P1 | Encuentre el conmutador $[\hat{x}, \hat{L}]$:

Sol: Dado que $\hat{L} = \hat{r} \times \hat{p}$:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \quad , \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \quad , \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

Entonces:

- $[\hat{x}, \hat{L}_x] = 0$
- $[\hat{x}, \hat{L}_y] = \hat{z}\underbrace{[\hat{x}, \hat{p}_x]}_{i\hbar} = i\hbar\hat{z}$
- $[\hat{x}, \hat{L}_z] = -\hat{y}[\hat{x}, \hat{p}_x] = -i\hbar\hat{y}$
- $[\hat{y}, \hat{L}_x] = -i\hbar\hat{z}$
- $[\hat{y}, \hat{L}_y] = 0$
- $[\hat{y}, \hat{L}_z] = i\hbar\hat{x}$
- $[\hat{z}, \hat{L}_x] = i\hbar\hat{y}$
- $[\hat{z}, \hat{L}_y] = -i\hbar\hat{x}$
- $[\hat{z}, \hat{L}_z] = 0$

De manera general: $[x_i, L_j] = \varepsilon_{ijk}\, i\hbar\, x_k$

P2 Muestre que;

$$(m-m')\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \text{sen}\,\theta \, Y^*_{l'm'}(\theta\phi) \cos\theta \, Y_{lm}(\theta,\phi) = 0$$

y luego que:

$$(m-m')^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \text{sen}\,\theta \, Y^*_{l'm'}(\theta,\phi) \, \text{sen}\,\theta \cos\phi \, Y_{lm}(\theta,\phi) =$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \text{sen}\,\theta \, Y^*_{l'm'}(\theta,\phi) \, \text{se}\,\theta \cos\phi \, Y_{lm}(\theta,\phi)$$

Usando el conmutador $[\hat{L}_z, \hat{x}_i]$

Sol: Primero sacamos elementos de matriz:

- $\langle l'm'|[\hat{L}_z,\hat{x}]|l\,m\rangle = i\hbar \langle l'm'|\hat{y}|l\,m\rangle \qquad (1a)$
- $\langle l'm'|[\hat{L}_z,\hat{y}]|l\,m\rangle = -i\hbar \langle l'm'|\hat{x}|l\,m\rangle \qquad (1b)$
- $\langle l'm'|[\hat{L}_z,\hat{z}]|l\,m\rangle = 0 \qquad (1c)$

Por otra parte:

- $\langle l'm'|[\hat{L}_z,\hat{x}]|l\,m\rangle = \langle l'm'|[\hat{L}_z\hat{x} - \hat{x}\hat{L}_z]|l\,m\rangle =$

$$= (m'-m)\langle l'm'|\hat{x}|l\,m\rangle \qquad (2a)$$

- $\langle l'm'|[\hat{L}_z,\hat{y}|l\,m\rangle = (m'-m)\langle l'm'|\hat{y}|l\,m\rangle \qquad (2b)$
- $\langle l'm'|[\hat{L}_z,\hat{z}|l\,m\rangle = (m'-m)\langle l'm'|\hat{z}|l\,m\rangle \qquad (2c)$

Igualando (1c) con (2c):

$$(m'-m)\langle l'm'|\hat{z}|l\,m\rangle = 0$$

$$(m'-m)\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \text{se}\,\theta \underbrace{\langle l'm'|z|\theta\,\phi\rangle}_{Y^*_{l'm'}\cos\theta} \underbrace{\langle\theta\phi|l\,m\rangle}_{Y_{lm}} = 0$$

de manera análoga:

$$(m'-m)\langle l'm'|\hat{x}|lm\rangle = i\hbar\langle l'm'|\hat{y}|lm\rangle$$

$$(m'-m)\langle l'm'|\hat{y}|lm\rangle = -i\hbar\langle l'm'|\hat{x}|lm\rangle$$

juntando ambos:

$$(m'-m)\underbrace{(m'-m)\langle l'm'|\hat{x}|lm\rangle/i\hbar}_{\langle l'm'|\hat{y}|lm\rangle} = -i\hbar\langle l'm|\hat{x}|lm\rangle$$

$$(m'-m)^2\langle l'm'|\hat{x}|lm\rangle = \hbar^2\langle l'm'|\hat{x}|lm\rangle$$

Completado con $\oint d\phi\int d\theta\,|\theta\phi\rangle\langle\phi\theta|$ se tiene:

$$(m'-m)^2\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\text{sen}\,\theta\,Y^*_{l'm'}\,\text{sen}\,\theta\cos\phi\,Y_{lm} = \int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\text{sen}\,\theta\cos\theta\,Y^*_{l'm'}Y_{lm}$$

