: $m\ddot{x}_3 = -k(x_3 - x_2 - b)$

Hacemos un cambio de variables:

$x_1^* = x_1 + b \longrightarrow \ddot{x}_1^* = \ddot{x}_1$

$x_3^* = x_3 - b \longrightarrow \ddot{x}_3^* = \ddot{x}_3$

llamando $\omega_0^2 = \frac{k}{m}$, las ecs. quedan:

$\ddot{x}_1^* = -\omega_0^2(x_1^* - x_2)$ (1)

$\ddot{x}_2 = -\omega_0^2(-x_1^* + 2x_2 - x_3^*)$ (2)

$\ddot{x}_3^* = -\omega_0^2(-x_2 + x_3^*)$ (3)

que de forma matricial queda:

$$\boxed{\underbrace{\frac{d^2}{dt^2}\begin{pmatrix} x_1^* \\ x_2 \\ x_3^* \end{pmatrix}}_{\vec{y}} = -\underbrace{\begin{pmatrix} \omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x_1^* \\ x_2 \\ x_3^* \end{pmatrix}}_{\vec{y}}}$$

b) Sacaremos los valores propios de $\det(M - \omega^2 \mathbb{1}) = 0$

$$\begin{vmatrix} \omega_0^2 - \omega^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = (\omega_0^2 - \omega^2)\left[(2\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - \omega_0^4\right] + \omega_0^2\left[-\omega_0^2(\omega_0^2 -\right.$$

$=$

$$= (\omega_0^2-\omega^2)[\underbrace{2\omega_0^4 - 2\omega_0^2\omega^2 - \omega^2\omega_0^2 + \omega^4 - \omega_0^4}_{\omega_0^4 - 3\omega_0^2\omega^2+\omega^4}] - \omega_0^6 + \omega_0^4\omega^2 = 0$$

$$\cancel{\omega_0^6} - 3\omega_0^4\omega^2 + \omega_0^2\omega^4 - \cancel{\omega_0^4\omega^2} + 3\omega_0^2\omega^4 - \omega^6 - \cancel{\omega_0^6} + \cancel{\omega_0^4\omega^2} = 0$$

$$\omega^6 - 4\omega_0^2\omega^4 + 3\omega_0^4\omega^2 = 0$$

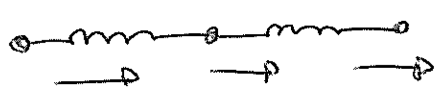
$$\omega^2(\omega^4 - 4\omega_0^2\omega^2 + 3\omega_0^4) = 0$$

$$\omega^2(\omega^2 - 3\omega_0^2)(\omega^2 - 1\omega_0^2) = 0$$

$$\Rightarrow \boxed{\begin{array}{l} \omega_1^2 = 0 \\ \omega_2^2 = 3\omega_0^2 \\ \omega_3^2 = \omega_0^2 \end{array}}$$

Los vectores propios:

- $\underline{\omega_1^2 = 0}$; Traslacional $\Rightarrow \boxed{\vec{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$



} molécula desplazándose

- $\underline{\omega_2^2 = 3\omega_0^2}$: $\begin{pmatrix} \omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\omega_0^2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

1ª fila $\rightarrow$ $\omega_0^2 x - \omega_0^2 y = 3\omega_0^2 x$

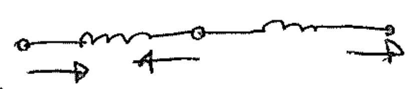
$-2x = y \Rightarrow y = -2x$

3ª fila $\rightarrow$ $-\omega_0^2 y + \omega_0^2 z = 3\omega_0^2 z$

$-2z = y \Rightarrow y = -2z$

$\Rightarrow \boxed{\vec{e}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$

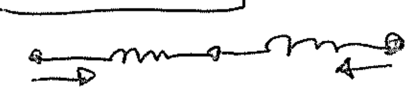
- $\underline{\omega_3^2 = \omega_0^2}$: $\begin{pmatrix} \omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \omega_0^2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

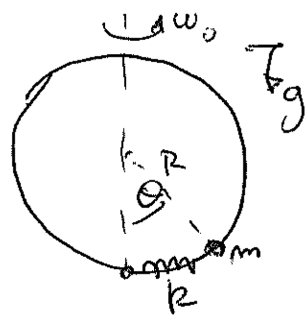


1ª fila: $\cancel{\omega_0^2 x} - \omega_0^2 y = \cancel{\omega_0^2 x} \Rightarrow y = 0$

2ª Fila: $-\omega_0^2 x - \omega_0^2 z = 0 \Rightarrow x = -z$

$\Rightarrow \boxed{\vec{e}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$



P3)  Posición de eq. y T

Sol: En esféricas:

$$\vec{v} = R\dot{\theta}\hat{\theta} + R\omega_0 \operatorname{sen}\theta\hat{\phi}$$

La energía:

$$E = \frac{m}{2}(R^2\dot{\theta}^2 + R^2\omega_0^2\operatorname{sen}^2\theta) + \frac{k}{2}R^2\theta^2 - mgR\cos\theta$$

$$= \frac{m}{2}R\dot{\theta}^2 + \underbrace{\frac{m}{2}R^2\omega_0^2\operatorname{sen}^2\theta + \frac{k}{2}R^2\theta^2 - mgR\cos\theta}_{U_{eff}(\theta)}$$

Derivamos:

$$\frac{\partial U_{eff}}{\partial\theta} = mR^2\omega_0^2\operatorname{sen}\theta\cos\theta + kR^2\theta + mgR\operatorname{sen}\theta$$

difícil sacar el pto. de equilibrio.

Si decimos que ω_0 es pequeño y $\theta\sim 0$:

$$U_{eff} = \frac{m}{2}R^2\omega_0^2\theta^2 + \frac{k}{2}R^2\theta^2 - mgR + \frac{mgR}{2}\theta^2$$

$$= -mgR + \frac{R}{2}[mR\omega_0^2 + kR + mg]\theta^2$$

ahora derivamos:

$$\frac{\partial U_{eff}}{\partial\theta} = R[mR\omega_0^2 + kR + mg] \quad \Rightarrow \boxed{\theta_{eq} = 0}$$ porque ω_0^2 es pequeño

$$\Rightarrow \frac{\partial^2 U_{eff}}{\partial\theta^2} = R[mR\omega_0^2 + kR + mg]$$

$$\rightarrow E = \frac{m}{2}R\dot{\theta}^2 + U_{eff}(\theta_{eq}) + \frac{R}{2}[mR\omega_0^2 + kR + mg]\theta^2 \quad / \frac{d}{dt}()$$

$$\Rightarrow \omega_0^2 = \frac{mR\omega_0^2 + kR + mg}{m} \quad \Rightarrow \boxed{T = \frac{2\pi}{\omega_0}}$$