AUXILIAR 1

P1 Normalizar, calcular transformada de Fourier, calcular $\langle x^2 \rangle$ y $\langle x \rangle$ de:

- $\psi(x) = A e^{-x^2/a^2}$

Para normalizar:

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = A^2 \int_{-\infty}^{\infty} e^{-2x^2/a^2} dx$$

Recordamos que: $\int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$

$$\rightarrow \quad 1 = A^2 \sqrt{\frac{a^2\pi}{2}} \quad \Rightarrow \quad \boxed{A = \left(\frac{2}{a^2\pi}\right)^{1/4}}$$

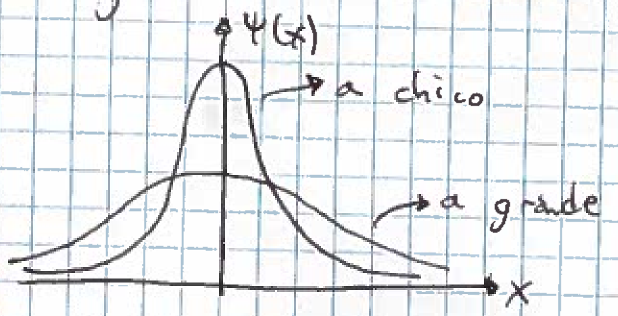
Calculamos la transformada de Fourier:

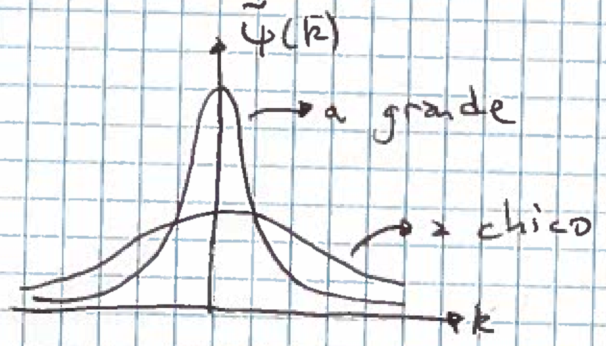
$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/a^2 - ikx} dx =$$

$$= \frac{A}{\sqrt{2\pi}} \left[\sqrt{\pi a^2}\, e^{-\frac{k^2}{4}a^2} \right]$$

$$\rightarrow \quad \boxed{\tilde{\psi}(k) = \left(\frac{a^2}{2\pi}\right)^{1/4} e^{-\frac{k^2}{4}a^2}}$$

Si graficamos:





Calculamos $\langle x \rangle$ y $\langle x^2 \rangle$:

$$\langle x \rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 x\, dx = A^2 \int_{-\infty}^{\infty} x\, e^{-2x^2/a^2} dx = 0$$

para $\langle x^2 \rangle$ usamos que: $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$

$$\rightarrow \langle x^2 \rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 x^2 dx = A^2 \int_{-\infty}^{\infty} x^2 e^{-2x^2/a^2} dx = \frac{A^2}{2}\sqrt{\frac{\pi a^6}{8}} =$$

$$= \frac{1}{2}\sqrt{\frac{\cancel{2}}{\cancel{a^2}\cancel{\pi}} \cdot \frac{\pi a^{\cancel{6}4}}{\cancel{8}4}} \Rightarrow \boxed{\langle x^2 \rangle = \frac{a^2}{4}}$$

- $\psi(x) = A e^{-ax^2+bx}$

Normalizar: $1 = A^2 \int_{-\infty}^{\infty} e^{-2ax^2+2bx} dx = A^2 e^{\frac{4b^2}{\cancel{2}8a}} \cdot \sqrt{\frac{\pi}{2a}}$

$$\rightarrow \boxed{A = \left(\frac{2a}{\pi}\right)^{1/4} e^{\frac{-b^2}{4a}}}$$

Transformada de Fourier:

$$\tilde{\psi}(k) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2+bx-ikx} dx =$$

$$= \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-b^2/4a}}{\sqrt{2\cancel{\pi}}} \sqrt{\frac{\cancel{\pi}}{a}} e^{\frac{(b-ik)^2}{4a}} =$$

$$= \left(\frac{1}{\sqrt{2\pi a}}\right)^{1/4} e^{\cancel{-b^2/4a}} \cdot e^{\frac{\cancel{b^2} - 2bik - k^2}{4a}}$$

$$\boxed{\tilde{\psi}(k) = \left(\frac{1}{2\pi a}\right)^{1/4} e^{-\frac{k^2}{4a}} e^{-\frac{2bki}{4a}}}$$

$$\langle x \rangle = A^2 \int_{-\infty}^{\infty} x e^{-ax^2+bx} dx = A^2 \int_{-\infty}^{\infty} x\, dx\, e^{-ax^2+bx+\frac{b^2}{4a}-\frac{b^2}{4a}} =$$