P3 Se tiene el estado:

$$|\psi\rangle = a|1\ 1\rangle + b|1\ 0\rangle + c|1\ {-1}\rangle$$ con la notación $|l\ m\rangle$

donde $|a|^2+|b|^2+|c|^2=1$

a) Encuentre $\langle L_x\rangle$

b) Encuentre $\langle L^2\rangle$

c) Encuentre a, b, c para que $\hat{L}_x|\psi\rangle = |\psi\rangle$

Sol: a) Dado que $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y \rightarrow \hat{L}_x = \frac{1}{2}\hat{L}_+ + \frac{1}{2}\hat{L}_-$

El valor de expectación $\langle \hat{L}_x\rangle = \frac{1}{2}\langle \hat{L}_+\rangle + \frac{1}{2}\langle \hat{L}_-\rangle$

- $\langle \hat{L}_+\rangle = \langle\psi|\hat{L}_+|\psi\rangle = \langle\psi|\left[a\overset{0}{\cancel{\sqrt{2-2}}}|1\ 2\rangle + b\sqrt{2}|1\ 1\rangle + c\sqrt{2}|1\ 0\rangle\right]$

$$= a^*b\sqrt{2} + b^*c\sqrt{2} = \sqrt{2}(a^*b + b^*c)$$

- $\langle \hat{L}_-\rangle = \langle\psi|\hat{L}_-|\psi\rangle = \langle\psi|\left[a\sqrt{2}|1\ 0\rangle + b\sqrt{2}|1\ {-1}\rangle + c\overset{0}{\cancel{\sqrt{2-2}}}|1\ {-2}\rangle\right]$

$$= b^*a\sqrt{2} + c^*b\sqrt{2} = \sqrt{2}(b^*a + c^*b)$$

Entonces:

$$\langle \hat{L}_x\rangle = \frac{\sqrt{2}}{2}\left(\underbrace{a^*b + b^*a}_{2\mathrm{Re}(a^*b)} + \underbrace{b^*c + c^*b}_{2\mathrm{Re}(b^*c)}\right)$$

$$\boxed{\langle \hat{L}_x\rangle = \sqrt{2}\left(\mathrm{Re}(a^*b) + \mathrm{Re}(b^*c)\right)}$$

b) Es más directo verlo:

$$\langle \hat{L}^2\rangle = \langle\psi|\hat{L}^2|\psi\rangle = \langle\psi|\left[2a|1\ 1\rangle + 2b|1\ 0\rangle + 2c|1\ {-1}\rangle\right] =$$

$$= 2\underbrace{(a^*a + b^*b + c^*c)}_{1} \Rightarrow \boxed{\langle \hat{L}^2\rangle = 2}$$

c) $$\hat{L}_x |\psi\rangle = |\psi\rangle$$

$$\frac{1}{2}(\hat{L}_+ + \hat{L}_-)|\psi\rangle = |\psi\rangle$$

$$\frac{1}{2}\left[b\sqrt{2}|1\ 1\rangle + c\sqrt{2}|1\ 0\rangle + a\sqrt{2}|1\ 0\rangle + b\sqrt{2}|1\ -1\rangle\right] = a|1\ 1\rangle + b|1\ 0\rangle + c|1\ -1\rangle$$

$$\rightarrow \frac{b\sqrt{2}}{2} = a \Rightarrow b = \sqrt{2}a$$

$$\rightarrow \frac{b\sqrt{2}}{2} = c \Rightarrow b = \sqrt{2}c$$

Normalizamos:

$$|a|^2 + |b|^2 + |c|^2 = b^2\left[\frac{1}{2} + 1 + \frac{1}{2}\right] = 2b^2 = 1 \Rightarrow \boxed{b = \frac{1}{\sqrt{2}}}$$

Entonces: $\boxed{a = \frac{1}{2}}$, $\boxed{c = \frac{1}{2}}$

P4 La función de onda de una partícula sujeta a un potencial central $V(r)$ es:

$$\psi(\vec{x}) = (x+y+3z)f(r)$$

a) ¿ψ es autofunción de $\hat{L}^2$? Si lo es, encuentre ℓ. Si no lo es, ¿qué valores posibles de ℓ se podrían obtener si se mide $\hat{L}^2$?

b) Encuentre la probabilidad de encontrar los estados m posibles

Sol: Se tiene que:

$$\hat{L}^2 = -\left[\frac{1}{\operatorname{sen}^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial}{\partial\theta}\right)\right]$$

