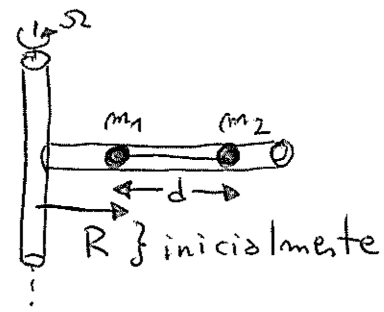
Auxiliar 5

P1



a) Ecs. de mov.

b) Encontrar $\rho_1(t)$ y $\rho_2(t)$

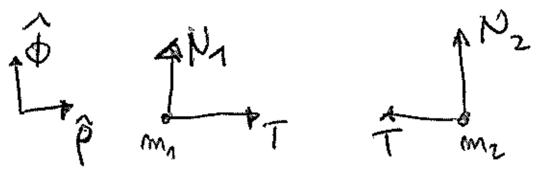
c) Tensión

Sol: a) En coordenadas cilíndricas:

$$\dot{\phi} = \Omega$$

$$\vec{r}_i = \rho_i \hat{\rho} \qquad \vec{v}_i = \dot{\rho}_i \hat{\rho} + \rho_i \Omega \hat{\phi} \qquad \vec{a}_i = (\ddot{\rho}_i - \rho_i \Omega^2)\hat{\rho} + (2\dot{\rho}_i \Omega + \cancel{\rho_i \ddot{\phi}}^{\,0})\hat{\phi}$$

DCL:



m_1: $\hat{\rho}$): $m_1(\ddot{\rho}_1 - \rho_1 \Omega^2) = T$ (1)

$\hat{\phi}$): $2m_1 \dot{\rho}_1 \Omega = N_1$ (2)

m_2: $\hat{\rho}$): $m_2(\ddot{\rho}_2 - \rho_2 \Omega^2) = -T$ (3)

$\hat{\phi}$): $2m_2 \dot{\rho}_2 \Omega = N_2$ (4)

b) Sumo (1)+(3)

$$m_1 \ddot{\rho}_1 + m_2 \ddot{\rho}_2 - \Omega^2(m_1 \rho_1 + m_2 \rho_2) = 0$$

llamo $u = m_1 \rho_1 + m_2 \rho_2$

$$\Rightarrow \ddot{u} - \Omega^2 u = 0$$

* Usamos polinomio característico para resolver, saltar si aún no se ve en EDO.

$$\frac{d^2u}{dt^2} - \Omega^2 u = 0$$

$$\lambda^2 - \Omega^2 \underbrace{\lambda^0}_{1} = 0 \Rightarrow \lambda = \pm\Omega \quad \begin{cases} \lambda_1 = \Omega \\ \lambda_2 = -\Omega \end{cases}$$

entonces la solución es:

$$u(t) = Ae^{\Omega t} + Be^{-\Omega t}$$

A y B se sacan de las condiciones iniciales.

$$u(t=0) = m_1\rho_1(t=0) + m_2\rho_2(t=0) = m_1 R + m_2(R+d) = A + B \qquad (5)$$

Derivamos u:

$$\dot{u}(t) = A\Omega e^{\Omega t} + B(-\Omega)e^{-\Omega t}$$

$$\dot{x}(t=0) = m_1 \underbrace{\dot{\rho}_1(0)}_{0} + m_2 \underbrace{\dot{\rho}_2(0)}_{0} = 0 = A\Omega - B\Omega \quad \Rightarrow \quad A = B$$

remplazando en ($):

$$2A = m_1 R + m_2 (R+d) \quad \Rightarrow A = \frac{1}{2}\left(R(m_1+m_2) + m_2 d\right)$$

finalmente remplazamos en x:

$$x(t) = \frac{1}{2}\left(R(m_1+m_2)+m_2 d\right)\left(e^{\Omega t} + e^{-\Omega t}\right) =$$

