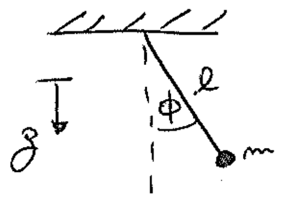
Auxiliar 28

P1)



Encontrar ecuación de movimiento

Sol: En cilíndricas: $\varphi = \phi$

$$K = \frac{m}{2} v^2 = \frac{m}{2} \ell^2 \dot{\phi}^2$$

$$U = -mg\ell\cos\phi \quad \Rightarrow \quad \mathcal{L} = K - U = \frac{m}{2}\ell^2\dot{\phi}^2 + mg\ell\cos\phi$$

Euler-Lagrange:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(m\ell^2\dot{\phi}\right) + mg\ell \operatorname{sen}\phi = 0$$

$$m\ell^2\ddot{\phi} + mg\ell \operatorname{sen}\phi = 0 \quad \Rightarrow \quad \boxed{\ddot{\phi} + \frac{g}{\ell}\operatorname{sen}\phi = 0}$$

¿Con cuántas coordenadas debo describir el sistema?

$$n = d \cdot \#\text{partículas} - \#\text{restricciones}$$

(d: dimensión)

En el problema: $d = 2$

$\#\text{part} = 1$

$\#\text{restr.} = 1 \; \} \; x^2 + y^2 = \ell^2 \quad (\rho = \ell)$

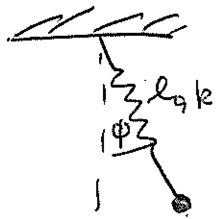
$\Rightarrow$ el sistema se describe en <u>una</u> coordenada φ

Las coordenadas q_i que describen el sistema se llaman <u>coordenadas generalizadas.</u>

Los pasos para los problemas con lagrangiano:

1) Imponer restricciones
2) Darse coordenadas generalizadas
3) Lagrangiano
4) Euler-Lagrange

P2



Ec. de mov:

1) Restricciones: No hay $\Rightarrow n = \underset{\substack{\uparrow \\ d}}{2} \cdot \underset{\substack{\uparrow \\ \#\,part}}{1} - \underset{\substack{\downarrow \\ \#\,restr.}}{0} = 2$

2) $q_1 = \rho \qquad q_2 = \phi$

3) $K = \frac{m}{2} v^2 = \frac{m}{2}(\dot\rho^2 + \rho^2\dot\phi^2)$

$\vec v = \dot\rho\hat\rho + \rho\dot\phi\hat\phi$

$U = -mg\cos\phi + \frac{k}{2}(\rho - l_0)^2$

$$\mathcal{L} = \frac{m}{2}(\dot\rho^2 + \rho^2\dot\phi^2) + \rho mg\cos\phi - \frac{k}{2}(\rho - l_0)^2$$

4) Son dos coordenadas, luego son 2 Euler-Lagrange:

$\hat\rho$: $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\rho}\right) - \frac{\partial\mathcal{L}}{\partial\rho} = 0$

$m\ddot\rho - m\rho\dot\phi^2 + k(\rho - l_0) - mg\cos\phi = 0$

$\rightarrow \boxed{\ddot\rho - \rho\dot\phi^2 + \frac{k}{m}(\rho - l_0) - g\cos\phi = 0}$

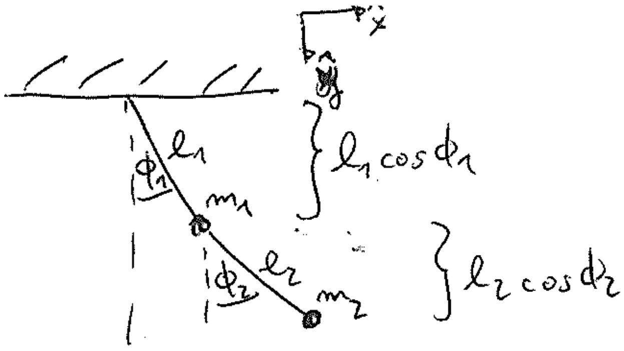
$\hat\phi$: $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$

$\frac{d}{dt}(m\rho^2\dot\phi) + mg\rho\,\text{sen}\,\phi = 0$

$m\rho^2\ddot\phi + 2m\rho\dot\rho\dot\phi + mg\rho\,\text{sen}\,\phi = 0$

$\rightarrow \boxed{\rho\ddot\phi + 2\dot\rho\dot\phi + g\,\text{sen}\,\phi = 0}$

P3



Ec. de movi

Sol: 1) $n = \underset{d}{2}\cdot\underset{\#\text{part}}{2} - \underset{\#\text{restricciones}}{2} = 2$

2) $q_1 = \phi_1 \quad , q_2 = \phi_2$

3) $U = -m_1 g l_1 \cos\phi_1 - m_2 g (l_2 \cos\phi_2 + l_1 \cos\phi_1)$

$K = K_1 + K_2$

$K_1 = \frac{m_1}{2} l_1^2 \dot{\phi}_1^2$

el K_2 es complicado

$\vec{r}_2 = (l_1 \operatorname{sen}\phi_1 + l_2 \operatorname{sen}\phi_2)\hat{x} + (l_1\cos\phi_1 + l_2\cos\phi_2)\hat{y}$