Escribiremos ψ en esféricas:

$$\psi(\vec{x}) = r\left(\cos\phi\operatorname{sen}\theta + \operatorname{sen}\phi\operatorname{sen}\theta + 3\cos\theta\right)f(r)$$

Como queremos estudiar $\hat{L}^2|\psi\rangle \to \langle\vec{x}|\hat{L}^2|\psi\rangle = \hat{L}^2\psi(\vec{x})$

- $$\frac{1}{\operatorname{sen}^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = rf(r)\frac{\partial}{\partial\phi}\left[\cos\phi - \operatorname{sen}\phi\right]\frac{\cancel{\operatorname{sen}\theta}}{\operatorname{sen}^{\cancel{2}}\theta} = \frac{rf(r)}{\operatorname{sen}\theta}\left[-\operatorname{sen}\phi - \cos\phi\right]$$

$$= -\frac{rf(r)}{\operatorname{sen}\theta}(\operatorname{sen}\phi + \cos\phi)$$

- $$\frac{1}{\operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial}{\partial\theta}\right)\psi = \frac{1}{\operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left[\operatorname{sen}\theta\left(\cos\phi\cos\theta + \operatorname{sen}\phi\cos\theta - 3\operatorname{sen}\theta\right)\right]rf(r) =$$

$$= \frac{rf(r)}{\operatorname{sen}\theta}\left[(\cos\phi + \operatorname{sen}\phi)\underbrace{(\cos^2\theta - \operatorname{sen}^2\theta)}_{1-2\operatorname{sen}^2\theta} - 6\operatorname{sen}\theta\cos\theta\right]$$

entonces:

$$L^2\psi = \frac{rf(r)}{\operatorname{sen}\theta}\left[\cancel{(\operatorname{sen}\phi+\cos\phi)} - (\operatorname{sen}\phi+\cos\phi)(\cancel{1}-2\operatorname{sen}^2\theta) + 6\operatorname{sen}\theta\cos\theta\right] =$$

$$= rf(r)\left[6\cos\theta + 2\operatorname{sen}\theta(\cos\phi+\operatorname{sen}\phi)\right] =$$

$$= 2r\left[\cos\phi\operatorname{sen}\theta + \operatorname{sen}\phi\operatorname{sen}\theta + 3\cos\theta\right]f(r) = 2\psi$$

Por tanto sí es autoestado. Además:

$$l(l+1) = 2 \implies \boxed{l=1}$$

b) Dado que $l=1$, se tiene $m=-1,0,1$.

Vamos escribir $\psi(\vec{x})$ como función de esféricos armónicos:

$$Y_{1,-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\, e^{-i\phi}\operatorname{sen}\theta = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\,\frac{(x-iy)}{r}$$

$$Y_{1,0} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta = \frac{1}{2}\sqrt{\frac{3}{\pi}}\,\frac{z}{r}$$

$$Y_{1,1} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\, e^{i\phi}\operatorname{sen}\theta = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\,\frac{(x+iy)}{r}$$

de donde sacamos:

$$z = r\sqrt{\frac{4\pi}{3}}\, Y_{1,0}$$

$$x = r\sqrt{\frac{2\pi}{3}}\left(Y_{1,-1} - Y_{1,1}\right)$$

$$y = ir\sqrt{\frac{2\pi}{3}}\left(Y_{1,-1} + Y_{1,1}\right)$$

Entonces:

$$\psi = \sqrt{\frac{2\pi}{3}}\, r f(r)\left[3\sqrt{2}\, Y_{1,0} + (1+i)Y_{1,-1} + (i-1)Y_{1,1}\right]$$

dado que nos interesa la parte de m:

$$|\psi_m\rangle = A\left[3\sqrt{2}\,|0\rangle + (1+i)\,|-1\rangle + (i-1)\,|+1\rangle\right]$$

Normalizando:

$$1 = A^2(18+2+2) = A^2 22 \Rightarrow A = \frac{1}{\sqrt{22}}$$

por tanto:

$$|\psi_m\rangle = \frac{3}{\sqrt{11}}|0\rangle + \frac{(1+i)}{\sqrt{22}}|-1\rangle + \frac{(i-1)}{\sqrt{22}}|1\rangle$$

luego las probabilidades que esté en cada estado de m:

$$\boxed{P_0 = \frac{9}{11}} \quad \boxed{P_{+1} = \frac{1}{11}} \quad \boxed{P_{-1} = \frac{1}{11}}$$