$$= \left(R(m_1+m_2)+m_2 d\right)\cosh(\Omega t) = m_1\rho_1 + m_2\rho_2$$

pero $\rho_2 = \rho_1 + d$

$$\Rightarrow \left(R(m_1+m_2)+m_2 d\right)\cosh(\Omega t) = m_1\rho_1 + m_2(\rho_1+d)$$

$$\Rightarrow \boxed{\rho_1(t) = \frac{\left(R(m_1+m_2)+m_2 d\right)\cosh(\Omega t) - m_2 d}{m_1+m_2}}$$

$$\boxed{\rho_2(t) = \frac{\left(R(m_1+m_2)+m_2 d\right)\cosh(\Omega t) + m_1 d}{m_1+m_2}}$$

b) Para la tensión usamos la ec. (1):

$$m_1(\ddot{\rho}_1 - \rho_1\Omega^2) = T$$

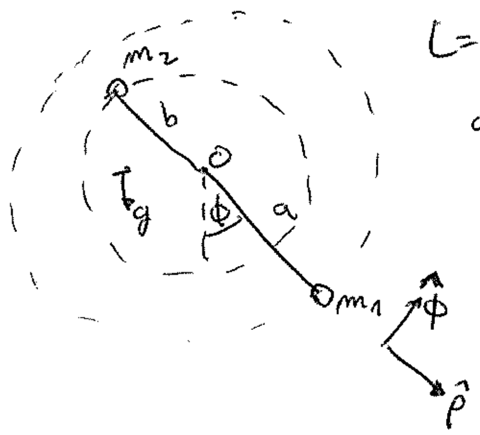
necesitamos $\ddot{\rho}_1$:

$$\dot{\rho}_1 = \frac{\left(R(m_1+m_2)+m_2 d\right)\Omega \operatorname{senh}(\Omega t)}{m_1+m_2}$$

$$\ddot{\rho}_1 = \frac{\left(R(m_1+m_2)+m_2 d\right)\Omega^2\cosh(\Omega t)}{m_1+m_2} = \rho_1\Omega^2 + \frac{m_2 d\Omega^2}{m_1+m_2}$$

$$\Rightarrow \boxed{m_1\left(\frac{m_2 d\Omega^2}{m_1+m_2}\right) = T}$$

P2



$L = a + b$

a) $\vec{\ell}$ y $\vec{\tau}$

b) Ecuación para ϕ e integrar

c) $\phi(0) \sim 0$, ¿se aleja o se acerca de $\phi = 0$?

Sol:

a) $\vec{r}_1 = a\hat{\rho}$ $\quad \vec{v}_1 = a\dot{\phi}\hat{\phi}$

$\vec{r}_2 = -b\hat{\rho}$ $\quad \vec{v}_2 = -b\dot{\phi}\hat{\phi}$

usando la definición de momentum angular:

$$\vec{\ell}_0 = \sum_i m_i (\vec{r}_i \times \vec{v}_i) = m_1 \underbrace{a\hat{\rho} \times a\dot{\phi}\hat{\phi}}_{\hat{z}} + m_2 \underbrace{b\hat{\rho} \times b\dot{\phi}\hat{\phi}}_{\hat{z}} =$$

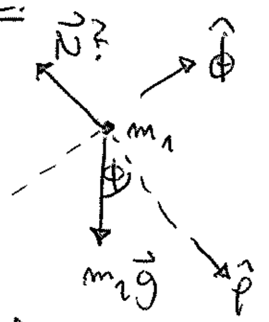
$$= m_1 a^2 \dot{\phi}\hat{z} + m_2 b^2 \dot{\phi}\hat{z}$$

$$\Rightarrow \boxed{\vec{\ell}_0 = \dot{\phi}(m_1 a^2 + m_2 b^2)\hat{z}}$$ } Suma de mom. angulares separados

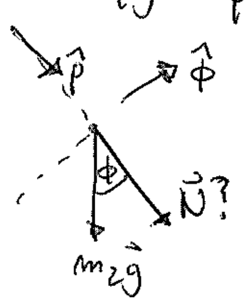
Ahora el torque:

$$\vec{\tau}_0 = \sum_i \vec{r}_i \times \vec{F}_i = a\hat{\rho} \times \vec{F}_1 - b\hat{\rho} \times \vec{F}_2$$ } sólo fuerzas en $\hat{\phi}$ y $\hat{z}$ hacen torque