$$= A^2 \int_{-\infty}^{\infty} x\, dx\, e^{-(\sqrt{a}x-\frac{b}{2\sqrt{a}})^2} \cdot e^{\frac{b^2}{4a}} = \frac{A^2 e^{\frac{b^2}{4a}}}{\sqrt{a}} \int_{-\infty}^{\infty} \sqrt{a}x\, dx\, e^{-(\sqrt{a}x-\frac{b}{2\sqrt{a}})^2} =$$

$$= \frac{A^2 e^{b^2/4a}}{a} \left[\underbrace{\int_{-\infty}^{\infty} x' dx' e^{-x'^2}}_{0} + \frac{b}{2\sqrt{a}} \underbrace{\int_{-\infty}^{\infty} dx' e^{-x'^2}}_{} \right] =$$

$x' = \sqrt{a}x - \frac{b}{2\sqrt{a}}$

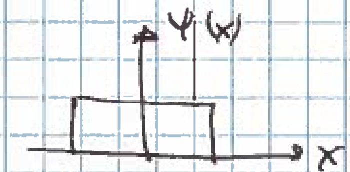
$$= \sqrt{\frac{2\cancel{a}}{\pi}}\, e^{-\frac{b^2}{2a}}\, \frac{e^{b^2/4a}}{a} \cdot \frac{b}{2\sqrt{\cancel{a}}} \cdot \sqrt{\cancel{\pi}}$$

$$\rightarrow \boxed{\langle x \rangle = \frac{b}{a\sqrt{2}}\, e^{-b^2/4a}}$$

$$\langle x^2 \rangle = A^2 \int_{-\infty}^{\infty} x^2\, dx\, e^{-ax^2+bx} = \frac{1}{a\sqrt{2}}\, e^{-\frac{b^2}{4a}} \left(1 + \frac{b^2}{2a}\right)$$

Tarea

$$\bullet\ \psi(x) = \begin{cases} A & |x| \le a/2 \\ 0 & \sim \end{cases}$$

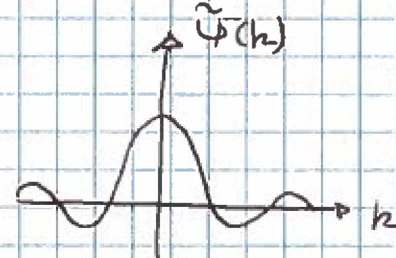


Normalizar: $1 = A^2 \int_{-a/2}^{a/2} dx = A^2 a \Rightarrow \boxed{A = \frac{1}{\sqrt{a}}}$

Transformada de Fourier: $\tilde{\psi}(k) = \frac{A}{\sqrt{2\pi}} \int_{-a/2}^{a/2} e^{-ikx}\, dx = \frac{A}{\sqrt{2\pi}} \frac{1}{(-ik)} \left[e^{-ik\frac{a}{2}} - e^{ik\frac{a}{2}} \right]$

$$= \frac{A}{\sqrt{2\pi}} \frac{1}{k} \frac{e^{ik\frac{a}{2}} - e^{-ik\frac{a}{2}}}{i}$$

$$\Rightarrow \boxed{\tilde{\psi}(k) = \frac{2}{\sqrt{2\pi a}} \frac{\operatorname{sen}\left(k\frac{a}{2}\right)}{k}}$$



$$\langle x \rangle = A \int_{-a/2}^{a/2} x\, dx = 0$$

$$\langle x^2 \rangle = A \int_{-a/2}^{a/2} x^2\, dx = \frac{2}{3\sqrt{a}} \frac{a^3}{8} = \frac{\sqrt{a^5}}{12}$$

P2 Velocidad de un $\bar{e}$ a 1 eV y de un p^+ a 1 MeV?

$\hbar c \approx 197$ MeV fm $= 197$ eV nm

$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$

$m_p c^2 \approx 938$ MeV

$m_e c^2 \approx 0.511$ MeV

Usamos: $K = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$

$$\rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{\left(1+\frac{K}{mc^2}\right)^2}$$

$$\approx 1 - \frac{1}{\left(1+\underbrace{\frac{1\,eV}{0.5\,MeV}}_{\sim\frac{1}{0.5\cdot 10^6}}\right)^2} \approx 1 - \frac{1}{(1+2\cdot 10^{-6})^2} \approx 4\cdot 10^{-6} \ll 1$$

en cambio, con 1 MeV:

$$\approx 1 - \frac{1}{\underbrace{\left(1+\underbrace{\frac{1\,MeV}{0.5\,MeV}}_{2}\right)^2}_{\frac{1}{9}}} = \frac{8}{9} \sim 1 \quad \Big\}\ \text{relativista}$$

Ahora con el protón:

$$\approx 1 - \frac{1}{\left(1+\frac{1\,MeV}{940\,MeV}\right)^2} \approx 1\cdot 10^{-3} \ll 1$$