$\Rightarrow \vec{v}_2 = (l_1\cos\phi_1\dot{\phi}_1 + l_2\cos\phi_2\dot{\phi}_2)\hat{x} - (l_1\operatorname{sen}\phi_1\dot{\phi}_1 + l_2\operatorname{sen}\phi_2\dot{\phi}_2)\hat{y}$

$$\Rightarrow K_2 = \frac{m_2}{2}v_2^2 = \frac{m_2}{2}\Big[l_1^2\cos^2\phi_1\dot{\phi}_1^2 + 2 l_1 l_2 \cos\phi_1\dot{\phi}_1\cos\phi_2\dot{\phi}_2 + l_2^2\cos^2\phi_2\dot{\phi}_2^2$$
$$+ l_1^2\operatorname{sen}^2\phi_1\dot{\phi}_1^2 + 2 l_1 l_2 \operatorname{sen}\phi_1\dot{\phi}_1\operatorname{sen}\phi_2\dot{\phi}_2 + l_2^2\operatorname{sen}^2\phi_2\dot{\phi}_2^2\Big] =$$
$$= \frac{m_2}{2}\Big[l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2 l_1 l_2 (\underbrace{\cos\phi_1\cos\phi_2 + \operatorname{sen}\phi_1\operatorname{sen}\phi_2}_{\cos(\phi_1-\phi_2)})\dot{\phi}_1\dot{\phi}_2\Big] =$$
$$= \frac{m_2}{2}(l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2 l_1 l_2 \cos(\phi_1-\phi_2))$$

$$\Rightarrow \mathcal{L} = \frac{m_1}{2}l_1^2\dot{\phi}_1^2 + \frac{m_2}{2}(l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2 l_1 l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2) + m_1 g l_1\cos\phi_1 + m_2 g l_2 \cos$$
$$+ m_2 g l_1 \cos\phi_1$$

4) Euler-Lagrange:

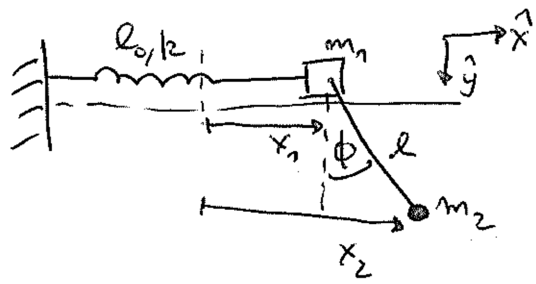
ϕ_1: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1}\right) = \frac{\partial \mathcal{L}}{\partial \phi_1}$

$$(m_1+m_2)l_1^2\ddot{\phi}_1 + m_2 l_2\cos(\phi_1-\phi_2)\ddot{\phi}_2 + m_2 l_2 \operatorname{sen}(\phi_1-\phi_2)\dot{\phi}_2^2 + (m_1+m_2) g \operatorname{sen}\phi_1 = 0$$

$\phi_2: \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi_2}\right) - \frac{\partial \mathcal{L}}{\partial \phi_2} = 0$

$$m_2 l_2 \ddot\phi_2 + m_2 l_1 \cos(\phi_1-\phi_2)\ddot\phi_1 - m_2 l_1 \operatorname{sen}(\phi_1-\phi_2)\dot\phi_1^2 + m_2 g \operatorname{sen}\phi_2 = 0$$

P4



a) Ecs. de movimiento

b) Frecuencias de oscilación

__Sol:__ Las coordenadas son: x_1, ϕ

• $U = \frac{k}{2}x_1^2 - m_2 g l \cos\phi$

Necesitamos la velocidad de m_2:

$$\vec{r}_2 = x_1\hat{x} + l\operatorname{sen}\phi\,\hat{x} + l\cos\phi\,\hat{y}$$

$$\vec{v}_2 = (\dot{x}_1 + l\cos\phi\,\dot\phi)\,\hat{x} - l\operatorname{sen}\phi\,\dot\phi\,\hat{y}$$

• $K = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\left(\dot{x}_1^2 + 2\dot{x}_1 l\cos\phi\,\dot\phi + l^2\cos^2\phi\,\dot\phi^2 + l^2\operatorname{sen}^2\phi\,\dot\phi^2\right) =$

$$= \frac{m_1+m_2}{2}\dot{x}_1^2 + \frac{m_2}{2}l^2\dot\phi^2 + m_2 l\cos\phi\,\dot\phi\,\dot{x}_1$$

$$\Rightarrow \mathcal{L} = \frac{m_1+m_2}{2}\dot{x}_1^2 + \frac{m_2}{2}l^2\dot\phi^2 + m_2 l\cos\phi\,\dot\phi\,\dot{x}_1 - \frac{k}{2}x_1^2 + m_2 g\, l\cos\phi$$

usando E-L:

$\hat\phi: \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) = \frac{d}{dt}\left(m_2 l^2\dot\phi + m_2 l\cos\phi\,\dot{x}_1\right) = m_2 l^2\ddot\phi + m_2 l\cos\phi\,\ddot{x}_1 - m_2 l\operatorname{sen}\phi\,\dot\phi\,\dot{x}_1$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m_2 l\operatorname{sen}\phi\,\dot\phi\,\dot{x}_1 - m_2 g\, l\operatorname{sen}\phi$$

$$\Rightarrow \boxed{\ddot\phi + \frac{\ddot{x}_1}{l}\cos\phi = -\frac{g}{l}\operatorname{sen}\phi}$$

$\hat{x}_1: \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) = \frac{d}{dt}\left((m_1+m_2)\dot{x}_1 + m_2 l\cos\phi\,\dot\phi\right) = (m_1+m_2)\ddot{x}_1 - m_2 l\operatorname{sen}\phi\,\dot\phi^2 + m_2 l\cos$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -k x_1$$

$$\Rightarrow \ddot{x}_1 + \frac{m_2}{m_1+m_2}l\cos\phi\,\ddot\phi - \frac{m_2}{m_1+m_2}l\operatorname{sen}\phi\,\dot\phi^2 = \frac{-k}{m_1+m_2}x_1$$

b) Vamos a tomar:

$\operatorname{sen}\phi \sim \phi$

$\cos\phi \sim 1$

$\dot{\phi}^2 \sim 0$

$$\Rightarrow (1).\quad \ddot{\phi} + \underbrace{\frac{\ddot{x}_1}{\ell}}_{\ddot{\alpha}} = -\omega_p^2 \phi \qquad \Big\}\ \omega_p^2 = \frac{g}{\ell}$$

$$(2).\quad \ddot{\phi} + \underbrace{\frac{(m_1+m_2)}{m_2}}_{A} \underbrace{\frac{\ddot{x}_1}{\ell}}_{\ddot{\alpha}} = -\omega_r^2 \underbrace{\frac{x_1}{\ell}}_{\alpha} \qquad \Big\}\ \omega_r^2 = \frac{k}{m_2}$$

$$\ddot{\phi} + A\ddot{\alpha} = -\omega_r^2 \alpha$$

Resto: $(1) - \frac{(2)}{A}$:

$$\underbrace{\left(1-\frac{1}{A}\right)}_{\frac{m_1}{m_1+m_2}} \ddot{\phi} = -\omega_p^2 \phi + \frac{\omega_r^2}{A}\alpha \quad \Big\}\quad \ddot{\phi} = -\left(\frac{(m_1+m_2)}{m_1}\omega_p^2 \phi - \frac{m_1 m_2}{(m_1+m_2)^2}\omega_r^2 \alpha\right)$$

Resto: $(1) - (2)$:

$$\underbrace{(1-A)}_{-\frac{m_1}{m_2}} \ddot{\alpha} = -\omega_p^2 \phi + \omega_r^2 \alpha \quad \Big\}\quad \ddot{\alpha} = -\left(-\frac{m_2}{m_1}\phi\omega_p^2 + \frac{m_2}{m_1}\alpha\omega_r^2\right) \qquad (4)$$

Escribo (3) y (4) de forma matricial:

$$\frac{d^2}{dt^2}\underbrace{\begin{pmatrix}\phi\\ \alpha\end{pmatrix}}_{y} = -\underbrace{\begin{pmatrix} \frac{m_1+m_2}{m_1}\omega_p^2 & -\frac{m_1 m_2}{(m_1+m_2)^2}\omega_r^2 \\ -\frac{m_2}{m_1}\omega_p^2 & \frac{m_2}{m_1}\omega_r^2 \end{pmatrix}}_{\mathbb{M}} \begin{pmatrix}\phi\\ \alpha\end{pmatrix}$$

Para simplificar, asumiremos $m_1 = m_2$:

$$\mathbb{M} = \begin{pmatrix} 2\omega_p^2 & -\frac{\omega_r^2}{4} \\ -\omega_p^2 & \omega_r^2 \end{pmatrix}$$

$$\rightarrow \det\begin{pmatrix} 2\omega_p^2-\omega^2 & -\frac{\omega_r^2}{4} \\ -\omega_p^2 & \omega_r^2-\omega^2 \end{pmatrix} = \underbrace{(2\omega_p^2-\omega^2)(\omega_r^2-\omega^2)}_{2\omega_p^2\omega_r^2 - 2\omega_p^2\omega^2 - \omega_r^2\omega^2 + \omega^4} - \frac{\omega_p^2\omega_r^2}{4} = 0$$

$$2\omega_p^2\omega_r^2 - 2\omega_p^2\omega^2 - \omega_r^2\omega^2 + \omega^4 - \frac{\omega_p^2\omega_r^2}{4} = 0$$

$$\rightarrow \quad \omega^4 - 2\omega_p^2\omega^2 + \frac{7\omega_p^2\omega_r^2}{4} = 0$$

resolviendo:

$$\boxed{\omega_{\pm}^2 = \omega_p^2 \pm \frac{1}{2}\sqrt{4\omega_p^4 - 7\omega_p^2\omega_r^2}}$$