DCL:



$$m_1\vec{g} = (\cos\phi\,\hat{\rho} - \text{sen}\phi\,\hat{\phi})m_1 g$$



$$m_2\vec{g} = (\cos\phi\,\hat{\rho} - \text{sen}\phi\,\hat{\phi})m_2 g$$

$$\Rightarrow \vec{\tau}_0 = -am_1 g\,\text{sen}\phi\,\hat{z} + bm_2 g\,\text{sen}\phi\,\hat{z} \Rightarrow \boxed{\vec{\tau}_0 = g\,\text{sen}\phi(m_2 b - m_1 a)\hat{z}}$$

b) La evolución viene de:

$$\dot{\vec{\ell}}_o = \vec{\tau}_o$$

$$\ddot{\phi}(m_1 a^2 + m_2 b^2) = g \operatorname{sen}\phi\, (m_2 b - m_1 a)$$

$$\Rightarrow \boxed{\ddot{\phi} = g \operatorname{sen}\phi \frac{m_2 b - m_1 a}{m_1 a^2 + m_2 b^2}}$$

usando: $\ddot{\phi} = \dot{\phi}\frac{d\dot{\phi}}{d\phi}$

$$\int_0^{\dot{\phi}} \dot{\phi}\, d\dot{\phi} = g \frac{m_2 b - m_1 a}{m_1 a^2 + m_2 b^2} \int_0^{\phi} d\phi \operatorname{sen}\phi$$

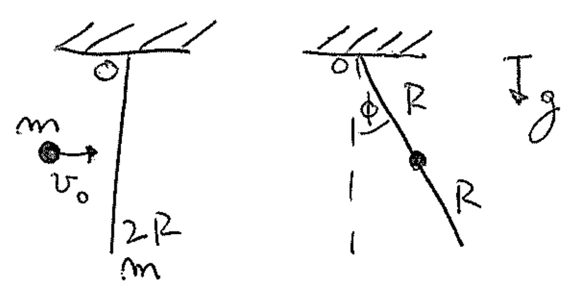
$$\frac{\dot{\phi}^2}{2} = g \frac{m_2 b - m_1 a}{m_1 a^2 + m_2 b^2} (-\cos\phi + \underbrace{\cos 0}_{1})$$

$$\Rightarrow \boxed{\dot{\phi}^2 = 2g \frac{m_2 b - m_1 a}{m_1 a^2 + m_2 b^2} (1 - \cos\phi)}$$

c) Si $m_2 b < m_1 a \Rightarrow$ se acerca

si $m_2 b > m_1 a \Rightarrow$ se aleja

P3



a) $\vec{l}$ antes de la colisión (justo antes)

b) $\vec{l}$ en un momento cualquier después de la colisión en términos de $\dot{\phi}$

c) $\dot{\phi}(\phi)$

Sol: a) Antes de la colisión sólo la partícula se mueve, luego es la única con momentum angular

$$\vec{l}_o = m\vec{r}\times\vec{v} = mR\hat{\rho}\times v_o\hat{\phi} = \boxed{mRv_o\hat{z} = \vec{l}_o}$$

b) Hay que sumar el momentum de la masa y la barra:

$$\vec{l}_o = \underbrace{m\vec{r}\times\vec{v}}_{\text{partícula}} + \int\vec{r}\times\vec{v}(\vec{r})\,dm(\vec{r})$$

$$\underbrace{}_{}\; mR\hat{\rho}\times\dot{\phi}R\hat{\phi} = mR^2\dot{\phi}\hat{z}$$

vemos la integral:

$$\int\vec{r}\times\vec{v}(\vec{r})\,dm(\vec{r}) \underset{\uparrow}{=} \int_0^{2R}\rho\hat{\rho}\times\rho\dot{\phi}\hat{\phi}\,\lambda d\rho = \frac{m}{2R}\int_0^{2R}\rho^2\dot{\phi}\hat{z}\,d\rho\,\hat{z} =$$

$$\lambda = \frac{m}{2R}\;\Big\}\; dm = \lambda d\rho$$

$$= \frac{m}{\cancel{2R}}\frac{(2R)^{\cancel{3}\,2}}{3}\dot{\phi}\hat{z} = \frac{4}{3}mR^2\dot{\phi}\hat{z} = I\dot{\phi}\hat{z}$$

el momento de inercia de una barra que gira en torno a su extremo es:

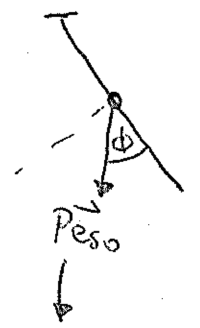
$$L\;\Big\}\; I = \frac{1}{3}ML^2 \overset{\text{en nuestro caso}}{=} \frac{1}{3}m(2R)^2 = \frac{4}{3}mR^2 \;\Big\}\; \text{está bien}$$

entonces: $\boxed{\vec{l}_o = \frac{7}{3}mR^2\dot{\phi}\hat{z}}$

c) Recordamos que:

$$\dot{\vec{\ell}}_0 = \vec{\tau}_0$$

el torque es:



$$\vec{\tau}_0 = \underbrace{\vec{r} \times \vec{F}}_{R\hat{\rho} \times (\cos\phi\hat{\rho} - \operatorname{sen}\phi\hat{\phi})mg = -mgR\operatorname{sen}\phi\hat{z}} + \int d\vec{r} \times \vec{F}(\vec{r})$$

$\vec{F} = mg\cos\phi\hat{\rho} - mg\operatorname{sen}\phi\hat{\phi}$

la integral:

$$\int_0^{2R} \rho\hat{\rho} \times (dm\, g\cos\phi\hat{\rho} - dm\, g\operatorname{sen}\phi\hat{\phi}) \underset{dm = \lambda d\rho}{=} -\lambda g \int_0^{2R} \rho \operatorname{sen}\phi\, d\rho\, \hat{z} =$$

$$= -\frac{m}{\cancel{2R}} g \frac{(2R)^{\cancel{2}}}{2} \operatorname{sen}\phi\hat{z} = -mgR\operatorname{sen}\phi\hat{z}$$

} la barra actúa como una partícu el peso actúa en el centro de mas

$$\Rightarrow \vec{\tau}_0 = -2mgR\operatorname{sen}\phi\hat{z}$$

Ahora $\dot{\vec{\ell}}_0$:

$$\dot{\vec{\ell}}_0 = \frac{7}{3} m R^2 \ddot{\phi}\hat{z}$$

finalmente: $\frac{7}{3}\cancel{m}R^2\ddot{\phi} + 2\cancel{m}gR\operatorname{sen}\phi = 0$

$$\ddot{\phi} + \frac{g}{R}\cdot\frac{6}{7}\operatorname{sen}\phi = 0$$

$$\dot{\phi}\frac{d\dot{\phi}}{d\phi}$$

$$\underbrace{\int_{\dot{\phi}_0}^{\dot{\phi}} \dot{\phi}\, d\dot{\phi}}_{\frac{\dot{\phi}^2}{2} - \frac{\dot{\phi}_0^2}{2}} = -\frac{6}{7}\frac{g}{R} \underbrace{\int_0^{\phi} \operatorname{sen}\phi}_{1-\cos\phi}$$

¿Cual es $\dot{\phi}_0$?

usamos conservación del momentum angular:

$$\vec{l}_0(t=0^-) = \vec{l}_0(t=0^+)$$

$$\cancel{m} R v_0 = \frac{7}{3}\cancel{m} R^2 \dot{\phi}_0 \Rightarrow \dot{\phi}_0 = \frac{3}{7}\frac{v_0}{R}$$

$$\Rightarrow \boxed{\dot{\phi}^2(\phi) = \left(\frac{3}{7}\frac{v_0}{R}\right)^2 - \frac{12}{7}\frac{g}{R}(1-\cos\phi)}$$